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WEAK APPROXIMATIONS FOR WIENER FUNCTIONALS

BY DORIVAL LEÃO AND ALBERTO OHASHI[1]

Universidade de São Paulo, and Universidade Federal da Paraiba and Insper Institute

In this paper we introduce a simple space-filtration discretization scheme on Wiener space which allows us to study weak decompositions and smooth explicit approximations for a large class of Wiener functionals. We show that any Wiener functional has an underlying robust semimartingale skeleton which under mild conditions converges to it. The discretization is given in terms of discrete-jumping filtrations which allow us to approximate nonsmooth processes by means of a stochastic derivative operator on the Wiener space. As a by-product, we provide a robust semimartingale approximation for weak Dirichlet-type processes.

The underlying semimartingale skeleton is intrinsically constructed in such way that all the relevant structure is amenable to a robust numerical scheme. In order to illustrate the results, we provide an easily implementable approximation scheme for the classical Clark–Ocone formula in full generality. Unlike in previous works, our methodology does not assume an underlying Markovian structure and does not require Malliavin weights. We conclude by proposing a method that enables us to compute optimal stopping times for possibly non-Markovian systems arising, for example, from the fractional Brownian motion.

1. Introduction. Discretization methods for stochastic systems have always been a topic of great interest in stochastic analysis and its applications. Since the pioneering work of Wong and Zakai we know that not every choice of discretization procedure leads to good stability properties of elementary processes such as Itô integrals and related stochastic equations. See, for example, the works [3, 12, 25, 28, 33] and other references therein.

In order to get those convergence results, one has to assume suitable compactness arguments which allow one to exchange the limits. On the one hand, one may interpret such assumptions as simple technical arguments imposed on the system to get the desirable robustness. On the other hand, Graversen and Rao [22] have shown a close relation between finite energy and the existence of Doob–Meyer-type decompositions. More recently, Coquet et al. [10] has proved the uniqueness of such decompositions by means of the so-called weak Dirichlet processes.

The classical Graversen–Rao theorem can be proved by means of compactness arguments on predictable compensators of simple time-discretizations of the

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original process. In general, approximating sequences arising from such compactness arguments are not intrinsically constructed and are not suitable for numerical schemes in nonstandard cases arising from non-Markovian and nonsemimartingale systems.

The primary goal of this work is to describe readable structural conditions on a given optional process adapted to the Brownian filtration (henceforth abbreviated by Wiener functional) so that one can construct an *explicit*, *robust* and *feasible* approximating skeleton of smooth semimartingales. In order to illustrate the basic idea, let us assume that a Wiener functional X has an abstract representation

$$X_t = X_0 + \int_0^t H_s\, dB_s + N_t, \tag{1.1}$$

where B is the standard Brownian motion under its natural filtration $\mathbb{F}$, N can be a nonsemimartingale $\mathbb{F}$-optional process and H is a progressive process which is completely unknown a priori. The main problem addressed in this paper is the following one: Construct an explicit and simple sequence of $\mathbb{F}^k$-special semimartingales given by

$$X^k = X_0 + \int H^k\, dA^k + N^k, \qquad \mathbb{F}^k \subset \mathbb{F},$$

where H^k is fully based on the information generated by the pair (X, B) such that

$$\begin{aligned} &H^k \to H, \qquad A^k \to B, \qquad \int H^k\, dA^k \to \int H\, dB, \\ &N^k \to N, \qquad \mathbb{F}^k \to \mathbb{F} \qquad \text{as } k \to \infty. \end{aligned} \tag{1.2}$$

The main difficulty in answering this question comes from the fact that when X is very rough, the joint convergence of $(H^k, \int H^k\, dA^k)$ to $(H, \int H\, dB)$ in general will not hold since H has no a priori path regularity. Similarly, N can be very irregular in such a way that $N^k \to N$ will not hold either. A similar type of problem was addressed by Jacod, Meleard and Protter [25] in a pure martingale and Markovian setup at a fixed terminal time $0 < T \le \infty$. In [25], they have provided reasonable explicit expressions for H^k when B and N are replaced by orthogonal square-integrable martingales w.r.t. an arbitrary filtration. More explicit expressions were obtained by imposing an underlying Markovian structure. In this paper, we are interested in somehow more irregular objects arising from non-Markovian and nonsemimartingale systems restricted to the Wiener space.

In order to study Wiener functionals of type (1.1), an abstract theory is developed based on an underlying smooth semimartingale skeleton induced by a suitable sequence of stopping times which measures the instants when the Brownian motion hits some a priori levels. More precisely, from a given Brownian motion B we shall define inductively a sequence of stopping times

$$T^k_n := \inf\{T^k_{n-1} < t < \infty; |B_t - B_{T^k_{n-1}}| = 2^{-k}\}, \qquad n \ge 1,$$

which induces an embedded semimartingale structure of the form

$$\delta^k X_t := X_0 + \sum_{n=1}^{\infty} \mathbb{E}[X_{T_n^k} | \mathcal{G}_n^k] \mathbb{1}_{\{T_n^k \le t < T_{n+1}^k\}}, \qquad 0 \le t \le T,$$

for a suitable family $(\mathcal{G}^k)$ of discrete-time filtrations. By the very definition, $\delta^k X$ should be interpreted as a space-filtration discretization scheme.

In this work, we prove that under mild conditions which are similar in nature to weak Dirichlet-type processes, $\delta^k X$ induces a robust skeleton $(A^k, \int H^k \, dA^k, N^k, \mathbb{F}^k)$ which realizes (1.2) in suitable topologies. Beyond that, and more importantly for applications, the skeleton is amenable to a feasible numerical analysis by means of perfect simulations of the first-passage times of the Brownian motion (see Burq and Jones [8]).

The second part of this article is devoted to the application of our abstract results to the pure martingale case. To illustrate the techniques developed in this paper, we present a step-by-step simulation method for the Clark–Ocone formula in full generality. Recall that if $Y \in L^2(\mathcal{F}_T)$, then

$$Y = \mathbb{E}[Y] + \int_0^T \mathbb{E}[D_s Y | \mathcal{F}_s] \, dB_s,$$

where D stands for the Gross–Sobolev derivative on the Gaussian space of the Brownian motion. The process $\mathbb{E}[DY|\mathcal{F}]$ has great importance in mathematical finance because it is the fundamental quantity for the hedging problem in a complete Brownian-based market (see, e.g., [32]). However, the practical implementation of the Clark–Ocone formula is still an open problem mainly because $D_t Y$ is only amenable to numerical schemes in very particular cases such as elliptic systems where the Malliavin weights can be efficiently used. See, for example, [4, 20, 21, 27] for a complete discussion on this matter.

In this article, we propose a rather different approach based on the sequence of stochastic ratios

$$\frac{\mathbb{E}[Y|\mathcal{G}_n^k] - \mathbb{E}[Y|\mathcal{G}_{n-1}^k]}{B_{T_n^k} - B_{T_{n-1}^k}}; \qquad k, n \ge 1. \tag{1.3}$$

Unlike in previous works (see, e.g., [17, 18]), the approximation scheme given in (1.3) is intrinsic and it is rather explicit without imposing smoothness in the sense of Malliavin calculus and no underlying Markovian structure is assumed (see also, e.g., [1, 21]). Moreover, no functional pathwise smoothness is required in the approximation of $\mathbb{E}[DY|\mathcal{F}]$ (see Dupire [19] and Cont and Fournie [9] for some results in this direction). More importantly for applications, $\mathbb{E}[DY|\mathcal{F}]$ is the limit of functionals of (1.3) which are fully described by the sequences of smooth i.i.d. stopping times $(T_n^k - T_{n-1}^k)_{n\ge1}$ and the Bernoulli variables $(B_{T_n^k} - B_{T_{n-1}^k})_{n\ge1}$. This makes our approximation explicit and easily implementable for a very large class of payoffs. Based on (1.3), we present a step-by-step simulation method for the

Clark–Ocone formula. To the best of our knowledge, the proposed methodology is the only one capable of simulating $\mathbb{E}[DY|\mathcal{F}]$ for arbitrary square-integrable $\mathcal{F}_T$-random variables.

In the last part of the article, we illustrate our discretization scheme with optimal stopping problems arising in non-Markovian systems. We propose an algorithm fully based on our discretization scheme which allows us to simulate value functions and the optimal stopping times for continuous Wiener functionals arising in genuinely non-Markovian cases such as, for example, the fractional Brownian motion.

The remainder of the article is structured as follows. In Section 2, we fix the notation and we give some preliminary results regarding the pre-limit sequence and its basic properties. In Section 3, we establish the convergence of the semimartingale skeleton. Section 4 is devoted to the stochastic derivative. In Section 5, a step-by-step algorithm to simulate the Clark–Ocone formula is presented. Section 6 presents an optimal stopping time algorithm based on the discretization scheme developed in this article.

2. Preliminaries. In this section we fix the basic notation and framework that we use in this paper and present some elementary results concerning our approximation scheme. Throughout this paper we are given the usual stochastic basis $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ of the standard Brownian motion B starting from 0, where Ω is the set $\mathcal{C}(\mathbb{R}_+;\mathbb{R}) := \{f : \mathbb{R}_+ \to \mathbb{R} \text{ continuous}; f(0) = 0\}$, $\mathcal{F}$ is the completed Borel sigma algebra, $\mathbb{P}$ is the Wiener measure on Ω and $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is the usual $\mathbb{P}$-augmentation of the natural filtration generated by the Brownian motion. We denote by $\mathcal{O}$ the optional sigma algebra with respect to $\mathbb{F}$.

For each positive integer k, we define $T_0^k = 0$ a.s. and

$$T_n^k := \inf\{T_{n-1}^k < t < \infty; |B_t - B_{T_{n-1}^k}| = 2^{-k}\}, \qquad n \geq 1. \tag{2.1}$$

One should notice that $(T_n^k)_{n\geq 0}$ is an exhaustive sequence of $\mathbb{F}$-stopping times for every k where $\{T_n^k - T_{n-1}^k\}_{n=1}^\infty$ is an i.i.d. sequence. Next we consider the following family of random variables:

$$\sigma_n^k := \begin{cases} 1; & \text{if } B_{T_n^k} - B_{T_{n-1}^k} = 2^{-k} \text{ and } T_n^k < \infty, \\ -1; & \text{if } B_{T_n^k} - B_{T_{n-1}^k} = -2^{-k} \text{ and } T_n^k < \infty, \\ 0; & \text{if } T_n^k = \infty. \end{cases} \tag{2.2}$$

We then define the following sequence of step processes as

$$A_t^k := \sum_{n=1}^\infty 2^{-k}\sigma_n^k \mathbb{1}_{\{T_n^k \leq t\}}, \qquad 0 \leq t < \infty; k \geq 1.$$

For each $k \geq 1$, let $(\mathcal{F}_t^k)_{t\geq 0}$ be the natural filtration generated by $\{A_t^k; 0 \leq t < \infty\}$. One should notice that $(\mathcal{F}_t^k)_{t\geq 0}$ is a discrete-type filtration (see, e.g., [23], page 321) in the sense that

$$\mathcal{F}_t^k = \bigcup_{i=0}^{\infty} \big(\mathcal{G}_i^k \cap \{T_i^k \le t < T_{i+1}^k\}\big), \qquad t \ge 0, \tag{2.3}$$

where $\mathcal{G}_0^k := \{\Omega, \varnothing\}$ and $\mathcal{G}_n^k := \mathcal{F}_{T_n^k}^k = \sigma(T_1^k, \dots, T_n^k, \sigma_1^k, \dots, \sigma_n^k)$. Moreover, since $\mathcal{G}_n^k = \sigma(A^k_{s\wedge T_n^k}; s \ge 0)$ then $\mathcal{G}_n^k$ and $\mathcal{F}_t^k$ coincide up to $\mathbb{P}$-null sets on $\{T_n^k \le t < T_{n+1}^k\}$. In other words, $(\mathcal{F}_t^k)_{t\ge 0}$ is a jumping filtration (e.g., [26]) with jumping sequence given by $(T_n^k)_{n\ge 1}$ for each $k \ge 1$. With a slight abuse of notation we write $\mathcal{F}_t^k$ to denote its $\mathbb{P}$-augmentation satisfying the usual conditions, where $\mathbb{F}^k := (\mathcal{F}_t^k)_{t\ge 0}$. We also denote by $\mathcal{O}^k$ and $\mathcal{P}^k$ the optional and predictable sigma algebras, respectively, with respect to $\mathbb{F}^k$.

In this work, the $\mathbb{F}^k$-dual predictable and optional projections of a real-valued measurable process Y will be denoted by $[Y]^{p,k}$ and $[Y]^{o,k}$, respectively. We also denote by $[X, Y]$ and $\langle X, Y\rangle$ the usual quadratic variation and predictable bracket of a pair of semimartingales, respectively. The usual jump of a process is denoted by $\Delta Y_t = Y_t - Y_{t-}$ where Y_{t-} is the left-hand limit of a càdlàg process Y. We set $Y_{0-} = Y_0$ for convenience. Moreover, if T and S are stopping times, then $[\![T, S]\!]$, $[\![T, S[\![$ and $]\!]T, S]\!]$ will denote the usual stochastic intervals. From now on we fix a terminal time $0 < T < \infty$.

We now give some elementary properties of our discretization scheme.

LEMMA 2.1. *For each $k \ge 1$, $\{A_t^k; 0 \le t \le T\}$ is an $\mathbb{F}^k$-martingale with locally integrable variation such that*

$$\sup_{0\le t\le T} \|B_t - A_t^k\|_\infty \le 2^{-k}, \tag{2.4}$$

where $\|\cdot\|_\infty$ denotes the usual norm on the space $L^\infty(\mathbb{P})$. Moreover, $\mathbb{F}^k$ is a quasi left-continuous filtration and it supports only martingales of bounded variation.

PROOF. The estimate (2.4) and the locally integrable variation property are immediate consequences of the definitions. For the martingale property we notice from (2.3) that we can write

$$\mathcal{F}_t^k = \left\{ \bigcup_{n=0}^{\infty} A_n \cap [T_n^k \le t < T_{n+1}^k]; A_n \in \mathcal{G}_n^k, n \ge 0 \right\}, \qquad t \ge 0,$$

where $A_s^k = B_{T_n^k}$ on $[T_n^k \le s < T_{n+1}^k]$ for each $n \ge 1$. In this case, the usual optional stopping theorem gives the representation (see also Remark 2.2)

$$\mathbb{E}\big[B_T | \mathcal{F}_t^k\big] = A_t^k \qquad \text{a.s.,} \qquad 0 \le t \le T,$$

and therefore we may conclude that A^k is an $\mathbb{F}^k$-martingale. For the second part, we notice that since T_1^k is an absolutely continuous random variable and A^k is a point process, then in this case it is well known that $\mathbb{F}^k$ is a quasi left-continuous filtration. The fact that every $\mathbb{F}^k$-martingale has bounded variation is a consequence of [26]. □

In the sequel, we denote by π the usual projection of $\mathbb{R}_+ \times \Omega$ onto Ω. For any measurable sets D and A we write $D - A$ to denote $D \cap A^c$, where A^c is the complement of the set A. Moreover, $\bigvee_{k\geq 0} \mathcal{A}_k$ denotes the sigma-algebra generated by $\bigcup_{k\geq 0} \mathcal{A}_k$ for a sequence of classes $\{\mathcal{A}_k; k \geq 0\}$.

LEMMA 2.2. *The natural filtration of A^k satisfies the following properties*:

(i) $\{\mathbb{F}^k; k \geq 1\}$ *is an increasing family of sigma-algebras such that* $\mathcal{F}_t = \bigvee_{k\geq 0} \mathcal{F}_t^k$ *for every* $t \geq 0$.

(ii) *The sequence of filtrations* $\mathbb{F}^k$ *converges weakly to* $\mathbb{F}$.

(iii) *For every* $O \in \mathcal{O}$ *there exists a sequence* $O^k \in \mathcal{O}^k$ *such that*

$$O^k \subset O \qquad \forall k \geq 1 \quad \textit{and} \quad \mathbb{P}[\pi(O) - \pi(O^k)] \to 0 \qquad \textit{as } k \to \infty.$$

PROOF. It is straightforward to check that $\mathcal{F}_t^k \subset \mathcal{F}_t^{k+1}$ for every k and $t \geq 0$. Moreover, each cylinder set of the form $\{b_1 < B_t \leq b_2\}$ can be approximated by

$$\begin{aligned} &\{b_1 + 2^{-k} < A_t^k \leq b_2 - 2^{-k}\} \\ &\quad \subset \{b_1 < B_t \leq b_2\} \\ &\quad \subset \{b_1 - 2^{-k} < A_t^k \leq b_2 + 2^{-k}\} \qquad \text{a.s.} \end{aligned} \tag{2.5}$$

for k large enough, thus proving part (i). To prove part (ii) we only need to show that for each $B \in \mathcal{F}_T$ the sequence of martingales $\mathbb{E}[\mathbb{1}_B|\mathcal{F}_\cdot^k]$ converges in probability to $\mathbb{E}[\mathbb{1}_B|\mathcal{F}_\cdot]$ on the space of càdlàg functions equipped with the usual Skorohod topology. But this is a simple application of [11], Proposition 4. Now let us fix an arbitrary $0 < t \leq T$. From (2.5) we know that for any cylinder set restricted on $[0, t]$ we may find two sequences $(D_i^k)_{k\geq 1}$, $i = 1, 2$, such that

$$D_1^k \subset D \subset D_2^k \tag{2.6}$$

for k large enough, where $D_1^m \subset D_1^{m+1}$ and $D_2^m \supset D_2^{m+1}; m \geq 1$. From (2.4) it follows that

$$\max\{\mathbb{P}[D - D_1^k]; \mathbb{P}[D_2^k - D]\} \to 0 \qquad \text{as } k \to \infty. \tag{2.7}$$

In fact, by a standard monotone class argument one can easily show that any set in $\mathcal{F}_t$ satisfies the above property. Now recall that $\mathcal{O} = \sigma(\mathcal{C})$ where

$$\mathcal{C} = \{E \times \{0\} : E \in \mathcal{F}_0\} \cup \{[s, t) \times E : s < t; s, t \in \mathbb{Q}_+ \cap [0, T], E \in \mathcal{F}_s\}.$$

From (2.6) and (2.7) it follows that for each $\Lambda \in \mathcal{C}$, there exist sequences O_i^k $i = 1, 2$ so that $O_1^k \subset \Lambda \subset O_2^k$ with k large enough and

$$\max\{\mathbb{P}[\pi(\Lambda) - \pi(O^k)]; \mathbb{P}[\pi(O^k) - \pi(\Lambda)]\} \to 0 \qquad \text{as } k \to \infty.$$

In order to recover any optional set in $\mathcal{O}$, we shall apply a routine argument based on the section theorem (see, e.g., [23], Theorem 4.5) so we omit the details. The proof of the lemma is complete. □

In the remainder of this paper, we will adopt the following terminology.

DEFINITION 2.1. We say that a real-valued process X is a *Wiener functional* if it is optional w.r.t. the Brownian filtration $\mathbb{F}$ and $\mathbb{E}|X_{T^k_n}| < \infty$ for every $k, n \geq 1$.

We now embed a given Wiener functional X into a sequence of $\mathbb{F}^k$ quasi-left continuous bounded variation processes as

$$\delta^k X_t := X_0 + \sum_{n=1}^{\infty} \mathbb{E}[X_{T^k_n} | \mathcal{G}^k_n] \mathbb{1}_{\{T^k_n \leq t < T^k_{n+1}\}}, \qquad 0 \leq t \leq T. \tag{2.8}$$

REMARK 2.1. The convergence $\delta^k X \to X$ is just a matter of path regularity. In fact, as a consequence of [11], Theorem 1, we know that if a given Wiener functional X has continuous paths, then

$$\mathbb{E}[X_\cdot | \mathcal{F}^k_\cdot] \to X_\cdot, \qquad \delta^k X_\cdot \to X_\cdot$$

uniformly in probability as $k \to \infty$.

REMARK 2.2. The usual optional stopping theorem implies that any $\mathbb{F}$-martingale M with $M_0 = 0$ a.s. admits the representation

$$\delta^k M_t = \mathbb{E}[M_T | \mathcal{F}^k_t], \qquad 0 \leq t \leq T. \tag{2.9}$$

In particular, $A^k = \delta^k B$.

Next, our goal is to establish an explicit decomposition for the embedded semimartingale skeleton $(\delta^k X)_{k\geq 1}$ in terms of a discrete-type derivative.

2.1. *The approximate decomposition.* In this section, we obtain an explicit Doob–Meyer decomposition for $\delta^k X$. At first, one should notice that $\{\delta^k X_t : 0 \leq t \leq T\}$ is an $\mathbb{F}^k$-adapted process with locally integrable variation for each $k \geq 1$. Moreover, there exists a unique $\mathbb{F}^k$-predictable process $N^{k,X}$ with locally integrable variation such that

$$\delta^k X_t - X_0 - N^{k,X}_t =: M^{k,X}_t, \qquad 0 \leq t \leq T, \tag{2.10}$$

is an $\mathbb{F}^k$-local martingale. The process $N^{k,X}$ is the $\mathbb{F}^k$-dual predictable projection of $\delta^k X_t - X_0$ which can be taken with continuous paths because $\mathbb{F}^k$ is quasi left-continuous.

Next we aim at characterizing the elements of the decomposition (2.10). One should notice that since $\mathbb{F}^k$ is not a completely continuous filtration [see (2.2)], then A^k cannot have a strong predictable representation.

REMARK 2.3. Since A^k is a quasi left-continuous martingale and a step process, then it has the so-called optional representation (see, e.g., [23], Theorem 13.19 and Example 13.9). That is, every $\mathbb{F}^k$-local martingale starting from zero is represented by an optional integral w.r.t. A^k.

In the remainder of this paper, we make use of the optional stochastic integration w.r.t. A^k. We refer the reader to [15, 23] for all details about optional integrals used in this paper. We just want to mention here that since the filtration $\mathbb{F}^k$ is quasi left-continuous, then the related optional integrals admit the usual operational properties of stochastic integrals with predictable integrands (see, e.g., [15], Remark 35, page 346). In this work, we denote by $\oint_0^t Y_s\, dA_s^k$ the optional integral of an $\mathbb{F}^k$-optional process Y.

We now introduce a process which will play a key role in this work. If $\delta^k X$ is the $\mathbb{F}^k$-projection of a Wiener functional X, then we define the following $\mathbb{F}^k$-optional process

$$\mathcal{D}\delta^k X := \sum_{n=1}^{\infty} \frac{\delta^k X_{T_n^k} - \delta^k X_{T_{n-1}^k}}{B_{T_n^k} - B_{T_{n-1}^k}} \mathbb{1}_{[\![T_n^k, T_n^k]\!]}. \tag{2.11}$$

If

$$\mathbb{E} \sum_{n=1}^{m} \left|\Delta \delta^k X_{T_n^k}\right|^2 < \infty \qquad \forall m, k \geq 1, \tag{2.12}$$

then

$$\left[\int_0^{\cdot} \mathcal{D}_s^2 \delta^k X\, d[A^k, A^k]_s\right]^{1/2} = \left[\sum_{n=1}^{\infty} (\delta^k X_{T_n^k} - \delta^k X_{T_{n-1}^k})^2 \mathbb{1}_{\{T_n^k \leq \cdot\}}\right]^{1/2}$$

is a locally integrable increasing process for every $k \geq 1$. In this case, there exists a unique $\mathbb{F}^k$-local martingale M such that for every bounded $\mathbb{F}^k$-martingale V, the process $[M, V] - \int_0^{\cdot} \mathcal{D}\delta^k X\, d[V, A^k]$ is an $\mathbb{F}^k$-local martingale and

$$M_t = \int_0^t \mathcal{D}_s \delta^k X\, dA_s^k - \left[\int_0^{\cdot} \mathcal{D}_s \delta^k X\, dA_s^k\right]_t^{p,k} = \oint_0^t \mathcal{D}_s \delta^k X\, dA_s^k,$$

where $\int_0^t \mathcal{D}_s \delta^k X\, dA_s^k$ is interpreted in the Lebesgue–Stieltjes sense. By observing that $\sum_{0 \leq s \leq t} \Delta \delta^k X_s = \sum_{0 \leq s \leq t} \Delta M_s^{k,X}$ and the fact that $\delta^k X$ is quasi left-continuous, we actually have the following optional representation for the martingale part in the decomposition (2.10):

$$M_t^{k,X} = \oint_0^t \mathcal{D}_s \delta^k X\, dA_s^k; \qquad 0 \leq t \leq T.$$

Of course, $\mathcal{D}\delta^k X$ is the unique $\mathbb{F}^k$-optional process which represents the martingale $M^{k,X}$ as an optional stochastic integral with respect to the martingale A^k. Let us characterize the remainder term in the decomposition (2.10).

LEMMA 2.3. *The $\mathbb{F}^k$-dual predictable projection of $\delta^k X - X_0$ is given by the continuous process*

$$\int_0^t U_s^{k,X} d\langle A^k, A^k\rangle_s, \qquad 0 \le t \le T,$$

where $U^{k,X} := \mathbb{E}_{[A^k]}[\mathcal{D}\delta^k X/\Delta A^k|\mathcal{P}^k]$. Here $\mathbb{E}_{[A^k]}[\cdot|\mathcal{P}^k]$ denotes the conditional expectation w.r.t. $\mathcal{P}^k$ under the Doléans measure generated by $[A^k, A^k]$. Moreover,

$$U_t^{k,X} = 0\mathbb{1}_{\{T_0^k = t\}} + \frac{1}{2^{-2k}} \sum_{n=1}^{\infty} \mathbb{E}\big[X_t - X_{T_{n-1}^k} | \mathcal{G}_{n-1}^k; T_n^k = t\big]\mathbb{1}_{\{T_{n-1}^k < t \le T_n^k\}}. \tag{2.13}$$

PROOF. The fact that $U^{k,X} = \mathbb{E}_{[A^k]}[\mathcal{D}\delta^k X/\Delta A^k|\mathcal{P}^k]$ is obvious. Let us now characterize $U^{k,X}$. For this, let us consider the sequence of sigma-algebras $\mathcal{G}_{n-}^k := \mathcal{G}_{n-1}^k \vee \sigma(T_n^k)$, $n \ge 1$. We recall that for every $C \in \mathcal{G}_{n-}^k$, there exists a predictable process H such that $H_{T_n^k} = \mathbb{1}_C$ and it is null outside the stochastic interval $]\!]T_{n-1}^k, T_n^k]\!]$ (see [7], Theorem 31, page 307). Then,

$$\mathbb{E}\big[\mathbb{1}_C \Delta\delta^k X_{T_n^k} \mathbb{1}_{\{T_n^k \le T\}}\big] = \mathbb{E}\big[\mathbb{1}_C U_{T_n^k}^{k,X} 2^{-2k} \mathbb{1}_{\{T_n^k \le T\}}\big].$$

Since C is arbitrary and $U^{k,X}$ a predictable process, it follows that

$$\mathbb{E}\big[\Delta\delta^k X_{T_n^k} \mathbb{1}_{\{T_n^k \le T\}} | \mathcal{G}_{n-}^k\big] = U_{T_n^k}^{k,X} 2^{-2k} \mathbb{1}_{\{T_n^k \le T\}}.$$

Then, one version of the conditional expectation can be written as (2.13). The proof of the lemma is complete. □

The next result describes an explicit expression for the predictable bracket of A^k in terms of the density f^k and the distribution function F^k of T_1^k (see, e.g., [8] for the corresponding formulas).

LEMMA 2.4. *The predictable bracket of A^k is an absolutely continuous process where the Radon–Nikodym derivative process is given by*

$$h_t^k = 2^{-2k} \sum_{n=1}^{\infty} \lambda_{t-T_{n-1}^k}^k \mathbb{1}_{\{T_{n-1}^k < t \le T_n^k\}}, \qquad 0 \le t \le T, \tag{2.14}$$

where $\lambda_t^k = \frac{f_t^k}{1-F_t^k}$, $0 \le t \le T$.

PROOF. Since A^k is a quasi left-continuous point process where the difference of the jumping times $\{(T_n^k - T_{n-1}^k); n \ge 1\}$ is a sequence of i.i.d. absolutely continuous random variables, then it is well known that $\langle A^k, A^k\rangle$ has absolutely continuous paths. A straightforward but lengthy calculation together with [29], Theorem 18.2, yields (2.14). □

Summing up all previous results of this section, we then arrive at the following representation.

PROPOSITION 2.1. *If X is a Wiener functional satisfying assumption* (2.12), *then the $\mathbb{F}^k$-special semimartingale decomposition* $(M^{k,X}, N^{k,X})$ *in* (2.10) *is actually given by*

$$\delta^k X_t = X_0 + \oint_0^t \mathcal{D}_s \delta^k X \, dA^k_s + \int_0^t U^{k,X}_s h^k_s \, ds, \qquad 0 \le t \le T. \tag{2.15}$$

3. Weak decomposition of Wiener functionals. In this section we are interested in providing readable conditions on a given Wiener functional X in such way that

$$X = \lim_{k\to\infty} \delta^k X; \qquad M^X = \lim_{k\to\infty} M^{k,X}; \qquad N^X = \lim_{k\to\infty} N^{k,X}$$

in a suitable topology. Under such assumptions, we are able to decompose X into a unique orthogonal decomposition which is similar in nature to weak Dirichlet processes (see, e.g., [13] and other references therein)

$$X_t = X_0 + M^X_t + N^X_t,$$

where M^X is a martingale and N^X is an adapted process whose specific type of *covariation* (see Definition 3.2) w.r.t. Brownian motion is null.

3.1. *Weak convergence and primary decomposition*. In this section we investigate the convergence of our preliminary decomposition (2.15) given in terms of the approximation scheme $(A^k, \mathbb{F}^k)$. By carefully choosing a suitable topology on the space of processes, our strategy will be fully based on the information given by the quadratic variation of the martingale component in (2.15).

Let $\mathrm{B}^p(\mathbb{F})$ be the set of all $\mathbb{F}$-optional processes and which are $1 \le p < \infty$ Böchner integrable in the sense that

$$\|X\|^p_{\mathrm{B}^p} = \mathbb{E}\big|X^*_T\big|^p < \infty, \tag{3.1}$$

where $X^*_T := \sup_{0\le t\le T}|X_t|$. Of course, $\mathrm{B}^p(\mathbb{F})$ endowed with the norm $\|\cdot\|_{\mathrm{B}^p}$ is a Banach space, where the subspace $\mathrm{H}^p(\mathbb{F})$ of the $\mathbb{F}$-martingales starting from zero is closed. Recall that the topological dual $\mathrm{M}^q(\mathbb{F})$ of $\mathrm{B}^p(\mathbb{F})$ is the space of processes $A = (A^{pr}, A^{pd})$ such that:

(i) A^{pr} and A^{pd} are right-continuous of bounded variation such that A^{pr} is $\mathbb{F}$-predictable with $A^{pr}_0 = 0$ and A^{pd} is $\mathbb{F}$-optional and purely discontinuous.

(ii) $\mathrm{Var}(A^{pd}) + \mathrm{Var}(A^{pr}) \in L^q$; $\frac{1}{p} + \frac{1}{q} = 1$,

where $\mathrm{Var}(\cdot)$ denotes the total variation of a bounded variation process on the interval $[0, T]$. The space $\mathrm{M}^q(\mathbb{F})$ has the strong topology given by

$$\|A\|_{M^q} := \big\|\mathrm{Var}\big(A^{pr}\big)\big\|_{L^q} + \big\|\mathrm{Var}\big(A^{pd}\big)\big\|_{L^q}.$$

The duality pair is given by

$$(A,X) := \mathbb{E}\int_0^T X_{s-}\,dA_s^{pr} + \mathbb{E}\int_0^T X_s\,dA_s^{pd}; \qquad X \in \mathrm{B}^p(\mathbb{F}),$$

where the following estimate holds:

$$\big|(A,X)\big| \le \|A\|_{M^q}\|X\|_{\mathrm{B}^p}$$

for every $A \in \mathrm{M}^q(\mathbb{F})$, $X \in \mathrm{B}^p(\mathbb{F})$ such that $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. We denote $\sigma(\mathrm{B}^p, \mathrm{M}^q)$ the weak topology of $\mathrm{B}^p(\mathbb{F})$.

In this work, the indexes $p = 1, 2$ will play a key role in our convergence results, in particular, the subspaces H^p for $p = 1, 2$. See the works [15, 16, 31] for detailed discussions on the weak topology of $\mathrm{B}^p(\mathbb{F})$ restricted to the subspace of martingales $\mathrm{H}^p(\mathbb{F})$.

In this article it will be also useful to work with the following notion of convergence. Actually, one can show that the set Λ^∞ of the $\mathbb{F}$-optional bounded variation processes of the form

$$C = g\mathbb{1}_{\{S\le\cdot\}}; \qquad g \in L^\infty(\mathcal{F}_S), S \text{ is an } \mathbb{F}\text{-stopping time (bounded by T)},$$

fulfills the Banach space $\mathrm{B}^1(\mathbb{F})$ in the sense that

$$\|X\|_{\mathrm{B}^1} = \sup\big\{\big|(X,C)\big|; C \in \Lambda^\infty, \|C\|_{\mathrm{M}^\infty} \le 1\big\}. \tag{3.2}$$

Relation (3.2) is given in [16], Lemma 1, and therefore we may also endow $\mathrm{B}^1(\mathbb{F})$ with the $\sigma(\mathrm{B}^1, \Lambda^\infty)$-topology induced by the family of seminorms

$$X \mapsto \big|(X,C)\big|; \qquad C \in \Lambda^\infty.$$

REMARK 3.1. Obviously, $\sigma(\mathrm{B}^1, \Lambda^\infty)$ is weaker then $\sigma(\mathrm{B}^1, \mathrm{M}^\infty)$. However, relation (3.2) says that Λ^∞ is a norming subset of M^∞ and therefore Λ^∞ is w^*-dense in M^∞.

REMARK 3.2. A result due to Mokobodzki [31] states that if X^n is a sequence of optional processes such that $\sup_{0\le t\le T}|X_t^n|$ is uniformly integrable and for every S stopping time the sequence X_S^n converges weakly in L^1 relatively to $\mathcal{F}_S$, then there exists an optional process X such that $X^n \to X$ in $\sigma(\mathrm{B}^1, \mathrm{M}^\infty)$. As a consequence, if $X^n \to X$ in $\sigma(\mathrm{B}^1, \Lambda^\infty)$ and $\sup_{0\le t\le T}|X_t^n|$ is uniformly integrable, we do have convergence in $\sigma(\mathrm{B}^1, \mathrm{M}^\infty)$ (see also Dellacherie, Meyer and Yor [16] for more details).

In the remainder of this paper, we shall write B^p (H^p) to denote the space of Böchner integrable process (p-integrable martingales starting from zero) satisfying (3.1) endowed with the Brownian filtration $\mathbb{F}$. We now introduce the following quantity which will play a crucial role in this work.

DEFINITION 3.1. We say that a given Wiener functional X has *finite energy* along the filtration family $(\mathbb{F}^k)_{k\geq 1}$ if

$$\mathcal{E}_2(X) := \sup_{k\geq 1} \mathbb{E} \sum_{n=1}^{\infty} \left|\Delta\delta^k X_{T_n^k}\right|^2 \mathbb{1}_{\{T_n^k \leq T\}} < \infty. \tag{3.3}$$

REMARK 3.3. The above definition is similar in spirit to the classical notion of energy (e.g., [10, 22]), but with one fundamental difference: The relevant information contained in the energy of X comes only from the sigma-algebras $\mathcal{G}_n^k$ which reveal the information generated by the jumps of the projected Brownian motion A^k up to the stopping time T_n^k. Moreover, $\mathcal{E}_2(X) = \sup_{k\geq 1} \mathbb{E}[M^{k,X}, M^{k,X}]_T$.

It is natural to ask what happens without conditioning on the information flow $\{\mathcal{G}_n^k; k, n \geq 1\}$. The following lemma answers this question.

LEMMA 3.1. *If X is a Wiener functional, then*

$$\mathcal{E}_2(X) \leq \sup_{k\geq 1} \mathbb{E} \sum_{n\geq 1} (X_{T_n^k} - X_{T_{n-1}^k})^2 \mathbb{1}_{\{T_n^k \leq T\}}. \tag{3.4}$$

PROOF. It is sufficient to check that $\mathbb{E}[\mathcal{H}_t^{k,X} | \mathcal{F}_t^k] = \mathcal{D}_t \delta^k X$ on $\{T_n^k \leq t < T_{n+1}^k\}$ for each $k, n \geq 1$ where

$$\mathcal{H}^{k,X} := \sum_{n=1}^{\infty} \frac{X_{T_n^k} - X_{T_{n-1}^k}}{B_{T_n^k} - B_{T_{n-1}^k}} [\![T_n^k, T_{n+1}^k[\![.$$

But this is a straightforward consequence of the strong Markov property of the Brownian motion. A simple application of Jensen inequality and the $\mathbb{F}^k$-optional duality establishes (3.4). $\square$

We are now in position to study convergence of the decomposition given in (2.15). In the sequel, we fix an element $X \in \mathrm{B}^1$ and let $(M^{k,X}, N^{k,X})$ be the associated canonical decomposition expressed in (2.15). In order to find a candidate for the limit of $M^{k,X}$, let us introduce the following family of $\mathbb{F}$-martingales:

$$Z_t^{k,X} := \mathbb{E}[M_T^{k,X} | \mathcal{F}_t]; \qquad 0 \leq t \leq T; k \geq 1. \tag{3.5}$$

In order to prove convergence of $M^{k,X}$ to an $\mathbb{F}$-martingale we may use some standard compactness arguments.

LEMMA 3.2. *The sequence of random variables* $\{[M^{k,X}, M^{k,X}]_T^{1/2} : k \geq 1\}$ *is uniformly integrable if, and only if, the sequence of stochastic process* $\{Z^{k,X} : k \geq 1\}$ *is weakly relatively compact in* H^1.

PROOF. The proof is a routine argument based on the Doob and Burkholder inequalities together with [16], Theorem 1, so we omit the details. □

REMARK 3.4. By the Doob maximal inequality one should notice that if $\mathcal{E}_2(X) < \infty$, then $\{Z^{k,X}; k \geq 1\}$ is a bounded sequence in H^2 which also implies that it is an H^2-weakly sequentially compact set.

LEMMA 3.3. *If S is an $\mathbb{F}$-stopping time, then there exists a sequence of positive random variables $(S_k)_{k\geq 1}$ such that S_k is an $\mathbb{F}^k$-stopping time for each $k \geq 1$ and $\lim_{k\to\infty} \mathbb{P}(S_k = S) = 1$. Moreover, for any $G \in \mathcal{F}_S$ there exists a sequence of sets $(G^k)_{k\geq 1}$ such that $G^k \in \mathcal{F}^k_{S_k}$, $G^k \subset G \cap \{S < \infty\}$ for every $k \geq 1$, and*

$$\lim_{k\to\infty} \mathbb{P}\big[G \cap \{S < \infty\} - G^k\big] = 0.$$

PROOF. Let S be an arbitrary $\mathbb{F}$-stopping time. Since the graph $[\![S]\!]$ belongs to $\mathcal{O}$, we may find a sequence $(O^k)_{k\geq 1}$ satisfying item (iii) in Lemma 2.2. For an arbitrary $\varepsilon > 0$, let k be large enough in such way that

$$\mathbb{P}\big[\pi\big([\![S]\!]\big) - \pi\big(O^k\big)\big] < \varepsilon/2.$$

From the standard section theorem there exists an $\mathbb{F}^k$-stopping time S_k such that

$$[\![S_k]\!] \subset O^k \subset [\![S]\!] \quad \text{and} \quad \mathbb{P}\big[\pi\big(O^k\big)\big] \leq \mathbb{P}[S_k < \infty] + \varepsilon/2.$$

Then it follows that $\mathbb{P}[\pi(S) - \pi([\![S_k]\!])] = \mathbb{P}[S_k \neq S] < \varepsilon$ for k large enough. This allows us to conclude the first part of the lemma. For the second part, let us recall that $\mathcal{F}_S \cap \{S < \infty\} = \{\Phi^{-1}(O) : O \in \mathcal{O}\}$, where $\Phi(w) = (S(w), w)$ for any $w \in \{S < \infty\}$. Then, for any $G \in \mathcal{F}_S$, there exists an optional set $J \in \mathcal{O}$ such that $G \cap \{S < \infty\} = \Phi^{-1}(J)$. We denote by J^k the sequence of sets satisfying item (iii) in Lemma 2.2 and $G^k = \Phi_k^{-1}(J^k)$, where $\Phi_k(w) = (S^k(w), w)$ for any $w \in \{S^k < \infty\}$ and $(S_k)_{k\geq 1}$ is the sequence of stopping times obtained from the first part. Then we conclude that $\mathbb{P}[G \cap \{S < \infty\} - G^k] \leq \mathbb{P}[\pi(J) - \pi(J^k)] \to 0$ as $k \to \infty$. □

Summing up the above lemmas we arrive at the following result.

PROPOSITION 3.1. *Assume that a Wiener functional X satisfies $\mathcal{E}_2(X) < \infty$. Then the set $\{M^{k,X}_\cdot; k \geq 1\}$ is $\sigma(B^2, \mathrm{M}^2)$—relatively sequentially compact where every limit point belongs to* H^2.

PROOF. From Remark 3.4, we know that $\mathcal{Z} = \{Z^{k,X}; k \geq 1\}$ is weakly relatively compact in H^2 and therefore any sequence in $\mathcal{Z}$ admits a weakly convergent subsequence in H^2. With a slight abuse of notation, let us denote by $Z^{k,X}$ this convergent subsequence in H^2 and Z the respective H^2-martingale limit point. Let us fix an $\mathbb{F}$-stopping time S which is bounded by the terminal time T. We claim that $M^{k,X} \to Z$ in $\sigma(\mathrm{B}^2, M^2)$. For this, at first we show that the convergence holds in the $\sigma(\mathrm{B}^1, \Lambda^\infty)$-topology. In other words,

$$\lim_{k\to\infty} \int M_S^{k,X} g \, d\mathbb{P} = \int Z_S g \, d\mathbb{P}$$

holds for every $g \in L^\infty(\mathcal{F}_S)$. Recall that it is sufficient to prove for indicator functions $g = \mathbb{1}_G$ where $G \in \mathcal{F}_S$. From Lemma 3.3 there exists a sequence of stopping times S_k and $G^k \in \mathcal{F}^k_{S^k}$ satisfying $\lim_{k\to\infty} \mathbb{P}[G - G^k] = 0$. By construction, one should notice from the proof of Lemma 3.3 that $S_k = S$ on G^k for every $k \geq 1$. Moreover, the martingale property yields

$$\begin{aligned} \int_G Z_S^{k,X} \, d\mathbb{P} &= \int_{G-G^k} M_T^{k,X} \, d\mathbb{P} + \int_{G^k} \mathbb{E}[M_T^{k,X} | \mathcal{F}^k_{S^k}] \, d\mathbb{P} \\ &= \int_{G-G^k} M_T^{k,X} \, d\mathbb{P} - \int_{G-G^k} M_S^{k,X} \, d\mathbb{P} + \int_G M_S^{k,X} \, d\mathbb{P}. \end{aligned}$$

Therefore, the uniform integrability assumption yields

$$\begin{aligned} &\left| \int_G Z_S^{k,X} \, d\mathbb{P} - \int_G M_S^{k,X} \, d\mathbb{P} \right| \\ &\qquad = \left| \int_{G-G^k} M_T^{k,X} \, d\mathbb{P} - \int_{G-G^k} M_S^{k,X} \, d\mathbb{P} \right| \longrightarrow 0 \qquad \text{as } k \to \infty. \end{aligned}$$

This shows that $\lim_{k\to\infty} (M^{k,X}, C) = (Z, C)$ for every $C \in \Lambda^\infty$ and therefore we may conclude that $\lim_{k\to\infty} M^{k,X} = Z$ in the $\sigma(\mathrm{B}^1, \Lambda^\infty)$-topology. The uniform integrability of $\{\sup_{0\leq t\leq T} |M_t^{k,X}|; k \geq 1\}$ and Remark 3.2 allow us to conclude that $M^{k,X} \to Z$ weakly in B^1. We now claim that the subsequence $M^{k,X}$ actually converges to Z weakly in B^2. For this, let us consider an arbitrary linear functional $Y \in \mathrm{M}^2$ given by $Y = (Y^{pr}, Y^{pd})$. Let $(S_n)_{n\geq 1}$ be a localizing sequence of $\mathbb{F}$-stopping times such that $\mathrm{Var}(Y^{pr}_{\wedge S_n})$ and $\mathrm{Var}(Y^{pd}_{\wedge S_n})$ are bounded for every $n \geq 1$. Let us denote by Y^n the respective stopped linear functional $Y^n \in \mathrm{M}^\infty$; $n \geq 1$. The finite energy assumption yields

$$\begin{aligned} \bigl|(Y, M^{k,X}) - (Y, Z)\bigr| &\leq \bigl|(Y^n, M^{k,X}) - (Y^n, Z)\bigr| \\ &\quad + \|Y^n - Y\|_{M^2} \bigl(\|Z\|_{B^2} + \mathcal{E}_2^{1/2}(X)\bigr). \end{aligned} \tag{3.6}$$

Since $\|Y^n - Y\|_{M^2} \to 0$ as $n \to \infty$ and $Y^n \in \mathrm{M}^\infty$ for every $n \geq 1$, we shall use the B^1-weak convergence of $M^{k,X}$ to Z and (3.6) to conclude that $\lim_{k\to\infty} M^{k,X} = Z$ weakly in B^2. In other words, $\{M^{k,X}; k \geq 1\}$ is a B^2-weakly, relatively, sequentially compact set where every limit point belongs to H^2. $\square$

In the sequel, we introduce a covariation notion which plays a key role in the numerical scheme of the stochastic derivative. We stress here that it is not our purpose to give a more general definition of a quadratic variation. Instead, we only need a slightly different type of approximation due to the (a priori) lack of regularity of the Wiener functionals.

DEFINITION 3.2. Let X and Y be Wiener functionals with $\mathbb{F}^k$-projections, $\delta^k X$ and $\delta^k Y$, respectively. We say that X admits the *δ-covariation* w.r.t. Y if the limit

$$\langle X, Y\rangle_t^\delta := \lim_{k\to\infty}[\delta^k X, \delta^k Y]_t \tag{3.7}$$

exists weakly in L^1 for every $t \in [0, T]$.

REMARK 3.5. In the particular case of Brownian semimartingales, one can easily check that the δ-covariation coincides with the usual quadratic variation by using Lemma 3.1 and Proposition 3.2.

REMARK 3.6. The reason for choosing the L^1-weak topology for the covariation is due to the lack of path regularity of processes which represents Brownian martingales. We will see that the L^1-weak topology is the correct one if one attempts to get a robust approximation scheme in full generality without requiring additional assumptions (see Remark 4.2).

Next, we prove some technical results which will allow us to state Theorem 3.1 which is the main result of this section. Not surprisingly, the quadratic variation and energy notions will play a key role in our result.

LEMMA 3.4. *Let $H_\cdot = \mathbb{E}[\mathbb{1}_G|\mathcal{F}_\cdot]$ and $H^k_\cdot = \mathbb{E}[\mathbb{1}_G|\mathcal{F}^k_\cdot]$ be positive and uniformly integrable martingales with respect to the filtrations $\mathbb{F}$ and $\mathbb{F}^k$, respectively, where $G \in \mathcal{F}_T$. If $W \in \mathrm{H}^\alpha(\mathbb{F})$ for some $\alpha > 2$ then*

$$\left\|\int_0^\cdot H_s\,dW_s - \oint_0^\cdot H_s^k\,d\delta^k W_s\right\|_{\mathrm{B}^2} \to 0 \qquad \textit{as } k \to \infty.$$

PROOF. Throughout the proof we write C to denote a positive constant which may differ from line to line. Let us write

$$\begin{aligned}\int_0^t H_s\,dW_s &- \oint_0^t H_s^k d\delta^k W_s \\ &= \int_0^t [H_s - H_s^k]\,dW_s + \left[\int_0^t H_s^k\,dW_s - \oint_0^t H_s^k\,d\delta^k W_s\right] \\ &=: T_k^1(t) + T_k^2(t), \qquad 0 \le t \le T.\end{aligned}$$

From the weak convergence of $\mathbb{F}^k$ to $\mathbb{F}$ [see (ii) in Lemma 2.2] and the fact that H is a continuous process it follows that

$$H^k \to H \qquad \text{uniformly in probability as } k \to \infty. \tag{3.8}$$

Burkholder and Hölder inequalities yield

$$\left\|\int_0^\cdot (H_t^k - H_t)\,dW_t\right\|_{B^2}^2 \le C\mathbb{E}^{1/p}\sup_{0\le t\le T}|H_t - H_t^k|^{2p}\mathbb{E}^{1/q}[W,W]_T^q$$

for $q = \frac{\alpha}{2}$ and $p = \frac{\alpha}{\alpha-2}$ with $\alpha > 2$. Therefore, we may conclude that $T_k^1 \to 0$ in B^2 as $k \to \infty$. In order to prove that T_k^2 vanishes when $k \to 0$, we split it into the following terms:

$$\begin{aligned}T_k^2(t) &= \int_0^t [H_s^k - H_{s-}^k]\,dW_s - \oint_0^t [H_s^k - H_{s-}^k]\,d\delta^k W_s \\ &\quad + \int_0^t H_{s-}^k\,dW_s - \int_0^t H_{s-}^k\,d\delta^k W_s.\end{aligned} \tag{3.9}$$

We shall estimate in the same way

$$\left\|\int_0^\cdot [H_s^k - H_{s-}^k]\,dW_s\right\|_{\mathrm{B}^2}^2 \le C\mathbb{E}^{1/p}\sup_{n\ge 1}\left|H_{T_n^k}^k - H_{T_{n-1}^k}^k\right|^{2p}\mathbb{1}_{\{T_n^k\le T\}}\mathbb{E}^{1/q}[W,W]_T^q$$

for p and q as above. One can easily check (see, e.g., Lemma 4.1) that $\sup_{n\ge1}|H_{T_n^k}^k - H_{T_{n-1}^k}^k| \to 0$ as $k\to\infty$ in L^r for any $r>1$. Therefore,

$$\left\|\int_0^\cdot [H_s^k - H_{s-}^k]\,dW_s\right\|_{\mathrm{B}^2} \to 0 \qquad \text{as } k\to\infty.$$

By using the representation $\delta^k W_t = \oint_0^t \mathcal{D}_s\delta^k W\,dA_s^k = \mathbb{E}[W_T|\mathcal{F}_t^k]$, we shall use the Doob maximal inequality to estimate in the same way

$$\left\|\oint_0^\cdot [H_s^k - H_{s-}^k]\,d\delta^k W_s\right\|_{\mathrm{B}^2}^2 \le C\mathbb{E}^{1/\alpha}|W|_T^\alpha\mathbb{E}^{1/p}\sup_{0\le t\le T}\left|H_t^k - H_{t-}^k\right|^{2p} \to 0$$

as $k\to\infty$. It remains to estimate the last part in (3.9). We claim that $(H_-^k, \delta^k W)$ satisfies the assumptions of [28], Theorem 2.7. To see this, one notices that the linearity of the conditional expectation, (3.8) and the path continuity of H and W yield

$$H^k - H + \delta^k W - W \to 0 \qquad \text{uniformly in probability.}$$

Since $\lim_{k\to\infty}\mathbb{E}\sup_{0\le t\le T}|H_t^k - H_{t-}^k| = 0$ we actually have $(H_-^k, \delta^k W) \to (H, W)$ in probability on the two-dimensional Skorohod space. Moreover, a simple application of the maximal Doob and Burkholder inequalities ensures that $\delta^k W$ satisfies [28], assumption C2.7. Therefore,

$$\int_0^\cdot H_{s-}^k\,d\delta^k W_s \to \int_0^\cdot H_s\,dW_s \qquad \text{uniformly in probability.}$$

Of course, $\int_0^\cdot H_{s-}^k\,dW_s \to \int_0^\cdot H_s\,dW_s$ uniformly in probability. By using the assumption that $W \in \mathrm{H}^\alpha$ for $\alpha > 2$, we have $\int_0^\cdot H_{s-}^k\,d\delta^k W_s + \int_0^\cdot H_{s-}^k\,dW_s$ is bounded in B^α. This shows that $T_k^2 \to 0$ in B^2 as $k \to \infty$ and therefore the proof is complete. □

The next result is fundamental for the approach taken in this work since it allows us to compute the δ-covariation under a compactness assumption.

LEMMA 3.5. *Let X be a finite energy Wiener functional with the $\mathbb{F}^k$-decomposition given by $(M^{k,X}, N^{k,X})$. Let $\{M^{k_i,X}; i \geq 1\}$ be a* B^2*-weakly convergent subsequence such that $\lim_{i\to\infty} M^{k_i,X} = Z$, where $Z \in H^2$. If $W \in H^2$, then*

$$\lim_{i\to\infty} [M^{k_i,X}, \delta^{k_i} W]_t = [Z, W]_t \qquad \text{weakly in } L^1 \tag{3.10}$$

for every $t \in [0, T]$.

PROOF. With a slight abuse of notation, let $Z^{k,X}$ be the $\mathbb{F}$-martingale subsequence obtained from (3.5), Remark 3.4 and Proposition 3.1 such that $\lim_{k\to\infty} Z^{k,X} = Z$ and $\lim_{k\to\infty} M^{k,X} = Z$ in $\sigma(B^2, M^2)$. By using representation (2.9) and the weak convergence $\mathbb{F}^k \to \mathbb{F}$, we notice that $\delta^k W \to W$ in $\sigma(B^2, M^2)$ as $k \to \infty$ for each $W \in H^2$. Thanks to [16], Theorem 7, we know that $[Z^{k,X}, U]_t \to [Z, U]_t$ weakly in $L^1(\mathbb{P})$ for every $t \in [0, T]$ and U a BMO $\mathbb{F}$-martingale. Given $G \in \mathcal{F}_T$, let us consider the $\mathbb{F}^k$-martingale $H^k_\cdot = \mathbb{E}[\mathbb{1}_G | \mathcal{F}^k_\cdot]$ and W a bounded Brownian martingale. At first, one should notice that the finite energy assumption gives $M^{k,X} \in H^2(\mathbb{F}^k)$ for every $k \geq 1$. By using the $\mathbb{F}^k$-dual optional projection property we shall write

$$\mathbb{E}[\mathbb{1}_G [M^{k,X}, \delta^k W]_t] = \mathbb{E}[M^{k,X}, J^k]_t = \mathbb{E}[M^{k,X}_t J^k_t], \qquad 0 \leq t \leq T,$$

where J^k is the $\mathbb{F}^k$-square integrable martingale given by the optional integral $\oint H^k \, d\delta^k W$. In the same manner, we have that

$$\mathbb{E}[\mathbb{1}_G [Z^{k,X}, W]_t] = \mathbb{E}[Z^{k,X}, J]_t = \mathbb{E}[Z^{k,X}_t J_t], \qquad 0 \leq t \leq T,$$

where J is the stochastic integral $\int H \, dW$ and $H = \mathbb{E}[\mathbb{1}_G | \mathcal{F}_\cdot]$. Moreover,

$$\begin{aligned} \mathbb{E}[Z^{k,X}_t J_t] - \mathbb{E}[M^{k,X}_t J^k_t] &= \mathbb{E}[M^{k,X}_t (J_t - J^k_t)] - \mathbb{E}[(M^{k,X}_t - Z^{k,X}_t) J_t] \\ &=: T^k_1(t) + T^k_2(t), \qquad 0 \leq t \leq T. \end{aligned}$$

We fix $t \in [0, T]$ and we notice that it is sufficient to prove that $T^k_1(t) + T^k_2(t) \to 0$ as $k \to \infty$. The first term $\lim_{k\to\infty} T^k_1(t) = 0$ because of the finite energy assumption and Lemma 3.4. By noting that both subsequences $Z^{k,X}$ and $M^{k,X}$ converge to the same limit in $\sigma(B^2, M^2)$, we shall take the linear functional $Y_\cdot = \mathbb{1}_{\{t \leq \cdot\}} J_t \in M^2$ to conclude that $T^k_2(t) \to 0$ as $k \to \infty$. Therefore, (3.10) holds for any bounded martingale W. If $W \in H^2$ then we shall take a sequence of bounded martingales

W^n such that $W^n \to W$ in H^2 as $n \to \infty$. Moreover, Burkholder and maximal Doob inequalities yield

$$
\begin{aligned}
&\big|\mathbb{E}\mathbb{1}_G\big[M^{k,X}, \delta^k W\big]_t - \mathbb{E}\mathbb{1}_G[Z, W]_t\big| \\
&\quad \le C\mathbb{E}\big|\big[M^{k,X}, \delta^k(W - W^n)\big]_t\big| \\
&\qquad + \big|\mathbb{E}\mathbb{1}_G\big[M^{k,X}, \delta^k W^n\big]_t - \mathbb{E}\mathbb{1}_G\big[Z, W^n\big]_t\big| \\
&\qquad + C\mathbb{E}\big|\big[Z, W^n - W\big]_t\big| \\
&\quad \le C\big(\mathcal{E}_2^{1/2}(X) + \|Z\|_{\mathrm{B}^2}\big)\big\|W_T^n - W_T\big\|_{L^2} \\
&\qquad + \big|\mathbb{E}\mathbb{1}_G\big[M^{k,X}, \delta^k W^n\big]_t - \mathbb{E}\mathbb{1}_G\big[Z, W^n\big]_t\big|.
\end{aligned}
\tag{3.11}
$$

Inequality (3.11) and the previous arguments allow us to conclude the proof. □

Next, we give a necessary and sufficient condition for the existence of the martingale limit.

PROPOSITION 3.2. *Let X be a Wiener functional such that $\mathcal{E}_2(X) < \infty$. Then $M^X := \lim_{k\to\infty} M^{k,X}$ exists weakly in H^2 if, and only if, the δ-covariation $\langle X, B\rangle^\delta$ exists. In this case, $\langle X, B\rangle^\delta_\cdot = [M^X, B]_\cdot$.*

PROOF. If X has finite energy, then by Proposition 3.1 we know that $\{M^{k,X}; k \ge 1\}$ is $\sigma(\mathrm{B}^2, \mathrm{M}^2)$—relatively sequentially compact where all limit points belong to H^2. By assumption, the δ-covariation $\langle X, B\rangle^\delta$ exists and therefore for every $t \in [0, T]$,

$$\lim_{i\to\infty}\big[M^{k_i,X}, A^{k_i}\big]_t = \lim_{m\to\infty}\big[M^{k_m,X}, A^{k_m}\big]_t = \langle X, B\rangle^\delta_t$$

weakly in L^1 for any two distinct B^2-weakly convergent subsequences $\{M^{k_i,X}\}_{i=1}^\infty$ and $\{M^{k_m,X}\}_{m=1}^\infty$. In particular, if $\lim_{i\to\infty} M^{k_i,X} = M$ and $\lim_{m\to\infty} M^{k_m,X} = M'$, then Lemma 3.5 yields

$$
\begin{aligned}
\lim_{i\to\infty}\big[M^{k_i,X}, A^{k_i}\big]_t &= \lim_{m\to\infty}\big[M^{k_m,X}, A^{k_m}\big]_t \\
&= \big[M', B\big]_t = [M, B]_t \qquad \text{weakly in } L^1
\end{aligned}
$$

for $0 \le t \le T$, and therefore $[M - M', B]_\cdot = 0$. The predictable representation of the Brownian motion yields $M = M'$. In this case, $M^{k,X}$ should be convergent and Lemma 3.5 yields $\langle X, B\rangle^\delta = [M^X, B]$, where $M^X := \lim_{k\to\infty} M^{k,X}$. Reciprocally, if $\lim_{k\to\infty} M^{k,X} = M^X \in \mathrm{H}^2$ exists weakly in B^2, then we may again invoke Lemma 3.5 to conclude that $\langle X, B\rangle^\delta$ exists. □

The main result of this section gives the structural conditions for our discretization scheme to work. In fact, those conditions are similar to weak Dirichlet-type processes where the notion of covariation is computed in terms of $\langle\cdot,\cdot\rangle^\delta$.

THEOREM 3.1. *Let X be a finite energy Wiener functional such that $\lim_{k\to\infty}\delta^k X = X$ weakly in* B^2 *and $\langle X, B\rangle^\delta_\cdot$ exists. Let $(M^{k,X}, N^{k,X})$ be the canonical decomposition of $\delta^k X$. Then there exists a unique martingale M^X in* H^2 *such that $N^X := X - X_0 - M^X$ satisfies the following orthogonality condition*:

$$\langle N^X, B\rangle^\delta \equiv 0.$$

If this is the case, *we may write*

$$X = X_0 + M^X + N^X \tag{3.12}$$

and this decomposition is unique. Moreover, $M^{k,X} \to M^X$ and $N^{k,X} \to N^X$ weakly in B^2 *as $k\to\infty$.*

PROOF. By Proposition 3.2 we know that

$$M^X := \lim_{k\to\infty} M^{k,X}$$

exists and $M^X \in \mathrm{H}^2$. From assumption $\lim_{k\to\infty}\delta^k X = X$, we shall define $N^X := \lim_{k\to\infty} N^{k,X}$ weakly in B^2. By the very definition, we have

$$\delta^k N^X = M^{k,X} - \delta^k M^X + N^{k,X}.$$

The path continuity of $N^{k,X}$ yields

$$[\delta^k N^X, A^k]_t = [M^{k,X} - \delta^k M^X, A^k]_t, \qquad 0 \le t \le T; k \ge 1. \tag{3.13}$$

The weak convergence of $\mathbb{F}^k$ to $\mathbb{F}$ [see Lemma 2.2, item (ii)] and relation (2.9) yield $\delta^k M^X = \mathbb{E}[M^X_T|\mathcal{F}^k_\cdot] \to M^X$ uniformly in probability as $k\to\infty$. By Lemma 3.1, we know that $\mathcal{E}_2(M^X) < \infty$ and therefore $\delta^k M^X \to M^X$ in $\sigma(\mathrm{B}^2, M^2)$. Lemma 3.5, Proposition 3.2 and (3.13) yield $\langle N^X, B\rangle^\delta = [M^X - M^X, B] = 0$.

The uniqueness of the decomposition is an immediate consequence of the orthogonality property of the nonmartingale component, the predictable representation property of the Brownian motion and the fact that $\langle W, B\rangle^\delta = [W, B]$ for every $W \in \mathrm{H}^2$. □

4. The stochastic derivative. In this section, we provide an explicit approximation scheme for the martingale representation in the decomposition given in Theorem 3.1. The approximation will be given in terms of $\mathcal{D}\delta^k$ which can be interpreted in the limit as a derivative operator on the Wiener space w.r.t. Brownian motion.

For a given Wiener functional X, we introduce the following family of $\mathbb{F}^k$-predictable processes:

$$\mathcal{D}^k X := 0\mathbb{1}_{[\![T^k_0, T^k_0]\!]} + \sum_{n=0}^{\infty} \mathcal{D}_{T^k_n}\delta^k X \mathbb{1}_{]\!]T^k_n, T^k_{n+1}]\!]}, \tag{4.1}$$

where

$$\mathcal{D}^k_s X = \frac{\delta^k X_{T^k_n} - \delta^k X_{T^k_{n-1}}}{B_{T^k_n} - B_{T^k_{n-1}}} \qquad \text{on } \{T^k_n < s \le T^k_{n+1}\}, n \ge 1.$$

In view of Theorem 3.1, the goal of this section is to show robustness of our approximation scheme in the sense that

$$\mathcal{D}X := \lim_{k\to\infty} \mathcal{D}^k X = H^X \qquad \text{weakly}$$

whenever X satisfies the assumptions of Theorem 3.1 such that the martingale component in (3.12) has a representation $M^X = \int H^X_s \, dB_s$. One should notice that since there is no a priori path regularity of X (in particular H^X), one has to choose an appropriate topology in order to get the existence of $\lim_{k\to\infty} \mathcal{D}^k X$. In the sequel, we denote by λ the usual Lebesgue measure on $[0, T]$.

Let us begin with the following technical lemmas. At first, the following remark proves to be very useful for the approach taken in this work. In fact, it will play a key role in the study of the limit $\lim_{k\to\infty} \mathcal{D}^k X$ because it allows us to control the quantity $(\Delta A^k_{T^k_n})^{-1}$ in (4.1). It is a straightforward consequence of the strong Markov property and the 1/2-self-similarity of the Brownian motion.

REMARK 4.1. The stopping time $T^k_n - T^k_{n-1}$ is independent from $\mathcal{G}^k_{n-1}$ for every $k, n \ge 1$. Moreover, $\mathbb{E}(T^k_n - T^k_{n-1}) = 2^{-2k}$ for every $k, n \ge 1$.

LEMMA 4.1. *If $g \in L^\infty$, then for every $1 < p < \infty$,*

$$\mathbb{E} \sup_{n\ge 1} \big|\mathbb{E}[g|\mathcal{G}^k_n] - \mathbb{E}[g|\mathcal{G}^k_{n-1}]\big|^p \mathbb{1}_{\{T^k_n \le T\}} \to 0 \qquad \textit{as } k \to \infty.$$

PROOF. For a given $g \in L^\infty$, let $X_t = E[g|\mathcal{F}_t], 0 \le t \le T$. Recall that $\delta^k X_t = E[X_T|\mathcal{F}^k_t], 0 \le t \le T$ and therefore $\mathbb{E}[g|\mathcal{G}^k_n] = \delta^k X_{T^k_n}$ on $\{T^k_n \le T\}$ for each k, $n \ge 1$. Moreover, X is a bounded $\mathbb{F}$-martingale with continuous paths. Remark 2.1 yields $\delta^k X \to X$ strongly in $\mathrm{B}^p(\mathbb{F})$ as $k \to \infty$, $p > 1$. We shall write

$$\begin{aligned} &\mathbb{E}^{1/p} \sup_{n\ge 1} \big|\mathbb{E}[g|\mathcal{G}^k_n] - \mathbb{E}[g|\mathcal{G}^k_{n-1}]\big|^p \mathbb{1}_{\{T^k_n \le T\}} \\ &\quad\le \mathbb{E}^{1/p} \sup_{n\ge 1} \big|\delta^k X_{T^k_n} - X_{T^k_n}\big|^p \mathbb{1}_{\{T^k_n \le T\}} \\ &\qquad + \mathbb{E}^{1/p} \sup_{n\ge 1} |X_{T^k_n} - X_{T^k_{n-1}}|^p \mathbb{1}_{\{T^k_n \le T\}} \\ &\qquad + \mathbb{E}^{1/p} \sup_{n\ge 1} \big|\delta^k X_{T^k_{n-1}} - X_{T^k_{n-1}}\big|^p \mathbb{1}_{\{T^k_n \le T\}}. \end{aligned} \tag{4.2}$$

The first and last terms in (4.2) vanish. Moreover, $\lim_{k\to\infty} \mathbb{E}^{1/p} \sup_{n\ge 1} |X_{T^k_n} - X_{T^k_{n-1}}|^p \mathbb{1}_{\{T^k_n \le T\}} = 0$ because of the path continuity of X together with the fact that $\sup_{n\ge 1} |T^k_n - T^k_{n-1}| \to 0$ a.s. as $k \to \infty$. □

Now we are in position to prove the existence of the stochastic derivative.

THEOREM 4.1. *Let X be a Wiener functional satisfying the assumptions of Theorem 3.1 with the weak decomposition represented by*

$$X = X_0 + \int H^X \, dB_s + N^X \tag{4.3}$$

for an adapted process H^X in $L^2(\lambda \times \mathbb{P})$ and $\langle N^X, B\rangle^\delta = 0$. Then H^X can be approximated by the $L^2(\lambda \times \mathbb{P})$-weak limit $\mathcal{D}X = \lim_{k\to\infty} \mathcal{D}^k X = H^X$.

PROOF. The unique orthogonal decomposition (4.3) represented by an adapted process H^X is a consequence of Theorem 3.1 together with the martingale representation of the Brownian motion. Therefore, it only remains to prove the existence of $\mathcal{D}X$. For this, let us consider a finite energy Wiener functional X and let us fix $0 \le t \le T$ and $g \in L^\infty$. In order to shorten notation, let us write $\xi^k_n := (T^k_n - T^k_{n-1})\mathbb{1}_{\{T^k_n \le T\}}$, $g^k_n := \mathbb{E}[g|\mathcal{G}^k_n] - \mathbb{E}[g|\mathcal{G}^k_{n-1}]$ for $k, n \ge 1$ and C is a constant which may differ from line to line. By the very definition, for every $k \ge 1$ and $t > 0$,

$$\begin{aligned} g\int_0^t \mathcal{D}^k_s X \, ds &= g\sum_{n=1}^{\infty} \mathcal{D}_{T^k_{n-1}}\delta^k X \xi^k_n \mathbb{1}_{\{T^k_{n-1}\le t\}} \\ &\quad - g\sum_{n=1}^{\infty} \mathcal{D}_{T^k_{n-1}}\delta^k X (T^k_n - t)\mathbb{1}_{\{T^k_{n-1}< t \le T^k_n\}}. \end{aligned} \tag{4.4}$$

At first, a simple application of Remark 4.1 and the very definition of $\mathcal{D}^k X$ yield

$$\mathbb{E}\int_0^T |\mathcal{D}^k_s X|^2 \, ds = \mathbb{E}\sum_{n=1}^{\infty} |\mathcal{D}_{T^k_{n-1}}\delta^k X|^2 \xi^k_n \mathbb{1}_{\{T^k_{n-1}\le T\}} \le \mathcal{E}_2(X), \qquad k \ge 1. \tag{4.5}$$

By Hölder inequality and (4.5), the second term in (4.4) vanishes as follows:

$$\begin{aligned} &\mathbb{E}\sum_{n=1}^{\infty} |g\mathcal{D}_{T^k_{n-1}}\delta^k X (T^k_n - t)|\mathbb{1}_{\{T^k_{n-1}< t \le T^k_n\}} \\ &\quad \le C\mathbb{E}\sum_{n=1}^{\infty} |\mathcal{D}_{T^k_{n-1}}\delta^k X|\xi^k_n \mathbb{1}_{\{T^k_{n-1}< t \le T^k_n\}} \\ &\quad \le C\mathcal{E}_2^{1/2}(X) \times \mathbb{E}^{1/2}\sup_{n\ge 1}|\xi^k_n|\mathbb{1}_{\{T^k_n \le T\}} \to 0 \qquad \text{as } k \to \infty. \end{aligned}$$

By using Remark 4.1, we shall write

$$\begin{aligned} \mathbb{E}g\sum_{n=1}^{\infty} \mathcal{D}_{T^k_{n-1}}\delta^k X \xi^k_n \mathbb{1}_{\{T^k_{n-1}\le t\}} &= \mathbb{E}\sum_{n=1}^{\infty} g^k_n \mathcal{D}_{T^k_{n-1}}\delta^k X \xi^k_n \mathbb{1}_{\{T^k_{n-1}\le t\}} \\ &\quad + \mathbb{E}g\sum_{n=1}^{\infty} \Delta\delta^k X_{T^k_{n-1}} \Delta A^k_{T^k_{n-1}} \mathbb{1}_{\{T^k_{n-1}\le t\}}. \end{aligned} \tag{4.6}$$

The first term in (4.6) vanishes as follows. By applying Lemma 4.1, (4.5) and Hölder inequality we have

$$\mathbb{E}\sum_{n=1}^{\infty}|g^k_n\mathcal{D}_{T^k_{n-1}}\delta^k X\xi^k_n|\mathbb{1}_{\{T^k_{n-1}\le t\}} \le \mathcal{E}_2^{1/2}(X)\mathbb{E}^{1/2}\sup_{n\ge 1}|g^k_n|^2\sum_{n=1}^{\infty}\xi^k_n\mathbb{1}_{\{T^k_{n-1}\le t\}}$$
$$\to 0 \qquad \text{as } k\to\infty.$$

Summing up the above arguments, we arrive at the following conclusion:

$$\lim_{k\to\infty}\mathbb{E}g\int_0^t\mathcal{D}^k_s X\,ds \text{ exists} \quad \text{if, and only if} \quad \lim_{k\to\infty}\mathbb{E}g[\delta^k X, A^k]_t \text{ exists.} \tag{4.7}$$

In other words, the estimate (4.5), the existence of $\langle X, B\rangle^\delta$ and (4.7) allow us to conclude that

$$\lim_{k\to\infty}\mathcal{D}^k X \qquad \text{exists weakly in } L^2(\lambda\times\mathbb{P}).$$

It follows from the above steps that

$$\begin{aligned}\lim_{k\to\infty}\mathbb{E}g\int_0^t\mathcal{D}^k_s X\,ds &= \mathbb{E}g\langle X, B\rangle^\delta_t = \mathbb{E}g[M^X, B]_t\\ &= \mathbb{E}g\int_0^t H^X_s\,ds, \qquad 0\le t\le T, g\in L^\infty,\end{aligned}\tag{4.8}$$

where the martingale component is represented by a progressive process H^X in $L^2(\lambda\times\mathbb{P})$, that is, $M^X_t = \int_0^t H^X_s\,dB_s$ for $0\le t\le T$. Identity (4.8) shows that $\mathcal{D}^k X \to H^X$ weakly in $L^2(\lambda\times\mathbb{P})$ as $k\to\infty$. The proof of the theorem is complete. □

REMARK 4.2. Under assumption $\mathcal{E}_2(X)<\infty$, relations (4.7) and (4.8) allow us to conclude that $\lim_{k\to\infty}\mathcal{D}^k X$ exists weakly in $L^2(\lambda\times\mathbb{P})$ if, and only if, the δ-covariation $\langle X, B\rangle^\delta$ exists. By Proposition 3.2, $\lim_{k\to\infty}\mathcal{D}^k X$ exists if, and only if, $\lim_{k\to\infty}M^{k,X}$ exists which shows a strong robustness of our approximation scheme. In this case, $\lim_{k\to\infty}\mathcal{D}^k X = H^X$ where $M^X = \int H^X_s\,dB_s = \lim_{k\to\infty}M^{k,X}$.

By applying Theorem 4.1 to the classical Itô representation theorem we arrive at the following result.

COROLLARY 4.1. *If F is an $\mathcal{F}_T$-square integrable random variable, then*

$$F = \mathbb{E}[F] + \int_0^T \mathcal{D}_s F\,dB_s,$$

where

$$\mathcal{D}F = \lim_{k\to\infty}\sum_{n=1}^{\infty}\frac{\mathbb{E}[F|\mathcal{G}^k_n]-\mathbb{E}[F|\mathcal{G}^k_{n-1}]}{B_{T^k_n}-B_{T^k_{n-1}}}\mathbb{1}_{]\!]T^k_n,T^k_{n+1}]\!]} \tag{4.9}$$

weakly in $L^2(\lambda \times \mathbb{P})$.

PROOF. If $X_t = \mathbb{E}[F|\mathcal{F}_t]$, $0 \le t \le T$, then $\delta^k X_t = \mathbb{E}[F] + \oint_0^t \mathcal{D}_s \delta^k X \, dA_s^k$ where $\delta^k X_{T_n^k} = \mathbb{E}[F|\mathcal{G}_n^k]$; $k, n \ge 1$. Since X is a square-integrable martingale, a simple application of Theorem 4.1 yields (4.9). □

REMARK 4.3. Corollary 4.1 and the classical Clark–Ocone formula yield $\mathcal{D}_t F = \mathbb{E}[D_t F|\mathcal{F}_t]$ where D denotes the Gross–Sobolev derivative of F in $L^2(\mathbb{P})$. If F is not differentiable in the sense of Malliavin calculus, the Gross–Sobolev derivative $D_t F$ is interpreted as a generalized process where $\mathbb{E}[D_t F|\mathcal{F}_t]$ can be interpreted as a real-valued process in $L^2(\lambda \times \mathbb{P})$ (see, e.g., [5] for more details).

5. The Clark–Ocone formula algorithm. In this section, we illustrate the theory developed in this article with some numerical examples. The goal here is to show that our approximation scheme can be easily implementable where a step-by-step algorithm for the Clark–Ocone formula is presented. We illustrate the method with the problem of hedging contingent claims in a complete market. For simplicity of exposition and comparison with exact known formulas, we will work on a simple diffusion setup together with well-known types of contingent claims. We stress that the algorithm presented in Section 5.1 holds for any square integrable $\mathcal{F}_T$-random variable.

In this section, the market consists of two assets: one riskless asset S^0 and one risky asset S with continuous paths. We will specify the evolution of the assets directly under the unique equivalent martingale measure $\mathcal{Q}$ together with the respective $\mathcal{Q}$-Brownian motion W. We assume that they are given by

$$dS_t^0 = rS_t^0 \, dt, \qquad S_0^0 = 1; \qquad dS_t = rS_t \, dt + \sigma S_t \, dW_t, \tag{5.1}$$

where $\sigma > 0$ and $r > 0$. It is well known (see, e.g., [5, 32]) that for any given contingent claim $F \in L^2(\mathcal{Q}, \mathcal{F}_T)$, the correspondent replicating strategy θ is derived by the Clark–Ocone–Karatzas [32] formula as

$$\theta_t = e^{-r(T-t)} \sigma^{-1} (S_t)^{-1} F_t^o, \tag{5.2}$$

where $F_\cdot^o := \mathbb{E}_\mathcal{Q}[D_\cdot F|\mathcal{F}_\cdot]$ and D is the Gross–Sobolev derivative. Since the filtration generated by W coincides with $\mathbb{F}$, Theorem 4.1 still holds under the correspondent martingale measure $\mathcal{Q}$ as well. In the sequel, with a slight abuse of notation, we also denote by $\mathbb{E}$ the expectation under the measure $\mathcal{Q}$.

REMARK 5.1. In Fournie et al. [21] and also in a series of papers [4, 14, 20, 27], the idea is to express the optional projection F^o by $\mathbb{E}[F.G|\mathcal{F}_\cdot]$ for a suitable random variable G which in general is represented by a Skorohod integral. In this case, a smooth underlying Markovian structure plays a key role. In this work, we take a rather different strategy which is fully based on the information generated by the stopping times $(T_n^k)_{k,n\ge 1}$ which allows us to treat any $L^2(\mathcal{F}_T)$-random variable (see also Remark 5.2).

To illustrate our method, we will study three types of derivatives: a European call option, a digital option and a barrier option given, respectively, by

$$\max\{S_T - K, 0\}, \qquad \mathbb{1}_{\{S_T \le K\}}, \qquad \mathbb{1}_{\{M^S_{0,T} \le K\}}, \tag{5.3}$$

where $M^S_{0,T} := \sup_{0\le t\le T} |S_t|$. It is well known that for these types of claims, there exist closed formulas for hedging (see, e.g., [5], examples 4.1 and 5.3).

5.1. *The algorithm.* The method is fully based on the space-filtration discretization scheme induced by the stopping times $\{T^k_n; k, n \ge 1\}$. In the sequel, we fix an $L^2(\mathcal{F}_T)$-random variable F and our goal is to describe an algorithm to calculate the optional projection F^o_0 which yields the hedging θ_t at time $t = 0$. The other times can be recovered from this case by a standard shift argument. From (4.9), it follows that for sufficiently small $\varepsilon > 0$ and k large enough,

$$\frac{1}{\varepsilon}\mathbb{E}\int_0^\varepsilon \mathcal{D}^k_s F\, ds = \frac{1}{\varepsilon}\mathbb{E}\int_{T^k_1}^\varepsilon \mathcal{D}^k_s F\, ds \sim \frac{1}{\varepsilon}\mathbb{E}\int_0^\varepsilon F^o_s\, ds \sim \mathbb{E}F^o_0 \tag{5.4}$$

as long as 0 is a Lebesgue point of $t \mapsto \mathbb{E}F^o_t$. For the purpose of hedging, we may assume that this is the case. Otherwise, we shall always find a point in a neighborhood of $t = 0$ such that (5.4) holds. In order to speed up the convergence of the algorithm we take

$$\mathbb{E}\frac{1}{\varepsilon - T^k_1}\int_{T^k_1}^\varepsilon \mathcal{D}^k_s F\, ds,$$

instead of $\frac{1}{\varepsilon}\mathbb{E}\int_{T^k_1}^\varepsilon \mathcal{D}^k_s F\, ds$ in (5.4). One should notice that $\frac{1}{\varepsilon - T^k_1}\mathbb{1}_{\{T^k_1 < \varepsilon\}} \to \frac{1}{\varepsilon}$ in L^2 as $k \to \infty$ and therefore

$$\mathbb{E}\frac{1}{\varepsilon - T^k_1}\int_{T^k_1}^\varepsilon \mathcal{D}^k_s F\, ds \sim \mathbb{E}F^o_0 \tag{5.5}$$

for k sufficiently large. By the very definition,

$$\begin{aligned} \frac{1}{\varepsilon - T^k_1}\int_{T^k_1}^\varepsilon \mathcal{D}^k_s F\, ds = \frac{1}{\varepsilon - T^k_1}\Bigg[&\sum_{n=1}^\infty \mathcal{D}^k_{T^k_n} F\big(T^k_{n+1} - T^k_n\big)\mathbb{1}_{\{T^k_{n+1} \le \varepsilon\}} \\ &+ \sum_{n=1}^\infty \mathcal{D}^k_{T^k_n} F\big(\varepsilon - T^k_n\big)\mathbb{1}_{\{T^k_n < \varepsilon \le T^k_{n+1}\}}\Bigg]. \end{aligned} \tag{5.6}$$

The whole structure of the algorithm is based on the perfect simulation of the first passage times $\{T^k_n - T^k_{n-1}; k, n \ge 1\}$. Based on the density of T^k_1, Burq and Jones [8] proposes a very simple and efficient algorithm. We refer the reader to this work for a detailed exposition of the perfect simulation method for the stopping times.

(S1) *Simulation of* $\{A^k; k \ge 1\}$.

- One chooses $k \geq 1$ which represents the discrimination level of the Brownian motion.
- One generates the stopping times $\{T_n^k - T_{n-1}^k; n \geq 1\}$ according to the algorithm described by [8].
- One simulates the family $\{\sigma_n^k; n \geq 1\}$ independently from $\{T_n^k - T_{n-1}^k; n \geq 1\}$. The i.i.d. family $\{\sigma_n^k; n \geq 1\}$ must be simulated according to the Bernoulli random variable σ_1^k such that $\mathbb{P}[\sigma_1^k = i] = 1/2$ for $i = -1, 1$. This simulates the jump process A^k.

The next step is the simulation of $\mathcal{D}^k F$ where the conditional expectations $\{\mathbb{E}[F|\mathcal{G}_n^k]; n, k \geq 1\}$ play a key role.

(S2) *Simulation of the stochastic derivative.*

- Fix a small $\varepsilon > 0$.
- Generate one sample of A^k according to (S1) for a large $k \geq 1$. From this sample, one takes $(t_1^k, \sigma_1^k); \dots; (t_n^k, \sigma_n^k)$ such that $t_n^k < \varepsilon \leq t_{n+1}^k$.
- For each $0 \leq j \leq n$, one applies Monte Carlo simulation to obtain an approximation of $\mathbb{E}[F|(t_1^k, \sigma_1^k), \dots, (t_j^k, \sigma_j^k)]$ (see Remark 5.2). This object is denoted by $\hat{\mathbb{E}}[F|(t_1^k, \sigma_1^k), \dots, (t_j^k, \sigma_j^k)]$.

Therefore, an approximation for the stochastic derivative $\mathcal{D}^k F$ is given by

$$\hat{\mathcal{D}}_{t_j^k}^k F := \frac{\hat{\mathbb{E}}[F|(t_1^k, \sigma_1^k), \dots, (t_j^k, \sigma_j^k)] - \hat{\mathbb{E}}[F|(t_1^k, \sigma_1^k), \dots, (t_{j-1}^k, \sigma_{j-1}^k)]}{2^{-k}\sigma_j^k}$$

for $1 \leq j \leq n$. Then one can define the following object according to (5.5) and (5.6):

$$\hat{F}_0^o(\varepsilon, k) := \frac{1}{\varepsilon - t_1^k}\left[\sum_{j=1}^{n-1} \hat{\mathcal{D}}_{t_j^k}^k F\big(t_{j+1}^k - t_j^k\big) + \hat{\mathcal{D}}_{t_n^k}^k F\big(\varepsilon - t_n^k\big)\right].$$

- From (5.5) and (5.2), the correspondent replicating strategy for this path can be approximated by
$$\hat{\theta}_0(\varepsilon, k) := e^{-r(T)}\sigma^{-1}(S_0)^{-1}\hat{F}_0^o(\varepsilon, k).$$
- Repeat these steps several times and take the mean of the strategies $\hat{\theta}_0(\varepsilon, k)$ as the estimative for the replicating strategy θ_t at the initial point $t = 0$.

REMARK 5.2. The methodology presented in this section is rather general in the sense that the only assumption that is made is the possibility to simulate the expectation $\mathbb{E}[F|(t_1^k, \sigma_1^k), \dots, (t_j^k, \sigma_j^k)]$ by a Monte Carlo method. In the classical Black–Scholes setup, one can simulate it by means of random samples generated by

$$S_t^k = \exp\big[\big(r - \tfrac{1}{2}\sigma^2\big)t + \sigma A_t^k\big], \qquad 0 \leq t \leq T. \tag{5.7}$$

In the general positive semimartingale case, the expectation can be simulated based on the Euler–Maruyama method, for example, for

$$dS_t = r_t S_t\,dt + \sigma_t S_t\,dW_t; \qquad 0 \le t \le T,$$

with the increments $(W_{T^k_n} - W_{T^k_{n-1}})$ computed in terms of (T^k_n) and σ can be random.

5.2. *Numerical examples.* As a simple illustration of our method, we consider three types of derivatives: a European call option, a digital option and a barrier option. The stock price is 49, the risk-free interest rate is 5% per annum, the stock price volatility is 20% per annum, the time to maturity is 20 weeks (0.3846 years) and the expected return from the stock is 13% per annum. We use strike price $K = 50$ for the European call option and digital option and $K = 55$ for the barrier option. In order to develop the simulation process we choose a discrimination level of order $k = 4$. The main point of the algorithm is the approximation of the conditional expectations by Monte Carlo simulation. For each case, we generate 10,000 paths of A^k, we evaluate the payoff function on each path based on (5.7) and we take the mean as the estimative of the conditional expectation. We choose $\varepsilon = 0.02$ and we generate 1000 samples of A^k stopped at 0.02.

For each sample, we calculate the respective hedging value at time $t = 0$. The estimative of the hedging at time $t = 0$ is given by the mean of the hedging values from the correspondent samples (see Figure 1). The *%Error* is the absolute value of the difference between estimated value and exact value divided by the exact value. With 1000 paths, we obtain an error of 0.57% for the European call option, 0.48% for the digital option and 0.1% for the barrier option.

At this stage, one can say that the method proposed in this work is rather general compared with the more classical ones based on the existence of densities (see, e.g., [27] and other references therein). Moreover, it does not require further smoothness assumptions and it can be easily implementable without requiring advanced mathematical calculations as in the classical literature in mathematical finance.

6. Optimal stopping. In this section, we illustrate the techniques developed in this paper with the optimal stopping time problem based on a Wiener functional X. Throughout this section we fix a bounded positive Wiener functional X with continuous paths. To shorten notation, in this section we shall extend the time domain of all stochastic processes X to $[0, \infty)$ as follows: $X_t = X_T$ for every $t \ge T$. In the sequel, we denote the set of all $\mathbb{F}$-stopping times by $\mathrm{ST}(\mathbb{F})$.

DEFINITION 6.1. For a fixed $\varepsilon > 0$, we say that the stopping time τ^ε is ε-optimal if

$$\mathbb{E}X_{\tau^\varepsilon} \ge \sup_{\tau \in \mathrm{ST}(\mathbb{F})} \mathbb{E}X_\tau - \varepsilon.$$

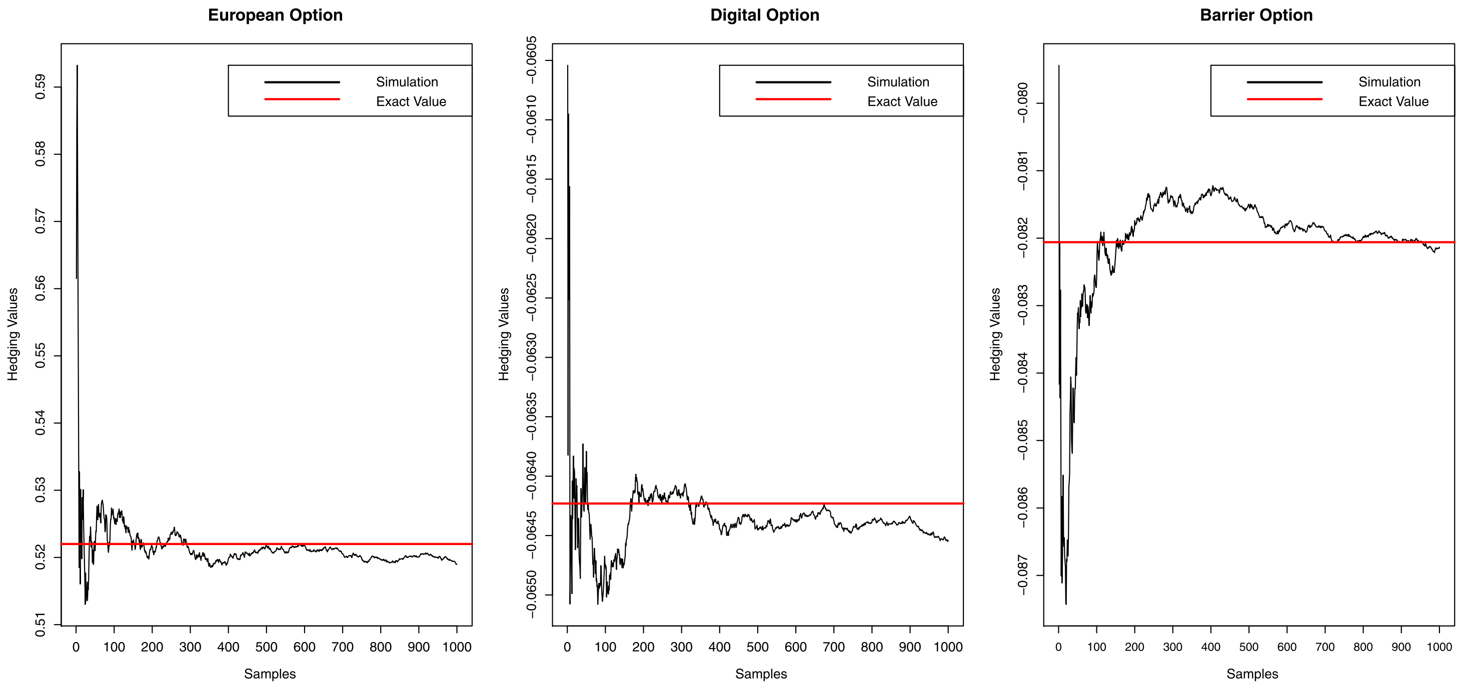

FIG. 1. *Monte Carlo simulations for hedging.*

From Remark 2.1, we know that $\lim_{k\to\infty}\delta^k X = X$ strongly in B^2 and therefore,

$$\sup_{\tau\in \mathrm{ST}(\mathbb{F})} \mathbb{E}\delta^k X_\tau \to \sup_{\tau\in \mathrm{ST}(\mathbb{F})} \mathbb{E}X_\tau \qquad \text{as } k\to\infty. \tag{6.1}$$

Let $\mathcal{U}^k$ be the class of totally inaccessible $\mathbb{F}^k$-stopping times. It is well known (see, e.g., [23]) that $S\in\mathcal{U}^k$ if, and only if, $S=\sum_{n=1}^{\infty} T_n^k \mathbb{1}_{\{S=T_n^k\}}$. The fact that $\delta^k X$ is constant on the stochastic interval $[\![T_n^k, T_{n+1}^k[\![$ for each $n\geq 0$ and convergence (6.1) allow us to state the following result.

LEMMA 6.1. *For each $\varepsilon>0$, we have*

$$\sup_{\tau\in \mathrm{ST}(\mathbb{F})} \mathbb{E}X_\tau \leq \sup_{\tau^k\in\mathcal{U}^k} \mathbb{E}\delta^k X_{\tau^k} + \varepsilon$$

for every k sufficiently large.

DEFINITION 6.2. For a given $\varepsilon\geq 0$, we say that $\tau^{k,\star}(\in\mathcal{U}^k)$ is (k,ε)-optimal, if

$$\mathbb{E}\delta^k X_{\tau^{k,\star}} \geq \sup_{\tau^k\in\mathcal{U}^k} \mathbb{E}\delta^k X_{\tau^k} - \varepsilon.$$

Summing up the above results together with Remark 2.1, the following proposition holds.

PROPOSITION 6.1. *Let X be a bounded positive continuous Wiener functional. For a given $\varepsilon>0$, each $(k,0)$-optimal stopping time $\tau^{k,\star}$ is ε-optimal for every k sufficiently large.*

6.1. *A dynamic programming principle.* In the sequel, we provide a dynamic programming principle to approximate a $(k,0)$-optimal stopping time for any Wiener functional X satisfying the assumptions of Proposition 6.1. Let S^k be the Snell envelope of $\delta^k X$, that is, the minimal positive $\mathbb{F}^k$-supermartingale which dominates $\delta^k X$. The dynamic programming principle can be written as follows. For a fixed $\omega\in\{T_n^k\leq T<T_{n+1}^k\}$, we shall write

$$\begin{cases} S^k_{T_n^k}(\omega) = \delta^k X_{T_n^k}(\omega), \\ S^k_{T_j^k}(\omega) = \max\{\delta^k X_{T_j^k}(\omega); \mathbb{E}[S^k_{T^k_{j+1}} | \mathcal{G}_j^k](\omega)\}, \qquad j\leq n. \end{cases}$$

By a backward induction argument, the dynamic programming principle can be written in terms of optimal stopping times as

$$\begin{cases} \tau_n^{k,\star}(\omega) := T_n^k(\omega), \\ \tau_{n-1}^{k,\star}(\omega) := T_{n-1}^k(\omega)\mathbb{1}_{G^k_{n-1}}(\omega) + \tau_n^{k,\star}(\omega)\mathbb{1}_{(G^k_{n-1})^c}(\omega), \\ \tau_j^{k,\star}(\omega) := T_j^k(\omega)\mathbb{1}_{G^k_j}(\omega) + \tau_{j+1}^{k,\star}(\omega)\mathbb{1}_{(G^k_j)^c}(\omega); \qquad \omega\in\Omega, \end{cases} \tag{6.2}$$

where

$$G_j^k := \{\delta^k X_{T_j^k} \geq \mathbb{E}[\delta^k X_{\tau_{j+1}^{k,\star}} | \mathcal{G}_j^k]\}, \qquad j \leq n-1. \tag{6.3}$$

In this case, the $\tau_0^{k,\star}$ is $(k,0)$-optimal and the value function is given by

$$\mathbb{E}S_0^k = \mathbb{E}\delta^k X_{\tau_0^{k,\star}}. \tag{6.4}$$

6.2. *Non-Markovian examples.* We shall consider a significant class of non-Markovian examples which fits into the assumptions of Proposition 6.1. For instance, for a given bounded continuous function $f : \mathbb{R} \to \mathbb{R}_+$, let us consider the following Wiener functionals:

$$f(B^H) \qquad \text{for } H \in (0,1), \tag{6.5}$$

where B^H is the fractional Brownian motion with parameter $H \in (0,1)$. Based on the simulation of $\{(T_n^k - T_{n-1}^k, \sigma_n^k); n \geq 1\}$ [see (S1) in Section 5] and the dynamic programming principle for $X = f(B^H)$ in the last section, it is straightforward to develop an algorithm to approximate a $(k,0)$-optimal stopping time. In the sequel, the index $\ell \in \{1, \dots, N\}$ encodes the ℓth iteration in a given dynamic programming procedure and $\{A^{k,\ell}\}_{\ell=1}^N$ are independent copies of A^k.

(A1) *Dynamic programming algorithm.*

- Fix a large $k \geq 1$ and generate one sample from $A^{k,\ell}$ based on (S1) and take $(t_1^{k,\ell}, \sigma_1^{k,\ell}), \dots, (t_n^{k,\ell}, \sigma_n^{k,\ell})$ such that $t_n^{k,\ell} \leq T < t_{n+1}^{k,\ell}$.
- One sets $\tau_n^{k,\ell,\star} = t_n^{k,\ell}$.
- One proceeds backward by taking the time $\tau_{j-1}^{k,\ell,\star}$ given by

$$\tau_{j-1}^{k,\ell,\star} = \begin{cases} t_{j-1}^{k,\ell}, & \text{if } f(A_{t_{j-1}^{k,\ell}}^{k,\ell,H}) \geq \mathbb{E}[f(A_{\tau_j^{k,\ell,\star}}^{k,H}) | (t_1^{k,\ell}, \sigma_1^{k,\ell}), \dots, (t_{j-1}^{k,\ell}, \sigma_{j-1}^{k,\ell})], \\ \tau_j^{k,\ell,\star}, & \text{if } f(A_{t_{j-1}^{k,\ell}}^{k,\ell,H}) < \mathbb{E}[f(A_{\tau_j^{k,\ell,\star}}^{k,H}) | (t_1^{k,\ell}, \sigma_1^{k,\ell}), \dots, (t_{j-1}^{k,\ell}, \sigma_{j-1}^{k,\ell})], \end{cases}$$

for any $j \leq n$. The value $A_t^{k,\ell,H}$ is obtained from $A^{k,\ell}$ via the Volterra representation of the fractional Brownian motion as

$$A_t^{k,\ell,H} := \int_0^t K(t,s)\, dA_s^{k,\ell}, \qquad 0 \leq t \leq T, H \in (0,1), \tag{6.6}$$

for a suitable square-integrable kernel $K(t,s)$ (see, e.g., [24]). The conditional expectations

$$\mathbb{E}[f(A_{\tau_j^{k,\ell,\star}}^{k,H}) | (t_1^{k,\ell}, \sigma_1^{k,\ell}), \dots, (t_{j-1}^{k,\ell}, \sigma_{j-1}^{k,\ell})], \qquad j \leq n,$$

are approximated by Monte Carlo methods via simulation of $A^{k,\ell}$ described in (S1), Section 5.1 and (6.6).

- One repeats the previous steps several times, $\ell = 1, \dots, N$ and the optimal value function (6.4) is approximated by $\frac{1}{N}\sum_{\ell=1}^{N} f(A^{k,\ell,H}_{\tau_0^{k,\ell,\star}})$ for large N.

The above formulation in terms of stopping rules (rather than in terms of value functions) is essential to our approach as well as in other probabilistic methods based on discretizations of the Snell envelope. The main feature of this methodology is the computation of conditional expectations which is generically based on the following alternatives: projections on $L^2(\mathbb{P})$ (see, e.g., [30]), quantization (as in [2]) and representation formulas based on Malliavin calculus (see, e.g., [6]). An important drawback of all these methodologies is that they essentially rely on an induced Markov chain arising from a time-discretization scheme of a continuous-time Markov process. Dynamic programming methods are not directly usable in genuinely non-Markovian cases due to the nontrivial time-correlation generated by the driving noise.

We circumvent this problem by introducing a space-filtration discretization scheme which allows us to write the original optimal stopping problem in terms of the information flow $(\mathcal{G}^k_n)_{k,n\ge 1}$. Conditional expectations appearing in the dynamic programming principle (6.2)–(6.4) can now be fairly simulated since most examples of interest can be viewed in terms of the process A^k as explained in Remark 5.2. A numerical study is needed in order to precisely evaluate our method with the more classical approaches, a topic which will be further explored in a forthcoming paper.

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DEPARTAMENTO DE MATEMÁTICA APLICADA
E ESTATÍSTICA
UNIVERSIDADE DE SÃO PAULO
13560-970 SÃO CARLOS, SP
BRAZIL
E-MAIL: leao@icmc.usp.br

DEPARTAMENTO DE MATEMATICA
UNIVERSIDADE FEDERAL DA PARAIBA
58051-900, JOAO PESSOA
BRAZIL
E-MAIL: ohashi@mat.ufpb.br