The Annals of Applied Probability
2015, Vol. 25, No. 6, 3571–3591
DOI: 10.1214/14-AAP1082

ASYMPTOTIC DISTRIBUTION OF THE MAXIMUM INTERPOINT DISTANCE IN A SAMPLE OF RANDOM VECTORS WITH A SPHERICALLY SYMMETRIC DISTRIBUTION

BY SREENIVASA RAO JAMMALAMADAKA AND SVANTE JANSON[1]

University of California and Uppsala University

Extreme value theory is part and parcel of any study of order statistics in one dimension. Our aim here is to consider such large sample theory for the maximum distance to the origin, and the related maximum "interpoint distance," in multidimensions. We show that for a family of spherically symmetric distributions, these statistics have a Gumbel-type limit, generalizing several existing results. We also discuss the other two types of limit laws and suggest some open problems. This work complements our earlier study on the minimum interpoint distance.

1. Introduction and main results. Let $X_1, X_2, \dots$ be an independently and identically distributed (i.i.d.) sequence of random vectors in $\mathbb{R}^d$ with a spherically symmetric distribution, where $d \geq 2$. (See Section 5.3 for some comments on the case $d = 1$; otherwise $d \geq 2$ will always be assumed.) We are interested in the maximum interpoint distance

$$M_n^{(2)} := \max_{1 \leq i < j \leq n} |X_i - X_j|, \tag{1.1}$$

where $|\cdot|$ is the usual Euclidean distance. This has previously been studied by several authors in various special cases, including Matthews and Rukhin [14] (symmetric normal distribution), Henze and Klein [5] (Kotz distribution), Appel, Najim and Russo [2] (uniform distribution in a ball), Appel and Russo [1] (uniform distribution on a sphere) and Mayer and Molchanov [15] (e.g., uniform distribution in a ball or on a sphere). We provide some general results here for the case of unbounded random vectors from any spherically symmetric distribution, which includes the work in [14] and [5].

The results for maximum distance can be considered as complementary to the results for the minimum interpoint distance; see, for example, Jammalamadaka and Janson [7]. One important difference is that the minimum distance is typically achieved by points in the bulk of the distribution, while the maximum distance is attained by outliers. This makes the maximum distance less useful for goodness of

Received December 2012; revised November 2014.
[1]Supported in part by the Knut and Alice Wallenberg Foundation.
MSC2010 subject classifications. 60D05, 60F05, 60G70, 62E20.
Key words and phrases. Maximum interpoint distance, extreme value distributions, Gumbel distribution.

fit tests, but might be suitable for detecting outliers. Some applications are given in Matthews and Rukhin [14].

The maximum pairwise distance $M_n^{(2)}$ is clearly related to the maximum distance to the origin

$$M_n := \max_{1\le i\le n} |X_i|. \tag{1.2}$$

We obviously have $M_n^{(2)} \le 2M_n$, and it seems reasonable to guess that this bound is rather sharp; this would mean that the maximum distance (1.1) is attained by two vectors X_i and X_j that have almost maximum lengths and are almost opposite in direction.

For the case $d = 1$, it is well known (see, e.g., Leadbetter, Lindgren and Rootzén [13]), that the asymptotic distribution of M_n after suitable normalization, may be of one of the three different types (assuming that the tail of the distribution of $|X_i|$ is so regular that there is an asymptotic distribution at all). The three types of limit distributions, called extreme value distributions, are known as Gumbel, Weibull and Fréchet distributions; they have the distribution functions

$$\exp(-e^{-x}), \qquad -\infty < x < \infty \text{ (Gumbel)}, \tag{1.3}$$

$$\exp(-|x|^{\alpha}), \qquad -\infty < x \le 0 \text{ (Weibull)}, \tag{1.4}$$

$$\exp(-x^{-\alpha}), \qquad 0 < x < \infty \text{ (Fréchet)}, \tag{1.5}$$

where for the last two cases α is a positive parameter.

For the multidimensional situation, we shall focus here mostly on the Gumbel limit which includes, for example, the important case of samples from a normal distribution; we show that under some regularity conditions $M_n^{(2)}$ in multidimensions also has an asymptotic Gumbel distribution. The Weibull case (including, e.g., the uniform distribution in a ball) was considered in [15]; in this case the asymptotic distribution of $M_n^{(2)}$ also turns out to be Weibull, although with a different parameter. We have not much to add to their results except to make a few comments in Section 5.1. The Fréchet case (e.g., power law tails) is more complicated; there is a limit distribution for $M_n^{(2)}$ in this case too, but it is not known explicitly. We explain this difference in Section 5.2.

1.1. *Notation*. All unspecified limits are as $n \to \infty$. In particular, $x_n \sim y_n$ means $x_n/y_n \to 1$ as $n \to \infty$ (allowing also $x_n = y_n = 0$ for some n). Convergence in probability or distribution is denoted by $\xrightarrow{p}$ and $\xrightarrow{d}$, respectively. We let $x_+ := \max(x, 0)$ for $x \in \mathbb{R}$.

1.2. *Main results*. Our main result is contained in the following theorem, whose proof is given in Section 3. We also provide two special versions of this main result (Theorem 1.4 and Theorem 1.5) which readily connect to useful applications.

THEOREM 1.1. *Suppose that $d \geq 2$ and that $X, X_1, X_2, \dots$ are i.i.d. $\mathbb{R}^d$-valued random vectors with a spherically symmetric distribution such that for some sequences a_n and b_n of positive numbers with $b_n = o(a_n)$,*

$$\mathbb{P}\big(|X| > a_n + tb_n\big) = \frac{1+o(1)}{n} e^{-t} \tag{1.6}$$

as $n \to \infty$, for all $t = t_n$ with $|t| \leq \frac{d-1}{2}\log(a_n/b_n)$. Let

$$c_d := (d-1)2^{d-4}\Gamma(d/2)/\sqrt{\pi}. \tag{1.7}$$

Then

$$\frac{M_n^{(2)} - 2a_n}{b_n} + \frac{d-1}{2}\log\frac{a_n}{b_n} - \log\log\frac{a_n}{b_n} - \log c_d \xrightarrow{d} V, \tag{1.8}$$

where V has the Gumbel distribution $\mathbb{P}(V \leq x) = e^{-e^{-x}}$.

REMARK 1.2. In particular, since $\log(a_n/b_n) \to \infty$, we assume that (1.6) holds for every fixed t. This is, by a standard argument (see, e.g., [13]), equivalent to

$$\mathbb{P}\big((M_n - a_n)/b_n \leq t\big) \to e^{-e^{-t}}, \tag{1.9}$$

that is,

$$\frac{M_n - a_n}{b_n} \xrightarrow{d} V, \tag{1.10}$$

where V has the Gumbel distribution. (This verifies our claim that we are dealing with the Gumbel case.) Conversely, if (1.10) holds, so (1.6) holds for every fixed t, then necessarily $b_n = o(a_n)$ (as is easily seen by considering large negative t). Thus the assumption $b_n = o(a_n)$ is redundant if we add the requirement that (1.6) holds for any fixed t.

We also note that our assumption is a bit stronger than just assuming (1.10), since we require (1.6) also for some $t = t_n \to \infty$. First, this restricts the choice of b_n. Indeed, in (1.10), b_n can be replaced by any $b_n' = b_n(1 + o(1))$, but for assumption (1.6) for our range of t one needs $b_n' = b_n(1 + o(1/\log(a_n/b_n)))$. Actually, the latter condition is also needed for replacing b_n by b_n' in the conclusion (1.8), giving some justification to our condition. Second, condition (1.6) for our range of t is satisfied for a suitable choice of a_n and b_n in sufficiently regular instances of (1.10), such as the examples in Section 2, but it does not always hold. A counterexample is given by $\mathbb{P}(|X| > x) = \exp(-\int_0^x h(t)\,\mathrm{d}t)$ with a function $h(t) > 0$ such that $h(t) \to 1$ as $t \to \infty$; this always satisfies (1.6) for fixed t (with $b_n \sim 1$ and some $a_n \sim \log n$), and thus (1.10), but for a suitably slowly oscillating h, for example, $h(x) = 1 + \sin(x/\log x)/\log\log x$, (1.6) does not hold for all t with $|t| \leq \frac{1}{2}\log(a_n/b_n)$, for any such a_n and b_n. We expect that it is possible to extend Theorem 1.1 to such cases, with some modification of (1.8), but we have not pursued this and leave it as an open problem.

REMARK 1.3. As a corollary we see that typically $2M_n - M_n^{(2)}$ is about $\frac{d-1}{2} b_n \log(a_n/b_n)$; more precisely, (1.8) and (1.10) imply

$$\frac{2M_n - M_n^{(2)}}{b_n \log(a_n/b_n)} \xrightarrow{p} \frac{d-1}{2}. \tag{1.11}$$

It can be seen from the proof below that if we order $X_1, \dots, X_n$ as $X_{(1)}, \dots, X_{(n)}$ with $M_n = |X_{(1)}| \geq \cdots \geq |X_{(n)}|$, then the probability that $M_n^{(2)}$ is attained by a pair including $X_{(1)}$ tends to 0; the reason is that the other large vectors $X_{(2)}, \dots$ probably are not almost opposite to $X_{(1)}$. However, if we consider points X_i such that $|X_i|$ is close to M_n, with a suitable margin, then there will be many such points, and it is likely that some pair will be almost opposite. There is a trade-off between what we lose in length and what we gain in angle, and the proof of the theorem is based on finding the right balance.

We now give two special versions of the main result that are more conveniently stated, and are most likely to be useful in applications. The proofs of these two theorems are given in Section 4.

THEOREM 1.4. *Suppose that $d \geq 2$ and that $X, X_1, X_2, \dots$ are i.i.d. $\mathbb{R}^d$-valued random vectors with a spherically symmetric distribution such that*

$$\mathbb{P}(|X| > x) = G(x) = e^{-g(x)+o(1)} \qquad \text{as } x \to \infty, \tag{1.12}$$

for some twice differentiable function $g(x)$ such that, as $x \to \infty$,

$$xg'(x) \to \infty, \tag{1.13}$$

$$\frac{g''(x)}{g'(x)^2} \log^2(xg'(x)) \to 0, \tag{1.14}$$

and that a_n and b_n are such that, as $n \to \infty$, $a_n \to \infty$ and

$$g(a_n) = \log n + o(1), \tag{1.15}$$

$$b_n = \frac{1 + o(1/\log(a_n g'(a_n)))}{g'(a_n)}. \tag{1.16}$$

Then (1.8) holds.

Note that Remark 1.2 gives an example of a distribution in the Gumbel domain of attraction such that (1.12) and (1.13) hold, but not the more technical assumption (1.14).

THEOREM 1.5. *Suppose that $d \geq 2$ and that $X_1, X_2, \dots$ are i.i.d. $\mathbb{R}^d$-valued random vectors with a spherically symmetric distribution with a density function $f(\mathbf{x})$ such that, as $|\mathbf{x}| \to \infty$,*

$$f(\mathbf{x}) \sim c|\mathbf{x}|^{\alpha} e^{-\beta|\mathbf{x}|^{\gamma}} \tag{1.17}$$

for some $c, \beta, \gamma > 0$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}(\beta^{1/\gamma}\gamma \log^{1-1/\gamma} n) \cdot M_n^{(2)} - \biggl(2\gamma \log n + \biggl(2\frac{\alpha+d}{\gamma} - \frac{d+3}{2}\biggr)\log\log n \\ + \log\log\log n + \log\bigl(c'_d \beta^{-2(\alpha+d)/\gamma}\gamma^{-(d+3)/2}c^2\bigr)\biggr) \\ \xrightarrow{d} V,\end{aligned}$$

where

$$c'_d = \frac{(d-1)2^{d-2}\pi^{d-1/2}}{\Gamma(d/2)}, \tag{1.18}$$

and V has the Gumbel distribution.

We give some specific examples in Section 2, and provide further comments as well as state some open problems in Section 5.

2. Examples.

EXAMPLE 2.1. Suppose that X_i has a standard multivariate normal distribution in R^d. The density function is

$$f(\mathbf{x}) = (2\pi)^{-d/2} e^{-|\mathbf{x}|^2/2}, \tag{2.1}$$

which satisfies (1.17) with $c = (2\pi)^{-d/2}$, $\alpha = 0$, $\beta = 1/2$ and $\gamma = 2$. Hence Theorem 1.5 yields, for $d \geq 2$,

$$\begin{aligned}\sqrt{2\log n}\, M_n^{(2)} - \biggl(4\log n + \frac{d-3}{2}\log\log n + \log\log\log n + \log\frac{(d-1)2^{(d-7)/2}}{\sqrt{\pi}\,\Gamma(d/2)}\biggr) \\ \xrightarrow{d} V.\end{aligned}$$

This was shown by Matthews and Rukhin [14] (with a correction by Henze and Klein [5]).

EXAMPLE 2.2. Henze and Klein [5] considered, more generally, the case when X_i has a symmetric Kotz-type distribution in $\mathbb{R}^d$, $d \geq 2$, with density

$$f(\mathbf{x}) = \frac{\kappa^{d/2+b-1}\Gamma(d/2)}{\pi^{d/2}\Gamma(d/2+b-1)}|\mathbf{x}|^{2(b-1)} e^{-\kappa|\mathbf{x}|^2}, \tag{2.2}$$

where $b \in \mathbb{R}$ and $\kappa > 0$. Theorem 1.5 applies with $c = \frac{\kappa^{d/2+b-1}\Gamma(d/2)}{\pi^{d/2}\Gamma(d/2+b-1)}$, $\alpha = 2(b-1)$, $\beta = \kappa$ and $\gamma = 2$, and yields

$$\sqrt{4\kappa \log n}\, M_n^{(2)} - \biggl(4\log n + \frac{4b+d-7}{2}\log\log n + \log\log\log n + \log \frac{(d-1)2^{(d-7)/2}\Gamma(d/2)}{\sqrt{\pi}\,\Gamma(d/2+b-1)^2}\biggr) \xrightarrow{d} V,$$

as shown by [5].

The case $\gamma = 1$ of Theorem 1.5 yields a similar result for a density $f(\mathbf{x}) = c|\mathbf{x}|^\alpha e^{-\beta|\mathbf{x}|}$.

EXAMPLE 2.3. Suppose that the points X_i are symmetrically distributed in the unit sphere with

$$\mathbb{P}\bigl(|X| > x\bigr) = e^{-x/(1-x)} = e \cdot e^{-1/(1-x)}, \qquad 0 \le x < 1. \tag{2.3}$$

It is easily verified that (1.6) holds for $t = O(\log\log n)$ with $a_n = 1 - \log^{-1} n + \log^{-2} n$ and $b_n = \log^{-2} n$; cf. [13], Example 1.7.5. Hence Theorem 1.1 yields a Gumbel limit for $M^{(2)}$ in this case too. (For some other distributions in the unit sphere, $M^{(2)}$ has an asymptotic Weibull distribution as shown by Mayer and Molchanov [15]; see Section 5.1.)

3. Proof of Theorem 1.1. Let λ be a fixed real number, and define two sequences r_n and s_n of positive numbers by

$$r_n := \frac{d-1}{2}\log\frac{a_n}{b_n} - \log\log\frac{a_n}{b_n} - \log c_d - \lambda, \tag{3.1}$$

$$s_n := \tfrac{1}{2}\log r_n. \tag{3.2}$$

(The value of r_n is determined by the argument below, but s_n could be any sequence that tends to ∞ sufficiently slowly.) Note that $r_n \to \infty$ and $s_n \to \infty$, and $s_n = o(r_n)$; furthermore, $r_n b_n = o(a_n)$. We assume below tacitly that n is so large that $r_n > s_n > 0$, and $(r_n + s_n)b_n < a_n$.

Further for convenience, we let

$$\tau_n := \frac{d-1}{2}\log\frac{a_n}{b_n}; \tag{3.3}$$

thus (1.6) is assumed to hold for $|t| \le \tau_n$ (and it then automatically holds uniformly for these t). Note that $r_n + s_n \le \tau_n$, at least for n large; it suffices to consider only such n, and thus (1.6) holds uniformly for $|t| \le r_n + s_n$.

In this section we prove the following result, which immediately implies Theorem 1.1, since W_n defined in this theorem is related to $M_n^{(2)}$ by the relation $M_n^{(2)} > 2a_n - r_n b_n \iff W_n \ne 0$.

THEOREM 3.1. *Let $X_1, X_2, \dots$ be as in Theorem 1.1, and let W_n be the number of pairs (i, j) with $1 \le i < j \le n$ such that $|X_i - X_j| > 2a_n - r_n b_n$. Then* $W_n \xrightarrow{d} \operatorname{Po}(e^{-\lambda})$.

We shall prove Theorem 3.1 by standard Poisson approximation techniques. However, some care is needed, since it turns out that the mean does not converge in Theorem 3.1; at least in typical cases, $\mathbb{E}W_n \to \infty$. The problem is that while (1.9)–(1.10) show that the largest $|X_i|$ typically is about a_n, the unlikely event that $\max |X_i|$ is substantially larger gives a significant contribution to $\mathbb{E}W_n$, since an exceptionally large X_i is likely to be part of many pairs with $|X_i - X_j| > 2a_n - r_n b_n$. (A formal proof can be made by the arguments below, but taking s_n to be a large constant times r_n.) We thus do a truncation (this is where we use s_n) and define, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, the indicator function

$$f_n(\mathbf{x}, \mathbf{y}) := \mathbf{1}\bigl\{|\mathbf{x} - \mathbf{y}| > 2a_n - r_n b_n \text{ and } |\mathbf{x}|, |\mathbf{y}| \le a_n + s_n b_n\bigr\} \tag{3.4}$$

and the corresponding sum

$$W_n' := \sum_{1 \le i < j \le n} f_n(X_i, X_j). \tag{3.5}$$

(This is somewhat similar to the proofs of [14] and [5] which also use Poisson approximation, but they use a decomposition with several terms.) Note that if $f_n(\mathbf{x}, \mathbf{y}) \ne 0$, then $|\mathbf{x}| + |\mathbf{y}| \ge |\mathbf{x} - \mathbf{y}| > 2a_n - r_n b_n$ and thus

$$a_n - (r_n + s_n) b_n < |\mathbf{x}|, |\mathbf{y}| \le a_n + s_n b_n. \tag{3.6}$$

REMARK 3.2. The fact that $\mathbb{E}W_n \to \infty$ shows also that the asymptotic distribution of $M_n^{(2)}$ is *not* the same as the asymptotic distribution of the maximum of $\binom{n}{2}$ independent random variables with the same distribution as $|X_1 - X_2|$. This is unlike the Weibull case (see Section 5.1), where Mayer and Molchanov [15] point out that such an equivalence holds.

We use the following estimates; proofs are given later in this section.

LEMMA 3.3. $\mathbb{P}(W_n \ne W_n') \to 0$.

LEMMA 3.4.

$$\mathbb{E} f_n(X_1, X_2) r_n e^{r_n} = \frac{2 + o(1)}{n^2} e^{-\lambda}.$$

LEMMA 3.5.

$$\mathbb{E}\bigl(f_n(X_1, X_2) f_n(X_1, X_3)\bigr) = o\bigl(n^{-3}\bigr).$$

PROOF OF THEOREM 3.1 (AND THUS OF THEOREM 1.1). Consider W'_n defined in (3.5), and note that Lemma 3.4 shows $\mathbb{E}W'_n \to e^{-\lambda}$. Moreover, the Poisson convergence

$$W'_n \xrightarrow{d} \mathrm{Po}(e^{-\lambda}) \tag{3.7}$$

follows from Lemmas 3.4 and 3.5 using a theorem by Silverman and Brown [16]; see also Barbour and Eagleson [3], Barbour, Holst and Janson [4], Theorem 2.N and Corollary 2.N.1, and Jammalamadaka and Janson [7], Theorem 3.1 and Remark 3.4.

The conclusion $W_n \xrightarrow{d} \mathrm{Po}(e^{-\lambda})$ now follows by Lemma 3.3. □

PROOF OF LEMMA 3.3. We have

$$\mathbb{P}(W_n \neq W'_n) \le \mathbb{P}\Bigl(\max_{i \le n} |X_i| > a_n + s_n b_n\Bigr) \le n\mathbb{P}(|X| > a_n + s_n b_n) \to 0$$

by (1.6), since $s_n \to \infty$. □

In order to prove Lemmas 3.4 and 3.5, we need some estimates.

LEMMA 3.6. *Let Y and Z be two independent random unit vectors in $\mathbb{R}^d$ such that Y is uniformly distributed on the unit sphere S^{d-1}, and let Θ be the angle between Y and Z. Then, as $\varepsilon \searrow 0$,*

$$\mathbb{P}(1 + \cos\Theta < \varepsilon) \sim \frac{2^{(d-3)/2}\Gamma(d/2)}{\sqrt{\pi}\Gamma((d+1)/2)} \varepsilon^{(d-1)/2}.$$

PROOF. By rotational invariance, we may assume that $Z = (1, 0, \dots, 0)$. In this case, if $Y = (\eta_1, \dots, \eta_d)$, then $\cos\Theta = \langle Y, Z\rangle = \eta_1$; moreover, it is well known (and easily seen) that η_1 has the density function

$$g(x) = c''_d(1 - x^2)^{(d-3)/2}, \qquad -1 < x < 1,$$

where

$$\begin{aligned} 1/c''_d &= \int_{-1}^{1} (1 - x^2)^{(d-3)/2}\,\mathrm{d}x = \int_0^1 (1 - y)^{(d-3)/2} y^{-1/2}\,\mathrm{d}y \\ &= \frac{\Gamma((d-1)/2)\Gamma(1/2)}{\Gamma(d/2)}. \end{aligned}$$

The result follows by a simple calculation. □

LEMMA 3.7. *If Y and Z are two independent random vectors in $\mathbb{R}^d$ such that Y is uniformly distributed on the sphere $|Y| = a_n + tb_n$, and Z has any distribution* *on the sphere* $|Z| = a_n + ub_n$, *with* $|t|, |u| \le r_n + s_n$, *then uniformly in all such* t *and* u,

$$\mathbb{P}\big(|Y - Z| > 2a_n - r_n b_n\big) \sim c_d''' \left(\frac{b_n}{a_n}\right)^{(d-1)/2} (r_n + t + u)_+^{(d-1)/2}$$

with

$$c_d''' := 2^{(d-1)/2} \frac{2^{(d-3)/2}\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d+1)/2)} = \frac{2^{d-2}\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d+1)/2)}.$$

PROOF. By the cosine formula, letting Θ be the angle between Y and Z,

$$|Y - Z|^2 = |Y|^2 + |Z|^2 - 2|Y||Z|\cos\Theta = \big(|Y| + |Z|\big)^2 - 2|Y||Z|(1 + \cos\Theta).$$

Hence, by the assumption $(r_n + s_n)b_n = o(a_n)$ and thus $tb_n, ub_n = o(a_n)$,

$$\begin{aligned} &|Y - Z| > 2a_n - r_n b_n \\ &\quad\iff \quad \big(2a_n + (t+u)b_n\big)^2 - 2(a_n + tb_n)(a_n + ub_n)(1 + \cos\Theta) \\ &\qquad\qquad\qquad > (2a_n - r_n b_n)^2 \\ &\quad\iff \quad 1 + \cos\Theta < \frac{(2a_n + (t+u)b_n)^2 - (2a_n - r_n b_n)^2}{2(a_n + tb_n)(a_n + ub_n)} \\ &\qquad\qquad\qquad\qquad = \frac{2b_n}{a_n}(r_n + t + u)\big(1 + o(1)\big). \end{aligned}$$

The result follows by Lemma 3.6 (applied to $Y/|Y|$ and $Z/|Z|$; the angle Θ remains the same), using again that $r_n b_n = o(a_n)$; the probability is obviously 0 when $r_n + t + u \le 0$. $\square$

REMARK 3.8. In this section we use fixed sequences a_n, b_n, r_n, s_n, but we note for future use that Lemma 3.7 more generally holds for any positive sequences with $(1 + r_n + s_n)b_n = o(a_n)$.

We let X be a random variable with $X \overset{d}{=} X_i$ and define

$$T_n := \big(|X| - a_n\big)/b_n; \tag{3.8}$$

thus $|X| = a_n + T_n b_n$ and (1.6) says that

$$\mathbb{P}(T_n > t) = \frac{1 + o(1)}{n} e^{-t}, \tag{3.9}$$

for all $t = t_n$ with $|t| \le \tau_n$, and in particular for all t with $|t| \le r_n + s_n$.

LEMMA 3.9. *Suppose that the function $h(t)$ is nonnegative, continuous and increasing in an interval $[t_0, t_1]$, with $[t_0, t_1] \subseteq [-\tau_n, \tau_n]$. Then, uniformly for all such intervals $[t_0, t_1]$ and functions h,*

$$\mathbb{E}\big(h(T_n)\mathbf{1}\{t_0 < T_n \le t_1\}\big) = \frac{1+o(1)}{n}\int_{t_0}^{t_1} h(t)e^{-t}\,\mathrm{d}t + o\bigg(\frac{h(t_1)e^{-t_1}}{n}\bigg).$$

PROOF. Let $\mu = \mu_n := \mathcal{L}(T_n)$ denote the distribution of T_n. Then, using (3.9) and two integrations by parts,

$$\begin{aligned}
&\mathbb{E}\big(h(T_n)\mathbf{1}\{t_0 < T_n \le t_1\}\big)\\
&\quad= \int_{t_0+}^{t_1} h(t)\,\mathrm{d}\mu(t) = -\int_{t_0+}^{t_1} h(t)\,\mathrm{d}\mathbb{P}(t < T_n \le t_1)\\
&\quad= h(t_0)\mathbb{P}(t_0 < T_n \le t_1) + \int_{t_0}^{t_1} \mathrm{d}h(u)\mathbb{P}(u < T_n \le t_1)\\
&\quad= \frac{1+o(1)}{n}\bigg(h(t_0)e^{-t_0} + \int_{t_0}^{t_1} \mathrm{d}h(u)e^{-u}\bigg)\\
&\qquad - \frac{1+o(1)}{n}\bigg(h(t_0)e^{-t_1} + \int_{t_0}^{t_1} \mathrm{d}h(u)e^{-t_1}\bigg)\\
&\quad= \frac{1+o(1)}{n}\int_{t_0}^{t_1} h(u)e^{-u}\,\mathrm{d}u + \frac{o(1)}{n}h(t_1)e^{-t_1},
\end{aligned}$$

with all $o(1)$ uniform in t_0, t_1 and h. □

LEMMA 3.10. *Let $\mathbf{x}$ be a vector in $\mathbb{R}^d$ with $|\mathbf{x}| = a_n + ub_n$ where $-r_n - s_n < u \le s_n$. Then, uniformly for all such $\mathbf{x}$,*

$$\mathbb{E} f_n(X, \mathbf{x}) = \frac{1+o(1)}{n}c''''_d\bigg(\frac{b_n}{a_n}\bigg)^{(d-1)/2}\big(e^{r_n+u} + O\big((r_n+s_n+u+1)^{(d-1)/2}e^{-s_n}\big)\big),$$

where $c''''_d := \Gamma((d+1)/2)c'''_d = 2^{d-2}\Gamma(d/2)/\sqrt{\pi}$.

PROOF. We use T_n defined by (3.8), and note that $f_n(X, \mathbf{x}) = 0$ unless $-r_n - s_n < T_n \le s_n$; see (3.6). Moreover, Lemma 3.7 shows that for $t \in (-r_n - s_n, s_n]$,

$$\mathbb{E}\big(f_n(X, \mathbf{x}) \mid T_n = t\big) \sim c'''_d\bigg(\frac{b_n}{a_n}\bigg)^{(d-1)/2}(r_n + t + u)_+^{(d-1)/2},$$

uniformly in these u and t, and thus

$$\mathbb{E} f_n(X, \mathbf{x}) \sim c'''_d\bigg(\frac{b_n}{a_n}\bigg)^{(d-1)/2}\mathbb{E}\big((r_n + u + T_n)_+^{(d-1)/2}\mathbf{1}\{-r_n - s_n < T_n \le s_n\}\big).$$

We apply Lemma 3.9 with $h(t)=(r_n+u+t)_+^{(d-1)/2}$ and obtain

$$
\begin{aligned}
&\mathbb{E}\big((r_n+u+T_n)_+^{(d-1)/2}\mathbf{1}\{-r_n-s_n<T_n\le s_n\}\big)\\
&\quad=\frac{1+o(1)}{n}\int_{-r_n-s_n}^{s_n}(r_n+u+t)_+^{(d-1)/2}e^{-t}\,\mathrm{d}t\\
&\qquad+o\bigg(\frac{(r_n+u+s_n)^{(d-1)/2}e^{-s_n}}{n}\bigg)\\
&\quad=\frac{1+o(1)}{n}\int_0^{r_n+s_n+u}x^{(d-1)/2}e^{r_n+u-x}\mathrm{d}x+o\bigg(\frac{(r_n+u+s_n)^{(d-1)/2}e^{-s_n}}{n}\bigg)\\
&\quad=\frac{1+o(1)}{n}\Gamma\bigg(\frac{d+1}{2}\bigg)\big(e^{r_n+u}+O\big((r_n+s_n+u+1)^{(d-1)/2}e^{-s_n}\big)\big),
\end{aligned}
$$

and the result follows. □

PROOF OF LEMMA 3.4. We condition on X_1 and apply Lemma 3.10, with X replaced by X_2 and $u=T_n$ given by (3.8) with $X=X_1$; thus, by (3.6),

$$
\begin{aligned}
&\mathbb{E}\big(f_n(X_1,X_2)\mid X_1\big)\\
&\quad=\mathbb{E}\big(f_n(X_2,X_1)\mid X_1\big)\\
&\quad=\frac{1+o(1)}{n}c_d''''\bigg(\frac{b_n}{a_n}\bigg)^{(d-1)/2}\\
&\qquad\times\big(e^{r_n+T_n}+O\big((r_n+s_n+T_n+1)^{(d-1)/2}e^{-s_n}\big)\big)\mathbf{1}\{-r_n-s_n<T_n\le s_n\}.
\end{aligned}
\tag{3.10}
$$

Hence

$$
\begin{aligned}
&\mathbb{E}f_n(X_1,X_2)\\
&\quad=\mathbb{E}\big(\mathbb{E}\big(f_n(X_1,X_2)\mid T_n\big)\big)\\
&\quad=\frac{1+o(1)}{n}c_d''''\bigg(\frac{b_n}{a_n}\bigg)^{(d-1)/2}\\
&\qquad\times\mathbb{E}\big(\big(e^{r_n+T_n}+O\big((r_n+s_n+T_n+1)^{(d-1)/2}e^{-s_n}\big)\big)\mathbf{1}\{-r_n-s_n<T_n\le s_n\}\big).
\end{aligned}
$$

By Lemma 3.9 with $h(t)=e^t$ we obtain, since $s_n=o(r_n)$ and $r_n\to\infty$,

$$
\mathbb{E}\big(e^{T_n}\mathbf{1}\{-r_n-s_n<T_n\le s_n\}\big)=\frac{1+o(1)}{n}(r_n+2s_n)+o\bigg(\frac{1}{n}\bigg)=\frac{1+o(1)}{n}r_n
$$

and by Lemma 3.9 with $h(t)=(r_n+s_n+t+1)^{(d-1)/2}$,

$$
\mathbb{E}\big((r_n+s_n+T_n+1)^{(d-1)/2}\mathbf{1}\{-r_n-s_n<T_n\le s_n\}\big)=O\bigg(\frac{e^{r_n+s_n}}{n}\bigg),
$$

and the result follows, using our choice of r_n in (3.1) and $c_d=(d-1)c_d''''/4$. □

PROOF OF LEMMA 3.5. By (3.10),

$$\mathbb{E}\big(f_n(X_1, X_2) \mid X_1\big) = O\biggl(\frac{1}{n}\Bigl(\frac{b_n}{a_n}\Bigr)^{(d-1)/2} e^{r_n+T_n}\mathbf{1}\{-r_n - s_n < T_n \le s_n\}\biggr),$$

where we used $(r_n + s_n + T_n + 1)^{(d-1)/2} = O(e^{r_n+s_n+T_n})$. Hence, using Lemma 3.9 with $h(t) = e^{2t}$,

$$\begin{aligned}
&\mathbb{E}\big(f_n(X_1, X_2) f_n(X_1, X_3)\big)\\
&\quad = \mathbb{E}\big(\mathbb{E}(f_n(X_1, X_2) \mid X_1)^2\big)\\
&\quad = O\biggl(\frac{1}{n^2}\Bigl(\frac{b_n}{a_n}\Bigr)^{d-1} \mathbb{E}\big(e^{2r_n+2T_n}\mathbf{1}\{-r_n - s_n < T_n \le s_n\}\big)\biggr)\\
&\quad = O\biggl(\frac{1}{n^3}\Bigl(\frac{b_n}{a_n}\Bigr)^{d-1} e^{2r_n+s_n}\biggr) = O\biggl(\frac{1}{n^3}\cdot\frac{e^{s_n}}{r_n^2}\biggr),
\end{aligned}$$

and the result follows by our choice (3.2) of s_n. □

4. Proofs of Theorems 1.4 and 1.5.

PROOF OF THEOREM 1.4. First, by (1.13) and $a_n \to \infty$, $g'(a_n) > 0$ for large n at least, so $1/g'(a_n) > 0$. Furthermore, $a_n g'(a_n) \to \infty$ by (1.13), so $b_n \sim 1/g'(a_n)$ and $b_n/a_n \to 0$ by (1.16).

We will prove that (1.6) holds, uniformly for all t with $|t| \le A\log(a_n/b_n)$, for any fixed A. The result then follows by Theorem 1.1. In order to prove (1.6), we may suppose that $b_n = 1/g'(a_n)$; the general case (1.16) follows easily. We may also suppose that n is large.

Let $A > 0$ be a constant, and let, for x so large that $xg'(x) > 1$,

$$\delta(x) := A\frac{\log(xg'(x))}{g'(x)} \tag{4.1}$$

and

$$I_x := \big[x - \delta(x), x + \delta(x)\big]. \tag{4.2}$$

Since $\delta(x)/x \to 0$ as $x \to \infty$ by (1.13), we may assume that $0 < \delta(x) < x/2$; hence $I_x \subset (x/2, 2x)$. We claim that, for large x,

$$\tfrac{1}{2}g'(x) < g'(y) < 2g'(x), \qquad y \in I_x. \tag{4.3}$$

To show this, assume that (4.3) fails for some x, and let y by the point in I_x nearest to x where (4.3) fails. (If there are two possible choices for y, take any of the points.) Then

$$\left|\frac{1}{g'(y)} - \frac{1}{g'(x)}\right| \ge \frac{1}{2g'(x)}. \tag{4.4}$$

By (4.4) and the mean value theorem, there exists $z \in [y, x]$ (if $y < x$) or $z \in [x, y]$ (if $y > x$) such that

$$\frac{1}{2g'(x)} \le |y-x|\left|\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{g'(z)}\right| = |y-x|\frac{|g''(z)|}{g'(z)^2} \le \delta(x)\frac{|g''(z)|}{g'(z)^2}. \tag{4.5}$$

On the other hand, $\frac{1}{2}g'(x) \le g'(z) \le 2g'(x)$ by the choice of y; furthermore, $z \in I_x$ so $x/2 < z < 2x$. Hence (1.14) implies, for large x, using (1.13),

$$\frac{|g''(z)|}{g'(z)^2} \le \frac{1}{\log^2(zg'(z))} \le \frac{2}{\log^2(xg'(x))}. \tag{4.6}$$

However, (4.5) and (4.6) combine to yield

$$\frac{1}{2g'(x)} \le \frac{2\delta(x)}{\log^2(xg'(x))} = \frac{2A}{g'(x)\log(xg'(x))},$$

which contradicts (1.13) for large x. This contradiction shows that (4.3) holds for large x.

Next, (4.1), (4.3) and (1.14) imply that, for large x,

$$\begin{aligned}\sup_{y\in I_x}|g''(y)|\delta(x)^2 &= A^2\frac{\sup_{y\in I_x}|g''(y)|}{g'(x)^2}\log^2(xg'(x))\\ &\le 5A^2\sup_{y\in I_x}\frac{|g''(y)|}{g'(y)^2}\log^2(yg'(y)) \to 0\end{aligned}$$

as $x \to \infty$. Consequently, a Taylor expansion yields, uniformly for $|u| \le \delta(x)$,

$$g(x+u) = g(x) + ug'(x) + o(1) \tag{4.7}$$

as $x \to \infty$. Taking $x = a_n$ and $u = tb_n = t/g'(a_n)$, with $|t| \le A\log(a_n/b_n) = A\log(a_n g'(a_n))$, we have $|u| \le \delta(a_n)$ by (4.1), and thus (4.7) applies and shows, by (1.15) and our choice $b_n = 1/g'(a_n)$,

$$g(a_n + tb_n) = g(a_n) + tb_n g'(a_n) + o(1) = \log n + t + o(1), \tag{4.8}$$

uniformly for such t. By (1.12), this yields

$$\mathbb{P}(|X_1| > a_n + tb_n) = \exp(-\log n - t + o(1))$$

uniformly for $|t| \le A\log(a_n/b_n)$, which is (1.6). The result follows by Theorem 1.1. $\square$

Before proving Theorem 1.5 we give an elementary lemma.

LEMMA 4.1. *If $\beta, \gamma > 0$ and $h(x)$ is a positive differentiable function such that $(\log h(x))' = o(x^{\gamma-1})$ as $x \to \infty$, then*

$$\int_x^\infty h(y)e^{-\beta y^\gamma}\,\mathrm{d}y \sim (\beta\gamma)^{-1}x^{1-\gamma}h(x)e^{-\beta x^\gamma} \qquad \textit{as } x \to \infty.$$

PROOF. Suppose first that $\gamma = 1$. Then we assume $(\log h(x))' = o(1)$. Let $\varepsilon(x) := \sup_{y \ge x} |(\log h(y))'|$; thus $\varepsilon(x) \to 0$ as $x \to \infty$. Furthermore, for $t > 0$,

$$\left|\log h(x+t) - \log h(x)\right| \le \varepsilon(x)t$$

and thus

$$e^{-(\beta+\varepsilon(x))t} \le \frac{h(x+t)e^{-\beta(x+t)}}{h(x)e^{-\beta x}} \le e^{-(\beta-\varepsilon(x))t}.$$

Integrating we obtain, for x so large that $\varepsilon(x) < \beta$,

$$\frac{h(x)e^{-\beta x}}{\beta + \varepsilon(x)} \le \int_0^\infty h(x+t)e^{-\beta(x+t)}\,\mathrm{d}t \le \frac{h(x)e^{-\beta x}}{\beta - \varepsilon(x)},$$

and (4.1) follows when $\gamma = 1$.

For a general γ we change variable by $y = z^{1/\gamma}$:

$$\int_x^\infty h(y)e^{-\beta y^\gamma} = \int_{x^\gamma}^\infty h(x^{1/\gamma})e^{-\beta z}\gamma^{-1}z^{1/\gamma-1}\,\mathrm{d}z.$$

The function $H(z) = \gamma^{-1}h(z^{1/\gamma})z^{1/\gamma-1}$ satisfies

$$(\log H(z))' = (\log h)'(z^{1/\gamma}) \cdot \gamma^{-1}z^{1/\gamma-1} + (\gamma^{-1} - 1)z^{-1} = o(1),$$

and thus the case $\gamma = 1$ applies and yields

$$\int_x^\infty h(y)e^{-\beta y^\gamma} = \int_{x^\gamma}^\infty H(z)e^{-\beta z}\,\mathrm{d}z \sim \beta^{-1}H(x^\gamma)e^{-\beta x^\gamma} \qquad \text{as } x \to \infty,$$

which is (4.1). $\square$

PROOF OF THEOREM 1.5. Let $\omega_{d-1} := 2\pi^{d/2}/\Gamma(d/2)$, the surface area of the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. By (1.17) and Lemma 4.1, with $h(x) = x^{\alpha+d-1}$,

$$\mathbb{P}(|X_1| > x) \sim \int_x^\infty cr^\alpha e^{-\beta r^\gamma}\omega_{d-1}r^{d-1}\,\mathrm{d}r \sim c\omega_{d-1}(\beta\gamma)^{-1}x^{\alpha+d-\gamma}e^{-\beta x^\gamma}.$$

Hence (1.12) holds with

$$g(x) = \beta x^\gamma - (\alpha + d - \gamma)\log x + \log(\beta\gamma/c\omega_{d-1}). \tag{4.9}$$

We have

$$g'(x) = \beta\gamma x^{\gamma-1} - (\alpha + d - \gamma)x^{-1}, \tag{4.10}$$

$$g''(x) = \beta\gamma(\gamma - 1)x^{\gamma-2} + (\alpha + d - \gamma)x^{-2}, \tag{4.11}$$

and (1.13)–(1.14) are easily verified.

In order to have (1.15), we need, since $g(x) \sim \beta x^\gamma$ as $x \to \infty$ by (4.9), $a_n \sim \beta^{-1/\gamma}\log^{1/\gamma} n$; furthermore, (1.16) then yields

$$b_n \sim \frac{1}{g'(a_n)} \sim \frac{1}{\beta\gamma a_n^{\gamma-1}} \sim \beta^{-1/\gamma}\gamma^{-1}\log^{1/\gamma-1} n.$$

We thus choose, for simplicity,

$$b_n := \beta^{-1/\gamma}\gamma^{-1}\log^{1/\gamma-1} n. \tag{4.12}$$

If $u_n = O(\log\log n)$, then, by a Taylor expansion and (4.9)–(4.11),

$$\begin{aligned} &g(\beta^{-1/\gamma}\log^{1/\gamma} n + u_n b_n) \\ &\quad = g(\beta^{-1/\gamma}\log^{1/\gamma} n) + u_n b_n g'(\beta^{-1/\gamma}\log^{1/\gamma} n) + o(1) \\ &\quad = \log n - (\alpha + d - \gamma)\gamma^{-1}(\log\log n - \log\beta) + \log(\beta\gamma/c\omega_{d-1}) \\ &\qquad + u_n + o(1). \end{aligned}$$

Hence we define

$$\begin{aligned} a_n := \beta^{-1/\gamma}\log^{1/\gamma} n + b_n\biggl(&\frac{\alpha+d-\gamma}{\gamma}\log\log n - (\alpha+d)\gamma^{-1}\log\beta \\ &- \log\gamma + \log(c\omega_{d-1})\biggr) \end{aligned} \tag{4.13}$$

and find that (1.15) holds. Furthermore, by another Taylor expansion,

$$g'(a_n) = \beta^{1/\gamma}\gamma\log^{1-1/\gamma} n \cdot \bigl(1 + O\bigl(\log^{-1} n + \log\log n \cdot \log^{-1/\gamma} n\bigr)\bigr),$$

and (1.16) follows easily. Hence Theorem 1.4 applies, and (1.8) holds. Moreover, by (4.13) and (4.12),

$$\begin{aligned} a_n/b_n &\sim \gamma\log n, \\ \log(a_n/b_n) &= \log\log n + \log\gamma + o(1), \\ \log\log(a_n/b_n) &= \log\log\log n + o(1), \end{aligned}$$

and the result follows from (1.8) by collecting terms, with $c'_d = c_d\omega_{d-1}^2$, which yields (1.18). $\square$

5. Further comments.

5.1. *Weibull-type extremes.* The Weibull-type extreme value distribution occurs for random variables that are bounded above; in our context this means that $|X|$ is bounded, so X takes values in a bounded set. (However, there are examples of Gumbel-type in this case too; see Example 2.3.) By scaling we may assume that the upper endpoint of the support of $|X|$ is 1, so X belongs to the unit ball, but not always to any smaller ball. The typical Weibull case is

$$\mathbb{P}(|X| > x) \sim c(1-x)^\alpha, \qquad x \nearrow 1, \tag{5.1}$$

for some $\alpha > 0$, in which case

$$\mathbb{P}\bigl(n^{1/\alpha}(1 - M_n) > x\bigr) \to \exp(-cx^\alpha), \tag{5.2}$$

which means that $c^{1/\alpha}n^{1/\alpha}(M_n - 1)$ converges to the (negative) Weibull distribution in (1.4).

Mayer and Molchanov [15] show that if (5.1) holds, then $M_n^{(2)}$ also has an asymptotic Weibull distribution, with a different parameter. More precisely, they show the following; see also Lao and Mayer [12] and Lao [11], which contain further related results.

THEOREM 5.1 (Mayer and Molchanov [15]). *Suppose that $d \geq 2$ and that $X_1, X_2, \dots$ are i.i.d. $\mathbb{R}^d$-valued random vectors with a spherically symmetric distribution such that* (5.1) *hold for some $\alpha \geq 0$ and $c > 0$. Then*

$$\mathbb{P}\bigl(n^{4/(d-1+4\alpha)}(2 - M_n^{(2)}) > x\bigr) \to \exp\bigl(-c_{d,\alpha}c^2 x^{(d-1+4\alpha)/2}\bigr), \tag{5.3}$$

with

$$c_{d,\alpha} := \frac{\Gamma(\alpha+1)^2\Gamma((d+1)/2)}{2\Gamma((d+1+4\alpha)/2)}c_d''' = \frac{2^{d-3}\Gamma(\alpha+1)^2\Gamma(d/2)}{\sqrt{\pi}\,\Gamma((d+1+4\alpha)/2)}. \tag{5.4}$$

Hence, $n^{4/(d-1+4\alpha)}(M_n^{(2)} - 2)$ has, apart from a constant factor, the (negative) Weibull distribution (1.4) with parameter $(d-1+4\alpha)/2$.

Note that Theorem 5.1 includes the case $\alpha = 0$, that is, when $\mathbb{P}(|X| = 1) = c > 0$, in particular the case $|X| = 1$ with X uniformly distributed on the unit sphere. (The latter case was earlier shown by Appel and Russo [1].) In the case $\alpha = 0$, (5.2) does not make sense; the asymptotic distribution of M_n is degenerate, since $\mathbb{P}(M_n = 1) \to 1$.

Theorem 5.1 can easily be proved by the method in Section 3, taking $a_n := 1$, $b_n := c^{-1/\alpha}n^{-1/\alpha}$ (with $b_n := 1$ when $\alpha = 0$), $r_n := xb_n^{-1}n^{-4/(d-1+4\alpha)}$ and $s_n = 0$. (We take $s_n = 0$ since no truncation is needed in this case; indeed, $W_n' = W_n$; cf. Remark 3.2.) We omit the details. (The authors of [15] and [1] also use Poisson approximation, but the details are different.)

REMARK 5.2. Since the normalizing factors in (5.2) and (5.3) have different powers of n, $2 - M_n^{(2)}$ is asymptotically much larger than $1 - M_n$, and thus $2M_n - M_n^{(2)}$ has the same asymptotic distribution as $2 - M_n^{(2)}$; see (5.3), and cf. Remark 1.3 for the Gumbel case.

We have here for simplicity considered only the standard case when (5.1) holds, and leave extensions to more general distributions with M_n asymptotically Weibull to the reader.

5.2. *Fréchet-type extremes.* The Fréchet-type extreme value distribution occurs for $|X|$ if (and only if, see [13], Theorem 1.6.2 and Corollary 1.6.3) there exists a sequence $\gamma_n \to \infty$ such that

$$\mathbb{P}\bigl(|X| > x\gamma_n\bigr) \sim \frac{1}{n}x^{-\alpha} \tag{5.5}$$

for every (fixed) $x > 0$; then

$$\gamma_n^{-1} M_n \xrightarrow{d} \tilde{V}, \tag{5.6}$$

where $\tilde{V}$ has the Fréchet distribution (1.5). The typical case is a power-law tail

$$\mathbb{P}(|X| > x) \sim cx^{-\alpha} \qquad \text{as } x \to \infty; \tag{5.7}$$

in this case, (5.5) and (5.6) hold with $\gamma_n = (cn)^{1/\alpha}$. We have the following result, independently found by Henze and Lao [6]. Let again $\omega_{d-1} := 2\pi^{d/2}/\Gamma(d/2)$, the surface area of the unit sphere in $\mathbb{R}^d$.

THEOREM 5.3 (Henze and Lao [6]). *Suppose that $d \geq 2$ and that $X_1, X_2, \ldots$ are i.i.d. $\mathbb{R}^d$-valued random vectors with a spherically symmetric distribution such that* (5.5) *hold for some $\gamma_n \to \infty$. Then*

$$\gamma_n^{-1} M_n^{(2)} \xrightarrow{d} Z_\alpha \tag{5.8}$$

for some random variable Z_α, which can be described as the maximum distance $\max_{i,j} |\xi_i - \xi_j|$ between the points in a Poisson point process $\Xi = \{\xi_i\}$ on $\mathbb{R}^d \setminus \{0\}$ with intensity $\alpha\omega_{d-1}^{-1}|\mathbf{x}|^{-\alpha-d}$.

SKETCH OF PROOF. It is easy to see that the scaled set of points $\{\gamma_n^{-1} X_i : 1 \leq i \leq n\}$, regarded as a point process on $\mathbb{R}^d \setminus 0$, converges in distribution to the Poisson process Ξ. It then follows that the maximum interpoint distance converges. We omit the details. See, for example, Kallenberg [9] or [10] for details on point processes, or Janson [8], Section 4, for a brief summary. $\square$

Note that the point process Ξ has infinite intensity, and thus a.s. an infinite number of points, clustering at 0, but a.s. only a finite number of points $|\xi| > \varepsilon$ for any $\varepsilon > 0$. (This is the reason for regarding the point processes on $\mathbb{R}^d \setminus 0$ only, since we want the point processes to be locally finite.)

We leave it as an open problem to find an explicit description of the limit distribution, that is, the distribution of Z_α. We do not believe that it is Fréchet, so $M_n^{(2)}$ and M_n will (presumably) not have the same type of asymptotic distribution in the Fréchet case, unlike the Gumbel and Weibull cases treated above.

One reason for the more complicated limit behavior in the Fréchet case is that the Poisson approximation argument in Section 3 fails. If we define W_n' as there, with a suitable threshold and a suitable truncation (avoiding small $|X_i|$ this time), we can achieve $\mathbb{E} f_n(X_1, X_2) \sim Cn^{-2}$ as in Lemma 3.4, for a constant $C > 0$, but then $\mathbb{E} f_n(X_1, X_2) f_n(X_1, X_3)$ will be of order n^{-3} and there is no analogue of Lemma 3.5; this ought to mean that W_n does not have an asymptotic Poisson distribution. In other words, the problem is that there is too much dependence between pairs with a large distance.

Furthermore, it can be seen from Theorem 5.3 that there is a positive limiting probability that the maximum distance $M_n^{(2)}$ is attained between the two vectors $X_{(1)}$ and $X_{(2)}$ with largest length, but it can also be attained (with probability bounded away from 0) by any other pair $X_{(k)}$ and $X_{(l)}$ with given $1 \leq k < l$. This is related to the preceding comment, and may thus also be a reason for the more complicated behavior of $M_n^{(2)}$ in the Fréchet case. (It shows also that there is an asymptotic dependence between $M_n^{(2)}$ and M_n which does not exist in the Gumbel and Weibull cases.) Moreover, it follows also that, again unlike the Gumbel and Weibull cases, the angle between the maximizing vectors X_i and X_j is not necessarily close to π (it can be any angle $> \pi/3$), so it is not enough to use asymptotic estimates as Lemma 3.6.

5.3. *The case $d = 1$.* The theorem above supposes $d > 1$, for example, because we need $r_n \to \infty$. In the case $d = 1$, there is a similar result, which is much simpler, but somewhat different; for comparison we give this result too. (For simplicity we continue to consider symmetric variables.)

THEOREM 5.4. *Suppose that $X_1, X_2, \dots$ are i.i.d. symmetric real-valued random variables such that for some sequences a_n and b_n of positive numbers, (1.6) holds as $n \to \infty$, for any fixed real t. Then*

$$\frac{M_n^{(2)} - 2a_n}{b_n} + 2\log 2 \xrightarrow{d} V_+ + V_-, \tag{5.9}$$

where V_1, V_2 are two independent random variables with the Gumbel distribution $\mathbb{P}(V_\pm \leq t) = e^{-e^{-t}}$.

PROOF. When $d = 1$,

$$M_n^{(2)} = M_n^+ - M_n^-, \tag{5.10}$$

where $M_n^+ := \max_{i \leq n} X_i$ and $M_n^- := \min_{i \leq n} X_i$.

There are about $n/2$ positive and $n/2$ negative X_i. More precisely, denoting these numbers by N_+ and $N_- = n - N_+$, where we assign a random sign also to any value that is 0, we have $N_+, N_- \sim \mathrm{Bi}(n, 1/2)$. Conditioned on N_+ and N_-, and assuming that both are nonzero, M_n^+ and M_n^- are independent, with $M_n^+ \overset{d}{=} M_{N_+}$ and $M_n^- \overset{d}{=} -M_{N_-}$. Moreover, if we assume (1.6) for every fixed t, then it is easy to see that for any random N with $N/n \xrightarrow{p} 1/2$ as $n \to \infty$, we have

$$\frac{M_N - a_n}{b_n} + \log 2 \xrightarrow{d} V; \tag{5.11}$$

cf. (1.10). Consequently,

$$\frac{\pm M_n^\pm - a_n}{b_n} + \log 2 \xrightarrow{d} V_\pm, \tag{5.12}$$

where $V_\pm$ are two random variables with the same Gumbel distribution; moreover, it is easy to see that this holds jointly with V_+ and V_- independently. The result follows from (5.10) and (5.12). $\square$

Comparing Theorem 5.4 to Theorem 1.1, we see that first of all the limit distribution is different. Furthermore, the term $\frac{d-1}{2}\log(a_n/b_n)$ in (1.8) disappears, which is expected, but also the term $\log\log(a_n/b_n)$ disappears, and the constant term is different, with $-\log c_d$ replaced by $2\log 2$. [c_d in (1.7) would be 0 for $d=1$, which does not make sense in (1.8).] A reason for the different behavior is that for $d=1$, there is no issue with the angles, and $M^{(2)}$ is the sum of two extreme value statistics (M_n^+ and $-M_n^-$ in the proof above).

Similarly, in the special case in Theorem 1.5, we obtain for $d=1$ from (4.13) and (4.12) (which hold also for $d=1$ by the proof above) the following, where the limit distribution again is different; furthermore, the $\log\log\log n$ term disappears, and the constant term is slightly different.

THEOREM 5.5. *Suppose that $X_1, X_2, \dots$ are i.i.d. symmetric real-valued random variables with a density function $f(x)$ such that, as $|x|\to\infty$,*

$$f(x) \sim c|x|^\alpha e^{-\beta|x|^\gamma} \tag{5.13}$$

for some $c, \beta, \gamma > 0$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}(\beta^{1/\gamma}\gamma\log^{1-1/\gamma} n)\cdot M_n^{(2)} - \biggl(2\gamma\log n + \biggl(2\frac{\alpha+1}{\gamma} - 2\biggr)\log\log n \\ + \log\bigl(\beta^{-2(\alpha+1)/\gamma}\gamma^{-2}c^2\bigr)\biggr) \\ \xrightarrow{d} V_+ + V_-,\end{aligned}$$

where $V_\pm$ are independent and have the Gumbel distribution. $\square$

Typical examples are given by $f(x) = (2\pi)^{-1/2}e^{-x^2/2}$ and $f(x) = \frac{1}{2}e^{-|x|}$; we leave the details to the reader.

The argument above applies also to the Weibull and Fréchet cases when $d=1$; we omit the details. (Furthermore, Theorem 5.3 holds also for $d=1$.)

5.4. *Nonsymmetric distributions.* We have assumed that the distribution of X is spherically symmetric. What happens if we relax that condition? Consider, for example, the case of a normal distribution with a nonisotropic covariance matrix, for example, with a simple largest eigenvalue so that there is a unique direction where the variance is largest. Will the asymptotic distribution of $M_n^{(2)}$ then be governed mainly by the component in that direction only, so that there is a limit law similar to the case $d=1$, or will the result still be similar to the theorems above for the spherically symmetric case, or is the result somewhere in between? We leave this as an open problem.

For the case of points distributed inside a bounded set, Appel, Najim and Russo [2], Mayer and Molchanov [15], Lao and Mayer [12] and Lao [11] have results also in the nonsymmetric case. As an example, consider points uniformly distributed inside an ellips with major axis 1 and minor axis $b < 1$. The maximum distance is obviously attained by some pair of points close to the endpoints of the major axis, and it can be shown, by arguments similar to the proof of Theorem 5.3, that $n^{2/3}(2 - M_n^{(2)}) \xrightarrow{d} Z$, where Z can be described as the distribution of $\pi^{2/3} \min_{i,j}(x_i' + x_j'' - b^2(y_i' - y_j'')^2/4)$, with $\{(x_i', y_i')\}$ and $\{(x_j'', y_j'')\}$, two independent Poisson processes with intensity 1 in the parabola $\{(x, y) : y^2 \leq 2x\}$. [If the endpoints of the major axis are $(\pm 1, 0)$, we represent points close to them as $(1 - x', by')$ and $(-1 + x'', by'')$, and note that the distance $|(1 - x', by') - (-1 + x'', by'')| \approx 2 - x' - x'' + b^2(y' - y'')^2/4$; we omit the details.] We do not know any explicit description of this limit distribution. It seems likely that limits of similar types arise also in other cases where $\max |X_i|$ is attained in a single direction, for example, a 3-dimensional ellipsoid with semiaxes $a > b \geq c$, while we believe that there is a Weibull limit similar to Theorem 5.1 if $a = b > c$, so that there is rotational symmetry around the shortest axis.

5.5. *Other norms.* We have considered here only the Euclidean distance. It seems to be an open problem to find similar results for other distances, for example, the ℓ^1-norm or the ℓ^∞-norm in $\mathbb{R}^d$.

Acknowledgments. We wish to thank Professor Norbert Henze for bringing this problem to the attention of one of us and two anonymous referees for very helpful comments.

This paper was largely written on the occasion of SRJ's visit to Uppsala in October 2012 to receive an honorary doctorate from the Swedish University of Agricultural Sciences.

REFERENCES

[1] APPEL, M. J. and RUSSO, R. P. (2006). Limiting distributions for the maximum of a symmetric function on a random point set. *J. Theoret. Probab.* **19** 365–375. MR2283381

[2] APPEL, M. J. B., NAJIM, C. A. and RUSSO, R. P. (2002). Limit laws for the diameter of a random point set. *Adv. in Appl. Probab.* **34** 1–10. MR1895327

[3] BARBOUR, A. D. and EAGLESON, G. K. (1984). Poisson convergence for dissociated statistics. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **46** 397–402. MR0790624

[4] BARBOUR, A. D., HOLST, L. and JANSON, S. (1992). *Poisson Approximation*. *Oxford Studies in Probability* **2**. Oxford Univ. Press, Oxford. MR1163825

[5] HENZE, N. and KLEIN, T. (1996). The limit distribution of the largest interpoint distance from a symmetric Kotz sample. *J. Multivariate Anal.* **57** 228–239. MR1391170

[6] HENZE, N. and LAO, W. (2010). The limit distribution of the largest interpoint distance for power-tailed spherically decomposable distributions and their affine images. Preprint.

[7] JAMMALAMADAKA, S. R. and JANSON, S. (1986). Limit theorems for a triangular scheme of U-statistics with applications to inter-point distances. *Ann. Probab.* **14** 1347–1358. MR0866355

[8] JANSON, S. (2003). Cycles and unicyclic components in random graphs. *Combin. Probab. Comput.* **12** 27–52. MR1967484

[9] KALLENBERG, O. (1983). *Random Measures*, 3rd ed. Akademie-Verlag, Berlin. MR0818219

[10] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. Springer, New York. MR1876169

[11] LAO, W. (2010). Some weak limits for the diameter of random point sets in bounded regions. Dissertation, Karlsruher Institut für Technologie, KIT Scientific Publishing.

[12] LAO, W. and MAYER, M. (2008). U-max-statistics. *J. Multivariate Anal.* **99** 2039–2052. MR2466550

[13] LEADBETTER, M. R., LINDGREN, G. and ROOTZÉN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer, New York. MR0691492

[14] MATTHEWS, P. C. and RUKHIN, A. L. (1993). Asymptotic distribution of the normal sample range. *Ann. Appl. Probab.* **3** 454–466. MR1221161

[15] MAYER, M. and MOLCHANOV, I. (2007). Limit theorems for the diameter of a random sample in the unit ball. *Extremes* **10** 129–150. MR2394205

[16] SILVERMAN, B. and BROWN, T. (1978). Short distances, flat triangles and Poisson limits. *J. Appl. Probab.* **15** 815–825. MR0511059

DEPARTMENT OF STATISTICS
AND APPLIED PROBABILITY
UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CALIFORNIA 93106
USA
E-MAIL: rao@pstat.ucsb.edu
URL: http://www.pstat.ucsb.edu/faculty/jammalam/

DEPARTMENT OF MATHEMATICS
UPPSALA UNIVERSITY
PO BOX 480
SE-751 06 UPPSALA
SWEDEN
E-MAIL: svante.janson@math.uu.se
URL: http://www2.math.uu.se/~svante/