

# QUANTITATIVE VERSION OF THE KIPNIS–VARADHAN THEOREM AND MONTE CARLO APPROXIMATION OF HOMOGENIZED COEFFICIENTS<sup>1</sup>

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This article is devoted to the analysis of a Monte Carlo method to approximate effective coefficients in stochastic homogenization of discrete elliptic equations. We consider the case of independent and identically distributed coefficients, and adopt the point of view of the random walk in a random environment. Given some final time  $t > 0$ , a natural approximation of the homogenized coefficients is given by the empirical average of the final squared positions re-scaled by  $t$  of  $n$  independent random walks in  $n$  independent environments. Relying on a quantitative version of the Kipnis–Varadhan theorem combined with estimates of spectral exponents obtained by an original combination of PDE arguments and spectral theory, we first give a sharp estimate of the error between the homogenized coefficients and the expectation of the re-scaled final position of the random walk in terms of  $t$ . We then complete the error analysis by quantifying the fluctuations of the empirical average in terms of  $n$  and  $t$ , and prove a large-deviation estimate, as well as a central limit theorem. Our estimates are optimal, up to a logarithmic correction in dimension 2.

## 1. Main result and structure of the proof.

**1.1. Main result.** We consider the discrete elliptic operator  $-\nabla^* \cdot A \nabla$ , where  $\nabla^*$  and  $\nabla$  are the discrete backward divergence and forward gradient, respectively. For all  $x \in \mathbb{Z}^d$ ,  $A(x)$  is the diagonal matrix whose entries are the conductances  $\omega_{x, x+\mathbf{e}_i}$  of the edges  $(x, x+\mathbf{e}_i)$  starting at  $x$ , where  $(\mathbf{e}_i)_{i \in \{1, \dots, d\}}$  denotes the canonical basis of  $\mathbb{R}^d$ . Let  $\mathbb{B}$  denote the set of *unoriented* edges of  $\mathbb{Z}^d$ . We call the family of conductances  $\omega = (\omega_e)_{e \in \mathbb{B}}$  the *environment*. This environment is *symmetric* in the sense that for all  $x, y \in \mathbb{Z}^d$  with  $|x - y| = 1$ , we have  $e = (x, y) = (y, x)$ , so that  $\omega_{x, y} = \omega_{y, x} = \omega_e$ . The environment  $\omega$  is random, and we write  $\mathbb{P}$  for its distribution (with corresponding expectation  $\mathbb{E}$ ). We make the following assumptions:

(H1) the measure  $\mathbb{P}$  is invariant under translations,

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(H2) the conductances are i.i.d.,<sup>2</sup>

(H3) there exists  $0 < \alpha < \beta$  such that  $\alpha \leq \omega_e \leq \beta$  almost surely.

Under these conditions, standard homogenization results ensure that there exists some *deterministic* symmetric matrix  $A_{\text{hom}}$  such that the solution operator of the deterministic continuous differential operator  $-\nabla \cdot A_{\text{hom}} \nabla$  describes the large scale behavior of the solution operator of the random discrete differential operator  $-\nabla^* \cdot A \nabla$  almost surely [for this statement, (H2) can in fact be replaced by the weaker assumption that the measure  $\mathbb{P}$  is ergodic with respect to the group of translations] (see [17]).

The operator  $-\nabla^* \cdot A \nabla$  is the infinitesimal generator of a stochastic process  $(X(t))_{t \in \mathbb{R}_+}$  which can be defined as follows. Given an environment  $\omega$ , it is the Markov process whose jump rate from a site  $x \in \mathbb{Z}^d$  to a neighboring site  $y$  is given by  $\omega_{x,y}$ . We write  $\mathbf{P}_x^\omega$  for the law of this process starting from  $x \in \mathbb{Z}^d$ .

It is proved in [13] that under the averaged measure  $\mathbb{P}\mathbf{P}_0^\omega$ , the re-scaled process  $\sqrt{\varepsilon}X(\varepsilon^{-1}t)$  converges in law, as  $\varepsilon$  tends to 0, to a Brownian motion whose infinitesimal generator is  $-\nabla \cdot A_{\text{hom}} \nabla$ , or in other words, a Brownian motion with covariance matrix  $2A_{\text{hom}}$  (see also [1, 15, 17] for prior results). We will use this fact to construct computable approximations of  $A_{\text{hom}}$ . As proved in [5], this invariance principle holds as soon as (H1) is true, (H2) is replaced by the ergodicity of the measure  $\mathbb{P}$  and (H3) by the integrability of the conductances. Under the assumptions (H1)–(H3), [23] strengthens this result in another direction, showing that for almost every environment,  $\sqrt{\varepsilon}X(\varepsilon^{-1}t)$  converges in law under  $\mathbf{P}_0^\omega$  to a Brownian motion with covariance matrix  $2A_{\text{hom}}$ . This has been itself extended to environments which do not satisfy the uniform ellipticity condition (H3) (see [2–4, 18, 19]).

Let  $(Y(t))_{t \in \mathbb{N}}$  denote the sequence of consecutive sites visited by the random walk  $(X(t))_{t \in \mathbb{R}_+}$  [note that the “times” are different in nature for  $X(t)$  and  $Y(t)$ ]. This sequence is itself a Markov chain that satisfies for any two neighbors  $x, y \in \mathbb{Z}^d$ ,

$$\mathbf{P}_x^\omega[Y(1) = y] = \frac{\omega_{x,y}}{p_\omega(x)},$$

where  $p_\omega(x) = \sum_{|z|=1} \omega_{x,x+z}$ . We simply write  $p(\omega)$  for  $p_\omega(0)$ . Let us introduce a “tilted” version of the law  $\mathbb{P}$  on the environments, that we write  $\tilde{\mathbb{P}}$  and define by

$$(1.1) \quad d\tilde{\mathbb{P}}(\omega) = \frac{p(\omega)}{\mathbb{E}[p]} d\mathbb{P}(\omega).$$

The reason why this measure is natural to consider is that it makes the environment seen from the position of the random walk  $Y$  a stationary process [see (3.2) for a definition of this process].

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<sup>2</sup>(H2) obviously implies (H1) in the present form. Yet for most qualitative (and some quantitative) results (H2) can be weakened and may not imply (H1) any longer.

Interpolating between two integers by a straight line, we can think of  $Y$  as a continuous function on  $\mathbb{R}_+$ . With this in mind, it is also true that there exists a matrix  $A_{\text{hom}}^{\text{disc}}$  such that, as  $\varepsilon$  tends to 0, the re-scaled process  $\sqrt{\varepsilon}Y(\varepsilon^{-1}t)$  converges in law under  $\tilde{\mathbb{P}}\mathbf{P}_0^\omega$  to a Brownian motion with covariance matrix  $2A_{\text{hom}}^{\text{disc}}$ . Moreover,  $A_{\text{hom}}^{\text{disc}}$  and  $A_{\text{hom}}$  are related by (see [5], Theorem 4.5(ii))

$$(1.2) \quad A_{\text{hom}} = \mathbb{E}[p]A_{\text{hom}}^{\text{disc}} = 2d\mathbb{E}[\omega_e]A_{\text{hom}}^{\text{disc}}.$$

Given that the numerical simulation of  $Y$  saves some operations compared to the simulation of  $X$  (there is no waiting time to compute, and the running time is equal to the number of steps), we will focus on approximating  $A_{\text{hom}}^{\text{disc}}$ . More precisely, we fix once and for all some  $\xi \in \mathbb{R}^d$  with  $|\xi| = 1$ , and define

$$(1.3) \quad \sigma_t^2 = t^{-1}\tilde{\mathbb{E}}\mathbf{E}_0^\omega[(\xi \cdot Y(t))^2],$$

$$(1.4) \quad \sigma^2 = 2\xi \cdot A_{\text{hom}}^{\text{disc}}\xi = \frac{2\xi \cdot A_{\text{hom}}\xi}{\mathbb{E}[p]}.$$

It follows from results of [13] (or [5], Theorem 2.1) that  $\sigma_t^2$  tends to  $\sigma^2$  as  $t$  tends to infinity. We now describe a Monte Carlo method to approximate  $\sigma_t^2$ . Using the definition of the tilted measure (1.1), one can see that

$$(1.5) \quad \sigma_t^2 = \frac{\tilde{\mathbb{E}}\mathbf{E}_0^\omega[(\xi \cdot Y(t))^2]}{t} = \frac{\mathbb{E}\mathbf{E}_0^\omega[p(\omega)(\xi \cdot Y(t))^2]}{t\mathbb{E}[p]}.$$

Assuming that we have easier access to the measure  $\mathbb{P}$  than to the tilted  $\tilde{\mathbb{P}}$ , we prefer to base our Monte Carlo procedure on the right-hand side of the second identity in (1.5). Let  $Y^{(1)}, Y^{(2)}, \dots$  be independent random walks evolving in the environments  $\omega^{(1)}, \omega^{(2)}, \dots$ , respectively. We write  $\mathbf{P}_0^\omega$  for their joint distribution, all random walks starting from 0, where  $\bar{\omega}$  stands for  $(\omega^{(1)}, \omega^{(2)}, \dots)$ . The family of environments  $\bar{\omega}$  is itself random, and we let  $\mathbb{P}^\otimes$  be the product distribution with marginal  $\mathbb{P}$ . In other words, under  $\mathbb{P}^\otimes$ , the environments  $\omega^{(1)}, \omega^{(2)}, \dots$  are independent and distributed according to  $\mathbb{P}$ . Our computable approximation of  $\sigma_t^2$  is defined by

$$(1.6) \quad \hat{A}_n(t) = \frac{p(\omega^{(1)})(\xi \cdot Y^{(1)}(t))^2 + \dots + p(\omega^{(n)})(\xi \cdot Y^{(n)}(t))^2}{nt\mathbb{E}[p]}.$$

In  $\hat{A}_n(t)$ , the expectation  $\mathbb{E}[p] = 2d\mathbb{E}[\omega_e]$  comes into play. This expectation can be easily computed, so we assumed that we did so beforehand.

The main result of this paper is the following optimal bounds on the distribution of the error  $|\hat{A}_n(t) - \sigma^2|$ .

**THEOREM 1.1.** *Under the assumptions (H1)–(H3), there exist  $C, c > 0$  such that, for any  $n \in \mathbb{N}^*$ , any  $\varepsilon > 0$  and any  $t$  large enough,*

$$(1.7) \quad \mathbb{P}^\otimes \mathbf{P}_0^\omega[|\hat{A}_n(t) - \sigma^2| \geq (C\mu_d(t) + \varepsilon)/t] \leq \exp\left(-\frac{n\varepsilon^2}{ct^2}\right),$$

where  $\sigma^2$  and  $\hat{A}_n(t)$  are defined, respectively, in (1.4) and (1.6), and

$$\mu_d(t) = \begin{cases} \ln^q t, & \text{for } d = 2, \\ 1, & \text{for } d > 2, \end{cases}$$

for some  $q > 0$  depending only on  $\alpha$  and  $\beta$ .

This result precisely quantifies the convergence rate of a method proposed by Papanicolaou in [21] in the beginning of the eighties to approximate the homogenized coefficients  $A_{\text{hom}}$  numerically.

For completeness of the analysis we also prove a central limit theorem (and identify the limiting variance) for the quantity  $\sqrt{n(t)}(\hat{A}_{n(t)}(t) - \sigma_t^2)$  for all  $n : \mathbb{N} \rightarrow \mathbb{N}$  such that  $n(t)$  tends to infinity with  $t$ .

Let us quickly discuss the sharpness of these results. If  $A$  was a periodic matrix (or even a constant matrix) we would get the same estimate as in Theorem 1.1, except in dimension 2 for which no logarithmic correction would be needed [in the setting of Theorem 1.1, we conjecture that  $q = 1$  is the optimal exponent in (1.7)]. Numerical tests illustrating (1.7) for  $d = 2$  are reported and commented on in Section 6 of this article.

**1.2. Structure of the proof.** Although the result of Theorem 1.1 is purely probabilistic (we estimate a distribution) its proof involves both nontrivial probabilistic arguments (martingale decomposition and Kipnis–Varadhan theory, large deviation estimates) and nontrivial arguments of elliptic theory (Harnack inequality, De Giorgi–Nash–Moser theory and  $L^p$ -theory). What allows us to combine these arguments is spectral theory. This makes the overall structure of the proof interesting and rather unusual.

The starting point of the proof is the observation that

$$|\hat{A}_n(t) - \sigma^2| \leq |\hat{A}_n(t) - \sigma_t^2| + |\sigma_t^2 - \sigma^2|.$$

The result then follows from the following two estimates:

$$(1.8) \quad |\sigma_t^2 - \sigma^2| \leq C \frac{\mu_d(t)}{t},$$

$$(1.9) \quad \mathbb{P}^{\otimes} \mathbf{P}_0^{\omega} [|\hat{A}_n(t) - \sigma_t^2| \geq \varepsilon/t] \leq \exp\left(-\frac{n\varepsilon^2}{ct^2}\right)$$

(see Theorems 3.1 and 4.1). The second estimate is a large deviation estimate. Its proof is standard once we are given sharp upper bounds on the transition probabilities of the random walk in the random environment—which are also by now standard under assumption (H3). The proof is given in Section 4 for completeness. The central limit theorem for the quantity  $\sqrt{n(t)}(\hat{A}_{n(t)}(t) - \sigma_t^2)$  is given in Proposition 5.1 and proved in Section 5.

The core of this article is the estimate (1.8). We call its left-hand side the systematic error. As proved in the celebrated paper [13] by Kipnis and Varadhan (see

also [5]), the systematic error vanishes as  $t$  goes to infinity as soon as the measure  $\mathbb{P}$  is ergodic under translations. The strategy to prove this result is to find a decomposition of  $Y(t) \cdot \xi$  into a martingale plus a remainder, in such a way that the remainder term becomes negligible in the limit, and conclude using the orthogonality of the increments of the martingale and ergodicity. The approach taken up by [13] is based on the spectral analysis of the (self-adjoint) operator of the environment viewed by the particle. More precisely, it is shown that in order for this decomposition with negligible remainder to exist, it suffices that the spectral measure of this operator, once projected on the “local drift”  $\mathfrak{d}$  [see (3.4)], satisfies some integrability condition (IC) at the edge of the spectrum. Condition (IC) is then seen to be equivalent to asking  $\mathfrak{d}$  to belong to the function space  $H^{-1}$ , a fact which is automatically true due to certain symmetry considerations that were systematized in [5].

Our proof of (1.8) consists of two steps. We first make the argument of Kipnis and Varadhan quantitative in Section 2. That is, we show that stronger integrability conditions than (IC) on the spectral measure can be turned into quantitative estimates on the systematic error—this is a general result of independent interest.

In the second step, addressed in Section 3, we prove that indeed (IC) can be strengthened to higher integrability properties, provided ergodicity is replaced by the stronger assumption that the conductances are i.i.d., the hypothesis (H2). This result is the main achievement of this article. In [9], we had taken advantage of spectral theory to turn results of [11] into bounds on spectral exponents. In the present paper we go the other way around, and make systematic use of the interplay between estimates on the spectral measure and iterates of the elliptic operator. There is a twist in the analysis at this point. In [9] spectral theory is somehow only used at the end of the argument to re-phrase in terms of spectral exponents the results on systematic errors obtained by PDE arguments in [11]. Here spectral theory enters the proof itself and is used in combination with PDE arguments. This approach has the advantage of revealing the very nice structure of the problem under consideration.

Let us point out that although the results of this paper are proved under assumptions (H1)–(H3), the assumption (H2) on the statistics of  $\omega$  is only used to obtain the variance estimate of [10], Lemma 2.3. In particular, (H2) can be weakened as follows:

- the distribution of  $\omega_{z, z+\mathbf{e}_i}$  may, in addition, depend on  $\mathbf{e}_i$ ,
- independence can be replaced by finite correlation length  $C_L > 0$ , that is, for all  $e, e' \in \mathbb{B}$ ,  $\omega_e$  and  $\omega_{e'}$  are independent if  $|e - e'| \geq C_L$ .

*Notation.* So far we have already introduced the probability measures  $\mathbf{P}_0^\omega$  (distribution of  $Y$ ),  $\mathbf{P}_0^{\bar{\omega}}$  (distribution of  $Y^{(1)}, Y^{(2)}, \dots$ ),  $\mathbb{P}$  [i.i.d. distribution for  $\omega = (\omega_e)_{e \in \mathbb{B}}$ ],  $\tilde{\mathbb{P}}$  [tilted measure defined in (1.1)] and  $\mathbb{P}^\otimes$  (product distribution of  $\bar{\omega}$  with marginal  $\mathbb{P}$ ). It will be convenient to define  $\tilde{\mathbb{P}}^\otimes$  as the product distribution

of  $\bar{\omega}$  with marginal  $\tilde{\mathbb{P}}$ . For convenience, we write  $\mathbb{P}_0$  as a shorthand notation for  $\mathbb{P}\mathbf{P}_0^\omega$ ,  $\tilde{\mathbb{P}}_0$  for  $\tilde{\mathbb{P}}\mathbf{P}_0^\omega$ ,  $\mathbb{P}_0^\otimes$  for  $\mathbb{P}^\otimes\mathbf{P}_0^\omega$  and  $\tilde{\mathbb{P}}_0^\otimes$  for  $\tilde{\mathbb{P}}^\otimes\mathbf{P}_0^\omega$ . The corresponding expectations are written accordingly, replacing “P” by “E” with the appropriate typography. We write  $|\cdot|$  for the Euclidian norm of  $\mathbb{R}^d$ .

Finally,  $\lesssim$  and  $\gtrsim$  stand, respectively, for  $\leq$  and  $\geq$  up to multiplicative constants (which depend only on the bounds  $\alpha$  and  $\beta$  on the conductances and the dimension  $d$ , if not otherwise stated).

**2. Quantitative version of the Kipnis–Varadhan theorem.** The Kipnis–Varadhan theorem [13] concerns additive functionals of reversible Markov processes. It gives conditions for such additive functionals to satisfy an invariance principle. The proof of the result relies on a decomposition of the additive functional as the sum of a martingale term plus a remainder term, the latter being shown to be negligible. In this section, which can be read independently of the rest of the paper, we give conditions that enable us to obtain some quantitative bounds on this remainder term.

We consider discrete and continuous times simultaneously. Let  $(\eta_t)_{t \geq 0}$  be a Markov process defined on some measurable state space  $\mathfrak{X}$  (here,  $t \geq 0$  stands either for  $t \in \mathbb{N}$  or for  $t \in \mathbb{R}_+$ ). We denote by  $P_x$  the distribution of the process started from  $x \in \mathfrak{X}$ , and by  $E_x$  the associated expectation. We assume that this Markov process is reversible and ergodic with respect to some probability measure  $\nu$ . We write  $P_\nu$  for the law of the process started from the distribution  $\nu$ , and  $E_\nu$  for the associated expectation.

To the Markov process is naturally associated a semi-group  $(P_t)_{t \geq 0}$  defined, for any  $f \in \mathbb{L}_2(\nu)$ , by

$$P_t f(x) = E_x[f(\eta_t)].$$

Each  $P_t$  is a self-adjoint contraction of  $\mathbb{L}^2(\nu)$ . In the continuous-time case, we assume further that the semi-group is strongly continuous, that is to say, that  $P_t f$  converges to  $f$  in  $\mathbb{L}^2(\nu)$  as  $t$  tends to 0, for any  $f \in \mathbb{L}^2(\nu)$ . We let  $L$  be the  $\mathbb{L}^2(\nu)$ -infinitesimal generator of the semi-group. It is self-adjoint in  $\mathbb{L}^2(\nu)$ , and we fix the sign convention so that it is a positive operator (i.e.,  $P_t = e^{-tL}$ ).

Note that, in general, one can see using spectral analysis that there exists a projection  $\bar{P}$  such that  $P_t f$  converges to  $\bar{P}f$  as  $t$  tends to  $\infty$ ,  $t > 0$ . Changing  $\mathbb{L}^2(\nu)$  to the image of the projection  $\bar{P}$ , and  $P_0$  for  $\bar{P}$ , one recovers a strongly continuous semigroup of contractions, and one can still carry the analysis below replacing  $\mathbb{L}^2(\nu)$  by the image of  $\bar{P}$  when necessary.

In discrete time, we set  $L = \text{Id} - P_1$ . Again,  $L$  is a positive self-adjoint operator on  $\mathbb{L}^2(\nu)$ . Note that we slightly depart from the custom of defining the generator as  $P_1$  in order to match more closely the continuous-time situation.

We denote by  $\langle \cdot, \cdot \rangle$  the scalar product in  $\mathbb{L}^2(\nu)$ . For any function  $f \in \mathbb{L}^2(\nu)$  we define the *spectral measure* of  $L$  projected on the function  $f$  as the measure  $e_f$  on

$\mathbb{R}_+$  that satisfies, for any bounded continuous  $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ , the relation

$$(2.1) \quad \langle f, \Psi(L)f \rangle = \int \Psi(\lambda) \, de_f(\lambda).$$

The Dirichlet form associated to  $L$  is given by

$$(2.2) \quad \|f\|_1^2 = \int \lambda \, de_f(\lambda).$$

We denote by  $H^1$  the completion of the space  $\{f \in \mathbb{L}^2(\nu) : \|f\|_1 < +\infty\}$  with respect to this  $\|\cdot\|_1$  norm, taken modulo functions of zero  $\|\cdot\|_1$  norm. This turns  $(H^1, \|\cdot\|_1)$  into a Hilbert space, and we let  $H^{-1}$  denote its dual. One can identify  $H^{-1}$  with the completion of the space  $\{f \in \mathbb{L}^2(\nu) : \|f\|_{-1} < +\infty\}$  with respect to the norm  $\|\cdot\|_{-1}$  defined by

$$\|f\|_{-1}^2 = \int \lambda^{-1} \, de_f(\lambda).$$

Indeed, for all  $f \in \mathbb{L}^2(\nu)$ , the linear form

$$\begin{cases} (\mathbb{L}^2(\nu) \cap H^1, \|\cdot\|_1) \rightarrow \mathbb{R}, \\ \phi \mapsto \langle f, \phi \rangle, \end{cases}$$

has norm  $\|f\|_{-1}$ , and thus defines an element of  $H^{-1}$  (with norm  $\|f\|_{-1}$ ) iff  $\|f\|_{-1}$  is finite. The notion of spectral measure introduced in (2.1) for functions of  $\mathbb{L}^2(\nu)$  can be extended to elements of  $H^{-1}$ . Indeed, let  $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}$  be a continuous function such that  $\Psi(\lambda) = O(\lambda^{-1})$  as  $\lambda \rightarrow +\infty$ . One can check that the map

$$\begin{cases} (\mathbb{L}^2(\nu) \cap H^{-1}, \|\cdot\|_{-1}) \rightarrow H^1, \\ f \mapsto \Psi(L)f, \end{cases}$$

extends to a bounded linear map on  $H^{-1}$ . One can then define the spectral measure of  $L$  projected on the function  $f$  as the measure  $e_f$  such that for any continuous  $\Psi$  with  $\Psi(\lambda) = O(\lambda^{-1})$ , (2.1) holds. With a slight abuse of notation, for all  $f \in H^{-1}$  and  $g \in H^1$ , we write  $\langle f, g \rangle$  for the  $H^{-1} - H^1$  duality product between  $f$  and  $g$ .

For any  $f \in H^{-1}$ , we define  $(Z_f(t))_{t \geq 0}$  as

$$(2.3) \quad Z_f(t) = \int_0^t f(\eta_s) \, ds \quad \text{or} \quad Z_f(t) = \sum_{s=0}^{t-1} f(\eta_s),$$

according to whether we consider the continuous or the discrete time cases. In the continuous case, the meaning of (2.3) is unclear a priori. Yet it is proved in [5], Lemma 2.4, that for any  $t \geq 0$  the map

$$\begin{cases} \mathbb{L}^2(\nu) \cap H^{-1} \rightarrow \mathbb{L}^2(P_\nu), \\ f \mapsto Z_f(t), \end{cases}$$

can be extended by continuity to a bounded linear map on  $H^{-1}$ , and moreover, that (2.3) coincides with the usual integral as soon as  $f \in \mathbb{L}^1(\nu)$ . The following theorem is due to [5], building on previous work of [13].

**THEOREM 2.1.** (i) For all  $f \in H^{-1}$ , there exists  $(M_t)_{t \geq 0}$ ,  $(\xi_t)_{t \geq 0}$  such that  $Z_f(t)$  defined in (2.3) satisfies the identity  $Z_f(t) = M_t + \xi_t$ , where  $(M_t)$  is a square-integrable martingale with stationary increments under  $P_v$  (and the natural filtration), and  $(\xi_t)$  is such that

$$(2.4) \quad t^{-1} E_v[(\xi_t)^2] \xrightarrow[t \rightarrow +\infty]{} 0.$$

As a consequence,  $t^{-1/2} Z_f(t)$  converges in law under  $P_v$  to a Gaussian random variable of variance  $\sigma^2(f) < +\infty$  as  $t$  goes to infinity, and

$$(2.5) \quad t^{-1} E_v[(Z_f(t))^2] \xrightarrow[t \rightarrow +\infty]{} \sigma^2(f).$$

(ii) If, moreover,  $f \in \mathbb{L}^1(v)$  and, for some  $\bar{t} > 0$ ,  $\sup_{0 \leq t \leq \bar{t}} |Z_f(t)|$  is in  $\mathbb{L}^2(v)$ , then the process  $t \mapsto \sqrt{\varepsilon} Z_f(\varepsilon^{-1}t)$  converges in law under  $P_v$  to a Brownian motion of variance  $\sigma^2(f)$  as  $\varepsilon$  goes to 0.

*Remarks.* The additional conditions appearing in statement (ii) are automatically satisfied in discrete time, due to the fact that  $H^{-1} \subseteq \mathbb{L}^2(v)$  in this case. In the continuous-time setting and when  $f \in \mathbb{L}^1(v)$ , the process  $t \mapsto Z_f(t)$  is almost surely continuous, and  $\sup_{0 \leq t \leq \bar{t}} |Z_f(t)|$  is indeed a well-defined random variable.

Under some additional information on the spectral measure of  $f$ , we can estimate the rates of convergence in the limits (2.4) and (2.5). For any  $\gamma > 1$  and  $q \geq 0$ , we say that the spectral exponents of a function  $f \in H^{-1}$  are at least  $(\gamma, -q)$  if

$$(2.6) \quad \int_0^\mu de_f(\lambda) = O(\mu^\gamma \ln^q(\mu^{-1})) \quad (\mu \rightarrow 0).$$

Note that the phrasing is consistent, since if  $(\gamma', -q') \leq (\gamma, -q)$  for the lexicographical order, and if the spectral exponents of  $f$  are at least  $(\gamma, -q)$ , then they are at least  $(\gamma', -q')$ . In [20], it was found more convenient to consider, instead of (2.6), a condition of the following form:

$$(2.7) \quad \int_0^\mu \lambda^{-1} de_f(\lambda) = O(\mu^{\gamma-1} \ln^q(\mu^{-1})) \quad (\mu \rightarrow 0).$$

One can easily check that conditions (2.6) and (2.7) are equivalent. Indeed, on the one hand, one has the obvious inequality

$$\int_0^\mu de_f(\lambda) \leq \mu \int_0^\mu \lambda^{-1} de_f(\lambda),$$

which shows that (2.7) implies (2.6). On the other hand, one may perform a kind of integration by parts (use Fubini's theorem),

$$\begin{aligned} \int_0^\mu \lambda^{-1} de_f(\lambda) &= \int_0^\mu \int_{\delta=\lambda}^{+\infty} \delta^{-2} d\delta de_f(\lambda) \\ &= \int_{\delta=0}^{+\infty} \delta^{-2} \int_{\lambda=0}^{\delta \wedge \mu} de_f(\lambda) d\delta \end{aligned}$$



and obtain the converse implication by examining separately the integration over  $\delta$  in  $[0, \mu)$  and in  $[\mu, +\infty)$ .

For all  $\gamma > 1$  and  $q \geq 0$ , we set

$$(2.8) \quad \psi_{\gamma,q}(t) = \begin{cases} t^{1-\gamma} \ln^q(t), & \text{if } \gamma < 2, \\ t^{-1} \ln^{q+1}(t), & \text{if } \gamma = 2, \\ t^{-1}, & \text{if } \gamma > 2. \end{cases}$$

The quantitative version of Theorem 2.1 is as follows.

**THEOREM 2.2.** *If the spectral exponents of  $f \in H^{-1}$  are at least  $(\gamma, -q)$ , then the decomposition  $Z_f(t) = M_t + \xi_t$  of Theorem 2.1 holds with the additional property that*

$$t^{-1} E_v[(\xi_t)^2] = O(\psi_{\gamma,q}(t)) \quad (t \rightarrow +\infty).$$

Moreover,

$$\sigma^2(f) - \frac{E_v[Z_f(t)^2]}{t} = O(\psi_{\gamma,q}(t)) \quad (t \rightarrow +\infty).$$

**PROOF.** In the continuous-time setting, the argument for the first estimate is very similar to the one of [20], Proposition 8.2, and we do not repeat the details here. It is based on the observation that

$$(2.9) \quad \frac{1}{t} E_v[(\xi_t)^2] = 2 \int \frac{1 - e^{-\lambda t}}{\lambda^2 t} d\mu_f(\lambda).$$

One needs to take into account the possible logarithmic terms that appear in (2.7) and which are not considered in [20]. Some care is also needed because we do not assume that  $f \in \mathbb{L}^2(\nu)$ . Yet one can easily replace the bound involving the  $\mathbb{L}^2(\nu)$  norm of  $f$  by its  $H^{-1}$  norm. The second part of the statement is given by [20], Proposition 8.3.

We now turn to the discrete time setting. In this context, identity (2.9) should be replaced by

$$\frac{1}{t} E_v[(\xi_t)^2] = 2 \int \frac{1 - (1 - \lambda)^t}{\lambda^2 t} d\mu_f(\lambda).$$

By definition,  $L = \text{Id} - P_1$ , where  $P_1$  is the semi-group at time 1. Hence, the spectrum of  $L$  is contained in  $[0, 2]$ . One can then follow the same computations as before to prove the first part of Theorem 2.2.

Somewhat surprisingly, the second part of the statement requires additional attention in the discrete time setting. Indeed, in the continuous case, the argument of [20], Proposition 8.3 (which already appears in [5]) is that  $Z_f(t)$  and  $\xi(t)$  are orthogonal in  $\mathbb{L}^2(P_\nu)$ , a fact obtained using the invariance under time symmetry. This orthogonality is only approximately valid in the discrete-time setting. Indeed,

let us recall that  $Z_f(t)$  is given by (2.3), while  $\xi_t$  is obtained as the limit in  $\mathbb{L}^2(P_\nu)$  of

$$-u_\varepsilon(\eta_t) + u_\varepsilon(\eta_0),$$

where  $u_\varepsilon = (\varepsilon + L)^{-1}f$ . Using time symmetry, what we obtain is that  $\xi_t$  is orthogonal to  $(Z_f(t) + f(\eta_t))$ . As a consequence, the cross-product  $E_\nu[Z_f(t)\xi_t]$ , which is equal to 0 in the proof of [20], Proposition 8.3, is in the present case equal to  $-E_\nu[f(\eta_t)\xi_t]$ . Yet spectral analysis ensures that this term is equal to

$$\int \frac{1 - (1 - \lambda)^t}{\lambda} d\mu_f(\lambda) = O(1) \quad (t \rightarrow +\infty),$$

which is what we need to obtain the second claim of the theorem.  $\square$

**3. The systematic error.** We now come back to the analysis of the Monte Carlo approximation of the homogenized coefficients within assumptions (H1)–(H3). The aim of this section is to estimate the difference between  $\sigma_t^2$  and the quantity  $\sigma^2$  we wish to approximate [both being defined in (1.3)]. This difference, that we refer to as the *systematic error* after [10], is shown to be of order  $1/t$  as  $t$  tends to infinity, up to a logarithmic correction in dimension 2.

**THEOREM 3.1.** *Under assumptions (H1)–(H3), there exists  $q \geq 0$  such that, as  $t$  tends to infinity,*

$$(3.1) \quad \sigma_t^2 - \sigma^2 = \begin{cases} O(t^{-1} \ln^q(t)), & \text{if } d = 2, \\ O(t^{-1}), & \text{if } d > 2. \end{cases}$$

Theorem 3.1 is a discrete-time version of [20], Corollary 2.6. Its proof makes use of an auxiliary process that we now introduce.

Let  $(\theta_x)_{x \in \mathbb{Z}^d}$  be the translation group that acts on the set of environments as follows: for any pair of neighbors  $y, z \in \mathbb{Z}^d$ ,  $(\theta_x \omega)_{y,z} = \omega_{x+y, x+z}$ . The *environment viewed by the particle* is the process defined by

$$(3.2) \quad \omega(t) = \theta_{Y(t)} \omega.$$

One can check that  $(\omega(t))_{t \in \mathbb{N}}$  is a Markov chain whose generator is given by

$$(3.3) \quad -\mathcal{L}f(\omega) = \frac{1}{p(\omega)} \sum_{|z|=1} \omega_{0,z} (f(\theta_z \omega) - f(\omega)),$$

so that  $\mathbf{E}_0^\omega[f(\omega(1))] = (I - \mathcal{L})f(\omega)$ . Moreover, the measure  $\tilde{\mathbb{P}}$  defined in (1.1) is reversible and ergodic for this process [5], Lemma 4.3(i). As a consequence, the operator  $\mathcal{L}$  is (positive and) self-adjoint in  $\mathbb{L}^2(\tilde{\mathbb{P}})$ .

The proof of Theorem 3.1 relies on spectral analysis. For any function  $f \in L^2(\tilde{\mathbb{P}})$ , let  $e_f$  be the spectral measure of  $\mathcal{L}$  projected on the function  $f$ . This measure is such that, for any positive continuous function  $\Psi : [0, +\infty) \rightarrow \mathbb{R}_+$ , one has

$$\tilde{\mathbb{E}}[f\Psi(\mathcal{L})f] = \int \Psi(\lambda) \, de_f(\lambda).$$

For any  $\gamma > 1$  and  $q \geq 0$ , we recall that we say that the spectral exponents of a function  $f$  are at least  $(\gamma, -q)$  if (2.6) holds.

Let us define the local drift  $\mathfrak{d}$  in direction  $\xi$  as

$$(3.4) \quad \mathfrak{d}(\omega) = \mathbf{E}_0^\omega[\xi \cdot Y(1)] = \frac{1}{p(\omega)} \sum_{|z|=1} \omega_{0,z} \xi \cdot z.$$

As we shall prove at the end of this section, we have the following bounds on the spectral exponents of  $\mathfrak{d}$ .

**PROPOSITION 3.2.** *Under assumptions (H1)–(H3), there exists  $q \geq 0$  such that the spectral exponents of the function  $\mathfrak{d}$  are at least*

$$(3.5) \quad \begin{cases} (2, -q), & \text{if } d = 2, \\ (d/2 + 1, 0), & \text{if } 3 \leq d \leq 5, \\ (4, -1), & \text{if } d = 6, \\ (4, 0), & \text{if } d \geq 7. \end{cases}$$

Let us see how this result implies Theorem 3.1. In order to do so, we also need the following information that is a consequence of Proposition 3.2.

**COROLLARY 3.3.** *Let*

$$(3.6) \quad \mathfrak{d}_t(\omega) = \mathbf{E}_0^\omega[\mathfrak{d}(\omega(t))]$$

*be the image of  $\mathfrak{d}$  by the semigroup at time  $t$  associated with the Markov chain  $(\omega(t))_{t \in \mathbb{N}}$ . There exists  $q \geq 0$  such that*

$$\tilde{\mathbb{E}}[(\mathfrak{d}_t)^2] = \begin{cases} O(t^{-2} \ln^q(t)), & \text{if } d = 2, \\ O(t^{-(d/2+1)}), & \text{if } 3 \leq d \leq 5, \\ O(t^{-4} \ln(t)), & \text{if } d = 6, \\ O(t^{-4}), & \text{if } d \geq 7. \end{cases}$$

**PROOF.** This result is the discrete-time analog of [9], Corollary 1. It is obtained the same way, noting that

$$\tilde{\mathbb{E}}[(\mathfrak{d}_t)^2] = \int (1 - \lambda)^{2t} \, de_{\mathfrak{d}}(\lambda)$$

and that the support of the measure  $e_{\mathfrak{d}}$  is contained in  $[0, 2]$ .  $\square$

We are now in position to prove Theorem 3.1.

PROOF OF THEOREM 3.1. The proof has the same structure as for the continuous-time case of [20], Proposition 8.4. Note that [5], Theorem 2.1, ensures that

$$(3.7) \quad \lim_{t \rightarrow \infty} \sigma_t^2 \stackrel{(\text{def})}{=} \lim_{t \rightarrow \infty} t^{-1} \mathbb{E}_0[(\xi \cdot Y(t))^2] = \sigma^2.$$

The starting point is the observation that, under  $\tilde{\mathbb{P}}_0$ , the process defined by

$$(3.8) \quad N_t = \xi \cdot Y(t) - \sum_{s=0}^{t-1} \mathfrak{d}(\omega(s))$$

is a square-integrable martingale with stationary increments. On the one hand, following (2.3), we denote by  $Z_{\mathfrak{d}}(t)$  the sum appearing in the right-hand side of (3.8). From Proposition 3.2 and Theorem 2.2, we learn that there exist  $\bar{\sigma}$  and  $q \geq 0$  such that

$$(3.9) \quad t\bar{\sigma}^2 - \tilde{\mathbb{E}}_0[(Z_{\mathfrak{d}}(t))^2] = \begin{cases} O(\ln^q(t)), & \text{if } d = 2, \\ O(1), & \text{if } d > 2. \end{cases}$$

On the other hand, since  $N_t$  is a martingale with stationary increments,

$$(3.10) \quad \tilde{\mathbb{E}}_0[(N_t)^2] = t\tilde{\mathbb{E}}_0[(N_1)^2].$$

As in the proof of Theorem 2.2 in the discrete time case, we then use that  $\xi \cdot Y(t)$  is orthogonal to  $(Z_{\mathfrak{d}}(t) + \mathfrak{d}(\omega(t)))$  to turn (3.8) into

$$(3.11) \quad \begin{aligned} t^{-1} \tilde{\mathbb{E}}_0[(N_t)^2] &= t^{-1} \tilde{\mathbb{E}}_0[(\xi \cdot Y(t))^2] + t^{-1} \tilde{\mathbb{E}}_0[(Z_{\mathfrak{d}}(t))^2] \\ &\quad + 2t^{-1} \tilde{\mathbb{E}}_0[\mathfrak{d}(\omega(t))(\xi \cdot Y(t))]. \end{aligned}$$

We already control the left-hand side and the second term of the right-hand side of (3.11). In order to quantify the convergence of  $t^{-1} \tilde{\mathbb{E}}_0[(\xi \cdot Y(t))^2]$  it remains to control the last term. In particular, provided we show that

$$(3.12) \quad \tilde{\mathbb{E}}_0[\mathfrak{d}(\omega(t))(\xi \cdot Y(t))] = \begin{cases} O(\ln^q(t)), & \text{if } d = 2, \\ O(1), & \text{if } d > 2, \end{cases}$$

(3.11), (3.9), (3.10) and (3.7) imply first that  $\sigma^2 = \tilde{\mathbb{E}}_0[(N_1)^2] - \bar{\sigma}^2$ , and then the desired quantitative estimate (3.1). We now turn to (3.12) and write

$$\begin{aligned} \tilde{\mathbb{E}}_0[\mathfrak{d}(\omega(t))(\xi \cdot Y(t))] &= \sum_{s=0}^{t-1} \tilde{\mathbb{E}}_0[\mathfrak{d}(\omega(t))(\xi \cdot (Y(s+1) - Y(s)))] \\ &= \sum_{s=0}^{t-1} \tilde{\mathbb{E}}_0[\mathfrak{d}_{t-s-1}(\omega(s+1))(\xi \cdot (Y(s+1) - Y(s)))], \end{aligned}$$

where we have used the Markov property at time  $s + 1$ , together with the definition (3.6) of  $\mathfrak{d}_{t-s-1}$ . Using Cauchy–Schwarz inequality and the stationarity of the process  $(\omega(t))_{t \in \mathbb{N}}$  under  $\mathbb{E}_0$ , this sum is bounded by

$$|\xi|^2 \sum_{s=0}^{t-1} \tilde{\mathbb{E}}[(\mathfrak{d}_{t-s-1})^2]^{1/2}.$$

Estimate (3.12) then follows from Corollary 3.3. The proof of the theorem is complete.  $\square$

Proposition 3.2 is a discrete-time counterpart of [9], Theorem 5. In [9], Theorem 5, however, we had proved, in addition, that the spectral exponents are at least  $(d/2 - 2, 0)$ , which is sharper than the exponents of Proposition 3.2 for  $d > 10$ . In particular, for  $d > 10$  the bounds of [9], Theorem 5, follow from results of [20], whose adaptation to the discrete time setting is not straightforward. As shown above, the present statement is sufficient to prove the optimal scaling of the systematic error, and we do not investigate further this issue (see, however, Remark 3.10). The proof of Proposition 3.2 is rather involved and one may wonder whether this is worth the effort in terms of the application we have in mind, namely, Theorem 3.1. In order to obtain the optimal convergence rate in Theorem 3.1 we need the spectral exponents to be larger than  $(2, 0)$ . Proving that the exponents are at least  $(2, 0)$  is rather direct using results of [10] (see the first three steps of the proof of Proposition 3.2). Yet proving that they are larger than  $(2, 0)$  for  $d > 2$  is as involved as proving Proposition 3.2 itself. This is the reason why we display the complete proof of Proposition 3.2—although the precise values of the spectral exponents are not that important in the context of this paper.

There are two new features in the proof of Proposition 3.2 with respect to our previous works:

- First, the discrete elliptic operator we consider here is slightly different than the operator considered in [10] since the zero-order term is now random as well—the adaptation of the results of [10] is only technical though;
- The string of arguments is different than in the proof of [9], Theorem 5. In particular, the starting point of [9] was an estimate obtained in [11] based on the crucial use of a covariance estimate. In [11] the main quantity of interest was a systematic error. In the present proof the main quantity of interest is the spectral exponents at the first place. This twist of points of view allows us to reduce the proof to a suitable use of the variance estimate only, and reveals the general structure of the problem.

This proof does not only complete the proof of Theorem 3.1 but allows us to shed some new light on our conjecture in [9] on the optimal values of the spectral exponents (see Remark 3.10).

As already mentioned, this proof makes extensive use of tools developed by the authors, and by Otto. For the reader's convenience, we recall five useful auxiliary results from [8, 10, 11]: a spectral gap estimate and bounds on Green's functions.

LEMMA 3.4 ([10], Lemma 2.3). *Let  $a = \{a_i\}_{i \in \mathbb{N}}$  be a sequence of i.i.d. random variables with range  $[\alpha, \beta]$ . Let  $X$  be a Borel measurable function of  $a \in \mathbb{R}^{\mathbb{N}}$  (i.e., measurable w.r.t. the smallest  $\sigma$ -algebra on  $\mathbb{R}^{\mathbb{N}}$  for which all coordinate functions  $\mathbb{R}^{\mathbb{N}} \ni a \mapsto a_i \in \mathbb{R}$  are Borel measurable; cf. [14], Definition 14.4).*

*Then we have*

$$(3.13) \quad \text{var}[X] \leq \left\langle \sum_{i=1}^{\infty} \sup_{a_i} \left| \frac{\partial X}{\partial a_i} \right|^2 \right\rangle \text{var}[a_1],$$

where  $\sup_{a_i} \left| \frac{\partial X}{\partial a_i} \right|$  denotes the supremum of the modulus of the  $i$ th partial derivative

$$\frac{\partial X}{\partial a_i}(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots)$$

of  $X$  with respect to the variable  $a_i \in [\alpha, \beta]$ .

Let  $h : \mathbb{Z}^d \rightarrow \mathbb{R}$  be some function. We define its forward and backward discrete gradients  $\nabla$  and  $\nabla^*$  as

$$\nabla h(x) := \begin{bmatrix} h(x + \mathbf{e}_1) - h(x) \\ \vdots \\ h(x + \mathbf{e}_d) - h(x) \end{bmatrix}, \quad \nabla^* h(x) := \begin{bmatrix} h(x) - h(x - \mathbf{e}_1) \\ \vdots \\ h(x) - h(x - \mathbf{e}_d) \end{bmatrix};$$

the discrete backward divergence of some vector field  $V : \mathbb{Z}^d \rightarrow \mathbb{R}^d$  is given by the “formal” scalar product between  $\nabla^*$  and  $V$ , that is,

$$\nabla^* \cdot V(x) = \sum_{i=1}^d (V_i(x + \mathbf{e}_i) - V_i(x)).$$

To avoid confusion, when a function  $h : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{R}$ ,  $(x, z) \mapsto h(x, z)$  depends on two variables, we denote by  $\nabla_1 h$  (resp.,  $\nabla_1^* h$ ) the forward (resp., backward) discrete gradient with respect to the first variable ( $x$  here) and by  $\nabla_2 h$  (resp.,  $\nabla_2^* h$ ) the forward (resp., backward) discrete gradient with respect to the second variable ( $z$  here). We further use the notation  $\nabla_{k,i} h := \nabla_k h \cdot \mathbf{e}_i$  for the forward discrete gradients in direction  $\mathbf{e}_i$  (and likewise for the backward gradients),  $i \in \{1, \dots, d\}$ .

We define discrete Green's functions as follows.

DEFINITION 3.5 (Discrete Green's function). Let  $d \geq 2$ . Let  $\omega$  be an environment,  $p_\omega : \mathbb{Z}^d \rightarrow \mathbb{R}$ ,  $x \mapsto \sum_{|z-x|=1} \omega_{x,z}$  and  $A$  be the associated diagonal matrix on  $\mathbb{Z}^d$  defined by  $A(x) = \text{diag}(\omega_{x,x+\mathbf{e}_1}, \dots, \omega_{x,x+\mathbf{e}_d})$ . For all  $T > 0$ , the Green function  $G_T(\cdot, \cdot; \omega) : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ ,  $(x, y) \mapsto G_T(x, y; \omega)$  associated with the

environment  $\omega$  is defined for all  $y \in \mathbb{Z}^d$  as the unique space square-integrable solution to

$$(3.14) \quad \int_{\mathbb{Z}^d} T^{-1} p_\omega(x) G_T(x, y; \omega) v(x) dx + \int_{\mathbb{Z}^d} \nabla v(x) \cdot A(x) \nabla_1 G_T(x, y; \omega) dx = v(y)$$

for all square-integrable functions  $v: \mathbb{Z}^d \rightarrow \mathbb{R}$ , where  $\int_{\mathbb{Z}^d} dy$  denotes the sum over all  $y \in \mathbb{Z}^d$ .

The existence and uniqueness of discrete Green's functions is a consequence of Riesz's representation theorem. In the rest of this article we use the shorthand notation  $G_T(x, y)$  for  $G_T(x, y; \omega)$ . Note that  $G_T$  is stationary in the sense that  $(x, y) \mapsto G_T(x + z, y + z)$  has the same statistics as  $(x, y) \mapsto G_T(x, y)$ . This will be used for the gradient of the Green function as follows: for all  $q > 0$ ,

$$(3.15) \quad \langle |\nabla_2 G_T(x, y)|^q \rangle = \langle |\nabla_1 G_T(x - y, 0)|^q \rangle.$$

The next two lemmas give estimates on the Green function and its derivatives.

LEMMA 3.6 ([8], Lemma 3.2). *There exists  $c > 0$  depending only on  $\alpha, \beta$  and  $d$  such that for every environment  $\omega$  and for all  $T > 0$ , the Green function  $G_T$  satisfies the pointwise estimates: for all  $x, y \in \mathbb{Z}^d$ ,*

$$(3.16) \quad \text{for } d > 2 \quad G_T(x, y) \lesssim (1 + |x - y|)^{2-d} \exp\left(-c \frac{|x - y|}{\sqrt{T}}\right),$$

$$(3.17) \quad \text{for } d = 2 \quad G_T(x, y) \lesssim \ln\left(\frac{\sqrt{T}}{1 + |x - y|}\right) \exp\left(-c \frac{|x - y|}{\sqrt{T}}\right).$$

LEMMA 3.7 ([10], Lemma 2.9). *Let  $\omega$  be an environment,  $T > 0$ , and let  $G_T$  be the associated Green function. Then, for  $d \geq 2$ , there exists  $p > 2$  depending only on  $\alpha, \beta$  and  $d$  such that for all  $T > 0$ ,  $p \geq r \geq 2$ ,  $k > 0$  and  $R \lesssim 1$ ,*

$$(3.18) \quad \int_{R \leq |z| \leq 2R} |\nabla_1 G_T(z, 0)|^r dz \lesssim R^d (R^{1-d})^r \min\{1, \sqrt{T} R^{-1}\}^k.$$

Note that this lemma shows that  $\nabla_1 G_T(z, 0)$  has the optimal decay  $(1 + |z|)^{1-d}$  (i.e., the decay of the Green function of the Laplace operator) when integrated on dyadic annuli (plus the exponential, or superalgebraic decay).

COROLLARY 3.8 ([10], Corollary 2.3). *For every environment  $\omega$  and for all  $T > 0$  and  $x, y \in \mathbb{Z}^d$ ,*

$$|\nabla_1 G_T(x, y; \omega)|, |\nabla_2 G_T(x, y; \omega)| \lesssim 1$$

(the multiplicative constant depending only on  $\alpha, \beta$  and  $d$ ).

Note that the versions of these lemmas proved in [10] and [8] cover the case when the zero-order term is constant, namely,  $T^{-1}$  in place of  $T^{-1}p_\omega(x)$ . The proofs adapt mutatis mutandis using the uniform bounds  $0 < 2^d\alpha \leq p_\omega \leq 2^d\beta$ .

The last lemma we shall need is the following double convolution estimate.

LEMMA 3.9 ([9], Lemma 6). *Let  $d > 2$ ,  $T \gg 1$  and let  $g_T : \mathbb{Z}^d \rightarrow \mathbb{R}^+$  be given by*

$$g_T(x) = (1 + |x|)^{2-d} \exp\left(-c \frac{|x|}{\sqrt{T}}\right)$$

for some  $c > 0$ . Let  $h_T : \mathbb{Z}^d \rightarrow \mathbb{R}^+$  be such that

$$\int_{|x| \leq R} h_T(x)^2 \lesssim 1$$

and for all  $R \gg 1$  and all  $j \in \mathbb{N}$ ,

$$\int_{2^j R \leq |x| < 2^{j+1} R} h_T(x)^2 dx \lesssim (2^j R)^{d-2(d-1)}.$$

Then,

$$(3.19) \quad \begin{aligned} & \int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} g_T(w) g_T(w') h_T(z-w) h_T(z-w') dz dw dw' \\ & \lesssim 1 + \begin{cases} T^{3-d/2}, & \text{if } 5 \geq d > 2, \\ \ln T, & \text{if } d = 6, \\ 1, & \text{if } d > 6. \end{cases} \end{aligned}$$

We are in position to prove Proposition 3.2.

PROOF OF PROPOSITION 3.2. Our starting point is the following inequality which holds for every nonnegative measure  $\kappa$ :

$$(3.20) \quad \int_0^{T^{-1}} d\kappa(\lambda) \lesssim T^{-4} \int_0^\infty \frac{1}{(T^{-1} + \lambda)^4} d\kappa(\lambda),$$

which follows from the fact that for  $\lambda \leq T^{-1}$ ,  $\frac{T^{-4}}{(T^{-1} + \lambda)^4} \gtrsim 1$ . The variable  $T^{-1}$  for  $T$  large plays the role of  $\mu$  in (2.6). In what follows we make the standard identification between stationary functions  $(z, \omega) \mapsto f(z, \omega)$  of both the space variable  $z \in \mathbb{Z}^d$  and the environment  $\omega$  and their translated versions at 0  $\omega \mapsto f(0, \theta_z \omega)$  depending on the environment only. We define  $\phi_T$  as the unique stationary solution to

$$(3.21) \quad T^{-1} \phi_T(x) - \frac{1}{p_\omega(x)} \nabla^* \cdot A(x) \nabla \phi_T = \frac{1}{p_\omega(x)} \nabla^* \cdot A(x) \xi,$$



whose existence and uniqueness follow from the Riesz representation theorem in  $\mathbb{L}^2(\tilde{\mathbb{P}})$  using the identification between the stationary function  $\phi_T$  and its version defined on the environment only (see a similar argument of [17]). In particular, with the notation  $\mathfrak{d} = \frac{1}{p_\omega(x)} \nabla^* \cdot A(x) \xi$ ,

$$\phi_T = (T^{-1} + \mathcal{L})^{-1} \mathfrak{d},$$

where  $\mathcal{L}$  is the operator defined in (3.3), and the spectral theorem ensures that

$$\tilde{\mathbb{E}}(\phi_T^2) = \tilde{\mathbb{E}}(\mathfrak{d}(T^{-1} + \mathcal{L})^{-2} \mathfrak{d}) = \int_0^\infty \frac{1}{(T^{-1} + \lambda)^2} \mathrm{d}e_{\mathfrak{d}}(\lambda),$$

where  $e_{\mathfrak{d}}$  is the spectral measure of  $\mathcal{L}$  projected on the drift  $\mathfrak{d}$ . We also let  $\psi_T$  be the unique stationary solution to

$$(3.22) \quad T^{-1} \psi_T(x) - \frac{1}{p_\omega(x)} \nabla^* \cdot A(x) \nabla \psi_T(x) = \phi_T(x),$$

whose existence and uniqueness also follows from the Riesz representation theorem in the probability space as well. This time,

$$\psi_T = (T^{-1} + \mathcal{L})^{-2} \mathfrak{d},$$

and the spectral theorem yields

$$\tilde{\mathbb{E}}(\psi_T^2) = \tilde{\mathbb{E}}(\mathfrak{d}(T^{-1} + \mathcal{L})^{-4} \mathfrak{d}) = \int_0^\infty \frac{1}{(T^{-1} + \lambda)^4} \mathrm{d}e_{\mathfrak{d}}(\lambda).$$

From now on, we shall use the shorthand notation  $\langle u \rangle := \tilde{\mathbb{E}}(u)$  and  $\mathrm{var}[u] = \langle (u - \langle u \rangle)^2 \rangle$  for all  $u \in \mathbb{L}^2(\tilde{\mathbb{P}})$ . In particular, the identity above turns into

$$(3.23) \quad \int_0^\infty \frac{1}{(T^{-1} + \lambda)^4} \mathrm{d}e_{\mathfrak{d}}(\lambda) = \mathrm{var}[\psi_T],$$

since  $\langle \psi_T \rangle = \frac{1}{\mathbb{E}[p]} \int \psi_T p \, \mathrm{d}\mathbb{P} = \frac{T}{\mathbb{E}[p]} \int \phi_T p \, \mathrm{d}\mathbb{P} = 0$  using equations (3.22) and (3.21).

The streamline of the proof is to obtain bounds on the spectral exponents via (3.20) and (3.23) by proving bounds on the variance of  $\psi_T$ .

The rest of the proof, which is dedicated to the estimate of  $\mathrm{var}[\psi_T]$ , is divided into five steps. As a starting point we appeal to the variance estimate of Lemma 3.4 that we apply to  $\psi_T$ . This requires us to estimate the susceptibility of  $\psi_T$  with respect to the random coefficients. In view of (3.22) it is not surprising that we will have to estimate not only the susceptibility of  $\psi_T$  but also of  $\phi_T$  and of some Green function with respect to the random coefficients. In the first step, we establish the susceptibility estimate for the Green function. In step 2 we turn to the susceptibility estimate for the approximate corrector  $\phi_T$ . We then show in step 3 that, relying on [10], this implies that the spectral exponents are at least

$$(3.24) \quad \begin{cases} (2, -q), & \text{for } d = 2, \\ (2, 0), & \text{for } d > 2. \end{cases}$$

In step 4 we estimate the susceptibility of  $\psi_T$ . We conclude the proof of the proposition in step 5.

*Step 1. Susceptibility of the Green function.*

We shall prove for all  $e = (z, z') \in \mathbb{B}$ ,  $z \in \mathbb{Z}^d$ ,  $z' = z + \mathbf{e}_i$ ,

$$(3.25) \quad \begin{aligned} \frac{\partial G_T}{\partial \omega_e}(x, y) = & -T^{-1}(G_T(z, y)G_T(x, z) + G_T(z', y)G_T(x, z')) \\ & - \nabla_{2,i}G_T(x, z)\nabla_{1,i}G_T(z, y) \end{aligned}$$

and

$$(3.26) \quad \begin{aligned} \sup_{\omega_e} |\nabla_{1,i}G_T(z, y)| & \lesssim |\nabla_{1,i}G_T(z, y)| + T^{-1}g_T(y - z), \\ \sup_{\omega_e} |\nabla_{2,i}G_T(y, z)| & \lesssim |\nabla_{2,i}G_T(y, z)| + T^{-1}g_T(y - z), \end{aligned}$$

where  $g_T : \mathbb{Z}^d \rightarrow \mathbb{R}^+$  satisfies for some constant  $c > 0$  (depending on  $\alpha, \beta, d$ )

$$(3.27) \quad g_T(x) = (1 + |x|)^{2-d} \exp\left(-c \frac{|x|}{\sqrt{T}}\right)$$

for  $d > 2$ , and

$$(3.28) \quad g_T(x) = \left| \ln\left(\frac{\sqrt{T}}{1 + |x|}\right) \right| \exp\left(-c \frac{|x|}{\sqrt{T}}\right)$$

for  $d = 2$ .

We define the elliptic operator  $L_T$  as

$$(L_T u)(x) = \sum_{x', |x-x'|=1} \omega_{x,x} T^{-1} u(x) + \sum_{x', |x-x'|=1} \omega_{x,x'} (u(x) - u(x')),$$

so that for all  $y \in \mathbb{Z}^d$ , (3.14) takes the form

$$(3.29) \quad (L_T G_T(\cdot, y))(x) = \delta(x - y).$$

Recalling that the edges are not oriented, a formal differentiation of this equation with respect to  $\omega_e = \omega_{z,z'} = \omega_{z',z}$  yields

$$\begin{aligned} L_T \left( \frac{\partial G_T}{\partial \omega_e}(\cdot, y) \right)(x) + T^{-1} G_T(x, y) (\delta(x - z) + \delta(x - z')) \\ + (G_T(z, y) - G_T(z', y)) \delta(x - z) + (G_T(z', y) - G_T(z, y)) \delta(x - z') \\ = 0. \end{aligned}$$

Using (3.29), this identity turns into

$$\begin{aligned} L_T \left( \frac{\partial G_T}{\partial \omega_e}(\cdot, y) + T^{-1} (G_T(z, y)G_T(\cdot, z) + G_T(z', y)G_T(\cdot, z')) \right. \\ \left. + \nabla_{2,i}G_T(\cdot, z)\nabla_{1,i}G_T(z, y) \right)(x) = 0. \end{aligned}$$

Provided that the argument of  $L_T$  is well defined (i.e.,  $G_T$  is differentiable w.r.t.  $\omega_e$ ) and that it is square-integrable on  $\mathbb{Z}^d$ , it vanishes identically by the Riesz representation theorem—which is the desired identity (3.25).

To turn this into a rigorous argument, one may first consider finite differences of parameter  $h > 0$  instead of a derivative w.r.t.  $\omega_e$ , use that  $L_T$  is bijective on the set of square-integrable functions on  $\mathbb{Z}^d$  and then pass to the limit  $h \rightarrow 0$ . We refer the reader to [10], proof of Lemma 2.5, for details, and directly turn to (3.26).

From (3.25) with  $x = z$  and  $x = z'$ , we infer that

$$\begin{aligned} (3.30) \quad & \frac{\partial \nabla_{1,i} G_T(z, y)}{\partial \omega_e} \\ &= -\nabla_{1,i} \nabla_{2,i} G_T(z, z) \nabla_{1,i} G_T(z, y) \\ & \quad - T^{-1} (G_T(z, y) \nabla_{1,i} G_T(z, z) + G_T(z', y) \nabla_{1,i} G_T(z, z')). \end{aligned}$$

Using the uniform pointwise estimate of Corollary 3.8 and the uniform estimate on the Green function of Lemma 3.6, we obtain (3.26) by considering (3.30) as an ODE for  $\nabla_{1,i} G_T(z, y)$  in function of  $\omega_e$ .

*Step 2. Susceptibility of  $\phi_T$ .*

In this step we shall prove that for  $e = (z, z') \in \mathbb{B}$ ,  $z \in \mathbb{Z}^d$  and  $z' = z + \mathbf{e}_i$ ,

$$\begin{aligned} (3.31) \quad & \frac{\partial \phi_T}{\partial \omega_e}(x) = -(\nabla_i \phi_T(z) + \xi_i) \nabla_{2,i} G_T(x, z) \\ & \quad - T^{-1} \phi_T(z) (G_T(x, z) + G_T(x, z')), \end{aligned}$$

$$\begin{aligned} (3.32) \quad & \sup_{\omega_e} |\phi_T(x)| \lesssim |\phi_T(x)| \\ & \quad + (|\nabla_i \phi_T(z)| + 1) \\ & \quad \times (|\nabla_{2,i} G_T(x, z)| + T^{-1/2} g_T(x - z)), \end{aligned}$$

$$(3.33) \quad \sup_{\omega_e} \left| \frac{\partial \phi_T}{\partial \omega_e}(x) \right| \lesssim (|\nabla_i \phi_T(z)| + 1) (|\nabla_{2,i} G_T(x, z)| + T^{-1/2} g_T(x - z))$$

and for all  $n \in \mathbb{N}$ ,

$$\begin{aligned} (3.34) \quad & \sup_{\omega_e} \left| \frac{\partial (\phi_T(x)^{n+1})}{\partial \omega_e} \right| \\ & \lesssim (|\nabla_i \phi_T(z)| + 1) (|\nabla_{2,i} G_T(x, z)| + T^{-1/2} g_T(x - z)) \\ & \quad \times (|\phi_T(x)| + (|\nabla_i \phi_T(z)| + 1) (|\nabla_{1,i} G_T(z, x)| + T^{-1/2} g_T(x - z)))^n. \end{aligned}$$

As for the Green function, we rewrite the defining equation for  $\phi_T$  as

$$(3.35) \quad (L_T \phi_T)(x) - \nabla^* \cdot A(x) \xi = 0.$$

Formally differentiating (3.35) w.r.t.  $\omega_e$  yields

$$\begin{aligned} L_T \frac{\partial \phi_T}{\partial \omega_e}(x) - (\nabla_i \phi_T(x) + \xi_i)(\delta(x - z) - \delta(x - z')) \\ + T^{-1} \phi_T(x)(\delta(x - z) + \delta(x - z')) = 0, \end{aligned}$$

which, using (3.29), turns into

$$\begin{aligned} L_T \left( \frac{\partial \phi_T}{\partial \omega_e} - (\nabla_i \phi_T + \xi_i)(G_T(\cdot, z) - G_T(\cdot, z')) \right. \\ \left. + T^{-1} \phi_T(G_T(\cdot, z) + G_T(\cdot, z')) \right)(x) = 0. \end{aligned}$$

This (formally) shows (3.31).

To turn this into a rigorous argument, we may combine (3.25) with the Green representation formula

$$\phi_T(x) = \int_{\mathbb{Z}^d} G_T(x, y) \nabla^* \cdot A(y) \xi \, dy,$$

which holds since  $G_T(x, \cdot)$  is integrable on  $\mathbb{Z}^d$  by Lemma 3.6, and use standard results of commutation of integration and differentiation.

We now turn to (3.33). This estimate follows from (3.31), (3.26) and the following two facts:

$$(3.36) \quad |\phi_T| \lesssim \sqrt{T}$$

and

$$(3.37) \quad \sup_{\omega_e} |\nabla_i \phi_T(z)| \lesssim |\nabla_i \phi_T(z)| + 1.$$

The starting point to prove (3.36) is the Green representation formula in the form of

$$\begin{aligned} (3.38) \quad |\phi_T(x)| &= \left| \int_{\mathbb{Z}^d} G_T(x, y) \nabla^* \cdot A(y) \xi \, dy \right| \\ &= \left| \int_{\mathbb{Z}^d} \nabla_2 G_T(x, y) \cdot A(y) \xi \, dy \right| \\ &\lesssim \int_{\mathbb{Z}^d} |\nabla_2 G_T(0, y)| \, dy. \end{aligned}$$

The claim would easily follow if we had the estimate

$$|\nabla_2 G_T(0, y)| \lesssim (1 + |y|)^{1-d} \exp\left(-c \frac{|y|}{\sqrt{T}}\right).$$

Although this estimate does not hold *pointwise*, it holds when *square-integrated on dyadic annuli*, as shows Lemma 3.7 with “ $p = 2$  and  $k$  large.” The claim

(3.36) thus follows from a dyadic decomposition of space in (3.38) combined with Cauchy–Schwarz’s inequality and Lemma 3.7 (a similar calculation is displayed, e.g., in [8], proof of Lemma 4).

For (3.37), we first note that (3.31) implies

$$\begin{aligned} \frac{\partial \nabla_i \phi_T(z)}{\partial \omega_e} &= -(\nabla_i \phi_T(z) + \xi_i)(\nabla_{2,i} G_T(z', z) - \nabla_{2,i} G_T(z, z)) \\ &\quad + T^{-1} \phi_T(z)(\nabla_{1,i} G_T(z, z) + \nabla_{1,i} G_T(z, z')), \end{aligned}$$

which (seen as an ODE w.r.t.  $\omega_e$ ) yields the claim using the uniform bound  $|\nabla_1 G_T|, |\nabla_2 G_T| \lesssim 1$  of Corollary 3.8 and (3.36).

Estimate (3.32) is a direct consequence of (3.33), whereas (3.34) follows from the Leibniz’s rule combined with (3.31), (3.32) and (3.33).

*Step 3. Proof of (3.24).*

The estimates (3.24) of the spectral exponents follow from the more general estimates; for all  $q > 0$  there exists  $\gamma(q) > 0$  such that

$$(3.39) \quad \langle |\phi_T|^q \rangle \lesssim \begin{cases} \ln^{\gamma(q)} T, & \text{for } d = 2, \\ 1, & \text{for } d > 2, \end{cases}$$

combined with the fact that

$$\int_0^{T^{-1}} de_{\mathfrak{d}}(\lambda) \lesssim T^{-2} \int_0^\infty \frac{1}{(T^{-1} + \lambda)^2} de_{\mathfrak{d}}(\lambda) = T^{-2} \langle \phi_T^2 \rangle.$$

The proof of (3.39) is an easy adaptation of [10], proof of Proposition 2.1, which already covers the case of a constant coefficient in the zero order term of  $L_T$ , that is, for  $T^{-1} \phi_T$  instead of  $T^{-1} p_\omega \phi_T$  (no randomness in the zero order term). We only point out what needs to be changed in [10], proof of Proposition 2.1.

The first step to apply the variance estimate of Lemma 3.4 is to show that  $\phi_T$  is measurable with respect to the cylindrical topology associated with the random variables. This is proved exactly as in [10], Lemma 2.6.

The auxiliary [10], Lemmas 2.4 and 2.5, are replaced by the susceptibility estimates (3.26), (3.32), (3.33) and (3.34) of steps 1 and 2, which have, however, the additional term  $T^{-1/2} g_T(x - z)$  next to  $|\nabla_{2,i} G_T(x, z)|$ .

In the proof of [10], Proposition 2.1, the terms  $|\nabla_{2,i} G_T(x, z)|$  are either estimated by the Green function  $G_T(x, z)$  itself [in which case the additional term  $T^{-1/2} g_T(x - z)$  is of higher order], or they are controlled on dyadic annuli by Lemma 3.7. By definition (3.27) for  $d > 2$  and (3.28) for  $d = 2$  of the function  $g_T$ , it is easy to see that for all  $r \geq 2$ ,  $k > 0$  and  $R \gg 1$ : for  $d > 2$ ,

$$\int_{R \leq |x-z| < 2R} (T^{-1/2} g_T(x - z))^r dz \lesssim R^d (R^{1-d})^r \min\{1, \sqrt{T} R^{-1}\}^k,$$

whereas for  $d = 2$

$$\int_{R \leq |x-z| < 2R} (T^{-1/2} g_T(x - z))^r dz \lesssim R^2 (R^{-1})^r \ln^q T \min\{1, \sqrt{T} R^{-1}\}^k.$$

These scalings coincide with those of Lemma 3.7 (with a possible additional logarithmic correction for  $d = 2$ ).

Hence, the proof of [10], Proposition 2.1, adapts mutatis mutandis to the present case, and we have (3.39).

*Step 4. Susceptibility of  $\psi_T$ .*

In this step we shall prove that for all  $e = (z, z')$ ,  $z \in \mathbb{Z}^d$  and  $z' = z + \mathbf{e}_i$  and for all  $x \in \mathbb{Z}^d$ ,

$$\begin{aligned}
 \frac{\partial \psi_T}{\partial \omega_e}(x) &= -\nabla_{2,i} G_T(x, z) \nabla_i \psi_T(z) - T^{-1} G_T(x, z) \psi_T(z) \\
 &\quad - T^{-1} G_T(x, z') \psi_T(z') \\
 (3.40) \quad &\quad - (\nabla_i \phi_T(z) + \xi_i) \int_{\mathbb{Z}^d} G_T(x, y) p_\omega(y) \nabla_{2,i} G_T(y, z) dy \\
 &\quad - T^{-1} \phi_T(z) \int_{\mathbb{Z}^d} G_T(x, y) p_\omega(y) (G_T(y, z) + G_T(y, z')) dy \\
 &\quad + G_T(x, z) \phi_T(z) + G_T(x, z') \phi_T(z')
 \end{aligned}$$

and

$$\begin{aligned}
 \sup_{\omega_e} \left| \frac{\partial \psi_T}{\partial \omega_e}(x) \right| &\lesssim g_T(z - x) (|\nabla_i \psi_T(z)| + T^{-1} |\psi_T(z)| + \nu_d(T) (1 + |\phi_T(z)| + |\phi_T(z')|)) \\
 (3.41) \quad &\quad + (1 + |\phi_T(z)| + |\phi_T(z')|) \\
 &\quad \times \int_{\mathbb{Z}^d} g_T(y - x) (|\nabla_{2,i} G_T(y, z)| + T^{-1} g_T(y - z)) dy,
 \end{aligned}$$

where

$$(3.42) \quad \nu_d(T) = \begin{cases} T, & \text{for } d = 2, \\ \sqrt{T}, & \text{for } d = 3, \\ \ln T, & \text{for } d = 4, \\ 1, & \text{for } d > 4. \end{cases}$$

The starting point is again the Green representation formula

$$\psi_T(x) = \int_{\mathbb{Z}^d} G_T(x, y) p_\omega(y) \phi_T(y) dy,$$

associated with (3.22) in the form

$$T^{-1} p_\omega \psi_T - \nabla^* \cdot A \nabla \psi_T = p_\omega \phi_T.$$

Differentiated w.r.t.  $\omega_e$  it turns into

$$\begin{aligned} \frac{\partial \psi_T(x)}{\partial \omega_e} &= \int_{\mathbb{Z}^d} \frac{\partial G_T(x, y)}{\partial \omega_e} p_\omega(y) \phi_T(y) dy \\ &\quad + \int_{\mathbb{Z}^d} G_T(x, y) \frac{\partial p_\omega(y)}{\partial \omega_e} \phi_T(y) dy \\ &\quad + \int_{\mathbb{Z}^d} G_T(x, y) p_\omega(y) \frac{\partial \phi_T(y)}{\partial \omega_e} dy. \end{aligned}$$

Combined with (3.25), (3.31) and the Green representation formula itself, this shows (3.40).

We now turn to (3.41) and treat each term of the right-hand side of (3.40) separately. We begin with the supremum of the third line of (3.40), and claim that

$$\begin{aligned} (3.43) \quad & \sup_{\omega_e} \left| (\nabla_i \phi_T(z) + \xi_i) \int_{\mathbb{Z}^d} G_T(x, y) p_\omega(y) \nabla_{2,i} G_T(y, z) dy \right| \\ & \lesssim (1 + |\phi_T(z)| + |\phi_T(z')|) \\ & \quad \times \int_{\mathbb{Z}^d} g_T(y - x) (|\nabla_{2,i} G_T(y, z)| + T^{-1} g_T(y - z)) dy, \end{aligned}$$

which is proved:

- using (3.37) to bound the supremum in  $\omega_e$  of  $|\nabla_i \phi_T(z)|$  by  $|\nabla_i \phi_T(z)|$  itself,
- bounding  $|\nabla_i \phi_T(z)|$  by the triangle inequality  $|\phi_T(z)| + |\phi_T(z')|$ ,
- replacing the Green function  $G_T$  by  $g_T$  using Lemma 3.6,
- and appealing to (3.26) to estimate the supremum in  $\omega_e$  of  $|\nabla_{2,i} G_T(y, x)|$ .

This shows that this term is controlled by the second term of the right-hand side of (3.41).

The supremum of the term in the fourth line of (3.40) is also estimated by the second term of the right-hand side of (3.41), namely,

$$\begin{aligned} (3.44) \quad & \sup_{\omega_e} \left| T^{-1} \phi_T(z) \int_{\mathbb{Z}^d} G_T(x, y) p_\omega(y) (G_T(y, z) + G_T(y, z')) dy \right| \\ & \lesssim (1 + |\phi_T(z)| + |\phi_T(z')|) T^{-1} \int_{\mathbb{Z}^d} g_T(y - x) g_T(y - z) dy. \end{aligned}$$

It is enough to bound the Green function by  $g_T$  using Lemma 3.6, and to apply (3.32) for  $x = z$  to control  $\sup_{\omega_e} |\phi_T(z)|$ , and use that  $|\nabla_1 G_T|, |\nabla_2 G_T|, T^{-1/2} \times G_T \lesssim 1$  by Corollary 3.8 and Lemma 3.6.

The suprema of the last two terms of (3.40) is bounded by

$$\begin{aligned} (3.45) \quad & \sup_{\omega_e} |G_T(x, z) \phi_T(z) + G_T(x, z') \phi_T(z')| \\ & \lesssim (1 + |\phi_T(z)| + |\phi_T(z')|) g_T(z - x) \end{aligned}$$

and therefore controlled by the first term of the right-hand side of (3.41). The argument is similar to the proof of (3.44).

The subtle terms are the first three ones, for which we have to estimate the suprema of  $|\nabla_i \psi_T(z)|$ ,  $|\psi_T(z)|$  and  $|\psi_T(z')|$  w.r.t.  $\omega_e$ .

We begin with the following two estimates

$$(3.46) \quad \sup_{\omega_e} |\psi_T(z)| \lesssim |\psi_T(z)| + (|\phi_T(z)| + |\phi_T(z')| + 1)v_d(T) \\ + \sup_{\omega_e} |\nabla_i \psi_T(z)|,$$

$$(3.47) \quad \sup_{\omega_e} |\nabla_i \psi_T(z)| \lesssim |\nabla_i \psi_T(z)| + (|\phi_T(z)| + |\phi_T(z')| + 1)v_d(T) \\ + T^{-1} \sup_{\omega_e} |\psi_T(z)|,$$

which (seen as a linear system) show that there exists some  $T_* > 0$  such that for all  $T \geq T_*$ ,

$$(3.48) \quad \sup_{\omega_e} |\psi_T(z)| \lesssim |\psi_T(z)| + (|\phi_T(z)| + |\phi_T(z')| + 1)v_d(T) \\ + |\nabla_i \psi_T(z)|,$$

$$(3.49) \quad \sup_{\omega_e} |\nabla_i \psi_T(z)| \lesssim |\nabla_i \psi_T(z)| + (|\phi_T(z)| + |\phi_T(z')| + 1)v_d(T) \\ + T^{-1} |\psi_T(z)|.$$

To prove (3.46) we consider (3.40) as an ODE on  $\psi_T(z)$ , bound  $\psi_T(z')$  by  $\psi_T(z) + |\nabla_i \psi_T(z)|$  and use (3.43), (3.44) and (3.45) (for  $x = z$ ), so that (3.40) turns into

$$\left| \frac{\partial \psi_T}{\partial \omega_e}(z) \right| \lesssim \sup_{\omega_e} \{ |\nabla_{2,i} G_T(z, z)| |\nabla_i \psi_T(z)| \} + T^{-1} G_T(z, z) |\psi_T(z)| \\ + T^{-1} G_T(z, z') (|\psi_T(z)| + \sup_{\omega_e} |\nabla_i \psi_T(z)|) \\ + (1 + |\phi_T(z)| + |\phi_T(z')|) \\ \times \left( 1 + \int_{\mathbb{Z}^d} g_T(y - z) (|\nabla_{2,i} G_T(y, z)| + T^{-1} g_T(y - z)) dy \right).$$

Using Corollary 3.8 and Lemma 3.6 in the form of  $|\nabla_1 G_T|$ ,  $|\nabla_2 G_T|$ ,  $T^{-1} G_T \lesssim 1$ , and bounding the gradient of the Green function by  $g_T$  in the integral, we obtain

$$\left| \frac{\partial \psi_T}{\partial \omega_e}(z) \right| \lesssim \sup_{\omega_e} \{ |\nabla_i \psi_T(z)| \} + |\psi_T(z)| \\ + (1 + |\phi_T(z)| + |\phi_T(z')|) \left( 1 + \int_{\mathbb{Z}^d} g_T(y)^2 dy \right).$$



Noting that by definitions (3.27) and (3.28) of  $g_T$  we have  $\int_{\mathbb{Z}^d} g_T(y)^2 dy \lesssim v_d(T)$ , this inequality turns into

$$\left| \frac{\partial \psi_T}{\partial \omega_e}(z) \right| \lesssim \sup_{\omega_e} \{ |\nabla_i \psi_T(z)| \} + |\psi_T(z)| + v_d(T)(1 + |\phi_T(z)| + |\phi_T(z')|).$$

Seen as an ODE for  $\psi_T$ , this implies (3.46).

We now turn to (3.47) and infer from (3.40) that

$$\begin{aligned} \frac{\partial \nabla_i \psi_T(z)}{\partial \omega_e} &= -\nabla_{1,i} \nabla_{2,i} G_T(z, z) \nabla_i \psi_T(z) - T^{-1} \nabla_{1,i} G_T(z, z) \psi_T(z) \\ &\quad - T^{-1} \nabla_{1,i} G_T(z, z') \psi_T(z') \\ &\quad - (\nabla_i \phi_T(z) + \xi_i) \int_{\mathbb{Z}^d} \nabla_{2,i} G_T(z, y) p_\omega(y) \nabla_{2,i} G_T(y, z) dy \\ &\quad - T^{-1} \phi_T(z) \int_{\mathbb{Z}^d} \nabla_{2,i} G_T(z, y) p_\omega(y) (G_T(y, z) + G_T(y, z')) dy \\ &\quad + \nabla_{1,i} G_T(z, z) \phi_T(z) + \nabla_{1,i} G_T(z, z') \phi_T(z'). \end{aligned}$$

Repeating the string of arguments leading from (3.40) to (3.46), we deduce (3.47) and therefore (3.48) and (3.49). Combining the inequality  $|\psi_T(z')| \leq |\psi_T(z)| + |\nabla_i \psi_T(z)|$  with (3.48) and (3.49) yields the last estimate we need:

$$(3.50) \quad \sup_{\omega_e} |\psi_T(z')| \lesssim |\psi_T(z)| + (|\phi_T(z)| + |\phi_T(z')| + 1) v_d(T) + |\nabla_i \psi_T(z)|.$$

We are finally in position to conclude the proof of (3.41). The last four terms are controlled by (3.43), (3.44) and (3.45). Using (3.49), (3.48) and (3.50), and Corollary 3.8 and Lemma 3.6, the first three terms of the right-hand side of (3.40) are controlled by the first term of the right-hand side of (3.41). Estimate (3.41) is proved.

*Step 5. Estimate of  $\text{var}[\psi_T]$  for  $d > 2$  and conclusion.*

We apply the variance estimate of Lemma 3.4 to  $\psi_T$

$$(3.51) \quad \text{var}[\psi_T] \lesssim \sum_{e \in \mathbb{B}} \left\langle \sup_{\omega_e} \left| \frac{\partial \psi_T(0)}{\partial \omega_e} \right|^2 \right\rangle$$

and appeal to (3.41). We distinguish two contributions in this sum and define

$$\begin{aligned} A_e &:= g_T(z) (|\nabla_i \psi_T(z)| + T^{-1} |\psi_T(z)| + v_d(T)(1 + |\phi_T(z)| + |\phi_T(z')|)), \\ B_e &:= (1 + |\phi_T(z)| + |\phi_T(z')|) \int_{\mathbb{Z}^d} g_T(y) (|\nabla_{2,i} G_T(y, z)| + T^{-1} g_T(y - z)) dy. \end{aligned}$$

The contribution associated with  $A_e$  is estimated as follows:

$$\begin{aligned} \sum_{e \in \mathbb{B}} \langle A_e^2 \rangle &\lesssim \sum_{e \in \mathbb{B}} \langle g_T(z)^2 (|\nabla_i \psi_T(z)|^2 + T^{-2} |\psi_T(z)|^2 \\ &\quad + \nu_d(T)^2 (1 + |\phi_T(z)|^2 + |\phi_T(z')|^2)) \rangle \\ &\lesssim \left( \sum_{z \in \mathbb{Z}^d} g_T(z)^2 \right) (\langle |\nabla \psi_T|^2 \rangle + T^{-2} \langle \psi_T^2 \rangle + \nu_d(T)^2 (1 + \langle \phi_T^2 \rangle)) \\ &\lesssim \nu_d(T) (\langle |\nabla \psi_T|^2 \rangle + T^{-2} \langle \psi_T^2 \rangle + \nu_d(T)^2 (1 + \langle \phi_T^2 \rangle)) \end{aligned}$$

by stationarity of  $\phi_T$ ,  $\psi_T$  and  $\nabla \psi_T$ . This is a nonlinear estimate since  $\langle \psi_T^2 \rangle$  and  $\langle |\nabla \psi_T|^2 \rangle$  appear in the right-hand side whereas we want to estimate  $\langle \psi_T^2 \rangle$ . We then appeal to the elementary a priori estimate

$$\langle |\nabla \psi_T|^2 \rangle \lesssim \langle \phi_T^2 \rangle^{1/2} \langle \psi_T^2 \rangle^{1/2},$$

which we obtain by testing (3.22) with the test solution  $\psi_T$ , integrating by parts, using the bounds on  $A$  and Cauchy–Schwarz’s inequality. Using, in addition, Young’s inequality, the estimate turns into

$$\sum_{e \in \mathbb{B}} \langle A_e^2 \rangle - \frac{1}{C} \langle \psi_T^2 \rangle \lesssim C \nu_d(T)^2 \langle \phi_T^2 \rangle + \nu_d(T)^3 (1 + \langle \phi_T^2 \rangle)$$

for all  $C > 0$  and  $T$  large enough.

Combined with (3.39) for  $q = 2$  and the definition of  $\nu_d(T)$ , this turns into, for all  $C > 0$ ,

$$(3.52) \quad \sum_{e \in \mathbb{B}} \langle A_e^2 \rangle - \frac{1}{C} \langle \psi_T^2 \rangle \lesssim C \begin{cases} T^{3/2}, & \text{for } d = 3, \\ \ln^3 T, & \text{for } d = 4, \\ 1, & \text{for } d > 4. \end{cases}$$

We now turn to the term associated with  $B_e$ , which we split into two terms  $B_e = B_{e,1} + B_{e,2}$ , where

$$\begin{aligned} B_{e,1} &= (1 + |\phi_T(z)| + |\phi_T(z')|) T^{-1} \int_{\mathbb{Z}^d} g_T(y) g_T(y - z) \, dy, \\ B_{e,2} &= (1 + |\phi_T(z)| + |\phi_T(z')|) \int_{\mathbb{Z}^d} g_T(y) |\nabla_{2,i} G_T(y, z)| \, dy. \end{aligned}$$

In particular, we shall prove that

$$(3.53) \quad \sum_{e \in \mathbb{B}} \langle B_e \rangle^2 \lesssim \sum_{e \in \mathbb{B}} \langle B_{e,1}^2 \rangle + \sum_{e \in \mathbb{B}} \langle B_{2,e}^2 \rangle \lesssim \begin{cases} T^{3/2}, & \text{for } d = 3, \\ T, & \text{for } d = 4, \\ \sqrt{T}, & \text{for } d = 5, \\ \ln T, & \text{for } d = 6, \\ 1, & \text{for } d > 6. \end{cases}$$

We start with the sum of  $B_{e,1}^2$  on  $\mathbb{B}$ . Since  $g_T$  is deterministic and  $\phi_T$  is stationary,

$$\begin{aligned} \sum_{e \in \mathbb{B}} \langle B_{e,1}^2 \rangle &\lesssim (1 + \langle \phi_T^2 \rangle^{1/2}) \\ &\times \int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} T^{-1} g_T(y) g_T(y') g_T(y-z) g_T(y'-z) dy dy' dz. \end{aligned}$$

Using (3.39) with  $q = 2$  and the definitions (3.28) and (3.27) of  $g_T$  to estimate the integral, we conclude that the first term of the left-hand side of (3.53) is controlled by the right-hand side of (3.53). A formal argument to estimate the triple integral is as follows. By the exponential decay of  $g_T$ , it is enough to integrate on the set  $|y|, |y'| \lesssim \sqrt{T}$  and  $|y-z| + |y'-z| \lesssim \sqrt{T}$ , and the integral essentially behaves as the integral on the ball of radius  $\sqrt{T}$  in  $\mathbb{Z}^{3d}$  of  $T^{-1}(1+|x|)^{4(2-d)}$ , whence the bounds

$$\sum_{e \in \mathbb{B}} \langle B_{e,1}^2 \rangle \lesssim \begin{cases} T^{3/2}, & \text{for } d = 3, \\ T, & \text{for } d = 4, \\ \sqrt{T}, & \text{for } d = 5, \\ 1, & \text{for } d = 6, \\ T^{-1/2}, & \text{for } d = 7, \\ T^{-1} \ln T, & \text{for } d = 8, \\ T^{-1}, & \text{for } d > 8. \end{cases}$$

To rigorously prove that the right-hand side of (3.53) is an upper bound for  $\sum_{e \in \mathbb{B}} \langle B_{e,1}^2 \rangle$ , we may simply note that for  $d > 2$ , if we define  $h_T(z) := \sqrt{T}^{-1} \times g_T(z)$ , then for all  $z \in \mathbb{Z}^d$ ,

$$h_T(z) \lesssim (1 + |z|)^{1-d} \exp\left(-c \frac{|z|}{\sqrt{T}}\right),$$

and  $g_T$  and  $h_T$  satisfy the assumptions of Lemma 3.9, which yields the desired upper bound.

We turn to the sum of  $B_{e,2}^2$  on  $\mathbb{B}$ :

$$\begin{aligned} \sum_{e \in \mathbb{B}} \langle B_{e,2}^2 \rangle &\lesssim \left\langle \sum_{i=1}^d \int_{\mathbb{Z}^d} (1 + |\phi_T(z)| + |\phi_T(z + \mathbf{e}_i)|) \right. \\ &\quad \times \int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} g_T(y) g_T(y') |\nabla_{2,i} G_T(y, z)| \\ &\quad \left. \times |\nabla_{2,i} G_T(y', z)| dy dy' dz \right\rangle. \end{aligned}$$

Since  $g_T$  is deterministic, one can take it out of the expectation. We then choose  $p > 2$  such that the higher integrability result of Lemma 3.7 applies, and use

Hölder's inequality in probability with exponents  $(p/(p-2), p, p)$ . By stationarity of  $\phi_T$  and of  $G_T$  [in the form of (3.15)], this estimate turns into

$$\begin{aligned} \sum_{e \in \mathbb{B}} \langle B_{e,2}^2 \rangle &\lesssim \langle |\phi_T|^{p/(p-2)} \rangle^{(p-2)/p} \\ &\quad \times \int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} g_T(y) g_T(y') \\ &\quad \times \langle |\nabla_1 G_T(y-z, 0)|^p \rangle^{1/p} \\ &\quad \times \langle |\nabla_1 G_T(y'-z, 0)|^p \rangle^{1/p} dy dy' dz. \end{aligned}$$

We then introduce the notation  $h_T(x) := \langle |\nabla_1 G_T(x, 0)|^p \rangle^{1/p}$ . By Lemma 3.7 and Corollary 3.8, we have for all  $R \lesssim 1$ ,

$$\int_{|z| \leq R} h_T(x)^2 dx \lesssim 1$$

and for all  $R \lesssim 1$  and  $j \in \mathbb{N}$ ,

$$\int_{2^j R \leq |z| < 2^{j+1} R} h_T(x)^2 dx \leq \left( \int_{2^j R \leq |z| < 2^{j+1} R} h_T(x)^p dx \right)^{2/p} \lesssim (2^j R)^{2-d}.$$

Hence,  $g_T$  and  $h_T$  satisfy the assumptions of Lemma 3.9, and  $\sum_{e \in \mathbb{B}} \langle B_{e,2}^2 \rangle$  is bounded by the right-hand side of (3.53).

We are in position to estimate  $\text{var}[\psi_T^2]$ . Choosing  $C$  large enough in (3.52) to absorb the term  $\frac{1}{C} \text{var}[\psi_T^2]$  in the left-hand side of (3.51), and using (3.53) we obtain the estimate

$$(3.54) \quad \text{var}[\psi_T^2] \lesssim \begin{cases} T^{3/2}, & \text{for } d = 3, \\ T, & \text{for } d = 4, \\ \sqrt{T}, & \text{for } d = 5, \\ \ln T, & \text{for } d = 6, \\ 1, & \text{for } d > 6. \end{cases}$$

We may conclude the proof. Estimate (3.24) proved in step 3 yields the desired spectral exponent for  $d = 2$ , whereas the combination of (3.54) with (3.20) and (3.23) yields the desired spectral exponents for  $d > 2$ .  $\square$

REMARK 3.10. The structure of the proof can be summarized as follows:

- (a) The starting point is the optimal estimates of  $\langle \phi_T^2 \rangle$  (up to logarithmic correction for  $d = 2$ ).
- (b) The variance estimate applied to  $\psi_T$  and combined with elliptic theory shows there exists a map  $F$  such that

$$\langle \psi_T^2 \rangle \leq F(T, \langle |\nabla \psi_T|^2 \rangle, \langle \psi_T^2 \rangle, \langle \phi_T^2 \rangle).$$

- (c) By an a priori estimate,  $\langle |\nabla \psi_T|^2 \rangle \lesssim \langle \psi_T^2 \rangle^{1/2} \langle \phi_T^2 \rangle^{1/2}$ .  
 (d) Combined with Young's inequality and (c), (b) turns into

$$\langle \psi_T^2 \rangle \leq \tilde{F}(T, \langle \phi_T^2 \rangle)$$

for some map  $\tilde{F}$ , and yields the claim.

In view of this, a possible strategy to prove optimal scalings of the spectral exponents in any dimension would be to proceed by induction. Set  $\phi_{1,T} \equiv \phi_T$ , and for all  $k \geq 1$  define  $\phi_{k,T}$  as the unique weak solution to

$$T^{-1} \phi_{k+1,T}(x) - \frac{1}{p_\omega(x)} \nabla^* \cdot A(x) \nabla \psi_{k+1,T}(x) = \psi_{k,T}(x),$$

and apply the strategy described above to obtain optimal bounds on  $\langle \phi_{k+1,T}^2 \rangle$  assuming optimal bounds on  $\langle \phi_{k,T}^2 \rangle$  (which would yield optimal spectral exponents up to dimension  $4k - 2$ —with a logarithmic correction in dimension  $4k - 2$ ). The main difficulty is to work out a suitable map  $F_k$  in step (b).

**4. The random fluctuations.** In this section, we show that the computable quantity  $\hat{A}_n(t)$  defined in (1.6) is a good approximation of  $\sigma_t^2$  in the sense that its random fluctuations are small as soon as  $n/t^2$  is large. We write  $\mathbb{N}^*$  for  $\mathbb{N} \setminus \{0\}$ .

**THEOREM 4.1.** *There exists  $c > 0$  such that, for any  $n \in \mathbb{N}^*$ ,  $\varepsilon > 0$  and  $t$  large enough,*

$$\mathbb{P}_0^\otimes[|\hat{A}_n(t) - \sigma_t^2| \geq \varepsilon/t] \leq \exp\left(-\frac{n\varepsilon^2}{ct^2}\right).$$

Note that  $\sigma_t^2$  is the mean value of  $\hat{A}_n(t)$ , and moreover,  $\hat{A}_n(t)$  consists of a sum of i.i.d. random variables. We will thus obtain Theorem 4.1 by using classical techniques from large deviation theory. The important point is that the i.i.d. random variables under consideration are uniformly exponentially integrable. To see this, we use a sharp upper bound on the transition probabilities of the random walk recalled in the following theorem. We refer the reader to [12] or [24], Theorem 14.12, for a proof.

**THEOREM 4.2.** *There exists a constant  $c_1 > 0$  such that, for any environment  $\omega$  with conductances in  $[\alpha, \beta]$ , any  $t \in \mathbb{N}^*$  and  $x \in \mathbb{Z}^d$ ,*

$$\mathbf{P}_0^\omega[Y(t) = x] \leq \frac{c_1}{t^{d/2}} \exp\left(-\frac{|x|^2}{c_1 t}\right).$$

From Theorem 4.2 we deduce the following result.

COROLLARY 4.3. *Let  $c_1$  be given by Theorem 4.2. For all  $\lambda < 1/c_1$ , one has*

$$\sup_{t \in \mathbb{N}^*} \tilde{\mathbb{E}}_0 \left[ \exp \left( \lambda \frac{|Y(t)|^2}{t} \right) \right] < +\infty.$$

PROOF. Let  $\delta = 1/c_1 - \lambda$ . By Theorem 4.2,

$$\mathbf{E}_0^\omega [e^{\lambda |Y(t)|^2/t}] \leq c_1 t^{-d/2} \sum_{x \in \mathbb{Z}^d} e^{-\delta |x|^2/t}.$$

If the sum ranges over all  $x \in (\mathbb{N}^*)^d$ , it is easy to bound it by a convergent integral,

$$t^{-d/2} \sum_{x \in (\mathbb{N}^*)^d} e^{-\delta |x|^2/t} \leq t^{-d/2} \int_{\mathbb{R}_+^d} e^{-\delta |x|^2/t} dx = \int_{\mathbb{R}_+^d} e^{-\delta |x|^2} dx.$$

By symmetry, the estimate carries over to the sum over all  $x \in (\mathbb{Z}^*)^d$ . The same argument applies for the sum over all  $x = (x_1, \dots, x_d)$  having exactly one component equal to 0, and so on.  $\square$

The following lemma shows that the log-Laplace transform of  $\frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2$  is bounded by a parabola in a neighborhood of 0, uniformly over  $t$ .

LEMMA 4.4. *There exist  $\lambda_1 > 0$  and  $c_2$  such that, for any  $\lambda < \lambda_1$  and any  $t \in \mathbb{N}^*$ ,*

$$\ln \tilde{\mathbb{E}}_0 \left[ \exp \left( \lambda \left( \frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2 \right) \right) \right] \leq c_2 \lambda^2.$$

PROOF. It is sufficient to prove that there exists  $c_3$  such that, for any  $\lambda$  small enough and any  $t$ ,

$$\tilde{\mathbb{E}}_0 \left[ \exp \left( \lambda \left( \frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2 \right) \right) \right] \leq 1 + c_3 \lambda^2.$$

We use the series expansion of the exponential to rewrite this expectation as

$$\sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \tilde{\mathbb{E}}_0 \left[ \left( \frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2 \right)^k \right].$$

The term corresponding to  $k = 0$  is equal to 1, whereas the term for  $k = 1$  vanishes. The remaining sum, for  $k$  ranging from 2 to infinity, can be controlled using Corollary 4.3 combined with the bound

$$\tilde{\mathbb{E}}_0 \left| \frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2 \right|^k \leq 2 \tilde{\mathbb{E}}_0 \left[ \frac{(\xi \cdot Y(t))^{2k}}{t^k} \right],$$

which follows from the definition of  $\sigma_t^2$  and Jensen's inequality.  $\square$

We are now in position to prove Theorem 4.1.

PROOF OF THEOREM 4.1. From the definition of  $\tilde{\mathbb{P}}$  given in (1.1), we can write

$$\mathbb{P}_0^\otimes[\hat{A}_n(t) - \sigma_t^2 \geq \varepsilon/t] = \tilde{\mathbb{P}}_0^\otimes\left[\frac{(\xi \cdot Y^{(1)}(t))^2 + \dots + (\xi \cdot Y^{(n)}(t))^2}{nt} - \sigma_t^2 \geq \varepsilon/t\right].$$

Let  $\lambda > 0$ . We bound the latter probability using Chebyshev's inequality,

$$\begin{aligned} \mathbb{P}_0^\otimes[\hat{A}_n(t) - \sigma_t^2 \geq \varepsilon/t] &\leq \tilde{\mathbb{E}}_0^\otimes\left[\exp\left(\lambda\left(\frac{(\xi \cdot Y^{(1)}(t))^2 + \dots + (\xi \cdot Y^{(n)}(t))^2}{t} - n\sigma_t^2\right)\right)\right] \\ (4.1) \quad &\times \exp\left(-\frac{n\lambda\varepsilon}{t}\right) \\ &\leq \tilde{\mathbb{E}}_0^\otimes\left[\exp\left(\lambda\left(\frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2\right)\right)\right]^n \exp\left(-\frac{n\lambda\varepsilon}{t}\right). \end{aligned}$$

By Lemma 4.4, the right-hand side of (4.1) is bounded by

$$\exp\left(n\left(c_2\lambda^2 - \frac{\lambda\varepsilon}{t}\right)\right)$$

for all  $\lambda$  small enough. Choosing  $\lambda = \varepsilon/2c_2t$  (which is small enough for  $t$  large enough), we obtain

$$(4.2) \quad \mathbb{P}_0^\otimes[\hat{A}_n(t) - \sigma_t^2 \geq \varepsilon/t] \leq \exp\left(-\frac{n\varepsilon^2}{4c_2t^2}\right).$$

The probability of the symmetric event

$$\mathbb{P}_0^\otimes[\sigma_t^2 - \hat{A}_n(t) \geq 2\varepsilon/t]$$

can be handled the same way, so the proof is complete.  $\square$

**5. Central limit theorem.** In this short section, we complete the analysis by showing that the quantity  $\sqrt{n(t)}(\hat{A}_{n(t)}(t) - \sigma_t^2)$  satisfies a central limit theorem.

PROPOSITION 5.1. *Let  $(n(t))_{t \in \mathbb{N}^*}$  be any sequence tending to infinity with  $t$ . Under the measure  $\mathbb{P}_0^\otimes$  and as  $t$  tends to infinity, the random variable*

$$\sqrt{n(t)}(\hat{A}_{n(t)}(t) - \sigma_t^2)$$

*converges in distribution to a Gaussian random variable of variance*

$$v = \left(3 \frac{\mathbb{E}[p^2]}{\mathbb{E}[p]^2} - 1\right) \sigma^4.$$

PROOF. Let us define

$$V(t) = \frac{p(\omega)(\xi \cdot Y_t)^2}{t\mathbb{E}[p]} - \sigma_t^2$$

and

$$V^{(k)}(t) = \frac{p(\omega^{(k)})(\xi \cdot Y_t^{(k)})^2}{t\mathbb{E}[p]} - \sigma_t^2,$$

so that

$$\hat{A}_{n(t)}(t) - \sigma_t^2 = \frac{1}{n(t)} \sum_{k=1}^{n(t)} V^{(k)}(t).$$

Let also  $v_t = \mathbb{E}_0[V(t)^2]$ . Note that for any  $t$ ,  $(V^{(k)}(t))_{k \in \mathbb{N}}$  are i.i.d. centered random variables under  $\mathbb{P}_0^\otimes$ . From the Lindeberg–Feller theorem (see, e.g., [6], Theorem 2.4.5), we know that in order to show

$$\frac{\sum_{k=1}^{n(t)} V^{(k)}(t)}{\sqrt{n(t)}} \xrightarrow[t \rightarrow +\infty]{(\text{distr.})} \text{Gaussian}(0, v),$$

it suffices to check that

$$(5.1) \quad v_t \xrightarrow[t \rightarrow +\infty]{} v$$

and that for any  $\varepsilon > 0$ ,

$$(5.2) \quad \mathbb{E}_0[V(t)^2 \mathbf{1}_{\{V(t) \geq \varepsilon \sqrt{n(t)}\}}] \xrightarrow[t \rightarrow +\infty]{} 0.$$

We learn from [23] that for almost every environment and as  $t$  tends to infinity,  $\xi \cdot Y_t / \sqrt{t}$  converges in distribution under  $\mathbb{P}_0^\omega$  to a Gaussian random variable of variance  $\sigma^2$ , that we write  $\sigma G$ , where  $G$  is a standard Gaussian random variable. In order to justify that for almost every environment,  $\xi \cdot Y_t / \sqrt{t}$  converges in distribution to  $(\sigma G)^2$ , we need some uniform integrability property, since the square function is unbounded. But this uniform integrability is a direct consequence of Theorem 4.2. Hence, under  $\mathbb{P}_0$  and as  $t$  tends to infinity, the random variable

$$(5.3) \quad \frac{p(\omega)(\xi \cdot Y_t)^2}{t\mathbb{E}[p]}$$

converges in distribution to

$$\frac{p(\omega)}{\mathbb{E}[p]} (\sigma G)^2,$$

where  $\omega$  follows the distribution  $\mathbb{P}$ , and is independent of  $G$ . For the foregoing reason, the squares of the random variables in (5.3) are uniformly integrable as  $t$  varies. Since we know, moreover, that  $\lim_{t \rightarrow +\infty} \sigma_t^2 = \sigma^2$ , we thus obtain

$$\lim_{t \rightarrow +\infty} v_t = \mathbb{E} \left[ \left( \frac{p(\omega)}{\mathbb{E}[p]} (\sigma G)^2 - \sigma^2 \right)^2 \right].$$



We obtain (5.1) by expanding this expectation, recalling that the fourth moment of  $G$  is equal to 3.

Similarly, Theorem 4.2 gives us sufficient control to guarantee that (5.2) holds, so the proof is complete.  $\square$

**6. Numerical validation and comments.** In this section, we illustrate on a simple two-dimensional example the sharpness of the estimates of the systematic error and of the random fluctuations obtained in Theorems 3.1 and 4.1.

In the numerical tests, each conductivity of  $\mathbb{B}$  takes the value  $\alpha = 1$  or  $\beta = 4$  with probability  $1/2$ . In this simple case, the homogenized matrix is given by Dykhne's formula, namely,  $A_{\text{hom}} = \sqrt{\alpha\beta}\text{Id} = 2\text{Id}$  (see, e.g., [8], Appendix A). For the simulation of the random walk, we generate (and store) the environment along the trajectory of the walk. In particular, this requires us to store up to a constant times  $t$  data. In terms of computational cost, the expensive part of the computations is the generation of the randomness. In particular, to compute one realization of  $\hat{A}_{t^2}(t)$  costs approximately the generation of  $t^2 \times 4t = 4t^3$  random variables. A natural advantage of the method is its full scalability; the  $t^2$  random walks used to calculate a realization of  $\hat{A}_{t^2}(t)$  are completely independent.

We first test the estimate of the systematic error: up to a logarithmic correction, the convergence is proved to be linear in time. In view of Theorem 4.1, typical fluctuations of  $t(\hat{A}_{n(t)}(t) - \sigma_t^2)$  are of order no greater than  $t/\sqrt{n(t)}$ , and thus become negligible when compared with the systematic error as soon as the number  $n(t)$  of realizations satisfies  $n(t) \gg t^2$ . We display in Table 1 an estimate of the systematic error  $t \mapsto |A_{\text{hom}} - \frac{\mathbb{E}[p]}{2} \hat{A}_{n(t)}(t)|$  obtained with  $n(t) = K(t)t^2$  realizations. The systematic error is plotted on Figure 1 in function of the time in logarithmic scale (crosses). It matches quite well the function  $f : t \mapsto Ct^{-1} \ln t$  [for  $C > 0$  chosen so that  $f(1280) = |A_{\text{hom}} - \frac{\mathbb{E}[p]}{2} \hat{A}_{n(640)}(1280)|$ ] which is plotted as a solid line. This is consistent with Theorem 3.1 and supports the fact that the spectral exponents are  $(2, 0)$  for  $d = 2$  [and not  $(2, -q)$  for some  $q > 0$ ].

We now turn to the random fluctuations of  $\hat{A}_{n(t)}(t)$ . Theorem 4.1 gives us a Gaussian upper bound on the tail of the fluctuations of  $t(\hat{A}_n(t) - \sigma_t^2)$ , measured in units of  $t/\sqrt{n}$ , whereas Proposition 5.1 proves the corresponding central limit

TABLE 1  
Systematic error  $|A_{\text{hom}} - \frac{\mathbb{E}[p]}{2} \hat{A}_{n(t)}(t)|$  in function of the final time  $t$  for  $K(t)t^2$  realizations

$t$	10	20	40	80	160	320	640	1280
$K(t)$	$10^4$	$10^4$	$10^4$	$10^4$	$10^4$	$10^4$	$4.010^3$	$10^3$
Systematic error	9.27E-02	5.31E-02	3.09E-02	1.71E-02	9.58E-03	5.45E-03	2.93E-03	1.66E-03

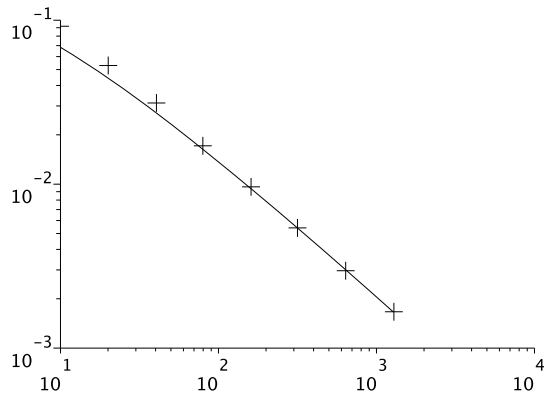


FIG. 1. Systematic error  $|A_{\text{hom}} - \frac{\mathbb{E}[p]}{2} \hat{A}_{n(t)}(t)|$  in function of the final time  $t$  for  $n(t) = K(t)t^2$  realizations (logarithmic scale).

theorem, that is, convergence in distribution of  $t(\hat{A}_{t^2}(t) - \sigma_t^2)$  to a Gaussian random variable. The Figures 2–7 display the histograms of  $t \frac{\mathbb{E}[p]}{2} (\hat{A}_{t^2}(t) - \sigma_t^2)$  for  $t = 10, 20, 40$  and  $80$  [with 10,000 realizations of  $\hat{A}_{t^2}(t)$  in each case, and  $\sigma_t^2$  approximated by the empirical mean of  $\hat{A}_{t^2}(t)$  over the 10,000 realizations]. As expected, they look Gaussian. In addition, Proposition 5.1 also gives the limiting variance. Table 2 displays the limiting variance  $(\mathbb{E}[p]/2)^2 v = 9.08$  and the empirical variances for  $t = 10, 20, 40, 80, 160$  and  $320$ , which are in good agreement.

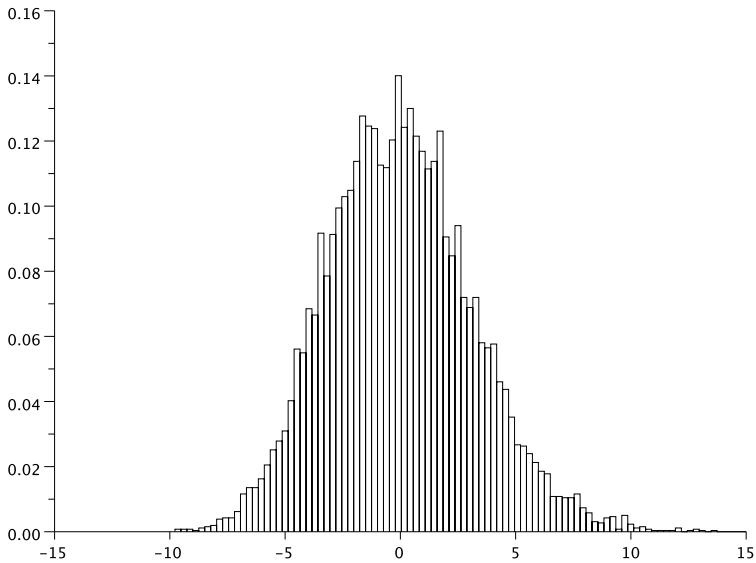


FIG. 2. Histogram of the re-scaled fluctuations for  $t = 10$ .

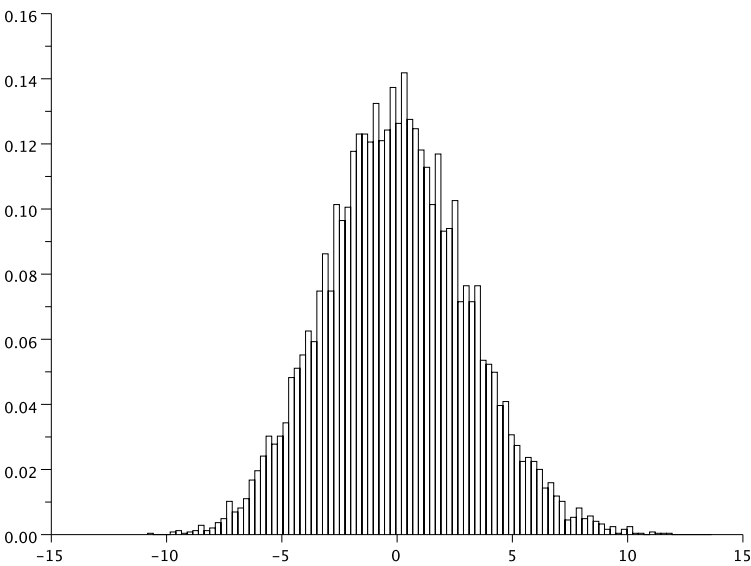


FIG. 3. *Histogram of the re-scaled fluctuations for  $t = 20$ .*

To conclude this article, let us quickly compare the Monte Carlo approach under consideration here to other approaches to approximate homogenized coefficients. Another possibility to approximate effective coefficients is to directly solve the

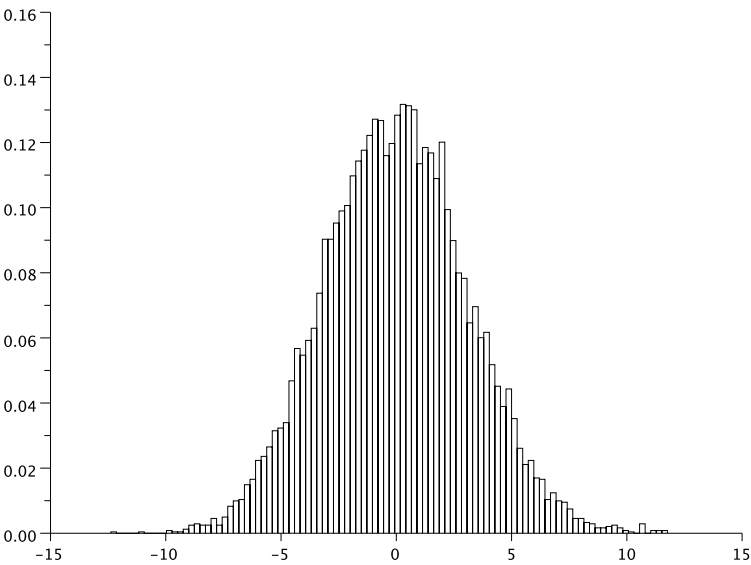


FIG. 4. *Histogram of the re-scaled fluctuations for  $t = 40$ .*

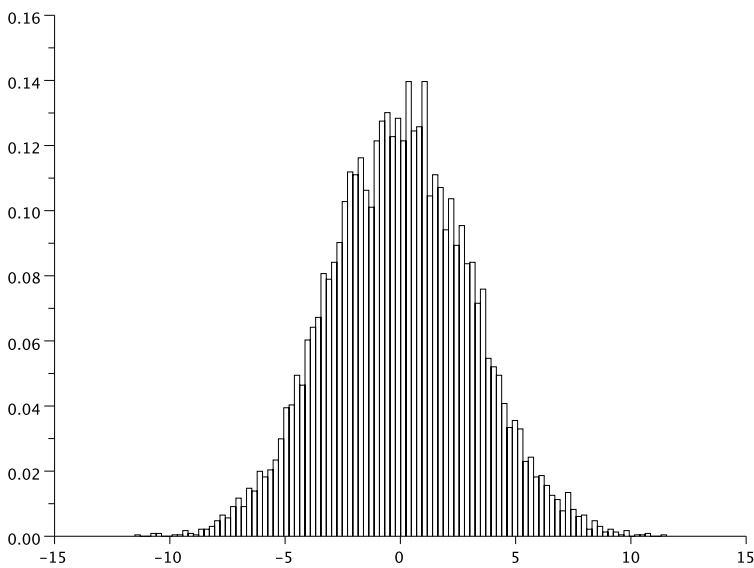


FIG. 5. Histogram of the re-scaled fluctuations for  $t = 80$ .

so-called corrector equation. In this approach, a first step toward the derivation of error estimates is a quantification of the qualitative results proved by Künnemann [17] (and inspired by Papanicolaou and Varadhan’s treatment of the continuous

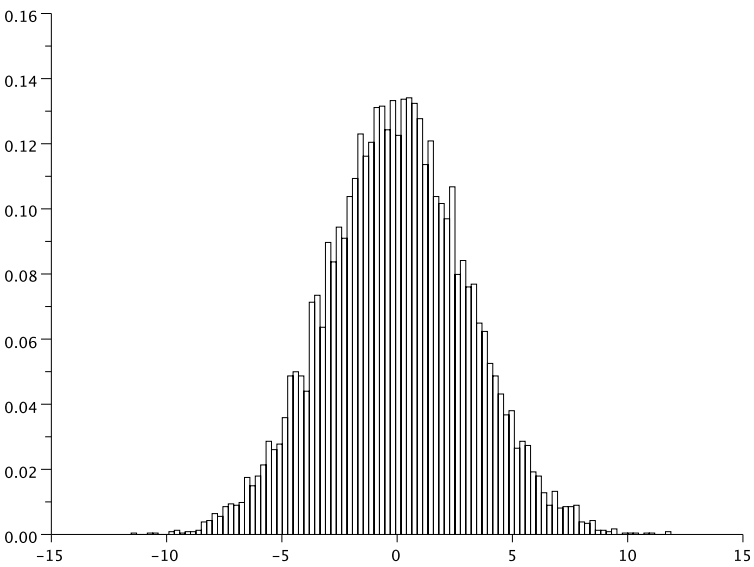


FIG. 6. Histogram of the re-scaled fluctuations for  $t = 160$ .

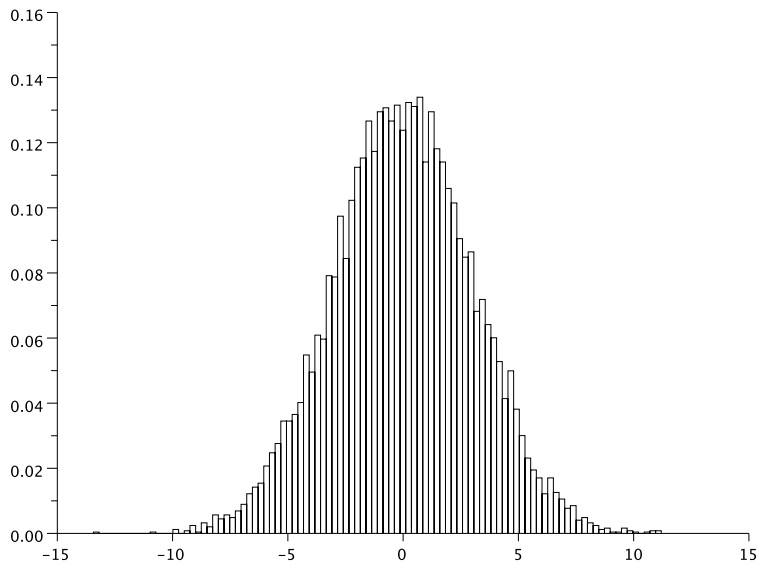


FIG. 7. Histogram of the re-scaled fluctuations for  $t = 320$ .

case [22]) and Kozlov [16]. In the stochastic case, such an equation is posed on the whole  $\mathbb{Z}^d$ , and we need to localize it on a bounded domain, say the hypercube  $Q_R$  of side  $R > 0$ . As shown in a series of papers by Otto and the first author [10, 11], and the first author [8], there are three contributions to the  $\mathbb{L}^2$ -error in probability between the true homogenized coefficients and its approximation. The dominant error in small dimensions takes the form of a variance: it measures the fact that the approximation of the homogenized coefficients by the average of the energy density of the corrector on a box  $Q_R$  fluctuates. This error decays at the rate of the central limit theorem  $R^{-d}$  in any dimension (with a logarithmic correction for  $d = 2$ ). The second error is a systematic error: it is due to the fact that we have modified the corrector equation by adding a zero-order term of strength  $T^{-1} > 0$  (as is standard in the analysis of the well-posedness of the corrector equation). The scaling of this error depends on the dimension and saturates at dimension 4. It is of higher order than the random error up to dimension 8. The last error is due to the use of boundary conditions on the bounded domain  $Q_R$ . Provided there is a buffer

TABLE 2  
Empirical variance of  $\frac{\mathbb{E}[p]}{2} \mathbf{1}_t(\hat{A}_{t,2}(t) - \sigma_t^2)$  and limiting variance from Proposition 5.1

$t$	10	20	40	80	160	320	$\infty$
Variance	9.86	9.46	9.49	9.46	9.36	9.06	9.08

region, this error is exponentially small in the distance to the buffer zone measured in units of  $\sqrt{T}$ .

This approach has two main drawbacks. First, the numerical method only converges at the central limit theorem (CLT) scaling in terms of  $R$  up to dimension 8, which is somehow disappointing from a conceptual point of view (although this is already fine in practice). Second, although the size of the buffer zone is roughly independent of the dimension, its cost with respect to the central limit theorem scaling dramatically increases with the dimension (recall that in dimension  $d$ , the CLT scaling is  $R^{-d}$ , so that in high dimension, we may consider smaller  $R$  for a given precision, whereas the use of boundary conditions requires  $R \gg \sqrt{T}$  in any dimension). Based on ideas of the second author in [20], we have taken advantage of the spectral representation of the homogenized coefficients (originally introduced by Papanicolaou and Varadhan to prove their qualitative homogenization result) in order to devise and analyze new approximation formulas for the homogenized coefficients in [9]. In particular, this has allowed us to get rid of the restriction on dimension, and exhibit refinements of the numerical method of [8] which converge at the central limit theorem scaling in any dimension (thus avoiding the first mentioned drawback). Unfortunately, the second drawback is inherent to the type of method used: if the corrector equation has to be solved on a bounded domain  $Q_R$ , boundary conditions need to be imposed on the boundary  $\partial Q_R$ . Since their values are actually also part of the problem, a buffer zone seems mandatory—with the notable exception of the periodization method, whose analysis is yet still unclear to us, especially when spatial correlations are introduced in the coefficients.

In this paper we have analyzed a method which does not suffer from the drawbacks mentioned above: the random walk in random environment approach. In particular, following [21] we have obtained an approximation of the homogenized coefficients by the numerical simulation of a random walk up to some large time. Compared to the deterministic approach based on the approximate corrector equation, the advantage of the present approach is that its convergence rate and computational costs are dimension-independent. In addition, the environment only needs to be generated along the trajectory of the random walker, so that much less information has to be stored during the calculation. This may be quite an important feature of the Monte Carlo method in view of the discussion of [8], Section 4.3.

A more thorough comparison of these numerical approaches in two and three dimensions, for correlated and uncorrelated examples, will be the object of a forthcoming work [7].

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