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STOCHASTIC TARGET GAMES WITH CONTROLLED LOSS

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We study a stochastic game where one player tries to find a strategy such that the state process reaches a target of controlled-loss-type, no matter which action is chosen by the other player. We provide, in a general setup, a relaxed geometric dynamic programming principle for this problem and derive, for the case of a controlled SDE, the corresponding dynamic programming equation in the sense of viscosity solutions. As an example, we consider a problem of partial hedging under Knightian uncertainty.

1. Introduction. We study a stochastic (semi) game of the following form. Given an initial condition (t, z) in time and space, we try to find a strategy $\mathfrak{u}[\cdot]$ such that the controlled state process $Z_{t,z}^{\mathfrak{u}[\nu],\nu}(\cdot)$ reaches a certain target at the given time T, no matter which control ν is chosen by the adverse player. The target is specified in terms of expected loss; that is, we are given a real-valued ("loss") function ℓ and try to keep the expected loss above a given threshold $p \in \mathbb{R}$,

$$\operatorname*{ess\,inf}_{\nu} \mathbb{E}\big[\ell\big(Z_{t,z}^{\mathfrak{u}[\nu],\nu}(T)\big)|\mathcal{F}_t\big] \geq p \qquad \text{a.s.} \tag{1.1}$$

Instead of a game, one may also see this as a target problem under Knightian uncertainty; then the adverse player has the role of choosing a worst-case scenario.

Our aim is to describe, for given t, the set $\Lambda(t)$ of all pairs (z, p) such that there exists a strategy $\mathfrak{u}$ attaining the target. We provide, in a general abstract framework, a geometric dynamic programming principle (GDP) for this set. To this end, p is seen as an additional state variable and formulated dynamically via a family $\{M^\nu\}$ of auxiliary martingales with expectation p, indexed by the adverse controls ν. Heuristically, the GDP then takes the following form: $\Lambda(t)$ consists of all (z, p) such that there exist a strategy $\mathfrak{u}$ and a family $\{M^\nu\}$ satisfying

$$\big(Z_{t,z}^{\mathfrak{u}[\nu],\nu}(\tau), M^\nu(\tau)\big) \in \Lambda(\tau) \qquad \text{a.s.}$$

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for all adverse controls ν and all stopping times $\tau \geq t$. The precise version of the GDP, stated in Theorem 2.1, incorporates several relaxations that allow us to deal with various technical problems. In particular, the selection of ε-optimal strategies is solved by a covering argument which is possible due to a continuity assumption on ℓ and a relaxation in the variable p. The martingale M^ν is constructed from the semimartingale decomposition of the adverse player's value process.

Our GDP is tailored such that the dynamic programming equation can be derived in the viscosity sense. We exemplify this in Theorem 3.4 for the standard setup where the state process is determined by a stochastic differential equation (SDE) with coefficients controlled by the two players; however, the general GDP applies also in other situations such as singular control. The solution of the equation, a partial differential equation (PDE) in our example, corresponds to the indicator function of (the complement of) the graph of Λ. In Theorem 3.8, we specialize to a case with a monotonicity condition, that is, particularly suitable for pricing problems in mathematical finance. Finally, in order to illustrate various points made throughout the paper, we consider a concrete example of pricing an option with partial hedging, according to a loss constraint, in a model where the drift and volatility coefficients of the underlying are uncertain. In a worst-case analysis, the uncertainty corresponds to an adverse player choosing the coefficients; a formula for the corresponding seller's price is given in Theorem 4.1.

Stochastic target (control) problems with almost-sure constraints, corresponding to the case where ℓ is an indicator function and ν is absent, were introduced in [24, 25] as an extension of the classical superhedging problem [9] in mathematical finance. Stochastic target problems with controlled loss were first studied in [3] and are inspired by the quantile hedging problem [13]. The present paper is the first to consider stochastic target games. The rigorous treatment of zero-sum stochastic differential games was pioneered in [12], where the mentioned selection problem for ε-optimal strategies was treated by a discretization and a passage to continuous-time limit in the PDEs. Let us remark, however, that we have not been able to achieve satisfactory results for our problem using such techniques. We have been importantly influenced by [7], where the value functions are defined in terms of essential infima and suprema, and then shown to be deterministic. The formulation with an essential infimum (rather than an infimum of suitable expectations) in (1.1) is crucial in our case, mainly because $\{M^\nu\}$ is constructed by a method of non-Markovian control, which raises the fairly delicate problem of dealing with one nullset for every adverse control ν.

The remainder of the paper is organized as follows. Section 2 contains the abstract setup and GDP. In Section 3 we specialize to the case of a controlled SDE and derive the corresponding PDE, first in the general case and then in the monotone case. The problem of hedging under uncertainty is discussed in Section 4.

2. Geometric dynamic programming principle. In this section, we obtain our geometric dynamic programming principle (GDP) in an abstract framework.

Some of our assumptions are simply the conditions we need in the proof of the theorem; we will illustrate later how to actually verify them in a typical setup.

2.1. *Problem statement*. We fix a time horizon $T > 0$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions of right-continuity and completeness. We shall consider two sets $\mathcal{U}$ and $\mathcal{V}$ of controls; for the sake of concreteness, we assume that each of these sets consists of stochastic processes on $(\Omega, \mathcal{F})$, indexed by $[0, T]$ and with values in some sets U and V, respectively. Moreover, let $\mathfrak{U}$ be a set of mappings $\mathfrak{u} : \mathcal{V} \to \mathcal{U}$. Each $\mathfrak{u} \in \mathfrak{U}$ is called a strategy, and the notation $\mathfrak{u}[\nu]$ will be used for the control it associates with $\nu \in \mathcal{V}$. In applications, $\mathfrak{U}$ will be chosen to consist of mappings that are nonanticipating; see Section 3 for an example. Furthermore, we are given a metric space $(\mathcal{Z}, d_{\mathcal{Z}})$ and, for each $(t, z) \in [0, T] \times \mathcal{Z}$ and $(\mathfrak{u}, \nu) \in \mathfrak{U} \times \mathcal{V}$, an adapted càdlàg process $Z^{\mathfrak{u}[\nu],\nu}_{t,z}(\cdot)$ with values in $\mathcal{Z}$ satisfying $Z^{\mathfrak{u}[\nu],\nu}_{t,z}(t) = z$. For brevity, we set

$$Z^{\mathfrak{u},\nu}_{t,z} := Z^{\mathfrak{u}[\nu],\nu}_{t,z}.$$

Let $\ell : \mathcal{Z} \to \mathbb{R}$ be a Borel-measurable function satisfying

$$\mathbb{E}\big[\big|\ell\big(Z^{\mathfrak{u},\nu}_{t,z}(T)\big)\big|\big] < \infty \qquad \text{for all } (t, z, \mathfrak{u}, \nu) \in [0, T] \times \mathcal{Z} \times \mathfrak{U} \times \mathcal{V}. \tag{2.1}$$

We interpret ℓ as a loss (or "utility") function and denote by

$$I(t, z, \mathfrak{u}, \nu) := \mathbb{E}\big[\ell\big(Z^{\mathfrak{u},\nu}_{t,z}(T)\big)|\mathcal{F}_t\big] \qquad (t, z, \mathfrak{u}, \nu) \in [0, T] \times \mathcal{Z} \times \mathfrak{U} \times \mathcal{V}$$

the expected loss given ν (for the player choosing $\mathfrak{u}$) and by

$$J(t, z, \mathfrak{u}) := \operatorname*{ess\,inf}_{\nu\in\mathcal{V}} I(t, z, \mathfrak{u}, \nu) \qquad (t, z, \mathfrak{u}) \in [0, T] \times \mathcal{Z} \times \mathfrak{U}$$

the worst-case expected loss. The main object of this paper is the reachability set

$$\begin{aligned}\Lambda(t) := \big\{(z, p) \in \mathcal{Z} \times \mathbb{R} :\ & \text{there exists } \mathfrak{u} \in \mathfrak{U} \\ & \text{such that } J(t, z, \mathfrak{u}) \geq p\ \mathbb{P}\text{-a.s.}\big\}.\end{aligned} \tag{2.2}$$

These are the initial conditions (z, p) such that starting at time t, the player choosing $\mathfrak{u}$ can attain an expected loss not worse than p, regardless of the adverse player's action ν. The main aim of this paper is to provide a geometric dynamic programming principle for $\Lambda(t)$. For the case without adverse player, a corresponding result was obtained in [24] for the target problem with almost-sure constraints and in [3] for the problem with controlled loss.

As mentioned above, the dynamic programming for problem (2.2) requires the introduction of a suitable set of martingales starting from $p \in \mathbb{R}$. This role will be played by certain families[4] $\{M^\nu, \nu \in \mathcal{V}\}$ of martingales which should be considered as additional controls. More precisely, we denote by $\mathcal{M}_{t,p}$ the set of all real-valued (right-continuous) martingales M satisfying $M(t) = p$ $\mathbb{P}$-a.s., and we

[4]Of course, there is no mathematical difference between families indexed by $\mathcal{V}$, like $\{M^\nu, \nu \in \mathcal{V}\}$, and mappings on $\mathcal{V}$, like $\mathfrak{u}$. We shall use both notions interchangeably, depending on notational convenience.

fix a set $\mathfrak{M}_{t,p}$ of families $\{M^\nu, \nu \in \mathcal{V}\} \subset \mathcal{M}_{t,p}$; further assumptions on $\mathfrak{M}_{t,p}$ will be introduced below. Since these martingales are not present in the original problem (2.2), we can choose $\mathfrak{M}_{t,p}$ at our convenience; see also Remark 2.2 below.

As usual in optimal control, we shall need to concatenate controls and strategies in time according to certain events. We use the notation

$$\nu \oplus_\tau \bar{\nu} := \nu \mathbf{1}_{[0,\tau]} + \bar{\nu} \mathbf{1}_{(\tau,T]}$$

for the concatenation of two controls $\nu, \bar{\nu} \in \mathcal{V}$ at a stopping time τ. We also introduce the set

$$\{\nu =_{(t,\tau]} \bar{\nu}\} := \{\omega \in \Omega : \nu_s(\omega) = \bar{\nu}_s(\omega) \text{ for all } s \in (t, \tau(\omega)]\}.$$

Analogous notation is used for elements of $\mathcal{U}$.

In contrast to the setting of control, strategies can be concatenated only at particular events and stopping times, as otherwise the resulting strategies would fail to be elements of $\mathfrak{U}$ (in particular, because they may fail to be nonanticipating, see also Section 3). Therefore, we need to formalize the events and stopping times which are admissible for this purpose: for each $t \le T$, we consider a set $\mathfrak{F}_t$ whose elements are families $\{A^\nu, \nu \in \mathcal{V}\} \subset \mathcal{F}_t$ of events indexed by $\mathcal{V}$, as well as a set $\mathfrak{T}_t$ whose elements are families $\{\tau^\nu, \nu \in \mathcal{V}\} \subset \mathcal{T}_t$, where $\mathcal{T}_t$ denotes the set of all stopping times with values in $[t, T]$. We assume that $\mathfrak{T}_t$ contains any deterministic time $s \in [t, T]$ (seen as a constant family $\tau^\nu \equiv s$, $\nu \in \mathcal{V}$). In practice, the sets $\mathfrak{F}_t$ and $\mathfrak{T}_t$ will not contain all families of events and stopping times, respectively; one will impose additional conditions on $\nu \mapsto A^\nu$ and $\nu \mapsto \tau^\nu$ that are compatible with the conditions defining $\mathfrak{U}$. Both sets should be seen as auxiliary objects which make it easier (if not possible) to verify the dynamic programming conditions below.

2.2. *The geometric dynamic programming principle.* We can now state the conditions for our main result. The first one concerns the concatenation of controls and strategies.

ASSUMPTION (C). The following hold for all $t \in [0, T]$:

(C1) Fix $\nu_0, \nu_1, \nu_2 \in \mathcal{V}$ and $A \in \mathcal{F}_t$. Then $\nu := \nu_0 \oplus_t (\nu_1 \mathbf{1}_A + \nu_2 \mathbf{1}_{A^c}) \in \mathcal{V}$.

(C2) Fix $(\mathfrak{u}_j)_{j \ge 0} \subset \mathfrak{U}$, and let $\{A_j^\nu, \nu \in \mathcal{V}\}_{j \ge 1} \subset \mathfrak{F}_t$ be such that $\{A_j^\nu, j \ge 1\}$ forms a partition of Ω for each $\nu \in \mathcal{V}$. Then $\mathfrak{u} \in \mathfrak{U}$ for

$$\mathfrak{u}[\nu] := \mathfrak{u}_0[\nu] \oplus_t \sum_{j \ge 1} \mathfrak{u}_j[\nu] \mathbf{1}_{A_j^\nu}, \qquad \nu \in \mathcal{V}.$$

(C3) Let $\mathfrak{u} \in \mathfrak{U}$ and $\nu \in \mathcal{V}$. Then $\mathfrak{u}[\nu \oplus_t \cdot] \in \mathfrak{U}$.

(C4) Let $\{A^\nu, \nu \in \mathcal{V}\} \subset \mathcal{F}_t$ be a family of events such that $A^{\nu_1} \cap \{\nu_1 =_{(0,t]} \nu_2\} = A^{\nu_2} \cap \{\nu_1 =_{(0,t]} \nu_2\}$ for all $\nu_1, \nu_2 \in \mathcal{V}$. Then $\{A^\nu, \nu \in \mathcal{V}\} \in \mathfrak{F}_t$.

(C5) Let $\{\tau^\nu, \nu \in \mathcal{V}\} \in \mathfrak{T}_t$. Then $\{\tau^{\nu_1} \le s\} \cap \{\nu_1 =_{(0,s]} \nu_2\} = \{\tau^{\nu_2} \le s\} \cap \{\nu_1 =_{(0,s]} \nu_2\}$ $\mathbb{P}$-a.s. for all $\nu_1, \nu_2 \in \mathcal{V}$ and $s \in [t, T]$.

(C6) Let $\{\tau^\nu, \nu \in \mathcal{V}\} \in \mathfrak{T}_t$. Then, for all $t \le s_1 \le s_2 \le T$, $\{\{\tau^\nu \in (s_1, s_2]\}, \nu \in \mathcal{V}\}$ and $\{\{\tau^\nu \notin (s_1, s_2]\}, \nu \in \mathcal{V}\}$ belong to $\mathfrak{F}_{s_2}$.

The second condition concerns the behavior of the state process.

ASSUMPTION (Z). The following hold for all $(t, z, p) \in [0, T] \times \mathcal{Z} \times \mathbb{R}$ and $s \in [t, T]$:

(Z1) $Z_{t,z}^{\mathfrak{u}_1,\nu}(s)(\omega) = Z_{t,z}^{\mathfrak{u}_2,\nu}(s)(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \{\mathfrak{u}_1[\nu] =_{(t,s]} \mathfrak{u}_2[\nu]\}$, for all $\nu \in \mathcal{V}$ and $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{U}$.

(Z2) $Z_{t,z}^{\mathfrak{u},\nu_1}(s)(\omega) = Z_{t,z}^{\mathfrak{u},\nu_2}(s)(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \{\nu_1 =_{(0,s]} \nu_2\}$, for all $\mathfrak{u} \in \mathfrak{U}$ and $\nu_1, \nu_2 \in \mathcal{V}$.

(Z3) $M^{\nu_1}(s)(\omega) = M^{\nu_2}(s)(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \{\nu_1 =_{(0,s]} \nu_2\}$, for all $\{M^\nu, \nu \in \mathcal{V}\} \in \mathfrak{M}_{t,p}$ and $\nu_1, \nu_2 \in \mathcal{V}$.

(Z4) There exists a constant $K(t, z) \in \mathbb{R}$ such that

$$\operatorname*{ess\,sup}_{\mathfrak{u}\in\mathfrak{U}} \operatorname*{ess\,inf}_{\nu\in\mathcal{V}} \mathbb{E}\big[\ell\big(Z_{t,z}^{\mathfrak{u},\nu}(T)\big)|\mathcal{F}_t\big] = K(t, z) \qquad \mathbb{P}\text{-a.s.}$$

The nontrivial assumption here is, of course, (Z4), stating that (a version of) the random variable $\operatorname{ess\,sup}_{\mathfrak{u}\in\mathfrak{U}} \operatorname{ess\,inf}_{\nu\in\mathcal{V}} \mathbb{E}[\ell(Z_{t,z}^{\mathfrak{u},\nu}(T))|\mathcal{F}_t]$ is *deterministic*. For the game determined by a Brownian SDE as considered in Section 3, this will be true by a result of [7], which, in turn, goes back to an idea of [21] (see also [16]). An extension to jump diffusions can be found in [6].

While the above assumptions are fundamental, the following conditions are of technical nature. We shall illustrate later how they can be verified.

ASSUMPTION (I). Let $(t, z) \in [0, T] \times \mathcal{Z}$, $\mathfrak{u} \in \mathfrak{U}$ and $\nu \in \mathcal{V}$.

(I1) There exists an adapted right-continuous process $N_{t,z}^{\mathfrak{u},\nu}$ of class (D) such that

$$\operatorname*{ess\,inf}_{\bar\nu\in\mathcal{V}} \mathbb{E}\big[\ell\big(Z_{t,z}^{\mathfrak{u},\nu\oplus_s\bar\nu}(T)\big)|\mathcal{F}_s\big] \geq N_{t,z}^{\mathfrak{u},\nu}(s) \qquad \mathbb{P}\text{-a.s. for all } s \in [t, T].$$

(I2) There exists an adapted right-continuous process $L_{t,z}^{\mathfrak{u},\nu}$ such that $L_{t,z}^{\mathfrak{u},\nu}(s) \in L^1$ and

$$\operatorname*{ess\,inf}_{\bar{\mathfrak{u}}\in\mathfrak{U}} \mathbb{E}\big[\ell\big(Z_{t,z}^{\mathfrak{u}\oplus_s\bar{\mathfrak{u}},\nu}(T)\big)|\mathcal{F}_s\big] \geq L_{t,z}^{\mathfrak{u},\nu}(s) \qquad \mathbb{P}\text{-a.s. for all } s \in [t, T].$$

Moreover, $L_{t,z}^{\mathfrak{u},\nu_1}(s)(\omega) = L_{t,z}^{\mathfrak{u},\nu_2}(s)(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \{\nu_1 =_{(0,s]} \nu_2\}$, for all $\mathfrak{u} \in \mathcal{U}$ and $\nu_1, \nu_2 \in \mathcal{V}$.

ASSUMPTION (R). Let $(t, z) \in [0, T] \times \mathcal{Z}$.

(R1) Fix $s \in [t, T]$ and $\varepsilon > 0$. Then there exist a Borel-measurable partition $(B_j)_{j\geq1}$ of $\mathcal{Z}$ and a sequence $(z_j)_{j\geq1} \subset \mathcal{Z}$ such that for all $\mathfrak{u} \in \mathfrak{U}$, $\nu \in \mathcal{V}$ and $j \geq 1$,

$$\left.\begin{aligned} \mathbb{E}\big[\ell\big(Z_{t,z}^{\mathfrak{u},\nu}(T)\big)|\mathcal{F}_s\big] &\geq I(s, z_j, \mathfrak{u}, \nu) - \varepsilon, \\ \operatorname*{ess\,inf}_{\bar\nu\in\mathcal{V}} \mathbb{E}\big[\ell\big(Z_{t,z}^{\mathfrak{u},\nu\oplus_s\bar\nu}(T)\big)|\mathcal{F}_s\big] &\leq J\big(s, z_j, \mathfrak{u}[\nu \oplus_s \cdot]\big) + \varepsilon, \\ K(s, z_j) - \varepsilon \leq K\big(s, Z_{t,z}^{\mathfrak{u},\nu}(s)\big) &\leq K(s, z_j) + \varepsilon \end{aligned}\right\}$$

$$\mathbb{P}\text{-a.s. on } \big\{Z_{t,z}^{\mathfrak{u},\nu}(s) \in B_j\big\}.$$

(R2) $\lim_{\delta\to 0}\sup_{\nu\in\mathcal{V},\tau\in\mathcal{T}_t}\mathbb{P}\{\sup_{0\le h\le\delta} d_{\mathcal{Z}}(Z_{t,z}^{\mathfrak{u},\nu}(\tau+h), Z_{t,z}^{\mathfrak{u},\nu}(\tau))\ge\varepsilon\} = 0$ for all $\mathfrak{u}\in\mathfrak{U}$ and $\varepsilon>0$.

Our GDP will be stated in terms of the closure

$$\bar{\Lambda}(t) := \left\{ \begin{array}{l} (z,p)\in\mathcal{Z}\times\mathbb{R}: \text{ there exist } (t_n,z_n,p_n)\to(t,z,p) \\ \text{such that } (z_n,p_n)\in\Lambda(t_n) \text{ and } t_n\ge t \text{ for all } n\ge 1 \end{array} \right\}$$

and the uniform interior

$$\Lambda_\iota(t) := \{(z,p)\in\mathcal{Z}\times\mathbb{R}: (t',z',p')\in B_\iota(t,z,p) \text{ implies } (z',p')\in\Lambda(t')\},$$

where $B_\iota(t,z,p)\subset[0,T]\times\mathcal{Z}\times\mathbb{R}$ denotes the open ball with center (t,z,p) and radius $\iota>0$ [with respect to the distance function $d_{\mathcal{Z}}(z,z')+|p-p'|+|t-t'|$]. The relaxation from Λ to $\bar{\Lambda}$ and Λ_ι essentially allows us to reduce to stopping times with countably many values in the proof of the GDP and thus to avoid regularity assumptions in the time variable. We shall also relax the variable p in the assertion of (GDP2); this is inspired by [4] and important for the covering argument in the proof of (GDP2), which, in turn, is crucial due to the lack of a measurable selection theorem for strategies. Of course, all our relaxations are tailored such that they will not interfere substantially with the derivation of the dynamic programming equation; cf. Section 3.

THEOREM 2.1. *Fix $(t,z,p)\in[0,T]\times\mathcal{Z}\times\mathbb{R}$ and let Assumptions (C), (Z), (I) and (R) hold true.*

(GDP1) *If $(z,p)\in\Lambda(t)$, then there exist $\mathfrak{u}\in\mathfrak{U}$ and $\{M^\nu,\nu\in\mathcal{V}\}\subset\mathcal{M}_{t,p}$ such that*

$$(Z_{t,z}^{\mathfrak{u},\nu}(\tau), M^\nu(\tau))\in\bar{\Lambda}(\tau) \qquad \mathbb{P}\text{-a.s. for all } \nu\in\mathcal{V} \text{ and } \tau\in\mathcal{T}_t.$$

(GDP2) *Let $\iota>0$, $\mathfrak{u}\in\mathfrak{U}$, $\{M^\nu,\nu\in\mathcal{V}\}\in\mathfrak{M}_{t,p}$ and $\{\tau^\nu,\nu\in\mathcal{V}\}\in\mathfrak{T}_t$ be such that*

$$(Z_{t,z}^{\mathfrak{u},\nu}(\tau^\nu), M^\nu(\tau^\nu))\in\Lambda_\iota(\tau^\nu) \qquad \mathbb{P}\text{-a.s. for all } \nu\in\mathcal{V}$$

and suppose that $\{M^\nu(\tau^\nu)^+:\nu\in\mathcal{V}\}$ and $\{L_{t,z}^{\mathfrak{u},\nu}(\tau')^-:\nu\in\mathcal{V},\tau'\in\mathcal{T}_t\}$ are uniformly integrable, where $L_{t,z}^{\mathfrak{u},\nu}$ is as in (I2). Then $(z,p-\varepsilon)\in\Lambda(t)$ for all $\varepsilon>0$.

The proof is stated in Sections 2.3 and 2.4 below.

REMARK 2.2. We shall see in the proof that the family $\{M^\nu,\nu\in\mathcal{V}\}\subset\mathcal{M}_{t,p}$ in (GDP1) can actually be chosen to be nonanticipating in the sense of (Z3). However, this will not be used when (GDP1) is applied to derive the dynamic programming equation. Whether $\{M^\nu,\nu\in\mathcal{V}\}$ is an element of $\mathfrak{M}_{t,p}$ will depend on the definition of the latter set; in fact, we did not make any assumption about its richness. In many application, it is possible to take $\mathfrak{M}_{t,p}$ to be the set of all nonanticipating families in $\mathcal{M}_{t,p}$; however, we prefer to leave some freedom for the definition of $\mathfrak{M}_{t,p}$ since this may be useful in ensuring the uniform integrability required in (GDP2).

We conclude this section with a version of the GDP for the case $\mathcal{Z} = \mathbb{R}^d$, where we show how to reduce from standard regularity conditions on the state process and the loss function to the Assumptions (R1) and (I).

COROLLARY 2.3. *Let Assumptions* (C), (Z) *and* (R2) *hold true. Assume also that ℓ is continuous and that there exist constants $C \geq 0$ and $\bar{q} > q \geq 0$ and a locally bounded function $\varrho : \mathbb{R}^d \mapsto \mathbb{R}_+$ such that*

$$|\ell(z)| \leq C(1 + |z|^q), \tag{2.3}$$

$$\operatorname*{ess\,sup}_{(\bar{\mathfrak{u}},\bar{\nu})\in\mathfrak{U}\times\mathcal{V}} \mathbb{E}\big[\big|Z^{\bar{\mathfrak{u}},\bar{\nu}}_{t,z}(T)\big|^{\bar{q}}|\mathcal{F}_t\big] \leq \varrho(z)^{\bar{q}} \qquad \mathbb{P}\text{-}a.s. \tag{2.4}$$

and

$$\begin{aligned} &\operatorname*{ess\,sup}_{(\bar{\mathfrak{u}},\bar{\nu})\in\mathfrak{U}\times\mathcal{V}} \mathbb{E}\big[\big|Z^{\mathfrak{u}\oplus_s\bar{\mathfrak{u}},\nu\oplus_s\bar{\nu}}_{t,z}(T) - Z^{\bar{\mathfrak{u}},\nu\oplus_s\bar{\nu}}_{s,z'}(T)\big||\mathcal{F}_s\big] \\ &\qquad \leq C\big|Z^{\mathfrak{u},\nu}_{t,z}(s) - z'\big| \qquad \mathbb{P}\text{-}a.s. \end{aligned} \tag{2.5}$$

for all $(t, z) \in [0, T] \times \mathbb{R}^d$, $(s, z') \in [t, T] \times \mathbb{R}^d$ and $(\mathfrak{u}, \nu) \in \mathfrak{U} \times \mathcal{V}$.

Let $(t, z) \in [0, T] \times \mathbb{R}^d$, and let $\{\tau^{\mathfrak{u},\nu}, (\mathfrak{u}, \nu) \in \mathfrak{U} \times \mathcal{V}\} \subset \mathcal{T}_t$ be such that the collection $\{Z^{\mathfrak{u},\nu}_{t,z}(\tau^{\mathfrak{u},\nu}), (\mathfrak{u}, \nu) \in \mathfrak{U} \times \mathcal{V}\}$ is uniformly bounded in L^∞.

(GDP1′) *If $(z, p + \varepsilon) \in \Lambda(t)$ for some $\varepsilon > 0$, then there exist $\mathfrak{u} \in \mathfrak{U}$ and $\{M^\nu, \nu \in \mathcal{V}\} \subset \mathcal{M}_{t,p}$ such that*

$$\big(Z^{\mathfrak{u},\nu}_{t,z}(\tau^{\mathfrak{u},\nu}), M^\nu(\tau^{\mathfrak{u},\nu})\big) \in \bar{\Lambda}(\tau^{\mathfrak{u},\nu}) \qquad \mathbb{P}\text{-}a.s. \text{ for all } \nu \in \mathcal{V}.$$

(GDP2′) *If $\iota > 0$, $\mathfrak{u} \in \mathfrak{U}$ and $\{M^\nu, \nu \in \mathcal{V}\} \in \mathfrak{M}_{t,p}$ are such that*

$$\big(Z^{\mathfrak{u},\nu}_{t,z}(\tau^{\mathfrak{u},\nu}), M^\nu(\tau^{\mathfrak{u},\nu})\big) \in \Lambda_\iota(\tau^{\mathfrak{u},\nu}) \qquad \mathbb{P}\text{-}a.s. \text{ for all } \nu \in \mathcal{V}$$

and $\{\tau^{\mathfrak{u},\nu}, \nu \in \mathcal{V}\} \in \mathfrak{T}_t$, then $(z, p - \varepsilon) \in \Lambda(t)$ for all $\varepsilon > 0$.

We remark that Corollary 2.3 is usually applied in a setting where $\tau^{\mathfrak{u},\nu}$ is the exit time of $Z^{\mathfrak{u},\nu}_{t,z}$ from a given ball, so that the boundedness assumption is not restrictive. (Some adjustments are needed when the state process admits unbounded jumps; see also [18].)

2.3. *Proof of* (*GDP*1). We fix $t \in [0, T]$ and $(z, p) \in \Lambda(t)$ for the remainder of this proof. By definition (2.2) of $\Lambda(t)$, there exists $\mathfrak{u} \in \mathfrak{U}$ such that

$$\mathbb{E}\big[G(\nu)|\mathcal{F}_t\big] \geq p \qquad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V}, \text{ where } G(\nu) := \ell\big(Z^{\mathfrak{u},\nu}_{t,z}(T)\big). \tag{2.6}$$

In order to construct the family $\{M^\nu, \nu \in \mathcal{V}\} \subset \mathcal{M}_{t,p}$ of martingales, we consider

$$S^\nu(r) := \operatorname*{ess\,inf}_{\bar{\nu}\in\mathcal{V}} \mathbb{E}\big[G(\nu \oplus_r \bar{\nu})|\mathcal{F}_r\big], \qquad t \leq r \leq T. \tag{2.7}$$

We shall obtain M^ν from a Doob–Meyer-type decomposition of S^ν. This can be seen as a generalization with respect to [3], where the necessary martingale was trivially constructed by taking the conditional expectation of the terminal reward.

Step 1. We have $S^\nu(r) \in L^1(\mathbb{P})$ and $\mathbb{E}[S^\nu(r)|\mathcal{F}_s] \ge S^\nu(s)$ for all $t \le s \le r \le T$ and $\nu \in \mathcal{V}$.

The integrability of $S^\nu(r)$ follows from (2.1) and (I1). To see the submartingale property, we first show that the family $\{\mathbb{E}[G(\nu \oplus_r \bar{\nu})|\mathcal{F}_r], \bar{\nu} \in \mathcal{V}\}$ is directed downward. Indeed, given $\bar{\nu}_1, \bar{\nu}_2 \in \mathcal{V}$, the set

$$A := \{\mathbb{E}[G(\nu \oplus_r \bar{\nu}_1)|\mathcal{F}_r] \le \mathbb{E}[G(\nu \oplus_r \bar{\nu}_2)|\mathcal{F}_r]\}$$

is in $\mathcal{F}_r$; therefore, $\bar{\nu}_3 := \nu \oplus_r (\bar{\nu}_1 \mathbf{1}_A + \bar{\nu}_2 \mathbf{1}_{A^c})$ is an element of $\mathcal{V}$ by Assumption (C1). Hence, (Z2) yields that

$$\begin{aligned}
\mathbb{E}[G(\nu \oplus_r \bar{\nu}_3)|\mathcal{F}_r] &= \mathbb{E}[G(\nu \oplus_r \bar{\nu}_1)\mathbf{1}_A + G(\nu \oplus_r \bar{\nu}_2)\mathbf{1}_{A^c}|\mathcal{F}_r] \\
&= \mathbb{E}[G(\nu \oplus_r \bar{\nu}_1)|\mathcal{F}_r]\mathbf{1}_A + \mathbb{E}[G(\nu \oplus_r \bar{\nu}_2)|\mathcal{F}_r]\mathbf{1}_{A^c} \\
&= \mathbb{E}[G(\nu \oplus_r \bar{\nu}_1)|\mathcal{F}_r] \wedge \mathbb{E}[G(\nu \oplus_r \bar{\nu}_2)|\mathcal{F}_r].
\end{aligned}$$

As a result, we can find a sequence $(\bar{\nu}_n)_{n \ge 1}$ in $\mathcal{V}$ such that $\mathbb{E}[G(\nu \oplus_r \bar{\nu}_n)|\mathcal{F}_r]$ decreases $\mathbb{P}$-a.s. to $S^\nu(r)$; cf. [19], Proposition VI-1-1. Recalling (2.1) and that $S^\nu(r) \in L^1(\mathbb{P})$, monotone convergence yields that

$$\begin{aligned}
\mathbb{E}[S^\nu(r)|\mathcal{F}_s] &= \mathbb{E}\Big[\lim_{n\to\infty} \mathbb{E}[G(\nu \oplus_r \bar{\nu}_n)|\mathcal{F}_r]|\mathcal{F}_s\Big] \\
&= \lim_{n\to\infty} \mathbb{E}[G(\nu \oplus_r \bar{\nu}_n)|\mathcal{F}_s] \\
&\ge \operatorname*{ess\,inf}_{\bar{\nu}\in\mathcal{V}} \mathbb{E}[G(\nu \oplus_r \bar{\nu})|\mathcal{F}_s] \\
&\ge \operatorname*{ess\,inf}_{\bar{\nu}\in\mathcal{V}} \mathbb{E}[G(\nu \oplus_s \bar{\nu})|\mathcal{F}_s] \\
&= S^\nu(s),
\end{aligned}$$

where the last inequality follows from the fact that any control $\nu \oplus_r \bar{\nu}$, where $\bar{\nu} \in \mathcal{V}$, can be written in the form $\nu \oplus_s (\nu \oplus_r \bar{\nu})$; cf. (C1).

Step 2. There exists a family of càdlàg martingales $\{M^\nu, \nu \in \mathcal{V}\} \subset \mathcal{M}_{t,p}$ such that $S^\nu(r) \ge M^\nu(r)$ $\mathbb{P}$-a.s. for all $r \in [t, T]$ and $\nu \in \mathcal{V}$.

Fix $\nu \in \mathcal{V}$. By step 1, $S^\nu(\cdot)$ satisfies the submartingale property. Therefore,

$$S_+(r)(\omega) := \lim_{u\in(r,T]\cap\mathbb{Q}, u\to r} S^\nu(u)(\omega) \qquad \text{for } 0 \le r < T \quad \text{and} \quad S_+(T) := S^\nu(T)$$

is well defined $\mathbb{P}$-a.s.; moreover, recalling that the filtration $\mathbb{F}$ satisfies the usual conditions, S_+ is a (right-continuous) submartingale satisfying $S_+(r) \ge S^\nu(r)$ $\mathbb{P}$-a.s. for all $r \in [t, T]$; cf. [8], Theorem VI.2. Let $H \subset [t, T]$ be the set of points where the function $r \mapsto \mathbb{E}[S^\nu(r)]$ is not right-continuous. Since this function is increasing, H is at most countable. (If H happens to be the empty set, then S_+

defines a modification of S^ν and the Doob–Meyer decomposition of S_+ yields the result.) Consider the process

$$\bar{S}(r) := S_+(r)\mathbf{1}_{H^c}(r) + S^\nu(r)\mathbf{1}_H(r), \qquad r \in [t, T].$$

The arguments (due to Lenglart) in the proof of [8], Theorem 10 of Appendix 1, show that $\bar{S}$ is an *optional modification* of S^ν and $\mathbb{E}[\bar{S}(\tau)|\mathcal{F}_\sigma] \geq \bar{S}(\sigma)$ for all $\sigma, \tau \in \mathcal{T}_t$ such that $\sigma \leq \tau$; that is, $\bar{S}$ is a strong submartingale. Let $N = N_{t,z}^{\mathfrak{u},\nu}$ be a right-continuous process of class (D) as in (I1); then $S^\nu(r) \geq N(r)$ $\mathbb{P}$-a.s. for all r implies that $S_+(r) \geq N(r)$ $\mathbb{P}$-a.s. for all r, and since both S_+ and N are right-continuous, this shows that $S_+ \geq N$ up to evanescence. Recalling that H is countable, we deduce that $\bar{S} \geq N$ up to evanescence, and as $\bar{S}$ is bounded from above by the martingale generated by $\bar{S}(T)$, we conclude that $\bar{S}$ is of class (D).

Now the decomposition result of Mertens [17], Theorem 3, yields that there exist a (true) martingale $\bar{M}$ and a nondecreasing (not necessarily càdlàg) predictable process $\bar{C}$ with $\bar{C}(t) = 0$ such that

$$\bar{S} = \bar{M} + \bar{C}$$

and in view of the usual conditions, $\bar{M}$ can be chosen to be càdlàg. We can now define $M^\nu := \bar{M} - \bar{M}(t) + p$ on $[t, T]$ and $M^\nu(r) := p$ for $r \in [0, t)$, then $M^\nu \in \mathcal{M}_{t,p}$. Noting that $\bar{M}(t) = \bar{S}(t) = S^\nu(t) \geq p$ by (2.6), we see that M^ν has the required property

$$M^\nu(r) \leq \bar{M}(r) \leq \bar{S}(r) = S^\nu(r) \qquad \mathbb{P}\text{-a.s. for all } r \in [t, T].$$

Step 3. Let $\tau \in \mathcal{T}_t$ have countably many values. Then

$$K\big(\tau, Z_{t,z}^{\mathfrak{u},\nu}(\tau)\big) \geq M^\nu(\tau) \qquad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V}.$$

Fix $\nu \in \mathcal{V}$ and $\varepsilon > 0$, let M^ν be as in step 2 and let $(t_i)_{i\geq 1}$ be the distinct values of τ. By step 2, we have

$$M^\nu(t_i) \leq \operatorname*{ess\,inf}_{\bar{\nu}\in\mathcal{V}} \mathbb{E}\big[\ell\big(Z_{t,z}^{\mathfrak{u},\nu\oplus_{t_i}\bar{\nu}}(T)\big)|\mathcal{F}_{t_i}\big] \qquad \mathbb{P}\text{-a.s.}, i \geq 1.$$

Moreover, (R1) yields that for each $i \geq 1$, we can find a sequence $(z_{ij})_{j\geq 1} \subset \mathcal{Z}$ and a Borel partition $(B_{ij})_{j\geq 1}$ of $\mathcal{Z}$ such that

$$\operatorname*{ess\,inf}_{\bar{\nu}\in\mathcal{V}} \mathbb{E}\big[\ell\big(Z_{t,z}^{\mathfrak{u},\nu\oplus_{t_i}\bar{\nu}}(T)\big)|\mathcal{F}_{t_i}\big](\omega) \leq J\big(t_i, z_{ij}, \mathfrak{u}[\nu \oplus_{t_i} \cdot]\big)(\omega) + \varepsilon$$
$$\text{for } \mathbb{P}\text{-a.e. } \omega \in C_{ij} := \big\{Z_{t,z}^{\mathfrak{u},\nu}(t_i) \in B_{ij}\big\}.$$

Since (C3) and the definition of K in (Z4) yield that $J(t_i, z_{ij}, \mathfrak{u}[\nu \oplus_{t_i} \cdot]) \leq K(t_i, z_{ij})$, we conclude by (R1) that

$$M^\nu(t_i)(\omega) \leq K(t_i, z_{ij}) + \varepsilon \leq K\big(t_i, Z_{t,z}^{\mathfrak{u},\nu}(t_i)(\omega)\big) + 2\varepsilon \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in C_{ij}.$$

Let $A_i := \{\tau = t_i\} \in \mathcal{F}_\tau$. Then $(A_i \cap C_{ij})_{i,j\geq 1}$ forms a partition of Ω, and the above shows that

$$M^\nu(\tau) - 2\varepsilon \leq \sum_{i,j\geq 1} K\big(t_i, Z_{t,z}^{\mathfrak{u},\nu}(t_i)\big)\mathbf{1}_{A_i\cap C_{ij}} = K\big(\tau, Z_{t,z}^{\mathfrak{u},\nu}(\tau)\big) \qquad \mathbb{P}\text{-a.s.}$$

As $\varepsilon > 0$ was arbitrary, the claim follows.

Step 4. We can now prove (GDP1). Given $\tau \in \mathcal{T}_t$, pick a sequence $(\tau_n)_{n\geq 1} \subset \mathcal{T}_t$ such that each τ_n has countably many values and $\tau_n \downarrow \tau$ $\mathbb{P}$-a.s. In view of the last statement of Lemma 2.4 below, step 3 implies that

$$\big(Z_{t,z}^{\mathfrak{u},\nu}(\tau_n), M^\nu(\tau_n) - n^{-1}\big) \in \Lambda(\tau_n) \qquad \mathbb{P}\text{-a.s. for all } n \geq 1.$$

However, using that $Z_{t,z}^{\mathfrak{u},\nu}$ and M^ν are càdlàg, we have

$$\big(\tau_n, Z_{t,z}^{\mathfrak{u},\nu}(\tau_n), M^\nu(\tau_n) - n^{-1}\big) \to \big(\tau, Z_{t,z}^{\mathfrak{u},\nu}(\tau), M^\nu(\tau)\big) \qquad \mathbb{P}\text{-a.s. as } n \to \infty,$$

so that, by the definition of $\bar{\Lambda}$, we deduce that $(Z_{t,z}^{\mathfrak{u},\nu}(\tau), M^\nu(\tau)) \in \bar{\Lambda}(\tau)$ $\mathbb{P}$-a.s.

LEMMA 2.4. *Let Assumptions* (C2), (C4), (Z1) *and* (Z4) *hold true. For each* $\varepsilon > 0$, *there exists a mapping* $\mu^\varepsilon : [0, T] \times \mathcal{Z} \to \mathfrak{U}$ *such that*

$$J\big(t, z, \mu^\varepsilon(t, z)\big) \geq K(t, z) - \varepsilon \qquad \mathbb{P}\textit{-a.s. for all } (t, z) \in [0, T] \times \mathcal{Z}.$$

In particular, if $(t, z, p) \in [0, T] \times \mathcal{Z} \times \mathbb{R}$, *then* $K(t, z) > p$ *implies* $(z, p) \in \Lambda(t)$.

PROOF. Since $K(t, z)$ was defined in (Z4) as the essential supremum of $J(t, z, \mathfrak{u})$ over $\mathfrak{u}$, there exists a sequence $(\mathfrak{u}^k(t, z))_{k\geq 1} \subset \mathfrak{U}$ such that

$$\sup_{k\geq 1} J\big(t, z, \mathfrak{u}^k(t, z)\big) = K(t, z) \qquad \mathbb{P}\text{-a.s.} \tag{2.8}$$

Set $\Delta_{t,z}^0 := \varnothing$ and define inductively the $\mathcal{F}_t$-measurable sets

$$\Delta_{t,z}^k := \big\{J\big(t, z, \mathfrak{u}^k(t, z)\big) \geq K(t, z) - \varepsilon\big\} \setminus \bigcup_{j=0}^{k-1} \Delta_{t,z}^j, \qquad k \geq 1.$$

By (2.8), the family $\{\Delta_{t,z}^k, k \geq 1\}$ forms a partition of Ω. Clearly, each $\Delta_{t,z}^k$ (seen as a constant family) satisfies the requirement of (C4) since it does not depend on ν and therefore belongs to $\mathfrak{F}_t$. Hence after fixing some $\mathfrak{u}_0 \in \mathfrak{U}$, (C2) implies that

$$\mu^\varepsilon(t, z) := \mathfrak{u}_0 \oplus_t \sum_{k\geq 1} \mathfrak{u}^k(t, z)\mathbf{1}_{\Delta_{t,z}^k} \in \mathfrak{U},$$

while (Z1) ensures that

$$\begin{aligned} J\big(t, z, \mu^\varepsilon(t, z)\big) &= \operatorname*{ess\,inf}_{\nu\in\mathcal{V}} \mathbb{E}\big[\ell\big(Z_{t,z}^{\mu^\varepsilon(t,z),\nu}(T)\big)|\mathcal{F}_t\big] \\ &= \operatorname*{ess\,inf}_{\nu\in\mathcal{V}} \mathbb{E}\bigg[\sum_{k\geq 1} \ell\big(Z_{t,z}^{\mathfrak{u}^k(t,z),\nu}(T)\big)\mathbf{1}_{\Delta_{t,z}^k}|\mathcal{F}_t\bigg] \\ &= \operatorname*{ess\,inf}_{\nu\in\mathcal{V}} \sum_{k\geq 1} \mathbb{E}\big[\ell\big(Z_{t,z}^{\mathfrak{u}^k(t,z),\nu}(T)\big)|\mathcal{F}_t\big]\mathbf{1}_{\Delta_{t,z}^k}, \end{aligned}$$

where the last step used that $\Delta^k_{t,z}$ is $\mathcal{F}_t$-measurable. Since

$$\mathbb{E}\big[\ell\big(Z^{\mathfrak{u}^k(t,z),\nu}_{t,z}(T)\big)|\mathcal{F}_t\big] \geq J\big(t,z,\mathfrak{u}^k(t,z)\big)$$

by the definition of J, it follows by the definition of $\{\Delta^k_{t,z}, k \geq 1\}$ that

$$J\big(t,z,\mu^\varepsilon(t,z)\big) \geq \sum_{k\geq 1} J\big(t,z,\mathfrak{u}^k(t,z)\big)\mathbf{1}_{\Delta^k_{t,z}} \geq K(t,z) - \varepsilon \qquad \mathbb{P}\text{-a.s.}$$

as required. $\square$

REMARK 2.5. Let us mention that the GDP could also be formulated using families of submartingales $\{S^\nu, \nu \in \mathcal{V}\}$ rather than martingales. Namely, in (GDP1), these would be the processes defined by (2.7). However, such a formulation would not be advantageous for applications as in Section 3 because we would then need an additional control process to describe the (possibly very irregular) finite variation part of S^ν. The fact that the martingales $\{M^\nu, \nu \in \mathcal{V}\}$ are actually sufficient to obtain a useful GDP can be explained heuristically as follows: the relevant situation for the dynamic programming equation corresponds to the adverse player choosing an (almost) optimal control ν, and then the value process S^ν will be (almost) a martingale.

2.4. *Proof of* (*GDP*2). In the sequel, we fix $(t,z,p) \in [0,T] \times \mathcal{Z} \times \mathbb{R}$ and let $\iota > 0$, $\mathfrak{u} \in \mathfrak{U}$, $\{M^\nu, \nu \in \mathcal{V}\} \in \mathfrak{M}_{t,p}$, $\{\tau^\nu, \nu \in \mathcal{V}\} \in \mathfrak{T}_t$ and $L^{\mathfrak{u},\nu}_{t,z}$ be as in (GDP2). We shall use the dyadic discretization for the stopping times τ^ν; that is, given $n \geq 1$, we set

$$\tau^\nu_n = \sum_{0\leq i\leq 2^n-1} t^n_{i+1}\mathbf{1}_{(t^n_i,t^n_{i+1}]}(\tau^\nu) \qquad \text{where } t^n_i = i2^{-n}T \text{ for } 0 \leq i \leq 2^n.$$

We shall first state the proof under the additional assumption that

$$M^\nu(\cdot) = M^\nu(\cdot \wedge \tau^\nu) \qquad \text{for all } \nu \in \mathcal{V}. \tag{2.9}$$

Step 1. Fix $\varepsilon > 0$ and $n \geq 1$. There exists $\mathfrak{u}^\varepsilon_n \in \mathfrak{U}$ such that

$$\mathbb{E}\big[\ell\big(Z^{\mathfrak{u}^\varepsilon_n,\nu}_{t,z}(T)\big)|\mathcal{F}_{\tau^\nu_n}\big] \geq K\big(\tau^\nu_n, Z^{\mathfrak{u},\nu}_{t,z}(\tau^\nu_n)\big) - \varepsilon \qquad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V}.$$

We fix $\varepsilon > 0$ and $n \geq 1$. It follows from (R1) and (C2) that, for each $i \leq 2^n$, we can find a Borel partition $(B_{ij})_{j\geq 1}$ of $\mathcal{Z}$ and a sequence $(z_{ij})_{j\geq 1} \subset \mathcal{Z}$ such that, for all $\bar{\mathfrak{u}} \in \mathfrak{U}$ and $\nu \in \mathcal{V}$,

$$\mathbb{E}\big[\ell\big(Z^{\mathfrak{u}\oplus_{t^n_i}\bar{\mathfrak{u}},\nu}_{t,z}(T)\big)|\mathcal{F}_{t^n_i}\big](\omega) \geq I\big(t^n_i, z_{ij}, \mathfrak{u}\oplus_{t^n_i}\bar{\mathfrak{u}}, \nu\big)(\omega) - \varepsilon \tag{2.10}$$

and

$$\begin{aligned} &K\big(t^n_i, z_{ij}\big) \geq K\big(t^n_i, Z^{\mathfrak{u},\nu}_{t,z}(t^n_i)(\omega)\big) - \varepsilon \\ &\qquad\qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in C^\nu_{ij} := \big\{Z^{\mathfrak{u},\nu}_{t,z}(t^n_i) \in B_{ij}\big\}. \end{aligned} \tag{2.11}$$

Let μ^ε be as in Lemma 2.4, $\mathfrak{u}^\varepsilon_{ij} := \mu^\varepsilon(t^n_i, z_{ij})$ and $A^\nu_{ij} := C^\nu_{ij} \cap \{\tau^\nu_n = t^n_i\}$, and consider the mapping

$$\nu \mapsto \mathfrak{u}^\varepsilon_n[\nu] := \mathfrak{u}[\nu] \oplus_{\tau^\nu_n} \sum_{j\geq 1, i\leq 2^n} \mathfrak{u}^\varepsilon_{ij}[\nu]\mathbf{1}_{A^\nu_{ij}}.$$

Note that (Z2) and (C4) imply that $\{C^\nu_{ij}, \nu \in \mathcal{V}\}_{j\geq 1} \subset \mathfrak{F}_{t^n_i}$ for each $i \leq 2^n$. Similarly, it follows from (C6) and the definition of τ^ν_n that the families $\{\{\tau^\nu_n = t^n_i\}, \nu \in \mathcal{V}\}$ and $\{\{\tau^\nu_n = t^n_i\}^c, \nu \in \mathcal{V}\}$ belong to $\mathfrak{F}_{t^n_i}$. Therefore, an induction (over i) based on (C2) yields that $\mathfrak{u}^\varepsilon_n \in \mathfrak{U}$. Using successively (2.10), (Z1), the definition of J, Lemma 2.4 and (2.11), we deduce that for $\mathbb{P}$-a.e. $\omega \in A^\nu_{ij}$,

$$\begin{aligned}
\mathbb{E}\big[\ell\big(Z^{\mathfrak{u}^\varepsilon_n,\nu}_{t,z}(T)\big)|\mathcal{F}_{\tau^\nu_n}\big](\omega) &\geq I\big(t^n_i, z_{ij}, \mathfrak{u}^\varepsilon_{ij}, \nu\big)(\omega) - \varepsilon \\
&\geq J\big(t^n_i, z_{ij}, \mu^\varepsilon\big(t^n_i, z_{ij}\big)\big)(\omega) - \varepsilon \\
&\geq K\big(t^n_i, z_{ij}\big) - 2\varepsilon \\
&\geq K\big(t^n_i, Z^{\mathfrak{u},\nu}_{t,z}\big(t^n_i\big)(\omega)\big) - 3\varepsilon \\
&= K\big(\tau^\nu_n(\omega), Z^{\mathfrak{u},\nu}_{t,z}\big(\tau^\nu_n\big)(\omega)\big) - 3\varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary and $\bigcup_{i,j} A^\nu_{ij} = \Omega$ $\mathbb{P}$-a.s., this proves the claim.

Step 2. Fix $\varepsilon > 0$ and $n \geq 1$. For all $\nu \in \mathcal{V}$, we have

$$\mathbb{E}\big[\ell\big(Z^{\mathfrak{u}^\varepsilon_n,\nu}_{t,z}(T)\big)|\mathcal{F}_{\tau^\nu_n}\big](\omega) \geq M^\nu\big(\tau^\nu_n\big)(\omega) - \varepsilon \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in E^\nu_n,$$

where

$$E^\nu_n := \big\{\big(\tau^\nu_n, Z^{\mathfrak{u},\nu}_{t,z}\big(\tau^\nu_n\big), M^\nu\big(\tau^\nu_n\big)\big) \in B_\iota\big(\tau^\nu, Z^{\mathfrak{u},\nu}_{t,z}\big(\tau^\nu\big), M^\nu\big(\tau^\nu\big)\big)\big\}.$$

Indeed, since $(Z^{\mathfrak{u},\nu}_{t,z}(\tau^\nu), M^\nu(\tau^\nu)) \in \Lambda_\iota(\tau^\nu)$ $\mathbb{P}$-a.s., the definition of Λ_ι entails that $(Z^{\mathfrak{u},\nu}_{t,z}(\tau^\nu_n), M^\nu(\tau^\nu_n)) \in \Lambda(\tau^\nu_n)$ for $\mathbb{P}$-a.e. $\omega \in E^\nu_n$. This, in turn, means that

$$K\big(\tau^\nu_n(\omega), Z^{\mathfrak{u},\nu}_{t,z}\big(\tau^\nu_n\big)(\omega)\big) \geq M^\nu\big(\tau^\nu_n\big)(\omega) \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in E^\nu_n.$$

Now the claim follows from step 1. [In all this, we actually have $M^\nu(\tau^\nu_n) = M^\nu(\tau^\nu)$ by (2.9), a fact we do not use here.]

Step 3. Let $L^\nu := L^{\mathfrak{u},\nu}_{t,z}$ be the process from (I2). Then

$$K(t,z) \geq p - \varepsilon - \sup_{\nu\in\mathcal{V}} \mathbb{E}\big[\big(L^\nu\big(\tau^\nu_n\big) - M^\nu\big(\tau^\nu_n\big)\big)^- \mathbf{1}_{(E^\nu_n)^c}\big].$$

Indeed, it follows from step 2 and (I2) that

$$\begin{aligned}
&\mathbb{E}\big[\ell\big(Z^{\mathfrak{u}^\varepsilon_n,\nu}_{t,z}(T)\big)|\mathcal{F}_t\big] \\
&\quad\geq \mathbb{E}\big[M^\nu\big(\tau^\nu_n\big)\mathbf{1}_{E^\nu_n}|\mathcal{F}_t\big] - \varepsilon + \mathbb{E}\big[\mathbb{E}\big[\ell\big(Z^{\mathfrak{u}^\varepsilon_n,\nu}_{t,z}(T)\big)|\mathcal{F}_{\tau^\nu_n}\big]\mathbf{1}_{(E^\nu_n)^c}|\mathcal{F}_t\big] \\
&\quad\geq \mathbb{E}\big[M^\nu\big(\tau^\nu_n\big)|\mathcal{F}_t\big] - \mathbb{E}\big[M^\nu\big(\tau^\nu_n\big)\mathbf{1}_{(E^\nu_n)^c}|\mathcal{F}_t\big] - \varepsilon + \mathbb{E}\big[L^\nu\big(\tau^\nu_n\big)\mathbf{1}_{(E^\nu_n)^c}|\mathcal{F}_t\big] \\
&\quad= p - \varepsilon + \mathbb{E}\big[\big(L^\nu\big(\tau^\nu_n\big) - M^\nu\big(\tau^\nu_n\big)\big)\mathbf{1}_{(E^\nu_n)^c}|\mathcal{F}_t\big].
\end{aligned}$$

By the definitions of K and J, we deduce that

$$\begin{aligned}K(t,z) &\geq J\big(t,z,\mathrm{u}_n^\varepsilon\big)\\ &\geq p-\varepsilon+\operatorname*{ess\,inf}_{\nu\in\mathcal{V}}\mathbb{E}\big[\big(L^\nu\big(\tau_n^\nu\big)-M^\nu\big(\tau_n^\nu\big)\big)\mathbf{1}_{(E_n^\nu)^c}|\mathcal{F}_t\big].\end{aligned}$$

Since K is deterministic, we can take expectations on both sides to obtain that

$$K(t,z)\geq p-\varepsilon+\mathbb{E}\big[\operatorname*{ess\,inf}_{\nu\in\mathcal{V}}\mathbb{E}\big[Y^\nu|\mathcal{F}_t\big]\big],$$
$$\text{where } Y^\nu := \big(L^\nu\big(\tau_n^\nu\big)-M^\nu\big(\tau_n^\nu\big)\big)\mathbf{1}_{(E_n^\nu)^c}.$$

The family $\{\mathbb{E}[Y^\nu|\mathcal{F}_t], \nu\in\mathcal{V}\}$ is directed downward; to see this, use (C1), (Z2), (Z3), (C5) and the last statement in (I2), and argue as in step 1 of the proof of (GDP1) in Section 2.3. It then follows that we can find a sequence $(\nu_k)_{k\geq 1}\subset\mathcal{V}$ such that $\mathbb{E}[Y^{\nu_k}|\mathcal{F}_t]$ decreases $\mathbb{P}$-a.s. to $\operatorname{ess\,inf}_{\nu\in\mathcal{V}}\mathbb{E}[Y^\nu|\mathcal{F}_t]$ (cf. [19], Proposition VI-1-1) so that the claim follows by monotone convergence.

Step 4. We have

$$\lim_{n\to\infty}\sup_{\nu\in\mathcal{V}}\mathbb{E}\big[\big(L^\nu\big(\tau_n^\nu\big)-M^\nu\big(\tau_n^\nu\big)\big)^- 1_{(E_n^\nu)^c}\big]=0 \qquad \mathbb{P}\text{-a.s.}$$

Indeed, since $M^\nu(\tau_n^\nu)=M^\nu(\tau^\nu)$ by (2.9), the uniform integrability assumptions in Theorem 2.1 yield that $\{(L^\nu(\tau_n^\nu)-M^\nu(\tau_n^\nu))^- : n\geq 1, \nu\in\mathcal{V}\}$ is again uniformly integrable. Therefore, it suffices to prove that $\sup_{\nu\in\mathcal{V}}\mathbb{P}\{(E_n^\nu)^c\}\to 0$. To see this, note that for n large enough, we have $|\tau_n^\nu-\tau^\nu|\leq 2^{-n}T\leq\iota/2$ and hence

$$\mathbb{P}\big\{\big(E_n^\nu\big)^c\big\}\leq\mathbb{P}\big\{d_{\mathcal{Z}}\big(Z_{t,z}^{\mathrm{u},\nu}\big(\tau_n^\nu\big), Z_{t,z}^{\mathrm{u},\nu}\big(\tau^\nu\big)\big)\geq\iota/2\big\},$$

where we have used that $M^\nu(\tau_n^\nu)=M^\nu(\tau^\nu)$. Using once more that $|\tau_n^\nu-\tau^\nu|\leq 2^{-n}T$, the claim then follows from (R2).

Step 5. The additional assumption (2.9) entails no loss of generality.

Indeed, let $\tilde{M}^\nu$ be the stopped martingale $M^\nu(\cdot\wedge\tau^\nu)$. Then $\{\tilde{M}^\nu, \nu\in\mathcal{V}\}\subset\mathcal{M}_{t,p}$. Moreover, since $\{M^\nu, \nu\in\mathcal{V}\}\in\mathfrak{M}_{t,p}$ and $\{\tau^\nu, \nu\in\mathcal{V}\}\in\mathfrak{T}_t$, we see from (Z3) and (C5) that $\{\tilde{M}^\nu, \nu\in\mathcal{V}\}$ again satisfies the property stated in (Z3). Finally, we have that the set $\{\tilde{M}^\nu(\tau^\nu)^+ : \nu\in\mathcal{V}\}$ is uniformly integrable like $\{M^\nu(\tau^\nu)^+ : \nu\in\mathcal{V}\}$, since these sets coincide. Hence, $\{\tilde{M}^\nu, \nu\in\mathcal{V}\}$ satisfies all properties required in (GDP2), and of course also (2.9). To be precise, it is not necessarily the case that $\{\tilde{M}^\nu, \nu\in\mathcal{V}\}\in\mathfrak{M}_{t,p}$; in fact, we have made no assumption whatsoever about the richness of $\mathfrak{M}_{t,p}$. However, the previous properties are all we have used in this proof and hence, we may indeed replace M^ν by $\tilde{M}^\nu$ for the purpose of proving (GDP2).

We can now complete the proof of (GDP2): in view of step 4, step 3 yields that $K(t,z)\geq p-\varepsilon$, which by Lemma 2.4 implies the assertion that $(z,p-\varepsilon)\in\Lambda(t)$.

2.5. *Proof of Corollary* 2.3. *Step* 1. Assume that ℓ is bounded and Lipschitz continuous. Then (I) and (R1) are satisfied.

Assumption (I) is trivially satisfied; we prove that (2.5) implies Assumption (R1). Let $t \leq s \leq T$ and $(\mathfrak{u}, \nu) \in \mathfrak{U} \times \mathcal{V}$. Let c be the Lipschitz constant of ℓ. By (2.5), we have

$$\begin{aligned}\left|\mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u},\nu}(T)\right) - \ell\left(Z_{s,z'}^{\mathfrak{u},\nu}(T)\right)|\mathcal{F}_s\right]\right| &\leq c\mathbb{E}\left[\left|Z_{t,z}^{\mathfrak{u},\nu}(T) - Z_{s,z'}^{\mathfrak{u},\nu}(T)\right||\mathcal{F}_s\right] \\ &\leq cC\left|Z_{t,z}^{\mathfrak{u},\nu}(s) - z'\right| \end{aligned} \tag{2.12}$$

for all $z, z' \in \mathbb{R}^d$. Let $(B_j)_{j\geq 1}$ be any Borel partition of $\mathbb{R}^d$ such that the diameter of B_j is less than $\varepsilon/(cC)$, and let $z_j \in B_j$ for each $j \geq 1$. Then

$$\left|\mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u},\nu}(T)\right) - \ell\left(Z_{s,z_j}^{\mathfrak{u},\nu}(T)\right)|\mathcal{F}_s\right]\right| \leq \varepsilon \qquad \text{on } C_j^{\mathfrak{u},\nu} := \left\{Z_{t,z}^{\mathfrak{u},\nu}(s) \in B_j\right\},$$

which implies the first property in (R1). In particular, let $\bar{\nu} \in \mathcal{V}$, then using (C1), we have

$$\left|\mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u},\nu\oplus_s\bar{\nu}}(T)\right) - \ell\left(Z_{s,z_j}^{\mathfrak{u},\nu\oplus_s\bar{\nu}}(T)\right)|\mathcal{F}_s\right]\right| \leq \varepsilon \qquad \text{on } C_j^{\mathfrak{u},\nu\oplus_s\bar{\nu}}.$$

Since $C_j^{\mathfrak{u},\nu\oplus_s\bar{\nu}} = C_j^{\mathfrak{u},\nu}$ by (Z2), we may take the essential infimum over $\bar{\nu} \in \mathcal{V}$ to conclude that

$$\operatorname*{ess\,inf}_{\bar{\nu}\in\mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u},\nu\oplus_s\bar{\nu}}(T)\right)|\mathcal{F}_s\right] \leq J\left(s, z_j, \mathfrak{u}[\nu \oplus_s \cdot]\right) + \varepsilon \qquad \text{on } C_j^{\mathfrak{u},\nu},$$

which is the second property in (R1). Finally, the last property in (R1) is a direct consequence of (2.12) applied with $t = s$.

Step 2. We now prove the corollary under the additional assumption that $|\ell(z)| \leq C$; we shall reduce to the Lipschitz case by inf-convolution. Indeed, if we define the functions ℓ_k by

$$\ell_k(z) = \inf_{z'\in\mathbb{R}^d}\left\{\ell(z') + k|z' - z|\right\}, \qquad k \geq 1$$

then ℓ_k is Lipschitz continuous with Lipschitz constant k, $|\ell_k| \leq C$, and $(\ell_k)_{k\geq 1}$ converges pointwise to ℓ. Since ℓ is continuous and the sequence $(\ell_k)_{k\geq 1}$ is monotone increasing, the convergence is uniform on compact sets by Dini's lemma. That is, for all $n \geq 1$,

$$\sup_{z\in\mathbb{R}^d, |z|\leq n} \left|\ell_k(z) - \ell(z)\right| \leq \epsilon_k^n, \tag{2.13}$$

where $(\epsilon_k^n)_{k\geq 1}$ is a sequence of numbers such that $\lim_{k\to\infty} \epsilon_k^n = 0$. Moreover, (2.4) combined with Chebyshev's inequality imply that

$$\operatorname*{ess\,sup}_{(\mathfrak{u},\nu)\in\mathfrak{U}\times\mathcal{V}} \mathbb{P}\left\{\left|Z_{t,z}^{\mathfrak{u},\nu}(T)\right| \geq n|\mathcal{F}_t\right\} \leq \left(\varrho(z)/n\right)^{\bar{q}}. \tag{2.14}$$

Combining (2.13) and (2.14) and using the fact that $\ell_k - \ell$ is bounded by $2C$ then leads to

$$\operatorname*{ess\,sup}_{(\mathfrak{u},\nu)\in\mathfrak{U}\times\mathcal{V}} \mathbb{E}\big[\big|\ell_k\big(Z_{t,z}^{\mathfrak{u},\nu}(T)\big) - \ell\big(Z_{t,z}^{\mathfrak{u},\nu}(T)\big)\big| \big| \mathcal{F}_t\big] \le \epsilon_k^n + 2C\big(\varrho(z)/n\big)^{\bar{q}}. \tag{2.15}$$

Let O be a bounded subset of $\mathbb{R}^d$, let $\eta > 0$ and let

$$I_k(t,z,\mathfrak{u},\nu) = \mathbb{E}\big[\ell_k\big(Z_{t,z}^{\mathfrak{u},\nu}(T)\big)|\mathcal{F}_t\big]. \tag{2.16}$$

Then we can choose an integer n_O^η such that $2C(\varrho(z)/n_O^\eta)^{\bar{q}} \le \eta/2$ for all $z \in O$ and another integer k_O^η such that $\epsilon_{k_O^\eta}^{n_O^\eta} \le \eta/2$. Under these conditions, (2.15) applied to $n = n_O^\eta$ yields that

$$\operatorname*{ess\,sup}_{(\mathfrak{u},\nu)\in\mathfrak{U}\times\mathcal{V}} \big|I_{k_O^\eta}(t,z,\mathfrak{u},\nu) - I(t,z,\mathfrak{u},\nu)\big| \le \eta \qquad \text{for } (t,z) \in [0,T]\times O. \tag{2.17}$$

In the sequel, we fix $(t,z,p) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}$ and a bounded set $O \subset \mathbb{R}^d$ containing z, and define $J_{k_O^\eta}$, $\Lambda_{k_O^\eta}$, $\Lambda_{k_O^\eta,\iota}$ and $\bar{\Lambda}_{k_O^\eta}$ in terms of $\ell_{k_O^\eta}$ instead of ℓ.

We now prove (GDP1$'$). To this end, suppose that $(z, p+2\eta) \in \Lambda(t)$. Then (2.17) implies that $(z, p+\eta) \in \Lambda_{k_O^\eta}(t)$. In view of step 1, we may apply (GDP1) with the loss function $\ell_{k_O^\eta}$ to obtain $\mathfrak{u} \in \mathfrak{U}$ and $\{M^\nu, \nu \in \mathcal{V}\} \subset \mathcal{M}_{t,p}$ such that

$$\big(Z_{t,z}^{\mathfrak{u},\nu}(\tau), M^\nu(\tau)+\eta\big) \in \bar{\Lambda}_{k_O^\eta}(\tau) \qquad \mathbb{P}\text{-a.s. for all } \nu\in\mathcal{V} \text{ and } \tau\in\mathcal{T}_t.$$

Using once more (2.17), we deduce that

$$\big(Z_{t,z}^{\mathfrak{u},\nu}(\tau), M^\nu(\tau)\big) \in \bar{\Lambda}(\tau)$$
$$\mathbb{P}\text{-a.s. for all } \nu\in\mathcal{V} \text{ and } \tau\in\mathcal{T}_t \text{ such that } Z_{t,z}^{\mathfrak{u},\nu}(\tau)\in O.$$

Recalling that $\{Z_{t,z}^{\mathfrak{u},\nu}(\tau^{\mathfrak{u},\nu}), (\mathfrak{u},\nu)\in\mathfrak{U}\times\mathcal{V}\}$ is uniformly bounded and enlarging O if necessary, we deduce that (GDP1$'$) holds for ℓ. [The last two arguments are superfluous as $\ell \ge \ell_{k_O^\eta}$ already implies $\bar{\Lambda}_{k_O^\eta}(\tau) \subset \bar{\Lambda}(\tau)$; however, we would like to refer to this proof in a similar situation below where there is no monotonicity.]

It remains to prove (GDP2$'$). To this end, let $\iota > 0$, $\mathfrak{u}\in\mathfrak{U}$, $\{M^\nu, \nu\in\mathcal{V}\}\in\mathfrak{M}_{t,p}$ and $\{\tau^\nu, \nu\in\mathcal{V}\}\in\mathfrak{T}_t$ be such that

$$\big(Z_{t,z}^{\mathfrak{u},\nu}(\tau^\nu), M^\nu(\tau^\nu)\big) \in \Lambda_{2\iota}(\tau^\nu) \qquad \mathbb{P}\text{-a.s. for all } \nu\in\mathcal{V}.$$

For $\eta < \iota/2$, we then have

$$\big(Z_{t,z}^{\mathfrak{u},\nu}(\tau^\nu), M^\nu(\tau^\nu)+2\eta\big) \in \Lambda_\iota(\tau^\nu) \qquad \mathbb{P}\text{-a.s. for all } \nu\in\mathcal{V}. \tag{2.18}$$

Let $\tilde{M}^\nu := M^\nu + \eta$. Since $\{Z_{t,z}^{\mathfrak{u},\nu}(\tau^\nu), \nu\in\mathcal{V}\}$ is uniformly bounded in L^∞, we may assume, by enlarging O if necessary, that $B_\iota(Z_{t,z}^{\mathfrak{u},\nu}(\tau^\nu)) \subset O$ $\mathbb{P}$-a.s. for all $\nu\in\mathcal{V}$. Then (2.17) and (2.18) imply that

$$\big(Z_{t,z}^{\mathfrak{u},\nu}(\tau^\nu), \tilde{M}^\nu(\tau^\nu)\big) \in \Lambda_{k_O^\eta,\iota}(\tau^\nu) \qquad \mathbb{P}\text{-a.s. for all } \nu\in\mathcal{V}.$$

Moreover, as $\ell \le C$, (2.18) implies that $\tilde{M}^\nu(\tau^\nu) \le C$; in particular, $\{\tilde{M}^\nu(\tau^\nu)^+, \nu \in \mathcal{V}\}$ is uniformly integrable. Furthermore, as $\ell \ge -C$, we can take $L_{t,z}^{\mathfrak{u},\nu} := -C$ for (I2). In view of step 1, (GDP2) applied with the loss function $\ell_{k_O^\eta}$ then yields that

$$(z, p+\eta-\varepsilon) \in \Lambda_{k_O^\eta}(t) \qquad \text{for all } \varepsilon > 0. \tag{2.19}$$

To be precise, this conclusion would require that $\{\tilde{M}^\nu, \nu \in \mathcal{V}\} \in \mathfrak{M}_{t,p+\eta}$, which is not necessarily the case under our assumptions. However, since $\{M^\nu, \nu \in \mathcal{V}\} \in \mathfrak{M}_{t,p}$, it is clear that $\{\tilde{M}^\nu, \nu \in \mathcal{V}\}$ satisfies the property stated in (Z3), so that, as in step 5 of the proof of (GDP2), there is no loss of generality in assuming that $\{\tilde{M}^\nu, \nu \in \mathcal{V}\} \in \mathfrak{M}_{t,p+\eta}$. We conclude by noting that (2.17) and (2.19) imply that $(z, p-\varepsilon) \in \Lambda(t)$ for all $\varepsilon > 0$.

Step 3. We turn to the general case. For $k \ge 1$, we now define $\ell_k := (\ell \wedge k) \vee (-k)$, while I_k is again defined as in (2.16). We also set

$$n_k = \max\{m \ge 0 : B_m(0) \subset \{\ell = \ell_k\}\} \wedge k$$

and note that the continuity of ℓ guarantees that $\lim_{k\to\infty} n_k = \infty$. Given a bounded set $O \subset \mathbb{R}^d$ and $\eta > 0$, we claim that

$$\begin{aligned} &\operatorname*{ess\,sup}_{(\mathfrak{u},\nu)\in\mathfrak{U}\times\mathcal{V}} \left| I_{k_O^\eta}(t,z,\mathfrak{u},\nu) - I(t,z,\mathfrak{u},\nu)\right| \le \eta \\ &\qquad\qquad\qquad\qquad \text{for all } (t,z) \in [0,T] \times O \end{aligned} \tag{2.20}$$

for any large enough integer k_O^η. Indeed, let $(\mathfrak{u},\nu) \in \mathfrak{U} \times \mathcal{V}$; then

$$\begin{aligned} &\left| I_k(t,z,\mathfrak{u},\nu) - I(t,z,\mathfrak{u},\nu)\right| \\ &\quad \le \mathbb{E}\big[|\ell - \ell_k|\big(Z_{t,z}^{\mathfrak{u},\nu}(T)\big)|\mathcal{F}_t\big] \\ &\quad = \mathbb{E}\big[|\ell - \ell_k|\big(Z_{t,z}^{\mathfrak{u},\nu}(T)\big)\mathbf{1}_{\{Z_{t,z}^{\mathfrak{u},\nu}(T)\notin\{\ell=\ell_k\}\}}|\mathcal{F}_t\big] \\ &\quad \le \mathbb{E}\big[\big|\ell\big(Z_{t,z}^{\mathfrak{u},\nu}(T)\big)\big|\mathbf{1}_{\{|Z_{t,z}^{\mathfrak{u},\nu}(T)|>n_k\}}|\mathcal{F}_t\big] \\ &\quad \le C\mathbb{E}\big[\big(1 + \big|Z_{t,z}^{\mathfrak{u},\nu}(T)\big|^q\big)\mathbf{1}_{\{|Z_{t,z}^{\mathfrak{u},\nu}(T)|>n_k\}}|\mathcal{F}_t\big] \end{aligned}$$

by (2.3). We may assume that $q > 0$, as otherwise we are in the setting of step 2. Pick $\delta > 0$ such that $q(1+\delta) = \bar{q}$. Then Hölder's inequality and (2.4) yield that

$$\begin{aligned} &\mathbb{E}\big[\big|\big(Z_{t,z}^{\mathfrak{u},\nu}(T)\big)\big|^q\mathbf{1}_{\{|Z_{t,z}^{\mathfrak{u},\nu}(T)|>n_k\}}|\mathcal{F}_t\big] \\ &\quad \le \mathbb{E}\big[\big|\big(Z_{t,z}^{\mathfrak{u},\nu}(T)\big)\big|^{\bar{q}}|\mathcal{F}_t\big]^{1/(1+\delta)}\mathbb{P}\big\{\big|Z_{t,z}^{\mathfrak{u},\nu}(T)\big| > n_k|\mathcal{F}_t\big\}^{\delta/(1+\delta)} \\ &\quad \le \rho(z)^{\bar{q}/(1+\delta)}\big(\rho(z)/n_k\big)^{\bar{q}\delta/(1+\delta)}. \end{aligned}$$

Since ρ is locally bounded and $\lim_{k\to\infty} n_k = \infty$, claim (2.20) follows. We can then obtain (GDP1′) and (GDP2′) by reducing to the result of step 2, using the same arguments as in the proof of step 2.

3. The PDE in the case of a controlled SDE. In this section, we illustrate how our GDP can be used to derive a dynamic programming equation and how its assumptions can be verified in a typical setup. To this end, we focus on the case where the state process is determined by a stochastic differential equation with controlled coefficients; however, other examples could be treated similarly.

3.1. *Setup.* Let $\Omega = C([0,T];\mathbb{R}^d)$ be the canonical space of continuous paths equipped with the Wiener measure $\mathbb{P}$, let $\mathbb{F} = (\mathcal{F}_t)_{t\le T}$ be the $\mathbb{P}$-augmentation of the filtration generated by the coordinate-mapping process W and let $\mathcal{F} = \mathcal{F}_T$. We define $\mathcal{V}$, the set of adverse controls, to be the set of all progressively measurable processes with values in a compact subset V of $\mathbb{R}^d$. Similarly, $\mathcal{U}$ is the set of all progressively measurable processes with values in a compact $U \subset \mathbb{R}^d$. Finally, the set of strategies $\mathfrak{U}$ consists of all mappings $\mathfrak{u}:\mathcal{V}\to\mathcal{U}$ which are nonanticipating in the sense that

$$\{\nu_1 =_{(0,s]} \nu_2\} \subset \{\mathfrak{u}[\nu_1] =_{(0,s]} \mathfrak{u}[\nu_2]\} \qquad \text{for all } \nu_1,\nu_2\in\mathcal{V} \text{ and } s\le T.$$

Given $(t,z)\in[0,T]\times\mathbb{R}^d$ and $(\mathfrak{u},\nu)\in\mathfrak{U}\times\mathcal{V}$, we let $Z_{t,z}^{\mathfrak{u},\nu}$ be the unique strong solution of the controlled SDE

$$Z(s) = z + \int_t^s \mu\big(Z(r),\mathfrak{u}[\nu]_r,\nu_r\big)\,dr + \int_t^s \sigma\big(Z(r),\mathfrak{u}[\nu]_r,\nu_r\big)\,dW_r, \qquad s\in[t,T], \tag{3.1}$$

where the coefficients

$$\mu:\mathbb{R}^d\times U\times V\to\mathbb{R}^d, \qquad \sigma:\mathbb{R}^d\times U\times V\to\mathbb{R}^{d\times d}$$

are assumed to be jointly continuous in all three variables, Lipschitz continuous with linear growth in the first variable, uniformly in the last two and Lipschitz continuous in the second variable, locally uniformly in the two others. Throughout this section, we assume that $\ell:\mathbb{R}^d\to\mathbb{R}$ is a continuous function of polynomial growth; that is, (2.3) holds true for some constants C and q. Since $Z_{t,z}^{\mathfrak{u},\nu}(T)$ has moments of all orders, this implies that the finiteness condition (2.1) is satisfied.

In view of the martingale representation theorem, we can identify the set $\mathcal{M}_{t,p}$ of martingales with the set $\mathcal{A}$ of all progressively measurable d-dimensional processes α such that $\int \alpha\,dW$ is a (true) martingale. Indeed, we have $\mathcal{M}_{t,p} = \{P_{t,p}^{\alpha}, \alpha\in\mathcal{A}\}$, where

$$P_{t,p}^{\alpha}(\cdot) = p + \int_t^{\cdot} \alpha_s\,dW_s.$$

We shall denote by $\mathfrak{A}$ the set of all mappings $\mathfrak{a}[\cdot]$: $\mathcal{V}\mapsto\mathcal{A}$ such that

$$\{\nu_1 =_{(0,s]} \nu_2\} \subset \{\mathfrak{a}[\nu_1] =_{(0,s]} \mathfrak{a}[\nu_2]\} \qquad \text{for all } \nu_1,\nu_2\in\mathcal{V} \text{ and } s\le T.$$

The set of all families $\{P_{t,p}^{\mathfrak{a}[\nu]}, \nu\in\mathcal{V}\}$ with $\mathfrak{a}\in\mathfrak{A}$ then forms the set $\mathfrak{M}_{t,p}$, for any given $(t,p)\in[0,T]\times\mathbb{R}$. Furthermore, $\mathfrak{T}_t$ consists of all families $\{\tau^\nu, \nu\in\mathcal{V}\}\subset\mathcal{T}_t$

such that, for some $(z, p) \in \mathbb{R}^d \times \mathbb{R}$, $(\mathfrak{u}, \mathfrak{a}) \in \mathfrak{U} \times \mathfrak{A}$ and some Borel set $O \subset [0, T] \times \mathbb{R}^d \times \mathbb{R}$,

$$\tau^\nu \text{ is the first exit time of } \big(\cdot, Z_{t,z}^{\mathfrak{u},\nu}, P_{t,p}^{\mathfrak{a}[\nu]}\big) \text{ from } O \qquad \text{for all } \nu \in \mathcal{V}.$$

(This includes the deterministic times $s \in [t, T]$ by the choice $O = [0, s] \times \mathbb{R}^d \times \mathbb{R}$.) Finally, $\mathfrak{F}_t$ consists of all families $\{A^\nu, \nu \in \mathcal{V}\} \subset \mathcal{F}_t$ such that

$$A^{\nu_1} \cap \{\nu_1 =_{(0,t]} \nu_2\} = A^{\nu_2} \cap \{\nu_1 =_{(0,t]} \nu_2\} \qquad \text{for all } \nu_1, \nu_2 \in \mathcal{V}.$$

PROPOSITION 3.1. *The conditions of Corollary 2.3 are satisfied in the present setup.*

PROOF. The above definitions readily yield that Assumptions (C) and (Z1)–(Z3) are satisfied. Moreover, Assumption (Z4) can be verified exactly as in [7], Proposition 3.3. Fix any $\bar{q} > q \vee 2$; then (2.4) can be obtained as follows. Let $(\mathfrak{u}, \nu) \in \mathfrak{U} \times \mathcal{V}$ and $A \in \mathcal{F}_t$ be arbitrary. Using the Burkholder–Davis–Gundy inequalities, the boundedness of U and V and the assumptions on μ and σ, we obtain that

$$\mathbb{E}\Big[\sup_{t \le s \le \tau} \big|Z_{t,z}^{\mathfrak{u},\nu}(s)\big|^{\bar{q}} \mathbf{1}_A\Big] \le c\mathbb{E}\bigg[\mathbf{1}_A + |z|^{\bar{q}} \mathbf{1}_A + \int_t^\tau \sup_{t \le s \le r} \big|Z_{t,z}^{\mathfrak{u},\nu}(s)\big|^{\bar{q}} \mathbf{1}_A \, dr\bigg],$$

where c is a universal constant, and τ is any stopping time such that $Z_{t,z}^{\mathfrak{u},\nu}(\cdot \wedge \tau)$ is bounded. Applying Gronwall's inequality and letting $\tau \to T$, we deduce that

$$\mathbb{E}\big[\big|Z_{t,z}^{\mathfrak{u},\nu}(T)\big|^{\bar{q}} \mathbf{1}_A\big] \le \mathbb{E}\Big[\sup_{t \le u \le T} \big|Z_{t,z}^{\mathfrak{u},\nu}(u)\big|^{\bar{q}} \mathbf{1}_A\Big] \le c\mathbb{E}\big[(1 + |z|^{\bar{q}}) \mathbf{1}_A\big].$$

Since $A \in \mathcal{F}_t$ was arbitrary, this implies (2.4). To verify condition (2.5), we note that the flow property yields

$$\mathbb{E}\big[\big|Z_{t,z}^{\mathfrak{u} \oplus_s \bar{\mathfrak{u}}, \nu \oplus_s \bar{\nu}}(T) - Z_{s,z'}^{\bar{\mathfrak{u}}, \nu \oplus_s \bar{\nu}}(T)\big| \mathbf{1}_A\big] = \mathbb{E}\big[\big|Z_{s, Z_{t,z}^{\mathfrak{u},\nu}(s)}^{\bar{\mathfrak{u}}, \nu \oplus_s \bar{\nu}}(T) - Z_{s,z'}^{\bar{\mathfrak{u}}, \nu \oplus_s \bar{\nu}}(T)\big| \mathbf{1}_A\big]$$

and estimate the right-hand side with the above arguments. Finally, the same arguments can be used to verify (R2). $\square$

REMARK 3.2. We emphasize that our definition of a strategy $\mathfrak{u} \in \mathfrak{U}$ does not include regularity assumptions on the mapping $\nu \mapsto \mathfrak{u}[\nu]$. This is in contrast to [2], where a continuity condition is imposed, enabling the authors to deal with the selection problem for strategies in the context of a stochastic differential game and use the traditional formulation of the value functions in terms of infima (not essential infima) and suprema. Let us mention, however, that such regularity assumptions may preclude existence of optimal strategies in concrete examples; see also Remark 4.3.

3.2. *PDE for the reachability set* Λ. In this section, we show how the PDE for the reachability set Λ from (2.2) can be deduced from the geometric dynamic programming principle of Corollary 2.3. This equation is stated in terms of the indicator function of the complement of the graph of Λ,

$$\chi(t, z, p) := 1 - \mathbf{1}_{\Lambda(t)}(z, p) = \begin{cases} 0, & \text{if } (z, p) \in \Lambda(t), \\ 1, & \text{otherwise} \end{cases}$$

and its lower semicontinuous envelope

$$\chi_*(t, z, p) := \liminf_{(t',z',p')\to(t,z,p)} \chi(t', z', p').$$

Corresponding results for the case without adverse player have been obtained in [3, 25]; we extend their arguments to account for the presence of ν and the fact that we only have a relaxed GDP. We begin by rephrasing Corollary 2.3 in terms of χ.

LEMMA 3.3. *Fix $(t, z, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$, and let $O \subset [0, T] \times \mathbb{R}^d \times \mathbb{R}$ be a bounded open set containing (t, z, p).*

(GDP1$_\chi$) *Assume that $\chi(t, z, p+\varepsilon) = 0$ for some $\varepsilon > 0$. Then there exist $\mathfrak{u} \in \mathfrak{U}$ and $\{\alpha^\nu, \nu \in \mathcal{V}\} \subset \mathcal{A}$ such that*

$$\chi_*\big(\tau^\nu, Z^{\mathfrak{u},\nu}_{t,z}(\tau^\nu), P^{\alpha^\nu}_{t,p}(\tau^\nu)\big) = 0 \qquad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V},$$

where τ^ν denotes the first exit time of $(\cdot, Z^{\mathfrak{u},\nu}_{t,z}, P^{\alpha^\nu}_{t,p})$ from O.

(GDP2$_\chi$) *Let φ be a continuous function such that $\varphi \ge \chi$ and let $(\mathfrak{u}, \mathfrak{a}) \in \mathfrak{U} \times \mathfrak{A}$ and $\eta > 0$ be such that*

$$\varphi\big(\tau^\nu, Z^{\mathfrak{u},\nu}_{t,z}(\tau^\nu), P^{\mathfrak{a}[\nu]}_{t,p}(\tau^\nu)\big) \le 1 - \eta \qquad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V}, \tag{3.2}$$

where τ^ν denotes the first exit time of $(\cdot, Z^{\mathfrak{u},\nu}_{t,z}, P^{\mathfrak{a}[\nu]}_{t,p})$ from O. Then $\chi(t, z, p - \varepsilon) = 0$ for all $\varepsilon > 0$.

PROOF. After observing that $(z, p + \varepsilon) \in \Lambda(t)$ if and only if $\chi(t, z, p + \varepsilon) = 0$ and that $(z, p) \in \bar{\Lambda}(t)$ implies $\chi_*(t, z, p) = 0$, (GDP1$_\chi$) follows from Corollary 2.3, whose conditions are satisfied by Proposition 3.1. We now prove (GDP2$_\chi$). Since φ is continuous and ∂O is compact, we can find $\iota > 0$ such that

$$\varphi < 1 \qquad \text{on a } \iota\text{-neighborhood of } \partial O \cap \{\varphi \le 1 - \eta\}.$$

As $\chi \le \varphi$, it follows that (3.2) implies

$$\big(Z^{\mathfrak{u},\nu}_{t,z}(\tau^\nu), M^\nu(\tau^\nu)\big) \in \Lambda_\iota(\tau^\nu) \qquad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V}.$$

Now Corollary 2.3 yields that $(z, p - \varepsilon) \in \Lambda(t)$; that is, $\chi(t, z, p - \varepsilon) = 0$. $\square$

Given a suitably differentiable function $\varphi = \varphi(t, z, p)$ on $[0, T] \times \mathbb{R}^{d+1}$, we shall denote by $\partial_t\varphi$ its derivative with respect to t and by $D\varphi$ and $D^2\varphi$ the Jacobian and the Hessian matrix with respect to (z, p), respectively. Given $u \in U$, $a \in \mathbb{R}^d$ and $v \in V$, we can then define the Dynkin operator

$$\mathcal{L}^{u,a,v}_{(Z,P)}\varphi := \partial_t\varphi + \mu_{(Z,P)}(\cdot, u, v)^\top D\varphi + \tfrac{1}{2}\operatorname{Tr}\big[\sigma_{(Z,P)}\sigma^\top_{(Z,P)}(\cdot, u, a, v)D^2\varphi\big]$$

with coefficients

$$\mu_{(Z,P)} := \begin{pmatrix}\mu\\ 0\end{pmatrix}, \qquad \sigma_{(Z,P)}(\cdot, a, \cdot) := \begin{pmatrix}\sigma\\ a\end{pmatrix}.$$

To introduce the associated relaxed Hamiltonians, we first define the relaxed kernel

$$\mathcal{N}_\varepsilon(z, q, v) = \big\{(u, a) \in U \times \mathbb{R}^d : \big|\sigma^\top_{(Z,P)}(z, u, a, v)q\big| \le \varepsilon\big\}, \qquad \varepsilon \ge 0$$

for $z \in \mathbb{R}^d$, $q \in \mathbb{R}^{d+1}$ and $v \in V$, as well as the set $N_{\mathrm{Lip}}(z, q)$ of all continuous functions

$$(\hat{u}, \hat{a}) : \mathbb{R}^d \times \mathbb{R}^{d+1} \times V \to U \times \mathbb{R}^d, \qquad \big(z', q', v'\big) \mapsto (\hat{u}, \hat{a})\big(z', q', v'\big)$$

that are locally Lipschitz continuous in (z', q'), uniformly in v' and satisfy

$$(\hat{u}, \hat{a}) \in \mathcal{N}_0 \qquad \text{on } B \times V \qquad \text{for some neighborhood } B \text{ of } (z, q).$$

The local Lipschitz continuity will be used to ensure the local wellposedness of the SDE for a Markovian strategy defined via $(\hat{u}, \hat{a})$. Setting

$$F(\Theta, u, a, v) := \big\{-\mu_{(Z,P)}(z, u, v)^\top q - \tfrac{1}{2}\operatorname{Tr}\big[\sigma_{(Z,P)}\sigma^\top_{(Z,P)}(z, u, a, v)A\big]\big\}$$

for $\Theta = (z, q, A) \in \mathbb{R}^d \times \mathbb{R}^{d+1} \times \mathbb{S}^{d+1}$ and $(u, a, v) \in U \times \mathbb{R}^d \times V$, we can then define the relaxed Hamiltonians

$$H^*(\Theta) := \inf_{v\in V} \limsup_{\varepsilon\searrow 0, \Theta'\to\Theta} \sup_{(u,a)\in\mathcal{N}_\varepsilon(\Theta',v)} F\big(\Theta', u, a, v\big), \tag{3.3}$$

$$H_*(\Theta) := \sup_{(\hat{u},\hat{a})\in N_{\mathrm{Lip}}(\Theta)} \inf_{v\in V} F\big(\Theta, \hat{u}(\Theta, v), \hat{a}(\Theta, v), v\big). \tag{3.4}$$

[In (3.4), it is not necessary to take the relaxation $\Theta' \to \Theta$ because $\inf_{v\in V} F$ is already lower semicontinuous.] The question whether $H^* = H_*$ is postponed to the monotone setting of the next section; see Remark 3.9.

We are now in the position to derive the PDE for χ; in the following, we write $H^*\varphi(t, z, p)$ for $H^*(z, D\varphi(t, z, p), D^2\varphi(t, z, p))$, and similarly for H_*.

THEOREM 3.4. *The function χ_* is a viscosity supersolution on $[0, T) \times \mathbb{R}^{d+1}$ of*

$$(-\partial_t + H^*)\varphi \ge 0.$$

The function χ^* is a viscosity subsolution on $[0, T) \times \mathbb{R}^{d+1}$ of

$$(-\partial_t + H_*)\varphi \le 0.$$

PROOF. *Step* 1. χ_* is a viscosity supersolution.

Let $(t_o, z_o, p_o) \in [0, T) \times \mathbb{R}^d \times \mathbb{R}$, and let φ be a smooth function such that

$$\text{(strict)} \min_{[0,T)\times\mathbb{R}^d\times\mathbb{R}} (\chi_* - \varphi) = (\chi_* - \varphi)(t_o, z_o, p_o) = 0. \tag{3.5}$$

We suppose that

$$\big(-\partial_t + H^*\big)\varphi(t_o, z_o, p_o) \le -2\eta < 0 \tag{3.6}$$

for some $\eta > 0$ and work toward a contradiction. Using the continuity of μ and σ and the definition of the upper-semicontinuous operator H^*, we can find $v_o \in V$ and $\varepsilon > 0$ such that

$$\begin{aligned} &-\mathcal{L}^{u,a,v_o}_{(Z,P)}\varphi(t, z, p) \le -\eta \\ &\qquad \text{for all } (u, a) \in \mathcal{N}_\varepsilon\big(z, D\varphi(t, z, p), v_o\big) \text{ and } (t, z, p) \in B_\varepsilon, \end{aligned} \tag{3.7}$$

where $B_\varepsilon := B_\varepsilon(t_o, z_o, p_o)$ denotes the open ball of radius ε around (t_o, z_o, p_o). Let

$$\partial B_\varepsilon := \{t_o + \varepsilon\} \times \overline{B_\varepsilon(z_o, p_o)} \cup [t_o, t_o + \varepsilon) \times \partial B_\varepsilon(z_o, p_o)$$

denote the parabolic boundary of B_ε, and set

$$\zeta := \min_{\partial B_\varepsilon}(\chi_* - \varphi).$$

In view of (3.5), we have $\zeta > 0$.

Next, we claim that there exists a sequence $(t_n, z_n, p_n, \varepsilon_n)_{n\ge1} \subset B_\varepsilon \times (0, 1)$ such that

$$\begin{aligned} &(t_n, z_n, p_n, \varepsilon_n) \to (t_o, z_o, p_o, 0) \quad \text{and} \\ &\chi(t_n, z_n, p_n + \varepsilon_n) = 0 \qquad \text{for all } n \ge 1. \end{aligned} \tag{3.8}$$

In view of $\chi \in \{0, 1\}$, it suffices to show that

$$\chi_*(t_o, z_o, p_o) = 0. \tag{3.9}$$

Suppose that $\chi_*(t_o, z_o, p_o) > 0$; then the lower semicontinuity of χ_* yields that $\chi_* > 0$ and therefore $\chi = 1$ on a neighborhood of (t_o, z_o, p_o), which implies that φ has a strict local maximum in (t_o, z_o, p_o) and thus

$$\partial_t\varphi(t_o, z_o, p_o) \le 0, \qquad D\varphi(t_o, z_o, p_o) = 0, \qquad D^2\varphi(t_o, z_o, p_o) \le 0.$$

This clearly contradicts (3.7), and so the claim follows.

For any $n \ge 1$, the equality in (3.8) and (GDP1$_\chi$) of Lemma 3.3 yield $\mathfrak{u}^n \in \mathfrak{U}$ and $\{\alpha^{n,\nu}, \nu \in \mathcal{V}\} \subset \mathcal{A}$ such that

$$\chi_*\big(t \wedge \tau_n, Z^n(t \wedge \tau_n), P^n(t \wedge \tau_n)\big) = 0, \qquad t \ge t_n, \tag{3.10}$$

where

$$\big(Z^n(s), P^n(s)\big) := \big(Z^{\mathfrak{u}^n, v_o}_{t_n, z_n}(s), P^{\alpha^{n,v_o}}_{t_n, p_n}(s)\big)$$

and

$$\tau_n := \inf\{s \geq t_n : \big(s, Z^n(s), P^n(s)\big) \notin B_\varepsilon\}.$$

(In the above, $v_o \in V$ is viewed as a constant element of $\mathcal{V}$.) By (3.10), (3.5) and the definitions of ζ and τ_n,

$$-\varphi\big(\cdot, Z^n, P^n\big)(t \wedge \tau_n) = (\chi_* - \varphi)\big(\cdot, Z^n, P^n\big)(t \wedge \tau_n) \geq \zeta \mathbf{1}_{\{t \geq \tau_n\}} \geq 0.$$

Applying Itô's formula to $-\varphi(\cdot, Z^n, P^n)$, we deduce that

$$S_n(t) := S_n(0) + \int_{t_n}^{t\wedge\tau_n} \delta_n(r)\, dr + \int_{t_n}^{t\wedge\tau_n} \Sigma_n(r)\, dW_r \geq -\zeta \mathbf{1}_{\{t<\tau_n\}}, \tag{3.11}$$

where

$$\begin{aligned} S_n(0) &:= -\zeta - \varphi(t_n, z_n, p_n), \\ \delta_n(r) &:= -\mathcal{L}^{\mathfrak{u}^n_r[v_o], \alpha^{n,v_o}_r, v_o}_{(Z,P)} \varphi\big(r, Z^n(r), P^n(r)\big), \\ \Sigma_n(r) &:= -D\varphi\big(r, Z^n(r), P^n(r)\big)^\top \sigma_{(Z,P)}\big(Z^n(r), \mathfrak{u}^n_r[v_o], \alpha^{n,v_o}_r, v_o\big). \end{aligned}$$

Define the set

$$A_n := [\![t_n, \tau_n]\!] \cap \{\delta_n > -\eta\};$$

then (3.7) and the definition of $\mathcal{N}_\varepsilon$ imply that

$$|\Sigma_n| > \varepsilon \qquad \text{on } A_n. \tag{3.12}$$

LEMMA 3.5. *After diminishing $\varepsilon > 0$ if necessary, the stochastic exponential*

$$E_n(\cdot) = \mathcal{E}\bigg(-\int_{t_n}^{\cdot\wedge\tau_n} \frac{\delta_n(r)}{|\Sigma_n(r)|^2} \Sigma_n(r) \mathbf{1}_{A_n}(r)\, dW_r\bigg)$$

is well defined and a true martingale for all $n \geq 1$.

This lemma is proved below; it fills a gap in the previous literature. Admitting its result for the moment, integration by parts yields

$$\begin{aligned} (E_n S_n)(t \wedge \tau_n) = S_n(0) &+ \int_{t_n}^{t\wedge\tau_n} E_n \delta_n \mathbf{1}_{A_n^c}\, dr \\ &+ \int_{t_n}^{t\wedge\tau_n} E_n\bigg(\Sigma_n - S_n \frac{\delta_n}{|\Sigma_n|^2} \Sigma_n \mathbf{1}_{A_n}\bigg)\, dW. \end{aligned}$$

As $E_n \geq 0$, it then follows from the definition of A_n that $E_n \delta_n \mathbf{1}_{A_n^c} \leq 0$ and so $E_n S_n$ is a local supermartingale; in fact, it is a true supermartingale since it is bounded from below by the martingale $-\zeta E_n$. In view of (3.11), we deduce that

$$-\zeta - \varphi(t_n, z_n, p_n) = (E_n S_n)(t_n) \geq \mathbb{E}\big[(E_n S_n)(\tau_n)\big] \geq -\zeta \mathbb{E}\big[\mathbf{1}_{\{\tau_n<\tau_n\}} E_n(\tau_n)\big] = 0,$$

which yields a contradiction due to $\zeta > 0$ and the fact that, by (3.9),

$$\varphi(t_n, z_n, p_n) \to \varphi(t_o, z_o, p_o) = \chi_*(t_o, z_o, p_o) = 0.$$

Step 2. χ^* is a viscosity subsolution.

Let $(t_o, z_o, p_o) \in [0, T) \times \mathbb{R}^d \times \mathbb{R}$ and let φ be a smooth function such that

$$\max_{[0,T)\times\mathbb{R}^d\times\mathbb{R}} (\chi^* - \varphi) = (\chi^* - \varphi)(t_o, z_o, p_o) = 0.$$

In order to prove that $(-\partial_t + H_*)\varphi(t_o, z_o, p_o) \le 0$, we assume for contradiction that

$$(-\partial_t + H_*)\varphi(t_o, z_o, p_o) > 0. \tag{3.13}$$

An argument analogous to the proof of (3.9) shows that $\chi^*(t_o, z_o, p_o) = 1$. Consider a sequence $(t_n, z_n, p_n, \varepsilon_n)_{n\ge1}$ in $[0, T) \times \mathbb{R}^d \times \mathbb{R} \times (0, 1)$ such that

$$(t_n, z_n, p_n - \varepsilon_n, \varepsilon_n) \to (t_o, z_o, p_o, 0)$$

and

$$\chi(t_n, z_n, p_n - \varepsilon_n) \to \chi^*(t_o, z_o, p_o) = 1.$$

Since χ takes values in $\{0, 1\}$, we must have

$$\chi(t_n, z_n, p_n - \varepsilon_n) = 1 \tag{3.14}$$

for all n large enough. Set

$$\tilde\varphi(t, z, p) := \varphi(t, z, p) + |t - t_o|^2 + |z - z_o|^4 + |p - p_o|^4.$$

Then inequality (3.13) and the definition of H_* imply that we can find $(\hat u, \hat a)$ in $N_{\mathrm{Lip}}(\cdot, D\tilde\varphi)(t_o, z_o, p_o)$ such that

$$\inf_{v\in V} \bigl(-\mathcal{L}_{(Z,P)}^{(\hat u,\hat a)(\cdot,D\tilde\varphi,v),v}\tilde\varphi\bigr) \ge 0 \qquad \text{on } B_\varepsilon := B_\varepsilon(t_o, z_o, p_o) \tag{3.15}$$

for some $\varepsilon > 0$. By the definition of N_{Lip}, after possibly changing $\varepsilon > 0$, we have

$$(\hat u, \hat a)(\cdot, D\tilde\varphi, \cdot) \in \mathcal{N}_0(\cdot, D\tilde\varphi, \cdot) \qquad \text{on } B_\varepsilon \times V. \tag{3.16}$$

Moreover, we have

$$\tilde\varphi \ge \varphi + \eta \qquad \text{on } \partial B_\varepsilon \tag{3.17}$$

for some $\eta > 0$. Since $\tilde\varphi(t_n, z_n, p_n) \to \varphi(t_o, z_o, p_o) = \chi^*(t_o, z_o, p_o) = 1$, we can find n such that

$$\tilde\varphi(t_n, z_n, p_n) \le 1 + \eta/2 \tag{3.18}$$

and such that (3.14) is satisfied. We fix this n for the remainder of the proof.

For brevity, we write $(\hat{u}, \hat{a})(t, z, p, v)$ for $(\hat{u}, \hat{a})(z, D\tilde{\varphi}(t, z, p), v)$ in the sequel. Exploiting the definition of N_{Lip}, we can then define the mapping $(\hat{\mathfrak{u}}, \hat{\mathfrak{a}})[\cdot]: \mathcal{V} \to \mathcal{U} \times \mathcal{A}$ implicitly via

$$(\hat{\mathfrak{u}}, \hat{\mathfrak{a}})[\nu] = (\hat{u}, \hat{a})\big(\cdot, Z_{t_n,z_n}^{\hat{\mathfrak{u}}[\nu],\nu}, P_{t_n,p_n}^{\hat{\mathfrak{a}}[\nu]}, \nu\big)\mathbf{1}_{[t_n,\tau^\nu]},$$

where

$$\tau^\nu := \inf\big\{r \geq t_n : \big(r, Z_{t_n,z_n}^{\hat{\mathfrak{u}}[\nu],\nu}(r), P_{t_n,p_n}^{\hat{\mathfrak{a}}[\nu]}(r)\big) \notin B_\varepsilon\big\}.$$

We observe that $\hat{\mathfrak{u}}$ and $\hat{\mathfrak{a}}$ are nonanticipating; that is, $(\hat{\mathfrak{u}}, \hat{\mathfrak{a}}) \in \mathfrak{U} \times \mathfrak{A}$. Let us write (Z^ν, P^ν) for $(Z_{t_n,z_n}^{\hat{\mathfrak{u}},\nu}, P_{t_n,p_n}^{\hat{\mathfrak{a}}[\nu]})$ to alleviate the notation. Since $\chi \leq \chi^* \leq \varphi$, the continuity of the paths of Z^ν and P^ν and (3.17) lead to

$$\varphi\big(\tau^\nu, Z^\nu(\tau^\nu), P^\nu(\tau^\nu)\big) \leq \tilde{\varphi}\big(\tau^\nu, Z^\nu(\tau^\nu), P^\nu(\tau^\nu)\big) - \eta.$$

On the other hand, in view of (3.15) and (3.16), Itô's formula applied to $\tilde{\varphi}$ on $[t_n, \tau^\nu]$ yields that

$$\tilde{\varphi}\big(\tau^\nu, Z^\nu(\tau^\nu), P^\nu(\tau^\nu)\big) \leq \tilde{\varphi}(t_n, z_n, p_n).$$

Therefore, the previous inequality and (3.18) show that

$$\varphi\big(\tau^\nu, Z^\nu(\tau^\nu), P^\nu(\tau^\nu)\big) \leq \tilde{\varphi}(t_n, z_n, p_n) - \eta \leq 1 - \eta/2.$$

By (GDP2$_\chi$) of Lemma 3.3, we deduce that $\chi(t_n, z_n, p_n - \varepsilon_n) = 0$, which contradicts (3.14). □

To complete the proof of the theorem, we still need to show Lemma 3.5. To this end, we first make the following observation.

LEMMA 3.6. *Let $\alpha \in L^2_{\mathrm{loc}}(W)$ be such that $M = \int \alpha \, dW$ is a bounded martingale and let β be an $\mathbb{R}^d$-valued, progressively measurable process such that $|\beta| \leq c(1 + |\alpha|)$ for some constant c. Then the stochastic exponential $\mathcal{E}(\int \beta \, dW)$ is a true martingale.*

PROOF. The assumption clearly implies that $\int_0^T |\beta_s|^2 \, ds < \infty$ $\mathbb{P}$-a.s. Since M is bounded, we have in particular that $M \in BMO$; that is,

$$\sup_{\tau \in \mathcal{T}_0} \bigg\| \mathbb{E}\bigg[\int_\tau^T |\alpha_s|^2 \, ds \Big| \mathcal{F}_\tau\bigg]\bigg\|_\infty < \infty.$$

In view of the assumption, the same holds with α replaced by β, so that $\int \beta \, dW$ is in *BMO*. This implies that $\mathcal{E}(\int \beta \, dW)$ is a true martingale; cf. [14], Theorem 2.3. □

PROOF OF LEMMA 3.5. Consider the process

$$\beta_n(r) := \frac{\delta_n(r)}{|\Sigma_n(r)|^2} \Sigma_n(r) \mathbf{1}_{A_n}(r);$$

we show that

$$|\beta_n| \le c\big(1 + |\alpha^{n,\nu_o}|\big) \qquad \text{on } [\![t_n, \tau_n]\!] \tag{3.19}$$

for some $c > 0$. Then the result will follow by applying Lemma 3.6 to $\alpha^{n,\nu_o}\mathbf{1}_{[\![t_n,\tau_n]\!]}$; note that the stochastic integral of this process is bounded by the definition of τ_n. To prove (3.19), we distinguish two cases.

Case 1. $\partial_p\varphi(t_o, z_o, p_o) \neq 0$. Using that μ and σ are continuous and that U and B_ε are bounded, tracing the definitions yields that

$$|\delta_n| \le c\big\{1 + |\alpha^{n,\nu_o}| + |\alpha^{n,\nu_o}|^2 \big|\partial_{pp}\varphi(\cdot, Z^n, P^n)\big|\big\} \qquad \text{on } [\![t_n, \tau_n]\!],$$

while

$$|\Sigma_n| \ge -c + |\alpha^{n,\nu_o}| \big|\partial_p\varphi(\cdot, Z^n, P^n)\big| \qquad \text{on } [\![t_n, \tau_n]\!]$$

for some $c > 0$. Since $\partial_p\varphi(t_o, z_o, p_o) \neq 0$ by assumption, $\partial_p\varphi$ is uniformly bounded away from zero on B_ε, after diminishing $\varepsilon > 0$ if necessary. Hence, recalling (3.12), there is a cancelation between $|\delta_n|$ and $|\Sigma_n|$ which allows us to conclude (3.19).

Case 2. $\partial_p\varphi(t_o, z_o, p_o) = 0$. We first observe that

$$\delta_n^+ \le c\big(1 + |\alpha^{n,\nu_o}|\big) - c^{-1}|\alpha^{n,\nu_o}|^2 \partial_{pp}\varphi(\cdot, Z^n, P^n) \qquad \text{on } [\![t_n, \tau_n]\!]$$

for some $c > 0$. Since δ_n^- and $|\Sigma_n|^{-1}$ are uniformly bounded on A_n, it therefore suffices to show that $\partial_{pp}\varphi \ge 0$ on B_ε. To see this, we note that (3.6) and the relaxation in the definition (3.3) of H^* imply that there exists $\iota > 0$ such that, for some $v \in V$ and all small $\varepsilon > 0$,

$$-\partial_t\varphi(t_o, z_o, p_o) + F\big(\Theta^\iota, u, a, v\big) \le -\eta \qquad \text{for all } (u, a) \in \mathcal{N}_\varepsilon\big(\Theta^\iota\big), \tag{3.20}$$

where $\Theta^\iota = (z_0, p_0, D\varphi, A^\iota)$ and A^ι is the same matrix as $D^2\varphi(t_o, z_o, p_o)$ except that the entry $\partial_{pp}\varphi(t_o, z_o, p_o)$ is replaced by $\partial_{pp}\varphi(t_o, z_o, p_o) - \iota$. Going back to the definition of $\mathcal{N}_\varepsilon$, we observe that $\mathcal{N}_\varepsilon(\Theta^\iota)$ does not depend on ι and, which is the crucial part, the assumption that $\partial_p\varphi(t_o, z_o, p_o) = 0$ implies that $\mathcal{N}_\varepsilon(\Theta^\iota)$ is of the form $\mathcal{N}^U \times \mathbb{R}^d$; that is, the variable a is unconstrained. Now (3.20) and the last observation show that

$$-\big(\partial_{pp}\varphi(t_o, z_o, p_o) - \iota\big)|a|^2 \le c\big(1 + |a|\big)$$

for *all* $a \in \mathbb{R}^d$, so we deduce that $\partial_{pp}\varphi(t_o, z_o, p_o) \ge \iota > 0$. Thus, after diminishing $\varepsilon > 0$ if necessary, we have $\partial_{pp}\varphi \ge 0$ on B_ε as desired. This completes the proof. $\square$

3.3. *PDE in the monotone case.* We now specialize the setup of Section 3.1 to the case where the state process Z consists of a pair of processes (X, Y) with values in $\mathbb{R}^{d-1} \times \mathbb{R}$, and the loss function

$$\ell : \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}, \qquad (x, y) \mapsto \ell(x, y)$$

is nondecreasing in the scalar variable y. This setting, which was previously studied in [3] for the case without adverse control, will allow for a more explicit description of Λ which is particularly suitable for applications in mathematical finance.

For $(t, x, y) \in [0, T] \times \mathbb{R}^{d-1} \times \mathbb{R}$ and $(\mathfrak{u}, \nu) \in \mathcal{U} \times \mathcal{V}$, let $Z^{\mathfrak{u},\nu}_{t,x,y} = (X^{\mathfrak{u},\nu}_{t,x}, Y^{\mathfrak{u},\nu}_{t,x,y})$ be the strong solution of (3.1) with

$$\mu(x, y, u, v) := \begin{pmatrix} \mu_X(x, u, v) \\ \mu_Y(x, y, u, v) \end{pmatrix}, \qquad \sigma(x, y, u, v) := \begin{pmatrix} \sigma_X(x, u, v) \\ \sigma_Y(x, y, u, v) \end{pmatrix},$$

where μ_Y and σ_Y take values in $\mathbb{R}$ and $\mathbb{R}^{1\times d}$, respectively. The assumptions from Section 3.1 remain in force; in particular, the continuity and growth assumptions on μ and σ. In this setup, we can consider the real-valued function

$$\gamma(t, x, p) := \inf\{y \in \mathbb{R} : (x, y, p) \in \Lambda(t)\}.$$

In mathematical finance, this may describe the minimal capital y such that the given target can be reached by trading in the securities market modeled by $X^{\mathfrak{u},\nu}_{t,x}$; an illustration is given in the subsequent section. In the present context, Corollary 2.3 reads as follows.

LEMMA 3.7. *Fix $(t, x, y, p) \in [0, T] \times \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}$, let $O \subset [0, T] \times \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}$ be a bounded open set containing (t, x, y, p) and assume that γ is locally bounded.*

(GDP1$_\gamma$) *Assume that $y > \gamma(t, x, p+\varepsilon)$ for some $\varepsilon > 0$. Then there exist $\mathfrak{u} \in \mathfrak{U}$ and $\{\alpha^\nu, \nu \in \mathcal{V}\} \subset \mathcal{A}$ such that*

$$Y^{\mathfrak{u},\nu}_{t,x,y}(\tau^\nu) \geq \gamma_*\big(\tau, X^{\mathfrak{u},\nu}_{t,x}(\tau^\nu), P^{\alpha^\nu}_{t,p}(\tau^\nu)\big) \qquad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V},$$

where τ^ν is the first exit time of $(\cdot, X^{\mathfrak{u},\nu}_{t,x}, Y^{\mathfrak{u},\nu}_{t,x,y}, P^{\alpha^\nu}_{t,p})$ from O.

(GDP2$_\gamma$) *Let φ be a continuous function such that $\varphi \geq \gamma$ and let $(\mathfrak{u}, \mathfrak{a}) \in \mathfrak{U} \times \mathfrak{A}$ and $\eta > 0$ be such that*

$$Y^{\mathfrak{u},\nu}_{t,x,y}(\tau^\nu) \geq \varphi\big(\tau^\nu, X^{\mathfrak{u},\nu}_{t,x}(\tau^\nu), P^{\mathfrak{a}[\nu]}_{t,p}(\tau^\nu)\big) + \eta \qquad \mathbb{P}\text{-a.s. for all } \nu \in \mathcal{V},$$

where τ^ν is the first exit time of $(\cdot, X^{\mathfrak{u},\nu}_{t,x}, Y^{\mathfrak{u},\nu}_{t,x,y}, P^{\mathfrak{a}[\nu]}_{t,p})$ from O. Then $y \geq \gamma(t, x, p-\varepsilon)$ for all $\varepsilon > 0$.

PROOF. Noting that $y > \gamma(t, x, p)$ implies $(x, y, p) \in \Lambda(t)$ and that $(x, y, p) \in \Lambda(t)$ implies $y \geq \gamma(t, x, p)$, the result follows from Corollary 2.3 by arguments similar to the proof of Lemma 3.3. $\square$

The Hamiltonians G^* and G_* for the PDE describing γ are defined like H^* and H_* in (3.3) and (3.4), but with

$$F(\Theta, u, a, v) \\ := \{\mu_Y(x, y, u, v) - \mu_{(X,P)}(x, u, v)^\top q - \tfrac{1}{2}\operatorname{Tr}[\sigma_{(X,P)}\sigma_{(X,P)}^\top(x, u, a, v)A]\},$$

where $\Theta := (x, y, q, A) \in \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ and

$$\mu_{(X,P)}(x, u, a, v) := \begin{pmatrix} \mu_X(x, u, v) \\ 0 \end{pmatrix}, \qquad \sigma_{(X,P)}(x, u, a, v) := \begin{pmatrix} \sigma_X(x, u, v) \\ a \end{pmatrix}$$

with the relaxed kernel $\mathcal{N}_\varepsilon$ replaced by

$$\mathcal{K}_\varepsilon(x, y, q, v) := \{(u, a) \in U \times \mathbb{R}^d : |\sigma_Y(x, y, u, v) - q^\top \sigma_{(X,P)}(x, u, a, v)| \le \varepsilon\}$$

and N_{Lip} replaced by a set K_{Lip}, defined like N_{Lip} but in terms of $\mathcal{K}_0$ instead of $\mathcal{N}_0$. We then have the following result for the semicontinuous envelopes γ^* and γ_* of γ.

THEOREM 3.8. *Assume that γ is locally bounded. Then γ_* is a viscosity supersolution on* $[0, T) \times \mathbb{R}^{d-1} \times \mathbb{R}$ *of*

$$(-\partial_t + G^*)\varphi \ge 0$$

and γ^* is a viscosity subsolution on $[0, T) \times \mathbb{R}^{d-1} \times \mathbb{R}$ *of*

$$(-\partial_t + G_*)\varphi \le 0.$$

PROOF. The result follows from Lemma 3.7 by adapting the proof of [3], Theorem 2.1, using the arguments from the proof of Theorem 3.4 to account for the game-theoretic setting and the relaxed formulation of the GDP. We therefore omit the details. □

We shall not discuss in this generality the boundary conditions as $t \to T$; they are somewhat complicated to state but can be deduced similarly as in [3]. Obtaining a comparison theorem at the present level of generality seems difficult, mainly due to the presence of the sets $\mathcal{K}_\varepsilon$ and K_{Lip} (which depend on the solution itself) and the discontinuity of the nonlinearities at $\partial_p\varphi = 0$. It seems more appropriate to treat this question on a case-by-case basis. In fact, once $G^* = G_*$ (see also Remark 3.9), the challenges in proving comparison are similar as in the case without adverse player. For that case, comparison results have been obtained, for example, in [5] for a specific setting; see also the references therein for more examples.

REMARK 3.9. Let us discuss briefly the question whether $G^* = G_*$. We shall focus on the case where U is convex and the (nondecreasing) function γ is strictly increasing with respect to p; in this case, we are interested only in test functions φ with $\partial_p\varphi > 0$. Under this condition, $(u, a) \in \mathcal{K}_\varepsilon(\cdot, \varphi, (\partial_x\varphi, \partial_p\varphi), v)$ if and only if there exists ζ with $|\zeta| \le 1$ such that $a = (\partial_p\varphi)^{-1}(\sigma_Y(\cdot, \varphi, u, v) - \partial_x\varphi^\top \sigma_X(\cdot, u, v) - \varepsilon\zeta)$. From this, it is not hard to see that for such functions, the relaxation $\varepsilon \searrow 0$, $\Theta' \to \Theta$ in (3.3) is superfluous as the operator is already continuous, so we are left with the question whether

$$\inf_{v\in V} \sup_{(u,a)\in\mathcal{K}_0(\Theta,v)} F(\Theta, u, a, v) = \sup_{(\hat{u},\hat{a})\in K_{\mathrm{Lip}}(\Theta)} \inf_{v\in V} F\big(\Theta, \hat{u}(\Theta, v), \hat{a}(\Theta, v), v\big).$$

The inequality "$\ge$" is clear. The converse inequality will hold if, say, for each $\varepsilon > 0$, there exists a locally Lipschitz mapping $(\hat{u}_\varepsilon, \hat{a}_\varepsilon) \in K_{\mathrm{Lip}}$ such that

$$F\big(\cdot, (\hat{u}_\varepsilon, \hat{a}_\varepsilon)(\cdot, v), v\big) \ge \sup_{(u,a)\in\mathcal{K}_0(\cdot,v)} F(\cdot, u, a, v) - \varepsilon \qquad \text{for all } v \in V.$$

Conditions for the existence of ε-optimal *continuous* selectors can be found in [15], Theorem 3.2. If $(u_\varepsilon, a_\varepsilon)$ is an ε-optimal continuous selector, the definition of $\mathcal{K}_0$ entails that $a_\varepsilon^\top(\Theta, v)q_p = -\sigma_X^\top(x, u_\varepsilon(\Theta, v), v)q_x + \sigma_Y(x, y, u_\varepsilon(\Theta, v), v)$, where we use the notation $\Theta = (x, y, p, (q_x^\top, q_p)^\top, A)$. Then u_ε can be further approximated, uniformly on compact sets, by a locally Lipschitz function $\hat{u}_\varepsilon$. We may restrict our attention to $q_p > 0$; so that, if we assume that $\sigma^\top$ is (jointly) locally Lipschitz, the mapping $\hat{a}_\varepsilon^\top(\Theta, v) := (q_p)^{-1}(-\sigma_X^\top(x, \hat{u}_\varepsilon(\Theta, v), v)q_x + \sigma_Y(x, y, \hat{u}_\varepsilon(\Theta, v), v))$ is locally Lipschitz, and then $(\hat{u}_\varepsilon, \hat{a}_\varepsilon)$ defines a sufficiently good, locally Lipschitz continuous selector: for all $v \in V$,

$$\begin{aligned} F\big(\cdot, (\hat{u}_\varepsilon, \hat{a}_\varepsilon)(\cdot, v), v\big) &\ge F\big(\cdot, (u_\varepsilon, a_\varepsilon)(\cdot, v), v\big) - O_\varepsilon(1) \\ &\ge \sup_{(u,a)\in\mathcal{K}_0} F(\cdot, u, a, v) - \varepsilon - O_\varepsilon(1) \end{aligned}$$

on a neighborhood of Θ, where $O_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. One can similarly discuss other cases, for example, when γ is strictly concave (instead of increasing) with respect to p and the mapping $(x, y, q_x, u, v) \mapsto -\sigma_X^\top(x, u, v)q_x + \sigma_Y(x, y, u, v)$ is invertible in u, with an inverse, that is, locally Lipschitz, uniformly in v.

4. Application to hedging under uncertainty. In this section, we illustrate our general results in a concrete example, and use the opportunity to show how to extend them to a case with unbounded strategies. To this end, we shall consider a problem of partial hedging under Knightian uncertainty. More precisely, the uncertainty concerns the drift and volatility coefficients of the risky asset, and we aim at controlling a function of the hedging error; the corresponding worst-case analysis is equivalent to a game where the adverse player chooses the coefficients. This problem is related to the G-expectation from [22, 23], the second order target problem from [26] and the problem of optimal arbitrage studied in [11]. We let

$$V = [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$$

be the possible values of the coefficients, where $\underline{\mu} \le 0 \le \overline{\mu}$ and $\overline{\sigma} \ge \underline{\sigma} \ge 0$. Moreover, $U = \mathbb{R}$ will be the possible values for the investment policy, so that, in contrast to the previous sections, U is not bounded.

The notation is the same as in the previous section, except for an integrability condition for the strategies that will be introduced below to account for the unboundedness of U; moreover, we shall sometimes write $\nu = (\mu, \sigma)$ for an adverse control $\nu \in \mathcal{V}$. Given $(\mu, \sigma) \in \mathcal{V}$ and $\mathfrak{u} \in \mathfrak{U}$, the state process $Z^{\mathfrak{u},\nu}_{t,x,y} = (X^\nu_{t,x}, Y^{\mathfrak{u},\nu}_{t,y})$ is governed by

$$\frac{dX^\nu_{t,x}(r)}{X^\nu_{t,x}(r)} = \mu_r\,dr + \sigma_r\,dW_r, \qquad X^\nu_{t,x}(t) = x$$

and

$$dY^{\mathfrak{u},\nu}_{t,y}(r) = \mathfrak{u}[\nu]_r(\mu_r\,dr + \sigma_r\,dW_r), \qquad Y^{\mathfrak{u},\nu}_{t,y}(t) = y.$$

To wit, the process $X^\nu_{t,x}$ represents the price of a risky asset with unknown drift and volatility coefficients (μ, σ), while $Y^{\mathfrak{u},\nu}_{t,y}$ stands for the wealth process associated to an investment policy $\mathfrak{u}[\nu]$, denominated in monetary amounts. (The interest rate is zero for simplicity.) We remark that it is clearly necessary to use strategies in this setup: even a simple stop-loss investment policy cannot be implemented as a control.

Our loss function is of the form

$$\ell(x, y) = \Psi\bigl(y - g(x)\bigr),$$

where $\Psi, g : \mathbb{R} \to \mathbb{R}$ are continuous functions of polynomial growth. The function Ψ is also assumed to be strictly increasing and concave, with an inverse $\Psi^{-1} : \mathbb{R} \to \mathbb{R}$, that is, again of polynomial growth. As a consequence, ℓ is continuous and (2.3) is satisfied for some $q > 0$; that is,

$$|\ell(z)| \le C\bigl(1 + |z|^q\bigr), \qquad z = (x, y) \in \mathbb{R}^2. \tag{4.1}$$

We interpret $g(X^\nu_{t,x}(T))$ as the random payoff of a European option written on the risky asset, for a given realization of the drift and volatility processes, while Ψ quantifies the disutility of the hedging error $Y^{\mathfrak{u},\nu}_{t,y}(T) - g(X^\nu_{t,x}(T))$. In this setup,

$$\begin{aligned}&\gamma(t, x, p)\\ &\quad = \inf\bigl\{y \in \mathbb{R} : \exists \mathfrak{u} \in \mathfrak{U} \text{ s.t. } \mathbb{E}\bigl[\Psi\bigl(Y^{\mathfrak{u},\nu}_{t,y}(T) - g\bigl(X^\nu_{t,x}(T)\bigr)\bigr)|\mathcal{F}_t\bigr] \ge p\ \mathbb{P}\text{-a.s. } \forall \nu \in \mathcal{V}\bigr\}\end{aligned}$$

is the minimal price for the option allowing to find a hedging policy such that the expected disutility of the hedging error is controlled by p.

We fix a finite constant $\bar{q} > q \vee 2$ and define $\mathfrak{U}$ to be the set of mappings $\mathfrak{u} : \mathcal{V} \to \mathcal{U}$ that are nonanticipating (as in Section 3) and satisfy the integrability condition

$$\sup_{\nu \in \mathcal{V}} \mathbb{E}\biggl[\biggl|\int_0^T \bigl|\mathfrak{u}[\nu]_r\bigr|^2\,dr\biggr|^{\bar{q}/2}\biggr] < \infty. \tag{4.2}$$

The conclusions below do not depend on the choice of $\bar{q}$. The main result of this section is an explicit expression for the price $\gamma(t, x, p)$.

THEOREM 4.1. *Let $(t,x,p) \in [0,T] \times (0,\infty) \times \mathbb{R}$. Then $\gamma(t,x,p)$ is finite and given by*

$$\gamma(t,x,p) = \sup_{\nu\in\mathcal{V}^0} \mathbb{E}\big[g\big(X^{\nu}_{t,x}(T)\big)\big] + \Psi^{-1}(p) \tag{4.3}$$
$$\text{where } \mathcal{V}^0 = \big\{(\mu,\sigma) \in \mathcal{V} : \mu \equiv 0\big\}.$$

In particular, $\gamma(t,x,p)$ coincides with the superhedging price for the shifted option $g(\cdot) + \Psi^{-1}(p)$ in the (driftless) uncertain volatility model for $[\underline{\sigma},\overline{\sigma}]$; see also below. That is, the drift uncertainty has no impact on the price, provided that $\underline{\mu} \le 0 \le \overline{\mu}$. Let us remark, in this respect, that the present setup corresponds to an investor who knows the present and historical drift and volatility of the underlying. It may also be interesting to study the case where only the trajectories of the underlying (and therefore the volatility, but not necessarily the drift) are observed. This, however, does not correspond to the type of game studied in this paper.

4.1. *Proof of Theorem 4.1.*

PROOF OF "$\ge$" IN (4.3). We may assume that $\gamma(t,x,p) < \infty$. Let $y > \gamma(t,x,p)$; then there exists $\mathfrak{u} \in \mathfrak{U}$ such that

$$\mathbb{E}\big[\Psi\big(Y^{\mathfrak{u},\nu}_{t,y}(T) - g\big(X^{\nu}_{t,x}(T)\big)\big)\big] \ge p \qquad \text{for all } \nu \in \mathcal{V}.$$

As Ψ is concave, it follows by Jensen's inequality that

$$\Psi\big(\mathbb{E}\big[Y^{\mathfrak{u},\nu}_{t,y}(T) - g\big(X^{\nu}_{t,x}(T)\big)\big]\big) \ge p \qquad \text{for all } \nu \in \mathcal{V}.$$

Since the integrability condition (4.2) implies that $Y^{\mathfrak{u},\nu}_{t,y}$ is a martingale for all $\nu \in \mathcal{V}^0$, we conclude that

$$\Psi\big(y - \mathbb{E}\big[g\big(X^{\nu}_{t,x}(T)\big)\big]\big) \ge p \qquad \text{for all } \nu \in \mathcal{V}^0$$

and hence $y \ge \sup_{\nu\in\mathcal{V}^0} \mathbb{E}[g(X^{\nu}_{t,x}(T))] + \Psi^{-1}(p)$. As $y > \gamma(t,x,p)$ was arbitrary, the claim follows. $\square$

We shall use Theorem 3.8 to derive the missing inequality in (4.3). Since $U = \mathbb{R}$ is unbounded, we introduce a sequence of approximating problems γ_n defined like γ, but with strategies bounded by n,

$$\gamma_n(t,x,p) := \inf\big\{y \in \mathbb{R} : \exists \mathfrak{u} \in \mathfrak{U}^n \text{ s.t. } \mathbb{E}\big[\ell\big(Z^{\mathfrak{u},\nu}_{t,x,y}(T)\big)|\mathcal{F}_t\big] \ge p \ \mathbb{P}\text{-a.s. } \forall \nu \in \mathcal{V}\big\},$$

where

$$\mathfrak{U}^n = \big\{\mathfrak{u} \in \mathfrak{U} : \big|\mathfrak{u}[\nu]\big| \le n \text{ for all } \nu \in \mathcal{V}\big\}.$$

Then clearly γ_n is decreasing in n and

$$\gamma_n \ge \gamma, \qquad n \ge 1. \tag{4.4}$$

LEMMA 4.2. *Let* $(t,z)\in[0,T]\times(0,\infty)\times\mathbb{R}$, $\mathfrak{u}\in\mathfrak{U}$, *and define* $\mathfrak{u}_n\in\mathfrak{U}$ *by*

$$\mathfrak{u}_n[\nu]:=\mathfrak{u}[\nu]\mathbf{1}_{\{|\mathfrak{u}[\nu]|\le n\}},\qquad \nu\in\mathcal{V}.$$

Then

$$\operatorname*{ess\,sup}_{\nu\in\mathcal{V}}\bigl|\mathbb{E}\bigl[\ell\bigl(Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\bigr)-\ell\bigl(Z^{\mathfrak{u},\nu}_{t,z}(T)\bigr)|\mathcal{F}_t\bigr]\bigr|\to 0\qquad \textit{in } L^1 \textit{ as } n\to\infty.$$

PROOF. Using monotone convergence and an argument as in the proof of step 1 in Section 2.3, we obtain that

$$\begin{aligned}\mathbb{E}\Bigl\{&\operatorname*{ess\,sup}_{\nu\in\mathcal{V}}\bigl|\mathbb{E}\bigl[\ell\bigl(Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\bigr)-\ell\bigl(Z^{\mathfrak{u},\nu}_{t,z}(T)\bigr)|\mathcal{F}_t\bigr]\bigr|\Bigr\}\\&=\sup_{\nu\in\mathcal{V}}\mathbb{E}\bigl\{\bigl|\ell\bigl(Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\bigr)-\ell\bigl(Z^{\mathfrak{u},\nu}_{t,z}(T)\bigr)\bigr|\bigr\}.\end{aligned}$$

Since V is bounded, the Burkholder–Davis–Gundy inequalities show that there is a universal constant $c>0$ such that

$$\begin{aligned}\mathbb{E}\bigl\{\bigl|Z^{\mathfrak{u}_n,\nu}_{t,z}(T)-Z^{\mathfrak{u},\nu}_{t,z}(T)\bigr|\bigr\}&\le c\mathbb{E}\biggl[\int_t^T\bigl|\mathfrak{u}[\nu]_r-\mathfrak{u}_n[\nu]_r\bigr|^2\,dr\biggr]^{1/2}\\&=c\mathbb{E}\biggl[\int_t^T\bigl|\mathfrak{u}[\nu]_r\mathbf{1}_{\{|\mathfrak{u}[\nu]_r|>n\}}\bigr|^2\,dr\biggr]^{1/2}\end{aligned}$$

and hence (4.2) and Hölder's inequality yield that, for any given $\delta>0$,

$$\begin{aligned}&\sup_{\nu\in\mathcal{V}}\mathbb{P}\bigl\{\bigl|Z^{\mathfrak{u}_n,\nu}_{t,z}(T)-Z^{\mathfrak{u},\nu}_{t,z}(T)\bigr|>\delta\bigr\}\\&\qquad\le\delta^{-1}\sup_{\nu\in\mathcal{V}}\mathbb{E}\bigl\{\bigl|Z^{\mathfrak{u}_n,\nu}_{t,z}(T)-Z^{\mathfrak{u},\nu}_{t,z}(T)\bigr|\bigr\}\to 0\end{aligned}\tag{4.5}$$

for $n\to\infty$. Similarly, the Burkholder–Davis–Gundy inequalities and (4.2) show that $\{|Z^{\mathfrak{u}_n,\nu}_{t,z}(T)|+|Z^{\mathfrak{u},\nu}_{t,z}(T)|,\nu\in\mathcal{V},n\ge 1\}$ is bounded in $L^{\bar q}$. This yields, on the one hand, that

$$\sup_{\nu\in\mathcal{V},n\ge1}\mathbb{P}\bigl\{\bigl|Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\bigr|+\bigl|Z^{\mathfrak{u},\nu}_{t,z}(T)\bigr|>k\bigr\}\to 0\tag{4.6}$$

for $k\to\infty$, and on the other hand, in view of (4.1) and $\bar q>q$, that

$$\bigl\{\ell\bigl(Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\bigr)-\ell\bigl(Z^{\mathfrak{u},\nu}_{t,z}(T)\bigr):\nu\in\mathcal{V},n\ge1\bigr\}\qquad\text{is uniformly integrable.}\tag{4.7}$$

Let $\varepsilon>0$; then (4.6) and (4.7) show that we can choose $k>0$ such that

$$\sup_{\nu\in\mathcal{V}}\mathbb{E}\bigl[\bigl|\ell\bigl(Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\bigr)-\ell\bigl(Z^{\mathfrak{u},\nu}_{t,z}(T)\bigr)\bigr|\mathbf{1}_{\{|Z^{\mathfrak{u}_n,\nu}_{t,z}(T)|+|Z^{\mathfrak{u},\nu}_{t,z}(T)|>k\}}\bigr]<\varepsilon$$

for all n. Using also that ℓ is uniformly continuous on $\{|z|\le k\}$, we thus find $\delta>0$ such that

$$\begin{aligned}&\sup_{\nu\in\mathcal{V}}\mathbb{E}\bigl[\bigl|\ell\bigl(Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\bigr)-\ell\bigl(Z^{\mathfrak{u},\nu}_{t,z}(T)\bigr)\bigr|\bigr]\\&\qquad\le 2\varepsilon+\sup_{\nu\in\mathcal{V}}\mathbb{E}\bigl[\bigl|\ell\bigl(Z^{\mathfrak{u}_n,\nu}_{t,z}(T)\bigr)-\ell\bigl(Z^{\mathfrak{u},\nu}_{t,z}(T)\bigr)\bigr|\mathbf{1}_{\{|Z^{\mathfrak{u}_n,\nu}_{t,z}(T)-Z^{\mathfrak{u},\nu}_{t,z}(T)|>\delta\}}\bigr].\end{aligned}$$

By (4.5) and (4.7), the supremum on the right-hand side tends to zero as $n \to \infty$. This completes the proof of Lemma 4.2. $\square$

PROOF OF "$\leq$" IN (4.3). It follows from the polynomial growth of g and the boundedness of V that the right-hand side of (4.3) is finite. Thus the already established inequality "$\geq$" in (4.3) yields that $\gamma(t,x,p) > -\infty$. We now show the theorem under the hypothesis that $\gamma(t,x,p) < \infty$ for all p; we shall argue at the end of the proof that this is automatically satisfied.

Step 1. Let $\gamma_\infty := \inf_n \gamma_n$. Then the upper semicontinuous envelopes of γ and γ_∞ coincide: $\gamma^* = \gamma_\infty^*$.

It follows from (4.4) that $\gamma_\infty^* \geq \gamma^*$. Let $\eta > 0$ and $y > \gamma(t,x,p+\eta)$. We show that $y \geq \gamma_n(t,x,p)$ for n large; this will imply the remaining inequality $\gamma_\infty^* \leq \gamma^*$. Indeed, the definition of γ and Lemma 4.2 imply that we can find $\mathfrak{u} \in \mathfrak{U}$ and $\mathfrak{u}_n \in \mathfrak{U}^n$ such that

$$J(t,x,y,\mathfrak{u}_n) \geq J(t,x,y,\mathfrak{u}) - \epsilon_n \geq p + \eta - \epsilon_n \qquad \mathbb{P}\text{-a.s.},$$

where $\epsilon_n \to 0$ in L^1. If K_n is defined like K, but with $\mathfrak{U}^n$ instead of $\mathfrak{U}$, then it follows that $K_n(t,x,y) \geq p + \eta - \epsilon_n$ $\mathbb{P}$-a.s. Recalling that K_n is deterministic (cf. Proposition 3.1), we may replace ϵ_n by $\mathbb{E}[\epsilon_n]$ in this inequality. Sending $n \to \infty$, we then see that $\lim_{n\to\infty} K_n(t,x,y) \geq p+\eta$, and therefore $K_n(t,x,y) \geq p + \eta/2$ for n large enough. The fact that $y \geq \gamma_n(t,x,p)$ for n large then follows from the same considerations as in Lemma 2.4.

Step 2. The relaxed semi-limit

$$\bar{\gamma}_\infty^*(t,x,p) := \limsup_{\substack{n\to\infty \\ (t',x',p')\to(t,x,p)}} \gamma_n^*(t',x',p')$$

is a viscosity subsolution on $[0,T) \times (0,\infty) \times \mathbb{R}$ of

$$-\partial_t\varphi + \inf_{\sigma\in[\underline{\sigma},\overline{\sigma}]} \left\{-\frac{1}{2}\sigma^2 x^2 \partial_{xx}\varphi\right\} \leq 0 \tag{4.8}$$

and satisfies the boundary condition $\bar{\gamma}_\infty^*(T,x,p) \leq g(x) + \Psi^{-1}(p)$.

We first show that the boundary condition is satisfied. Fix $(x,p) \in (0,\infty) \times \mathbb{R}$ and let $y > g(x) + \Psi^{-1}(p)$; then $\ell(x,y) > p$. Let $(t_n,x_n,p_n) \to (T,x,p)$ be such that $\gamma_n(t_n,x_n,p_n) \to \bar{\gamma}_\infty^*(T,x,p)$. We consider the strategy $\mathfrak{u} \equiv 0$ and use the arguments from the proof of Proposition 3.1 to find a constant c independent of n such that

$$\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}\big[\big|Z^{0,\nu}_{t_n,x_n,y}(T) - (x,y)\big|^{\bar{q}} \big| \mathcal{F}_{t_n}\big] \leq c\big(|T-t_n|^{\bar{q}/2} + |x-x_n|^{\bar{q}}\big).$$

Similar to the proof of Lemma 4.2, this implies that there exist constants $\varepsilon_n \to 0$ such that

$$J(t_n,x_n,y,0) \geq \ell(x,y) - \varepsilon_n \qquad \mathbb{P}\text{-a.s.}$$

In view of $\ell(x,y) > p$, this shows that $y \geq \gamma_n(t_n, x_n, p_n)$ for n large enough, and hence that $y \geq \bar{\gamma}^*_\infty(T,x,p)$. As a result, we have $\bar{\gamma}^*_\infty(T,x,p) \leq g(x) + \Psi^{-1}(p)$.

It remains to show the subsolution property. Let φ be a smooth function, and let $(t_o, x_o, p_o) \in [0,T) \times (0,\infty) \times \mathbb{R}$ be such that

$$(\bar{\gamma}^*_\infty - \varphi)(t_o, x_o, p_o) = \max(\bar{\gamma}^*_\infty - \varphi) = 0.$$

After passing to a subsequence, [1], Lemma 4.2, yields $(t_n, x_n, p_n) \to (t_o, x_o, p_o)$ such that

$$\lim_{n\to\infty}(\gamma^*_n - \varphi)(t_n, x_n, p_n) = (\bar{\gamma}^*_\infty - \varphi)(t_o, x_o, p_o)$$

and such that (t_n, x_n, p_n) is a local maximizer of $(\gamma^*_n - \varphi)$. Applying Theorem 3.8 to γ^*_n, we deduce that

$$\sup_{(\hat{u},\hat{a})\in K^n_{\mathrm{Lip}}(\cdot,D\varphi)} \inf_{(\mu,\sigma)\in V} G\varphi(\cdot,(\hat{u},\hat{a})(\mu,\sigma),(\mu,\sigma))(t_n,x_n,p_n) \leq 0, \tag{4.9}$$

where

$$G\varphi(\cdot,(u,a),(\mu,\sigma))$$
$$:= u\mu - \partial_t\varphi - \mu x\partial_x\varphi - \tfrac{1}{2}(\sigma^2x^2\partial_{xx}\varphi + a^2\partial_{pp}\varphi + 2\sigma x a\partial_{xp}\varphi)$$

and $K^n_{\mathrm{Lip}}(\cdot,D\varphi)(t_n,x_n,p_n)$ is the set of locally Lipschitz mappings $(\hat{u},\hat{a})$ with values in $[-n,n]\times\mathbb{R}$ such that

$$\sigma\hat{u}(x,q_x,q_p,\mu,\sigma) = x\sigma q_x + q_p\hat{a}(x,q_x,q_p,\mu,\sigma) \qquad \text{for all } \sigma\in[\underline{\sigma},\overline{\sigma}]$$

for all $(x,(q_x,q_p))$ in a neighborhood of $(x_n, D\varphi(t_n,x_n,p_n))$. Since the mapping

$$(0,\infty)\times\mathbb{R}^2\times[\underline{\mu},\overline{\mu}]\times[\underline{\sigma},\overline{\sigma}]\to\mathbb{R}^2 \qquad (x,q_x,q_p,\mu,\sigma)\mapsto(xq_x,0)$$

belongs to $K^n_{\mathrm{Lip}}(\cdot,D\varphi)(t_n,x_n,p_n)$ for n large enough, (4.9) leads to

$$-\partial_t\varphi + \inf_{\sigma\in[\underline{\sigma},\overline{\sigma}]}\left\{-\frac{1}{2}\sigma^2x^2\partial_{xx}\varphi\right\}(t_n,x_n,p_n) \leq 0$$

for n large. Here the nonlinearity is continuous; therefore, sending $n\to\infty$ yields (4.8).

Step 3. We have $\bar{\gamma}^*_\infty \leq \pi$ on $[0,T]\times(0,\infty)\times\mathbb{R}$, where

$$\pi(t,x,p) := \sup_{\nu\in\mathcal{V}^0}\mathbb{E}[g(X^\nu_{t,x}(T))] + \Psi^{-1}(p)$$

is the right-hand side of (4.3).

Indeed, our assumptions on g and Ψ^{-1} imply that π is continuous with polynomial growth. It then follows by standard arguments that π is a viscosity supersolution on $[0,T)\times(0,\infty)\times\mathbb{R}$ of

$$-\partial_t\varphi + \inf_{\sigma\in[\underline{\sigma},\overline{\sigma}]}\left\{-\frac{1}{2}\sigma^2x^2\partial_{xx}\varphi\right\} \geq 0$$

and clearly the boundary condition $\pi(T, x, p) \geq g(x) + \Psi^{-1}(p)$ is satisfied. The claim then follows from step 2 by comparison.

We can now deduce the theorem: we have $\gamma \leq \gamma^*$ by the definition of γ^* and $\gamma^* = \gamma^*_\infty$ by step 1. As $\gamma^*_\infty \leq \bar{\gamma}^*_\infty$ by construction, step 3 yields the result.

It remains to show that $\gamma < \infty$. Indeed, this is clearly satisfied when g is bounded from above. For the general case, we consider $g_m = g \wedge m$ and let γ_m be the corresponding value function. Given $\eta > 0$, we have $\gamma_m(t, x, p + \eta) < \infty$ for all m and so (4.3) holds for g_m. We see from (4.3) that $y := 1 + \sup_m \gamma_m(t, x, p + \eta)$ is finite. Thus, there exist $\mathfrak{u}_m \in \mathfrak{U}$ such that

$$\mathbb{E}\big[\Psi\big(Y^{\mathfrak{u}_m,\nu}_{t,y}(T) - g_m\big(X^\nu_{t,x}(T)\big)\big)|\mathcal{F}_t\big] \geq p + \eta \qquad \text{for all } \nu \in \mathcal{V}.$$

Using once more the boundedness of V, we see that for m large enough,

$$\mathbb{E}\big[\Psi\big(Y^{\mathfrak{u}_m,\nu}_{t,y}(T) - g\big(X^\nu_{t,x}(T)\big)\big)|\mathcal{F}_t\big] \geq p \qquad \text{for all } \nu \in \mathcal{V},$$

which shows that $\gamma(t, x, p) \leq y < \infty$. $\square$

REMARK 4.3. We sketch a probabilistic proof for the inequality "$\leq$" in Theorem 4.1, for the special case without drift ($\underline{\mu} = \overline{\mu} = 0$) and $\underline{\sigma} > 0$. We focus on $t = 0$, and recall that $y_0 := \sup_{\nu \in \mathcal{V}^0} \mathbb{E}[g(X^\nu_{0,x}(T))]$ is the superhedging price for $g(\cdot)$ in the uncertain volatility model. More precisely, if B is the coordinate-mapping process on $\Omega = C([0, T]; \mathbb{R})$, there exists an $\mathbb{F}^B$-progressively measurable process ϑ such that

$$y_0 + \int_0^T \vartheta_s \frac{dB_s}{B_s} \geq g(B_T) \qquad P^\nu\text{-a.s. for all } \nu \in \mathcal{V}^0,$$

where P^ν is the law of $X^\nu_{0,x}$ under P; see, for example, [20]. Seeing ϑ as an adapted functional of B, this implies that

$$y_0 + \int_0^T \vartheta_s\big(X^\nu_{0,x}\big) \frac{dX^\nu_{0,x}(s)}{X^\nu_{0,x}(s)} \geq g\big(X^\nu_{0,x}(T)\big) \qquad P\text{-a.s. for all } \nu \in \mathcal{V}^0.$$

Since $X^\nu_{0,x}$ is nonanticipating with respect to ν, we see that $\mathfrak{u}[\nu]_s := \vartheta_s(X^\nu_{0,x})$ defines a nonanticipating strategy such that, with $y := y_0 + \Psi^{-1}(p)$,

$$y + \int_0^T \mathfrak{u}[\nu]_s \frac{dX^\nu_{0,x}(s)}{X^\nu_{0,x}(s)} \geq g\big(X^\nu_{0,x}(T)\big) + \Psi^{-1}(p);$$

that is,

$$\Psi\big(Y^{\mathfrak{u},\nu}_{0,y}(T) - g\big(X^\nu_{0,x}(T)\big)\big) \geq p$$

holds even P-almost surely, rather than only in expectation, for all $\nu \in \mathcal{V}^0$, and $\mathcal{V}^0 = \mathcal{V}$ because of our assumption that $\underline{\mu} = \overline{\mu} = 0$. In particular, we have the *existence of an optimal strategy* $\mathfrak{u}$. (We notice that, in this respect, it is important that our definition of strategies does not contain regularity assumptions on $\nu \mapsto \mathfrak{u}[\nu]$.)

Heuristically, the case with drift uncertainty (i.e., $\underline{\mu} \neq \overline{\mu}$) can be reduced to the above by a Girsanov change of measure argument; for example, if μ is deterministic, then we can take $\mathfrak{u}[(\mu, \sigma)] := \mathfrak{u}[(0, \sigma^{\mu})]$, where $\sigma^{\mu}(\omega) := \sigma(\omega + \int \mu_t \, dt)$. However, for general μ, there are difficulties related to the fact that a Girsanov Brownian motion need not generate the original filtration (see, e.g., [10]), and we shall not enlarge on this.

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REFERENCES

[1] BARLES, G. (1994). *Solutions de Viscosité des Équations de Hamilton–Jacobi*. Springer, Paris. MR1613876

[2] BAYRAKTAR, E. and YAO, S. (2011). On zero-sum stochastic differential games. Preprint. Available at arXiv:1112.5744v3.

[3] BOUCHARD, B., ELIE, R. and TOUZI, N. (2010). Stochastic target problems with controlled loss. *SIAM J. Control Optim.* **48** 3123–3150. MR2599913

[4] BOUCHARD, B. and NUTZ, M. (2012). Weak dynamic programming for generalized state constraints. *SIAM J. Control Optim.* **50** 3344–3373. MR3024163

[5] BOUCHARD, B. and VU, T. N. (2012). A stochastic target approach for P&L matching problems. *Math. Oper. Res.* **37** 526–558. MR2971628

[6] BUCKDAHN, R., HU, Y. and LI, J. (2011). Stochastic representation for solutions of Isaacs' type integral-partial differential equations. *Stochastic Process. Appl.* **121** 2715–2750. MR2844538

[7] BUCKDAHN, R. and LI, J. (2008). Stochastic differential games and viscosity solutions of Hamilton–Jacobi–Bellman–Isaacs equations. *SIAM J. Control Optim.* **47** 444–475. MR2373477

[8] DELLACHERIE, C. and MEYER, P.-A. (1982). *Probabilities and Potential. B*. North-Holland, Amsterdam. MR0745449

[9] EL KAROUI, N. and QUENEZ, M.-C. (1995). Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim.* **33** 29–66. MR1311659

[10] FELDMAN, J. and SMORODINSKY, M. (1997). Simple examples of non-generating Girsanov processes. In *Séminaire de Probabilités, XXXI. Lecture Notes in Math.* **1655** 247–251. Springer, Berlin. MR1478733

[11] FERNHOLZ, D. and KARATZAS, I. (2011). Optimal arbitrage under model uncertainty. *Ann. Appl. Probab.* **21** 2191–2225. MR2895414

[12] FLEMING, W. H. and SOUGANIDIS, P. E. (1989). On the existence of value functions of two-player, zero-sum stochastic differential games. *Indiana Univ. Math. J.* **38** 293–314. MR0997385

[13] FÖLLMER, H. and LEUKERT, P. (1999). Quantile hedging. *Finance Stoch.* **3** 251–273. MR1842286

[14] KAZAMAKI, N. (1994). *Continuous Exponential Martingales and BMO. Lecture Notes in Math.* **1579**. Springer, Berlin. MR1299529

[15] KUCIA, A. and NOWAK, A. (1987). On ϵ-optimal continuous selectors and their application in discounted dynamic programming. *J. Optim. Theory Appl.* **54** 289–302. MR0895740

[16] LI, J. and PENG, S. (2009). Stochastic optimization theory of backward stochastic differential equations with jumps and viscosity solutions of Hamilton–Jacobi–Bellman equations. *Nonlinear Anal*. **70** 1776–1796. MR2483598

[17] MERTENS, J.-F. (1972). Théorie des processus stochastiques généraux applications aux surmartingales. *Z. Wahrsch. Verw. Gebiete* **22** 45–68. MR0346895

[18] MOREAU, L. (2011). Stochastic target problems with controlled loss in jump diffusion models. *SIAM J. Control Optim*. **49** 2577–2607. MR2873197

[19] NEVEU, J. (1975). *Discrete-Parameter Martingales*. North-Holland, Amsterdam. MR0402915

[20] NUTZ, M. and SONER, H. M. (2012). Superhedging and dynamic risk measures under volatility uncertainty. *SIAM J. Control Optim*. **50** 2065–2089. MR2974730

[21] PENG, S. (1997). BSDE and stochastic optimizations. In *Topics in Stochastic Analysis* (J. Yan, S. Peng, S. Fang and L. Wu, eds.). Science Press, Beijing.

[22] PENG, S. (2007). G-expectation, G-Brownian motion and related stochastic calculus of Itô type. In *Stochastic Analysis and Applications*. *Abel Symp*. **2** 541–567. Springer, Berlin. MR2397805

[23] PENG, S. (2008). Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation. *Stochastic Process. Appl*. **118** 2223–2253. MR2474349

[24] SONER, H. M. and TOUZI, N. (2002). Dynamic programming for stochastic target problems and geometric flows. *J. Eur. Math. Soc*. (*JEMS*) **4** 201–236. MR1924400

[25] SONER, H. M. and TOUZI, N. (2002). Stochastic target problems, dynamic programming, and viscosity solutions. *SIAM J. Control Optim*. **41** 404–424. MR1920265

[26] SONER, H. M., TOUZI, N. and ZHANG, J. (2013). Dual formulation of second order target problems. *Ann. Appl. Probab.* **23** 308–347. MR3059237

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