

## CONVOLUTION EQUIVALENT LÉVY PROCESSES AND FIRST PASSAGE TIMES

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We investigate the behavior of Lévy processes with convolution equivalent Lévy measures, up to the time of first passage over a high level  $u$ . Such problems arise naturally in the context of insurance risk where  $u$  is the initial reserve. We obtain a precise asymptotic estimate on the probability of first passage occurring by time  $T$ . This result is then used to study the process conditioned on first passage by time  $T$ . The existence of a limiting process as  $u \rightarrow \infty$  is demonstrated, which leads to precise estimates for the probability of other events relating to first passage, such as the overshoot. A discussion of these results, as they relate to insurance risk, is also given.

**1. Introduction.** Let  $X = (X_t)_{t \geq 0}$ , be a Lévy process with characteristics  $(\gamma, \sigma^2, \Pi_X)$ . Thus the characteristic function of  $X$  is given by the Lévy–Khintchine representation,  $Ee^{i\theta X_t} = e^{t\Psi_X(\theta)}$ , where

$$\Psi_X(\theta) = i\theta\gamma - \sigma^2\theta^2/2 + \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x \mathbf{1}_{\{|x| < 1\}}) \Pi_X(dx) \quad \text{for } \theta \in \mathbb{R}.$$

Historically, a number of different types of Lévy processes have arisen in the context of stochastic modeling depending on the phenomenon under investigation. This has motivated the detailed study of several different classes of processes. In this paper we will investigate one such class, those with convolution equivalent Lévy measure. This class has recently been proposed as a model for insurance risk, although its study certainly predates that. Convolution equivalent distributions were first introduced by Chistiakov [8] and later by Chover, Ney and Wainger [9]. Their properties have been investigated by several authors including [10, 14, 21, 25] and [26] where background and further information on this class of distributions can be found. We will restrict ourselves to the nonlattice case, with the understanding that the alternative can be handled by obvious modifications. A distribution  $F$  on  $[0, \infty)$  with tail  $\bar{F}$  belongs to the class  $\mathcal{L}^{(\alpha)}$ ,  $\alpha \geq 0$ , if  $\bar{F}(u) > 0$  for all  $u > 0$  and

$$\lim_{u \rightarrow \infty} \frac{\bar{F}(u+x)}{\bar{F}(u)} = e^{-\alpha x} \quad \text{for } x \in (-\infty, \infty).$$

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Received July 2011; revised May 2012.

*MSC2010 subject classifications.* Primary 60G51, 60F17; secondary 91B30, 62P05.

*Key words and phrases.* Lévy process, convolution equivalence, first passage time, insurance risk, probability of ruin in finite time.

$F$  belongs to the class  $\mathcal{S}^{(\alpha)}$  if, in addition,

$$(1.1) \quad \lim_{u \rightarrow \infty} \frac{\overline{F^{2*}}(u)}{\overline{F}(u)} \quad \text{exists and is finite,}$$

where  $F^{2*} = F * F$ . When  $F \in \mathcal{S}^{(\alpha)}$ ,

$$(1.2) \quad \delta_\alpha^F := \int_{[0, \infty)} e^{\alpha x} F(dx) < \infty,$$

and the limit in (1.1) is given by  $2\delta_\alpha^F$ . Distributions in  $\mathcal{S}^{(0)}$  are called subexponential, and those in  $\mathcal{S}^{(\alpha)}$ ,  $\alpha > 0$ , are called convolution equivalent with index  $\alpha > 0$ . The class  $\mathcal{S}^{(\alpha)}$  has several nice properties, including closure under tail equivalence, that is, if  $F \in \mathcal{S}^{(\alpha)}$  and  $G$  is a distribution on  $[0, \infty)$  for which

$$(1.3) \quad \lim_{u \rightarrow \infty} \frac{\overline{G}(u)}{\overline{F}(u)} = c \quad \text{for some } c \in (0, \infty),$$

then  $G \in \mathcal{S}^{(\alpha)}$ . This particular property also holds, trivially, for the class  $\mathcal{L}^{(\alpha)}$ .

The right tail of any Lévy measure, which is nonzero on an interval  $[x_0, \infty)$ ,  $x_0 > 0$ , may be taken as the tail of a distribution function on  $[x_0, \infty)$ , after renormalization. With this convention, we say that the Lévy measure (or its tail) is in  $\mathcal{S}^{(\alpha)}$ , respectively,  $\mathcal{L}^{(\alpha)}$ , if this is true of the corresponding renormalized tail. By closure under tail equivalence, this does not depend on the choice of  $x_0$ . A convolution equivalent Lévy process is one for which  $\overline{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$  for some  $\alpha > 0$ , where  $\Pi_X^+$  is the restriction of  $\Pi_X$  to  $(0, \infty)$  and, as above,  $\overline{\Pi}_X^+$  denotes its tail. Examples include, for appropriate choices of parameters, the CGMY, generalized inverse Gaussian (GIG) and generalized hyperbolic (GH) processes.

Let

$$(1.4) \quad \tau(u) = \inf\{t > 0 : X_t > u\}$$

denote the first passage time over level  $u$ . The behavior of

$$(1.5) \quad \lim_{u \rightarrow \infty} \frac{P(\tau(u) < T)}{\overline{\Pi}_X^+(u)}$$

has been investigated under various conditions on  $\Pi_X$ , for example, when  $\overline{\Pi}_X^+(u)$  is regularly varying (see Berman [3] and Marcus [24]), and more generally when  $\overline{\Pi}_X^+(u)$  is subexponential (see Rosiński and Samorodnitsky [27]). In the case of interest in this paper, when  $X$  is convolution equivalent, Braverman and Samorodnitsky [7] proved that the limit in (1.5) exists but were unable to identify its value. Later, Braverman [6] obtained a complicated description of the limit, which unfortunately lends little practical insight as to its actual value. Albin and Sundén [1] gave a much simpler proof of existence, but again their characterization of the limit is highly inexplicit. When  $T = \infty$ , Klüppelberg, Kyprianou and Maller [22] were

able to evaluate the limit in (1.5) under the additional assumption  $Ee^{\alpha X_1} < 1$ . As will become apparent in Section 4 (see Remark 4.3), when this condition fails the limit in (1.5) is infinite for  $T = \infty$ .

The assumption  $Ee^{\alpha X_1} < 1$  was introduced in [22] in the context of modeling insurance risk. Here  $u$  represents the initial reserve and  $X$  the excess in claims over premium. Ruin occurs when  $X$  exceeds  $u$ . Our first result, which evaluates the limit in (1.5), may thus be viewed in this context as providing a sharp asymptotic estimate for the probability of ruin in finite time.

**THEOREM 1.1.** *Assume  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ , then*

$$(1.6) \quad \lim_{u \rightarrow \infty} \frac{P(\tau(u) < T)}{\bar{\Pi}_X^+(u)} = \int_{[0, T)} e^{\psi(\alpha)t} Ee^{\alpha \bar{X}_{T-t}} dt,$$

where  $\bar{X}_t = \sup_{0 \leq s \leq t} X_s$  and  $\psi(\alpha) = \ln Ee^{\alpha X_1}$ .

The limit in (1.6) is finite, since  $Ee^{\alpha \bar{X}_T} < \infty$  for every  $T < \infty$  when  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$  (see Lemma 2.1). It yields a simple and transparent formula which allows further investigation of the limit as a function of  $T$ , as will be illustrated in Section 4. Formally, setting  $\alpha = 0$ , (1.6) reduces to the subexponential result of [27]. However, our interest here is in the convolution equivalent case, so throughout this paper, it will be tacitly assumed, without further mention, that  $\alpha > 0$ .

Building on work begun in [16] in the  $Ee^{\alpha X_1} < 1$  and  $T = \infty$  case, we investigate not only when, but how first passage occurs in finite time, that is, what do sample paths look like that result in first passage by time  $T$ ? Our main result is a functional limit theorem yielding an asymptotic description of the process conditioned on  $\tau(u) < T$  as  $u \rightarrow \infty$ . Roughly speaking, the conditioned process behaves like an Esscher transform  $Z$  of  $X$  up to independent time  $\tau$  when it jumps from  $Z_{\tau-}$  to a neighborhood of  $u$ . Let its position after the jump be  $u + W_0$ . If  $W_0 > 0$  the conditioned process then behaves like  $X$  started at  $u + W_0$ . If  $W_0 \leq 0$ , the conditioned process  $X - u$  behaves like  $X$  started at  $W_0$  and conditioned on  $\tau(0) < T - \tau$ . The precise descriptions of  $Z$ ,  $\tau$  and  $W_0$  are contained in (6.4), (6.5) and (6.8) and the functional limit theorem in Theorem 6.2. This result may be used to obtain precise asymptotic estimates for the probability of many other events relating to first passage. As one example, we derive the joint limiting distribution of the first passage time and the overshoot of  $X$  conditional on  $\tau(u) < T$  (see Theorems 7.1 and 7.2). It will be clear from this example how other limiting distributions relating to first passage may be found. Previous work in this area has been restricted to the  $T = \infty$  and  $Ee^{\alpha X_1} < 1$  case. Our results are the first that we are aware of that considers the finite time horizon problem and removes the condition that  $Ee^{\alpha X_1} < 1$ . The case  $Ee^{\alpha X_1} = 1$  is of particular interest, being the classical Cramér–Lundberg condition. This is discussed further in Sections 5 and 7.

We conclude the [Introduction](#) with a brief outline of the paper. Section 2 contains various notation and introduces two measures related to the description of the limiting process given above. Section 3 adapts a convergence result from [16] in the  $T = \infty$  case to the  $T < \infty$  case. Section 4 then contains the proof of (1.6). A further discussion of the meaning of (1.6) in the context of insurance risk is given in Section 5. Section 6 contains the functional limit theorem and Section 7 applies it to the overshoot. Finally, the [Appendix](#) justifies several formulas used in the paper relating to the measures introduced in Section 2.

**2. Notation.** We follow much of the notation laid out in [16]. This is briefly summarized in the next few paragraphs for the convenience of the reader. Let  $E = \mathbb{R} \cup \{\Delta\}$  where  $\Delta$  is a cemetery state. Define a metric  $d$  on  $E$  by

$$d(x, y) = \begin{cases} |x - y| \wedge 1, & x, y \in \mathbb{R}, \\ 1, & x \in \mathbb{R}, y = \Delta, \\ 0, & x = y = \Delta. \end{cases}$$

Thus  $\Delta$  is an isolated point and for  $x, y \in \mathbb{R}$ ,  $|x - y| \rightarrow 0$  if and only if  $d(x, y) \rightarrow 0$ . Let  $D$  be the Skorohod space of functions on  $[0, \infty)$ , taking values in the metric space  $E$ , and which are right-continuous with left limits. Let

$$\tau_z = \tau_z(w) = \inf\{t > 0 : w_t > z\}, \quad \tau_\Delta = \tau_\Delta(w) = \inf\{t > 0 : w_t = \Delta\}.$$

Thus, in the notation of (1.4),  $\tau(z) = \tau_z(X)$ . To avoid any possible confusion we reserve the notation  $\tau(z)$  exclusively for  $\tau_z(X)$ . When considering the passage time of a process other than  $X$ , say  $W$ , we will write  $\tau_z(W)$ .

For a given function  $w = (w_t)_{t \geq 0} \in D$ , and  $r \geq 0$ , let  $w_{[0,r]} = (w_{[0,r]}(t))_{t \geq 0} \in D$  denote the killed path

$$w_{[0,r]}(t) = \begin{cases} w_t, & 0 \leq t < r, \\ \Delta, & t \geq r. \end{cases}$$

Observe that for any  $t \geq 0$  and  $w \in D$

$$\tau_\Delta(w_{[0,t]}) = t \quad \text{if } \tau_\Delta(w) \geq t.$$

For  $x \in E$  let  $c^x \in D$  be the constant path  $c_t^x = x$  for all  $t \geq 0$ . If  $w, w' \in D$ , then  $w - w'$  denotes the path in  $D$  given by

$$(w - w')_t = \begin{cases} w_t - w'_t, & \text{if } t < \tau_\Delta(w) \wedge \tau_\Delta(w'), \\ \Delta, & \text{otherwise.} \end{cases}$$

It is convenient to assume that  $X$  is given as the coordinate process on  $D$ . The usual right-continuous completion of the filtration generated by the coordinate maps will be denoted by  $\{\mathcal{F}_t\}_{t \geq 0}$ .  $P_z$  denotes the probability measure induced on  $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$  by the Lévy process starting at  $z \in \mathbb{R}$ . We usually write just  $P$  for  $P_0$ . The shift operators  $\theta_t : D \rightarrow D$ ,  $t \geq 0$ , are defined by  $(\theta_t(w))_s = w(t + s)$ .

Let  $\mathcal{B}$  denote the Borel sets on  $\mathbb{R}$  and  $\mathcal{B}([0, \infty))$  the Borel sets on  $[0, \infty)$ . Let  $\mathcal{D} = D \otimes [0, \infty) \otimes (-\infty, \infty)$ , and for  $T \in (0, \infty)$ , set  $\mathcal{D}_T = D \otimes [0, T) \otimes (-\infty, \infty)$ . For  $K \in (-\infty, \infty]$  and  $x \in [0, \infty]$ , define measures  $\mu_K$  and  $\nu_x$  on  $\mathcal{F} \otimes \mathcal{B}([0, \infty)) \otimes \mathcal{B}$  by

$$\mu_K(dw, dt, d\phi) = I(\phi < K)e^{\alpha\phi} P(X_{[0,t]} \in dw, X_{t-} \in d\phi) dt$$

and

$$\nu_x(dw', dr, dz) = I(z > -x)\alpha e^{-\alpha z} dz P_z(X \in dw', \tau(0) \in dr).$$

We will write  $\mu$  and  $\nu$  for  $\mu_\infty$  and  $\nu_\infty$ , respectively. The [Appendix](#) contains a brief discussion of these measures, and several formulas involving them, which will be used in the body of the paper. Their probabilistic meaning will be discussed below after some preliminary observations have been made.

Without any assumptions on the Lévy process,  $\mu_K$  and  $\nu_x$  may be infinite measures, but on  $\mathcal{D}_T$  they are finite if  $K < \infty$  and  $x < \infty$ , respectively. This is because

$$(2.1) \quad \mu_K(\mathcal{D}_T) = \int_0^T E(e^{\alpha X_t}; X_t < K) dt$$

and

$$(2.2) \quad \begin{aligned} \nu_x(\mathcal{D}_T) &= \int_{z > -x} \alpha e^{-\alpha z} P_z(\tau(0) < T) dz \\ &= 1 + \int_{0 < z < x} \alpha e^{\alpha z} P(\tau(z) < T) dz \\ &= E(e^{\alpha \bar{X}_T}; \bar{X}_T \leq x) + e^{\alpha x} P(\bar{X}_T > x). \end{aligned}$$

Here, and elsewhere, we make use of the fact that  $X_t = X_{t-}$  a.s. for every  $t > 0$ . From (2.1) and (2.2) we also see that  $\mu$  and  $\nu$  are finite on  $\mathcal{D}_T$  whenever  $E e^{\alpha \bar{X}_T} < \infty$ . This condition clearly implies  $E e^{\alpha X_1} < \infty$ , and, as we show below, is equivalent to it. This will allow us to conclude that  $\mu$  and  $\nu$  are finite on  $\mathcal{D}_T$  when  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ .

Let  $(L_t)_{t \geq 0}$  be the local time of  $X$  at its maximum and  $H$  the corresponding ascending ladder height process (see [4, 11] or [23]). The renewal function of  $H$  is

$$V(z) = \int_{t \geq 0} P(H_t \leq z; t < L_\infty) dt,$$

with associated renewal measure  $V(dz)$ . When  $X_t \rightarrow -\infty$  a.s.,  $L_\infty$  has an exponential distribution with some parameter  $q > 0$ ,  $V$  is a finite measure of mass  $q^{-1}$  and the following version of the *Pollacek–Khintchine formula* holds (see [4], Proposition VI.17) for  $z \geq 0$ ,

$$(2.3) \quad P(\tau(z) < \infty) = q \bar{V}(z),$$

where

$$\bar{V}(z) = \int_{x>z} V(dx).$$

Thus

$$(2.4) \quad Ee^{\alpha\bar{X}_\infty} = q \int e^{\alpha z} V(dz).$$

LEMMA 2.1. *If  $Ee^{\alpha X_1} < \infty$  then  $Ee^{\alpha\bar{X}_T} < \infty$  for every  $T < \infty$ . If, in addition,  $Ee^{\alpha X_1} < 1$ , then  $Ee^{\alpha\bar{X}_\infty} < \infty$ . The condition  $Ee^{\alpha X_1} < \infty$  holds when  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ .*

PROOF. Assume  $Ee^{\alpha X_1} < 1$ . Then  $X_t \rightarrow -\infty$  a.s. since  $e^{\alpha X_t}$  is a nonnegative supermartingale. Further, by [22], Proposition 5.1,

$$\int_z e^{\alpha z} V(dz) < \infty$$

(their condition  $\Pi_X^+ \neq 0$  is not needed for this). Hence,  $Ee^{\alpha\bar{X}_\infty} < \infty$  by (2.4).

Now assume  $1 \leq Ee^{\alpha X_1} < \infty$ . Then we may choose  $\delta > 0$  so that  $Ee^{\alpha(X_1-\delta)} < 1$ . In that case  $Y_t = X_t - \delta t$  is a Lévy process with  $Ee^{\alpha Y_1} < 1$ . Since  $\bar{X}_T \leq \bar{Y}_T + \delta T$ , it then follows that  $Ee^{\alpha\bar{X}_T} < \infty$ .

Finally, if  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ , then  $\int_{[1,\infty)} e^{\alpha x} \Pi_X(dx) < \infty$  by (1.2), and so  $Ee^{\alpha X_1} < \infty$  by [28], Theorem 25.17.  $\square$

It will be convenient to introduce measures  $\mu_K^T$  and  $\nu_x^T$  on  $\mathcal{D}$  defined by

$$\mu_K^T(\cdot) = \mu_K(\cdot \cap \mathcal{D}_T), \quad \nu_x^T(\cdot) = \nu_x(\cdot \cap \mathcal{D}_T).$$

From the above discussion, these are finite measures if  $K$  and  $x$  are finite, or if  $Ee^{\alpha X_1} < \infty$ . Observe that

$$(2.5) \quad \begin{aligned} \mu_K^T(dw, dt, d\phi) &= I(t < T) \mu_K(dw, dt, d\phi) \\ &= I(\tau_\Delta(w) < T) \mu_K(dw, dt, d\phi), \\ \nu_x^T(dw', dr, dz) &= I(r < T) \nu_x(dw', dr, dz) \\ &= I(\tau_0(w') < T) \nu_x(dw', dr, dz). \end{aligned}$$

The first equalities are trivial and the second follow from Lemma A.1. The marginal measures will be denoted in the obvious way, for example,

$$(2.6) \quad \begin{aligned} \mu_K^T(dw) &= I(\tau_\Delta(w) < T) \mu_K(dw) \\ &= \int_0^T \int_{\phi < K} e^{\alpha\phi} P(X_{[0,t]} \in dw, X_{t-} \in d\phi) dt, \\ \nu_x^T(dr) &= I(r < T) \nu_x(dr) \\ &= I(r < T) \int_{z > -x} \alpha e^{-\alpha z} dz P_z(\tau(0) \in dr). \end{aligned}$$

This minor abuse of notation should not cause any confusion.

The precise probabilistic meaning of  $\mu$  and  $\nu$  in the  $\mathcal{S}^{(\alpha)}$  case can now be given. Define processes  $\tilde{Z}$  and  $\tilde{W}$  by

$$(2.7) \quad P(\tilde{Z} \in dw) = \frac{\mu^T(dw)}{\mu^T(\mathcal{D})} = \frac{1}{\mu^T(\mathcal{D})} \int_0^T E(e^{\alpha X_{t-}}; X_{[0,t)} \in dw) dt$$

and

$$(2.8) \quad \begin{aligned} P(\tilde{W} \in dw') &= \frac{\nu^T(dw')}{\nu^T(\mathcal{D})} \\ &= \frac{1}{\nu^T(\mathcal{D})} \int_{(-\infty, \infty)} \alpha e^{-\alpha z} P_z(X \in dw', \tau(0) < T) dz. \end{aligned}$$

It will be shown that  $\tilde{Z}$  is an Esscher transform of  $X$  killed at an independent time  $\tau$  where

$$P(\tau \in dt) = \frac{\mu^T(dt)}{\mu^T(\mathcal{D})} = \frac{I(t < T)e^{\psi(\alpha)t} dt}{\mu^T(\mathcal{D})},$$

while  $\tilde{W}$  is the process  $X$  conditioned on  $\tau(0) < T$ , and started with initial distribution

$$P(\tilde{W}_0 \in dz) = \frac{1}{\nu^T(\mathcal{D})} \alpha e^{-\alpha z} P_z(\tau(0) < T) dz.$$

Roughly speaking, in terms of the description of the limiting conditioned process given in the [Introduction](#),  $\mu$  describes the behavior of the conditioned process prior to the time of the jump into the neighborhood of  $u$ , and  $\nu$  describes the behavior after the jump.

**3. Preliminary convergence result.** In this section we prove a preliminary convergence result describing the behavior of the process for large  $u$  when it jumps from a neighborhood of the origin into a neighborhood of  $u$  before time  $T$ , and then passes over level  $u$  before a further time  $T$ . With this aim in mind, we begin by introducing a broad class of functions to which this and other convergence results apply.

Let  $H : D \otimes D \rightarrow \mathbb{R}$  be measurable with respect to the product  $\sigma$ -algebra and set

$$G(w, z) = E_z[H(w, X); \tau(0) < \infty], \quad w \in D, z \in \mathbb{R}.$$

We denote by  $\mathcal{H}$  the class of such functions  $H$  which satisfy

$$(3.1) \quad H(w, w')e^{\theta w_{\tau_\Delta} - I(w_{\tau_\Delta} \leq 0)} \quad \text{is bounded for some } \theta \in [0, \alpha);$$

$$(3.2) \quad G(w, \cdot) \quad \text{is continuous a.e. on } (-\infty, \infty) \text{ for every } w \in D.$$

For  $T > 0$ , let  $\mathcal{H}_T$  be the class of functions  $H$  for which  $H(w, w')I(\tau_0(w') < T) \in \mathcal{H}$ . Conditions (3.1) and (3.2) hold, for example, if  $H$  is bounded and continuous in the product Skorohod topology on  $D \otimes D$ . More general conditions on  $H$ , which ensure that (3.2) holds, are given below. Taking  $\theta > 0$  in (3.1) allows for certain unbounded functions  $H$ .

The following result is the starting point of our investigation. It is a consequence of [16], Remark 4.1. Let

$$A(u, x, T) = \{\tau(u - x) < T, \tau(u) - \tau(u - x) < T\}.$$

**THEOREM 3.1.** *Assume  $\overline{\Pi}_X^+ \in \mathcal{L}^{(\alpha)}$  and fix  $T > 0$ ,  $x \in [0, \infty)$  and  $K \in (-\infty, \infty)$ . Then, for any  $H \in \mathcal{H}_T$ ,*

$$(3.3) \quad \lim_{u \rightarrow \infty} \frac{E[H(X_{[0, \tau(u-x)]}, X \circ \theta_{\tau(u-x)} - c^u); X_{\tau(u-x)-} < K, A(u, x, T)]}{\overline{\Pi}_X^+(u)} = \int_{D \otimes D} H(w, w') \mu_K^T(dw) \otimes \nu_x^T(dw').$$

**PROOF.** Fix  $T > 0$ ,  $x \in [0, \infty)$  and  $K \in (-\infty, \infty)$ . We first note that the limiting expression is finite, since by (3.1), for some constant  $C$  and some  $\theta \in [0, \alpha)$ ,

$$(3.4) \quad \begin{aligned} & \int_{w \in D} \int_{w' \in D} |H(w, w')| \mu_K^T(dw) \nu_x^T(dw') \\ & \leq C \int_{w \in D} \int_{w' \in D} e^{-\theta w_{\tau_\Delta} - I(w_{\tau_\Delta} \leq 0)} \mu_K^T(dw) \nu_x^T(dw') \\ & \leq C \int_{w \in D} \int_{w' \in D} (1 + e^{-\theta w_{\tau_\Delta^-}}) \mu_K^T(dw) \nu_x^T(dw') \\ & = C \nu_x(\mathcal{D}_T) \int_0^T E(e^{\alpha X_{t-}} (1 + e^{-\theta X_{t-}}); X_{t-} < K) dt < \infty, \end{aligned}$$

where the last equality follows from (A.5).

For  $w, w' \in D$ ,  $z > -x$ ,  $\phi < K$  and  $t \geq 0$  let

$$(3.5) \quad \begin{aligned} \tilde{H}(w, w') &= H(w, w')I(\tau_\Delta(w) < T)I(\tau_0(w') < T), \\ \tilde{G}(w, z) &= E_z[\tilde{H}(w, X); \tau(0) < \infty] \\ &= I(\tau_\Delta(w) < T)E_z[H(w, X); \tau(0) < T], \end{aligned}$$

$$\tilde{\Lambda}_u(w, \phi) = \int_{z > -x} \tilde{G}(w, z) \frac{\Pi_X^+(u - \phi + dz)}{\overline{\Pi}_X^+(u)},$$

$$\tilde{\Phi}_u(t) = \int_{w \in D} \int_{\phi < K} \tilde{\Lambda}_u(w, \phi) P(X_{[0, t]} \in dw, X_{t-} \in d\phi; \tau(u - x) \geq t).$$



Then trivially  $\tilde{H} \in \mathcal{H}$ . Next, note that  $\tilde{\mu}_K$  and  $\tilde{\nu}_x$  in [16], Remark 4.1, are simply  $\mu_K$  and  $\nu_x$ , respectively. Thus by (2.5),

$$(3.6) \quad \tilde{H}(w, w') \tilde{\mu}_K(dw) \tilde{\nu}_x(dw') = H(w, w') \mu_K^T(dw) \nu_x^T(dw').$$

In particular, by (3.4),

$$(3.7) \quad \int_{w \in D} \int_{w' \in D} |\tilde{H}(w, w')| \tilde{\mu}_K(dw) \tilde{\nu}_x(dw') < \infty.$$

It then follows from [16], Remark 4.1, that

$$(3.8) \quad \begin{aligned} & \lim_{u \rightarrow \infty} \frac{E[\tilde{H}(X_{[0, \tau(u-x)]}, X \circ \theta_{\tau(u-x)} - c^u); X_{\tau(u-x)-} < K, \tau(u) < \infty]}{\bar{\Pi}_X^+(u)} \\ &= \int_{D \otimes D} \tilde{H}(w, w') \tilde{\mu}_K(dw) \otimes \tilde{\nu}_x(dw'), \end{aligned}$$

if  $\tilde{\Phi}_u, u \geq u_0$ , are dominated by an integrable function on  $[0, \infty)$ , for some  $u_0 < \infty$ . Once this is checked, the proof will be complete since (3.8) is the same as (3.3) because of (3.6) and the observation that

$$\tau_\Delta(X_{[0, \tau(u-x)])} = \tau(u-x), \quad \tau_0(X \circ \theta_{\tau(u-x)} - c^u) = \tau(u) - \tau(u-x)$$

on  $\{\tau(u) < \infty\}$ .

To prove the required domination, we modify an argument from the proof of [16], Theorem 4.1. Fix  $\varepsilon > 0$  so that  $\theta + \varepsilon \leq \alpha$ , and write

$$\frac{\Pi_X^+(u - \phi + dz)}{\bar{\Pi}_X^+(u)} = \frac{\Pi_X^+(u - \phi + dz)}{\bar{\Pi}_X^+(u - \phi - x)} \frac{\bar{\Pi}_X^+(u - \phi - x)}{\bar{\Pi}_X^+(u)}.$$

If  $u > \phi + x$ , the first term in the product is a probability measure on  $(-x, \infty)$ , while for the second, by a version of Potter’s bounds for regularly varying functions (see [5], Theorem 1.5.6(ii)) there exists an  $A = A_\varepsilon$  so that

$$(3.9) \quad \frac{\bar{\Pi}_X^+(u - \phi - x)}{\bar{\Pi}_X^+(u)} \leq A[e^{(\alpha-\varepsilon)(\phi+x)} \vee e^{(\alpha+\varepsilon)(\phi+x)}]$$

if  $u \geq 1$  and  $\phi + x \leq u - 1$ . Thus if  $u_0 =: (K + x + 1) \vee 1$ , then by (3.1), (3.5) and (3.9), for some constant  $C$  depending on  $H, K, x, \alpha$  and  $\varepsilon$ ,

$$(3.10) \quad \sup_{u \geq u_0} \tilde{\Lambda}_u(w, \phi) \leq CI(\tau_\Delta(w) < T) e^{(\alpha-\varepsilon)\phi} e^{-\theta w \tau_\Delta - I(w \tau_\Delta \leq 0)}$$

all  $w \in D, \phi < K$ ,

where we have used that  $e^{2\varepsilon K} e^{(\alpha-\varepsilon)\phi} \geq e^{(\alpha+\varepsilon)\phi}$  if  $\phi < K$  when applying (3.9). In particular, for every  $t \geq 0$ ,

$$\begin{aligned} & \sup_{u \geq u_0} \tilde{\Lambda}_u(X_{[0,t]}, X_{t-}) I(X_{t-} < K) \\ & \leq CI(t < T) [e^{(\alpha-\varepsilon-\theta)X_{t-}} I(X_{t-} \leq 0) + e^{(\alpha-\varepsilon)X_{t-}} I(0 < X_{t-} < K)] \\ & \leq C_1 I(t < T), \end{aligned}$$

where  $C_1 = C(1 + e^{(\alpha-\varepsilon)K})$ , since  $\alpha - \varepsilon - \theta \geq 0$ . Thus for  $u \geq u_0$

$$\tilde{\Phi}_u(t) = E[\tilde{\Lambda}_u(X_{[0,t]}, X_{t-}); X_{t-} < K, \tau(u-x) \geq t] \leq C_1 I(t < T).$$

Hence,  $\tilde{\Phi}_u$  for  $u \geq u_0$  are dominated, and the proof is complete.  $\square$

Conditions on  $H$  that ensure  $H \in \mathcal{H}$  are discussed in [16]. In particular, by [16], Proposition 4.2, if  $H$  satisfies (3.1), and for all  $w \in D$  and  $z \in \mathbb{R}$ ,

$$(3.11) \quad \lim_{\varepsilon \downarrow 0} H(w, w' - c^\varepsilon) = H(w, w') \quad \text{a.s. } P_z(dw') \text{ on } \{\tau_0(w') < \infty\},$$

then  $H \in \mathcal{H}$ . Observe that the function  $H_1(w, w') = I(\tau_0(w') < T)$  satisfies (3.11), because

$$\tau_0(w' - c^\varepsilon) = \tau_\varepsilon(w') \downarrow \tau_0(w') \quad \text{as } \varepsilon \downarrow 0$$

on  $\{\tau_0(w') < \infty\}$  by right-continuity. Since the class of functions satisfying (3.11) is clearly closed under products, it follows that if  $H$  satisfies (3.1) and (3.11), then  $H \in \mathcal{H}_T$  for every  $T > 0$ . For example, if  $H$  is bounded and  $H(w, \cdot)$  is continuous in any of the usual Skorohod topologies for every  $w \in D$ , then  $H$  satisfies (3.11) and hence,  $H \in \mathcal{H}_T$  for every  $T > 0$ . Thus  $\mathcal{H}_T$  is a broad class, containing essentially all functions that are likely to be of interest.

In subsequent sections, we will investigate convergence of the first passage time and the overshoot. Similar methods could be applied to other variables related to first passage, such as, for example, the undershoot or the time of the maximum prior to first passage. In applying Theorem 3.1 to the first passage time and the overshoot, the following class of functions will prove useful. Let  $f : \mathbb{R}^4 \rightarrow \mathbb{R}$  be a bounded Borel function which is jointly continuous in the last two arguments and set

$$(3.12) \quad H(w, w') = f(\tau_\Delta(w), \bar{w}_{\tau_\Delta-}, \tau_0(w'), w'_{\tau_0})$$

on  $\{\tau_\Delta(w) < \infty, \tau_0(w') < \infty\}$ , where  $\bar{w}_t = \sup_{0 \leq s \leq t} w_s$ . Since we only consider such  $H$  on this set, it's definition elsewhere does not much matter. For completeness though, here and below, we take any  $H$  of the form (3.12) to be 0 off this set. Then by [16], Proposition 5.1,  $H$  satisfies (3.11) and hence,  $H \in \mathcal{H}_T$  for every  $T > 0$ .

**4. First passage time.** To study the first passage time, we begin by applying Theorem 3.1 with

$$H(w, w') = h(\tau_\Delta(w), \tau_0(w')),$$

where  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a bounded Borel function such that  $h(t, \cdot)$  is continuous for every  $t \geq 0$ . Then  $H$  is of the form (3.12), and so  $H \in \mathcal{H}_T$ . Thus if  $\bar{\Pi}_X^+ \in \mathcal{L}^{(\alpha)}$ ,

then by (3.3)

$$\begin{aligned}
 & \lim_{u \rightarrow \infty} \frac{E[h(\tau(u-x), \tau(u) - \tau(u-x)), X_{\tau(u-x)-} < K, A(u, x, T)]}{\overline{\Pi}_X^+(u)} \\
 (4.1) \quad &= \int_{D \otimes D} h(\tau_\Delta(w), \tau_0(w')) \mu_K^T(dw) \otimes v_x^T(dw') \\
 &= \int_0^\infty \int_0^\infty h(t, r) \mu_K^T(dt) v_x^T(dr),
 \end{aligned}$$

where the final equality comes from applying (A.9) to the positive and negative parts of  $h$ . In particular,

$$\begin{aligned}
 & \lim_{u \rightarrow \infty} \frac{P(\tau(u-x) \in dt, \tau(u) - \tau(u-x) \in dr, X_{\tau(u-x)-} < K)}{\overline{\Pi}_X^+(u)} \\
 &= \mu_K^T(dt) \otimes v_x^T(dr)
 \end{aligned}$$

in the sense of weak convergence of measures on  $[0, T] \otimes [0, T]$ .

To obtain the asymptotic behavior of  $P(\tau(u) < T)$  when  $\overline{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$  we need a version of (4.1) in which  $X_{\tau(u-x)-} < K$  is replaced by  $\overline{X}_{\tau(u-x)-} < K$ . For this we introduce a measure  $\overline{\mu}_K^T$  on  $[0, \infty)$  by

$$\overline{\mu}_K^T(dt) = I(t < T) E(e^{\alpha X_{t-}}; \overline{X}_{t-} < K) dt.$$

This may be compared with the marginal measure [cf. (2.6)]

$$\mu_K^T(dt) = I(t < T) \mu_K(dt) = I(t < T) E(e^{\alpha X_{t-}}; X_{t-} < K) dt.$$

**PROPOSITION 4.1.** *Let  $h: \mathbb{R}^2 \rightarrow \mathbb{R}$  be a bounded Borel function such that  $h(t, \cdot)$  is continuous for every  $t \geq 0$ . If  $\overline{\Pi}_X^+ \in \mathcal{L}^{(\alpha)}$ , then*

$$\begin{aligned}
 & \lim_{u \rightarrow \infty} \frac{E[h(\tau(u-x), \tau(u) - \tau(u-x)); \overline{X}_{\tau(u-x)-} < K, A(u, x, T)]}{\overline{\Pi}_X^+(u)} \\
 (4.2) \quad &= \int_0^\infty \int_0^\infty h(t, r) \overline{\mu}_K^T(dt) v_x^T(dr).
 \end{aligned}$$

**PROOF.** Let  $H(w, w') = h(\tau_\Delta(w), \tau_0(w')) I(\overline{w}_{\tau_\Delta-} < K)$ . Then  $H$  is of the form (3.12), and so by Theorem 3.1, the limit in (4.2) is given by

$$\begin{aligned}
 & \int_{D \otimes D} h(\tau_\Delta(w), \tau_0(w')) I(\overline{w}_{\tau_\Delta-} < K) \mu_K^T(dw) \otimes v_x^T(dw') \\
 &= \int_0^\infty \int_0^\infty h(t, r) \overline{\mu}_K^T(dt) v_x^T(dr)
 \end{aligned}$$

by (A.10).  $\square$

Recall that  $\mu = \mu_\infty$  and  $\nu = \nu_\infty$ . If  $Ee^{\alpha X_1} < \infty$  then, as noted following (2.2), the marginal measures

$$(4.3) \quad \mu(dt) = Ee^{\alpha X_t} dt, \quad \nu(dr) = \int_z \alpha e^{-\alpha z} dz P_z(\tau(0) \in dr)$$

are finite on  $[0, T)$ . Equivalently,  $\mu^T(dt) = \mu(dt \cap [0, T))$  and  $\nu^T(dr) = \nu(dr \cap [0, T))$  are finite measures. For notational convenience, we will write  $\nu(t)$  for  $\nu([0, t))$  and similarly for other measures.

In the next two propositions and elsewhere, we consider limits as  $K, x \rightarrow \infty$ . By this we will always mean that the manner in which  $K$  and  $x$  approach infinity is irrelevant. In particular, they can do so in either order. Letting  $K, x \rightarrow \infty$  in Propositions 4.2 and 4.3 indicates that the most probable paths along which  $X$  can reach level  $u$  by time  $T$  are those in which the process jumps from a neighborhood of 0 to a neighborhood of  $u$ . This will be elucidated upon further in Theorem 6.2.

PROPOSITION 4.2. *If  $\bar{\Pi}_X^+ \in \mathcal{L}^{(\alpha)}$  and  $Ee^{\alpha X_1} < \infty$ , then*

$$(4.4) \quad \lim_{K, x \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{P(\tau(u) < T, \bar{X}_{\tau(u-x)-} < K)}{\bar{\Pi}_X^+(u)} = \int_{[0, T)} \nu(T-t)\mu(dt).$$

PROOF. Fix  $\varepsilon > 0$  and let  $g$  be continuous with  $g \equiv 1$  on  $[0, T]$ ,  $g \equiv 0$  on  $[T + \varepsilon, \infty)$  and  $g$  linear on  $[T, T + \varepsilon]$ . Since  $\{\tau(u) < T\} \subset A(u, x, T)$ , we have by (4.2) with  $h(t, r) = g(t + r)$ , for every  $x$  and  $K$ ,

$$(4.5) \quad \begin{aligned} & \limsup_{u \rightarrow \infty} \frac{P(\tau(u) < T, \bar{X}_{\tau(u-x)-} < K)}{\bar{\Pi}_X^+(u)} \\ & \leq \int_0^\infty \int_0^\infty g(t+r) \bar{\mu}_K^T(dt) \nu_x^T(dr) \\ & \leq \int_{[0, \infty)} \nu_x^T(T + \varepsilon - t) \bar{\mu}_K^T(dt) \\ & \leq \int_{[0, T)} E(e^{\alpha X_{t-}}; \bar{X}_{t-} < K) dt \\ & \quad \times \int_z I(z > -x) \alpha e^{-\alpha z} P_z(\tau(0) < T + \varepsilon - t) dz. \end{aligned}$$

Similarly,

$$(4.6) \quad \begin{aligned} & \liminf_{u \rightarrow \infty} \frac{P(\tau(u) < T, \bar{X}_{\tau(u-x)-} < K)}{\bar{\Pi}_X^+(u)} \\ & \geq \int_{[0, T)} E(e^{\alpha X_{t-}}; \bar{X}_{t-} < K) dt \\ & \quad \times \int_z I(z > -x) \alpha e^{-\alpha z} P_z(\tau(0) < T - \varepsilon - t) dz. \end{aligned}$$

The result now follows by letting  $K, x \rightarrow \infty$  and then  $\varepsilon \downarrow 0$  in (4.5) and (4.6) and noting that

$$\int_{[0,T)} \nu([0, T - t))\mu(dt) = \int_{[0,T)} \nu([0, T - t])\mu(dt)$$

since the integrands agree except on an at most countable set, and  $\mu$  has no atoms. □

REMARK 4.1. If  $\overline{\Pi}_X^+ \in \mathcal{L}^{(\alpha)}$  but  $Ee^{\alpha X_1} = \infty$ , it follows from (4.6) that Proposition 4.2 remains valid provided we interpret the integral in (4.4) as infinite.

If  $Ee^{\alpha X_1} < 1$  then  $\mu$  and  $\nu$  given by (4.3) are finite measures on  $[0, \infty)$ , since

$$(4.7) \quad \mu(\infty) = \int_0^\infty Ee^{\alpha X_t} dt = \int_0^\infty (Ee^{\alpha X_1})^t dt < \infty$$

and

$$(4.8) \quad \nu(\infty) = \int_z \alpha e^{-\alpha z} dz P_z(\tau_0 < \infty) = Ee^{\alpha \overline{X}_\infty} < \infty$$

by (2.2) and Lemma 2.1.

PROPOSITION 4.3. If  $\overline{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ , then

$$(4.9) \quad \lim_{K,x \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{P(\tau(u) < T, \overline{X}_{\tau(u-x)-} \geq K)}{\overline{\Pi}_X^+(u)} = 0.$$

PROOF. By Lemma 2.1,  $Ee^{\alpha X_1} < \infty$ . First assume that  $Ee^{\alpha X_1} < 1$ . Then, by (4.4),

$$\begin{aligned} \lim_{K,x \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{P(\tau(u) < \infty, \overline{X}_{\tau(u-x)-} < K)}{\overline{\Pi}_X^+(u)} &\geq \lim_{T \rightarrow \infty} \int_{[0,T)} \nu(T - t)\mu(dt) \\ &= \mu(\infty)\nu(\infty). \end{aligned}$$

On the other hand, by Theorem 4.1 and [22], Proposition 5.3,

$$(4.10) \quad \lim_{u \rightarrow \infty} \frac{P(\tau(u) < \infty)}{\overline{\Pi}_X^+(u)} = \mu(\infty)\nu(\infty),$$

from which (4.9) immediately follows (with  $T$  even replaced by  $\infty$ ).

Now assume  $Ee^{\alpha X_1} \geq 1$ . Choose  $\delta > 0$  so that  $Ee^{\alpha(X_1 - \delta)} < 1$  and set  $Y_t = X_t - \delta t$ . Then  $\Pi_Y = \Pi_X$  and  $Ee^{\alpha Y_1} < 1$ . Hence, (4.9) holds with  $X$  replaced by  $Y$ . Next, recalling  $\tau_u(Y) = \inf\{t > 0 : Y_t > u\}$ , it is clear that

$$(4.11) \quad \{\tau(u) < T\} \subset \{\tau_{(u-x)}(Y) < T\} \quad \text{for } x \geq \delta T.$$

Hence, if  $x \wedge K > \delta T$ , then

$$\begin{aligned}
 & \{\tau(u) < T, \bar{Y}_{\tau_{(u-x)}(Y)-} < K - \delta T\} \\
 &= \{\tau(u) < T, \bar{Y}_{\tau_{(u-x)}(Y)-} < K - \delta T, \tau_{(u-x)}(Y) < T\} \\
 (4.12) \quad & \subset \{\tau(u) < T, \bar{Y}_{\tau_{(u-x)}(Y)-} + \delta \tau_{(u-x)}(Y) < K\} \\
 & \subset \{\tau(u) < T, \bar{X}_{\tau_{(u-x)}(Y)-} < K\}.
 \end{aligned}$$

On the other hand, we trivially have

$$X_{\tau_{(u-x)}(Y)} > Y_{\tau_{(u-x)}(Y)} \geq u - x.$$

Thus, if additionally  $u - x > K$ , then  $\tau_{(u-x)}(Y) = \tau(u - x)$  on  $\{\tau(u) < T, \bar{Y}_{\tau_{(u-x)}(Y)-} < K - \delta T\}$ . Consequently, by (4.12),

$$(4.13) \quad \{\tau(u) < T, \bar{Y}_{\tau_{(u-x)}(Y)-} < K - \delta T\} \subset \{\tau(u) < T, \bar{X}_{\tau_{(u-x)}-} < K\}.$$

Hence, from (4.11) and (4.13) we may conclude that if  $x \wedge K > \delta T$  and  $x + K < u$ , then

$$\{\tau(u) < T, \bar{X}_{\tau_{(u-x)}-} \geq K\} \subset \{\tau_{(u-\delta T)}(Y) < T, \bar{Y}_{\tau_{(u-x)}(Y)-} \geq K - \delta T\}.$$

Thus by (4.9), with  $X$  replaced by  $Y$ ,  $u$  by  $u - \delta T$ ,  $x$  by  $x - \delta T$  and  $K$  by  $K - \delta T$ , we have

$$\begin{aligned}
 & \lim_{K,x \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{P(\tau(u) < T, \bar{X}_{\tau_{(u-x)}-} \geq K)}{\bar{\Pi}_X^+(u)} \\
 & \leq e^{\alpha \delta T} \lim_{K,x \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{P(\tau_{(u-\delta T)}(Y) < T, \bar{Y}_{\tau_{(u-\delta T)-(x-\delta T)}(Y)-} \geq K - \delta T)}{\bar{\Pi}_X^+(u - \delta T)} \\
 & = 0,
 \end{aligned}$$

which completes the proof.  $\square$

REMARK 4.2. As noted in the proof, if  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$  and  $Ee^{\alpha X_1} < 1$  then

$$\lim_{K,x \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{P(\tau(u) < \infty, \bar{X}_{\tau_{(u-x)}-} \geq K)}{\bar{\Pi}_X^+(u)} = 0.$$

Recall that  $\psi$ , introduced in (1.6), denotes the exponent of the mgf of  $X_1$ , that is,

$$e^{\psi(\beta)} = Ee^{\beta X_1}.$$

Note that  $\psi(\alpha) < \infty$  and  $\psi(\beta) = \infty$  for  $\beta > \alpha$  when  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ . Combining the two previous propositions yields the following result.

**THEOREM 4.1** (Probability of ruin in finite time). *Assume that  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ , then*

$$(4.14) \quad \begin{aligned} \lim_{u \rightarrow \infty} \frac{P(\tau(u) < T)}{\bar{\Pi}_X^+(u)} &= \int_{[0, T)} v(T - t) \mu(dt) \\ &= \int_{[0, T)} e^{\psi(\alpha)t} E e^{\alpha \bar{X}_{T-t}} dt. \end{aligned}$$

**PROOF.** The first equality follows from Propositions 4.2 and 4.3, and the second from (4.3) and (2.2).  $\square$

**REMARK 4.3.** If  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$  and  $E e^{\alpha X_1} \geq 1$ , then trivially

$$\lim_{T \rightarrow \infty} \int_{[0, T)} e^{\psi(\alpha)t} E e^{\alpha \bar{X}_{T-t}} dt = \infty.$$

Thus  $\bar{\Pi}_X^+(u) = o(P(\tau(u) < \infty))$  in contrast to the case  $E e^{\alpha X_1} < 1$ , where (4.10) holds and the limit is finite by (4.7) and (4.8).

The existence of the limit in (4.14) was first proved by Braverman and Samorodnitsky [7]. Later, Braverman [6] obtained a complicated description of the limit (see also Albin and Sundén [1], (6.1) and (6.6) and Hao and Tang [18], (4.8)). Albin and Sundén’s approach involved showing

$$\lim_{u \rightarrow \infty} \frac{P(\bar{X}_T > u)}{P(X_T > u)} = L(T)$$

exists. Their description of  $L(T)$  is highly inexplicit, but they were able to show  $L(T) > 1$  for all  $T > 0$  when  $X$  is not a subordinator. Since

$$(4.15) \quad \lim_{u \rightarrow \infty} \frac{P(X_T > u)}{\bar{\Pi}_X^+(u)} = T e^{\psi(\alpha)T}$$

when  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$  (see [29]) it follows from (4.14) that

$$L(T) = \frac{1}{T} \int_{[0, T)} e^{-\psi(\alpha)t} E e^{\alpha \bar{X}_t} dt,$$

providing an alternative proof that  $L(T) > 1$  precisely when  $X$  is not a subordinator.

We turn now to the limit in (4.14) and investigate its behavior as a function of  $T$ . When  $\psi(\alpha) \geq 0$ , a change variable gives

$$(4.16) \quad \int_{[0, T)} e^{\psi(\alpha)t} E e^{\alpha \bar{X}_{T-t}} dt = e^{\psi(\alpha)T} \int_{[0, T)} e^{-\psi(\alpha)t} E e^{\alpha \bar{X}_t} dt.$$

The integral on the RHS diverges as  $T \rightarrow \infty$  since  $E e^{\alpha \bar{X}_t} \geq E e^{\alpha X_t} = e^{\psi(\alpha)t}$ . To determine the correct exponential rate of growth, we note that  $\ln E e^{\alpha \bar{X}_t}$  is subadditive, hence, by Fekete’s lemma ([19], Theorem 7.6.1),

$$\lim_{t \rightarrow \infty} \frac{\ln E e^{\alpha \bar{X}_t}}{t} = C$$

for some  $C$ , where clearly  $C \in [\psi(\alpha), \infty)$ . It then easily follows that

$$(4.17) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \ln \left( \lim_{u \rightarrow \infty} \frac{P(\tau(u) < T)}{\bar{\Pi}_X^+(u)} \right) = C.$$

In general, when  $\bar{\Pi}_X^+ \in \mathcal{S}(\alpha)$ , we only know that  $E e^{\alpha X_1} < \infty$ , but now assume for the remainder of this paragraph that, in addition to  $\psi(\alpha) \geq 0$ , we have

$$(4.18) \quad E(X_1 e^{\alpha X_1}) < \infty.$$

This assumption arises in connection with the Cramér–Lundberg large deviation estimate in the  $\psi(\alpha) = 0$  case [see (5.3) below]. Then, using Doob’s  $L^1$ -maximal inequality (see [13], Exercise 5.4.6), it is easy to check that for some constant  $C < \infty$ ,

$$(4.19) \quad E e^{\alpha \bar{X}_t} \leq C(1+t)e^{\psi(\alpha)t}.$$

Hence,

$$\lim_{t \rightarrow \infty} \frac{\ln E e^{\alpha \bar{X}_t}}{t} = \psi(\alpha).$$

Thus, in particular, we are able to identify the correct rate of exponential growth as

$$(4.20) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \ln \left( \lim_{u \rightarrow \infty} \frac{P(\tau(u) < T)}{\bar{\Pi}_X^+(u)} \right) = \psi(\alpha).$$

When  $\psi(\alpha) = 0$ , more precise information is available from (4.19). In the special case that  $X$  is a subordinator, it follows from (4.14) or (4.15) that (4.20) can be sharpened to

$$\lim_{u \rightarrow \infty} \frac{P(\tau(u) < T)}{\bar{\Pi}_X^+(u)} = T e^{\psi(\alpha)T}.$$

When  $\psi(\alpha) > 0$ , (4.16) may be rewritten

$$(4.21) \quad \int_{[0,T)} e^{\psi(\alpha)t} E e^{\alpha \bar{X}_{T-t}} dt = \psi(\alpha)^{-1} e^{\psi(\alpha)T} E(e^{\alpha \bar{X}} \mathbf{e}; \mathbf{e} < T),$$

where  $\mathbf{e}$  is exponentially distributed with parameter  $\psi(\alpha)$  and independent of  $X$ . For fixed  $T$  this provides a formula which appears well suited to Monte Carlo simulation. It gives the more precise, than (4.17), asymptotic estimate

$$\lim_{u \rightarrow \infty} \frac{P(\tau(u) < T)}{e^{\psi(\alpha)T} \bar{\Pi}_X^+(u)} \sim \frac{E(e^{\alpha \bar{X}} \mathbf{e}; \mathbf{e} < T)}{\psi(\alpha)} \quad \text{as } T \rightarrow \infty,$$



where note

$$Ee^{\alpha\bar{X}_e} \geq Ee^{\alpha X_e} = \infty.$$

When  $\psi(\alpha) < 0$ , that is,  $Ee^{\alpha X_1} < 1$ ,

$$\int_{[0,T)} e^{\psi(\alpha)t} Ee^{\alpha\bar{X}_{T-t}} dt = -\psi(\alpha)^{-1} E[e^{\alpha\bar{X}_{T-e}}; \mathbf{e} < T],$$

where  $\mathbf{e}$  is exponentially distributed with parameter  $-\psi(\alpha)$  and independent of  $X$ . In this case

$$(4.22) \quad \lim_{T \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{P(\tau(u) < T)}{\bar{\Pi}_X^+(u)} = \frac{Ee^{\alpha\bar{X}_\infty}}{-\psi(\alpha)} < \infty$$

by Lemma 2.1. By way of comparison, observe that from (4.7) and (4.8), (4.10) may be written

$$(4.23) \quad \lim_{u \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{P(\tau(u) < T)}{\bar{\Pi}_X^+(u)} = \frac{Ee^{\alpha\bar{X}_\infty}}{-\psi(\alpha)}.$$

The asymptotic behavior as  $T \rightarrow 0$ , irrespective of the value of  $\psi(\alpha)$ , also follows easily from (4.14):

$$\lim_{T \rightarrow 0} \frac{1}{T} \lim_{u \rightarrow \infty} \frac{P(\tau(u) < T)}{\bar{\Pi}_X^+(u)} = 1.$$

**5. An application to insurance risk.** A popular model in insurance risk is the Cramér–Lundberg model in which

$$(5.1) \quad X_t = \sum_1^{N_t} U_i - pt,$$

where  $N_t$  is a rate  $\lambda$  Poisson process, and  $U_i > 0$  form an independent i.i.d. sequence. Here  $p$  represents the rate of premium inflow and  $U_i$  the size of the  $i$ th claim. Thus  $X$ , called the claim surplus process, represents the excess in claims over premium. The insurance company starts with a positive reserve  $u$ , and ruin occurs if this level is exceeded by  $X$ . It is assumed that  $EU_1 = \mu$  is finite and that  $p = (1 + \theta)\lambda\mu$  where  $\theta > 0$  is called the safety loading. This ensures  $X_t \rightarrow -\infty$ , and so the probability of eventual ruin  $P(\tau(u) < \infty) \rightarrow 0$  as  $u \rightarrow \infty$ .

A common assumption on  $X$  is the Cramér–Lundberg condition

$$(5.2) \quad Ee^{\nu X_1} = 1 \quad \text{for some } \nu > 0.$$

The  $\nu$  which satisfies (5.2) is often referred to as the Lundberg exponent or adjustment coefficient. This condition results in a well-known large deviation estimate for the probability of eventual ruin:

$$(5.3) \quad \lim_{u \rightarrow \infty} e^{\nu u} P(\tau(u) < \infty) = C =: \frac{-EX_1}{E(X_1 e^{\nu X_1})},$$

where  $C > 0$  if (4.18) holds with  $\alpha$  replaced by  $\nu$ .

The problem of a sharp estimate for the probability of ruin in finite time when an exponential moment exists is much more difficult. In the special case that the claim size distribution is itself exponential, an exact formula for  $P(\tau(u) < T)$  is available (see [2], Proposition V.1.3). Other than this, little is known, although several approximations have been proposed (see [2], Chapter V). One typical such approximation is the classical Segerdahl approximation; if (5.2) holds and  $E(X_1^2 e^{\nu X_1}) < \infty$ , then

$$(5.4) \quad P(\tau(u) < T) = C e^{-\nu u} \Phi\left(\frac{T - au}{b\sqrt{u}}\right) + o(e^{-\nu u})$$

uniformly in  $T$ , where  $a$  and  $b$  are known constants and  $\Phi$  is the standard normal distribution function. Considerable care must be taken in using (5.4). The only time (5.4) is guaranteed to provide a valid estimate is when  $T \geq au + O(\sqrt{u})$ . For  $T$  of smaller order, the estimate is of smaller order than  $e^{-\nu u}$ . For example, for fixed  $T$ , (5.4) gives

$$(5.5) \quad P(\tau(u) < T) = \frac{Cb}{a(2\pi u)^{1/2}} e^{-(\nu+a^2/(2b^2))u+(a/b^2)T} + o(e^{-\nu u}),$$

and it is quite likely that the error term will exceed the estimate itself. While some improvements to this estimate are possible, and alternative approximations such as the (corrected) diffusion approximation have been proposed, none can lay claim to giving a sharp estimate for the probability of ruin in finite time.

Recently a more general Lévy risk insurance model has been proposed in which (5.1) is replaced by a spectrally positive Lévy process  $X$ , that is,  $\Pi_X = \Pi_X^+$ , for which  $X_t \rightarrow -\infty$ . Theorem 4.1 then solves the problem of a sharp estimate for the probability of ruin in finite time in this more general model (even without the spectrally positive assumption) when  $\overline{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ :

$$(5.6) \quad P(\tau(u) < T) = \overline{\Pi}_X^+(u)B(T) + o(\overline{\Pi}_X^+(u)),$$

where

$$(5.7) \quad B(T) = \int_{[0,T)} e^{\psi(\alpha)(T-t)} E e^{\alpha \overline{X}_t} dt.$$

Since  $B$  is continuous and the limit of monotone functions, the estimate is uniform on compacts in  $T$ , and when  $B$  is bounded, that is, when  $E e^{\alpha X_1} < 1$ , the estimate is uniform over all  $T$ . There seems little hope of evaluating  $B$  explicitly and so in practice numerical techniques will be needed. One possibility is to approximate  $B$  using Monte Carlo simulation, for which formulations like (4.21) appear well suited. An alternative is to approximate  $B$  by (numerically) inverting its Laplace transform. For  $\delta > \psi(\alpha) \vee 0$ , this is given by

$$\int_{s \geq 0} e^{-\delta s} B(s) ds = \frac{1}{\delta - \psi(\alpha)} \int_{t \geq 0} e^{-\delta t} E e^{\alpha \overline{X}_t} dt = \frac{E e^{\alpha \overline{X}_{e(\delta)}}}{\delta(\delta - \psi(\alpha))},$$

where  $e(\delta)$  is an independent exponential with parameter  $\delta$ . In the spectrally positive case this may be written equivalently as

$$\int_{s \geq 0} e^{-\delta s} B(s) ds = \frac{(\phi(\delta) - \alpha)}{(\delta - \psi(\alpha))^2 \phi(\delta)},$$

where  $\phi$  is the inverse of the restriction of  $\psi$  to  $(-\infty, 0]$  (see [11], (4.3.7) and (9.2.9) (which note applies to spectrally negative processes)). It would be interesting to investigate how successfully these, and possibly other approximation methods, could be implemented in concrete classes of examples, such as those mentioned in the [Introduction](#) or the GTSC class of models introduced by Hubalek and Kyprianou [20] and further investigated in [17].

As an illustration we compare estimates (5.5) and (5.6) in the context of the Cramér–Lundberg model (5.1) when  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ . This is equivalent to the assumption  $U_1 \in \mathcal{S}^{(\alpha)}$ . One may regard (5.1) as giving a family of models indexed by the premium rate. Let  $p_0 = \lambda\mu$  be the premium rate corresponding to zero safety loading, and write  $X_t^{(p)} = Y_t - pt$ , where  $Y_t = \sum_1^{N_t} U_i$ . Since  $Ee^{\alpha Y_1} > 1$ , there is a unique  $p = p_L$  such that  $Ee^{\alpha X_1^{(p_L)}} = 1$ . Observe that  $X_t^{(p_L)} \rightarrow -\infty$  a.s. since  $e^{\alpha X_t^{(p_L)}}$  is a nonnegative martingale. Thus  $p_L > p_0$ . In comparing (5.5) and (5.6), we consider three different regimes for  $p$ . The first is large premiums;  $p > p_L$ . In that case  $Ee^{\alpha X_1^{(p)}} < 1$ , and since  $Ee^{\beta X_1^{(p)}} = \infty$  for every  $\beta > \alpha$  when  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ , the Lundberg exponent does not exist. Thus the classical Segerdahl approximation has nothing to say in this case. The second regime is when  $p = p_L$ . Then the Segerdahl approximation yields (5.5) with  $\nu = \alpha$ . However, not surprisingly, the estimate is of completely the wrong order, since  $e^{-\beta u} = o(\bar{\Pi}_X^+(u))$  for every  $\beta > \alpha$ . Finally, the third regime is small premiums;  $p_0 < p < p_L$ . In this case the Lundberg exponent  $\nu$  exists and  $\nu < \alpha$ , however, again one can show that the estimate is of the wrong order. In this third regime an alternative approximation, the (corrected) diffusion approximation, is often suggested. This is a heavy traffic limit, that is, an approximation as  $p \downarrow p_0$ . Asmussen and Albrecher [2], Chapter V.6, report that numerical evidence indicates it provides quite good estimates when  $p$  is close to  $p_0$ , but again it cannot expect to match the sharp estimate (5.6) which is valid for every  $p$ .

In concluding this section it should be pointed out that one would not expect the estimates for the probability of ruin in finite time that have been proposed in the literature to be as good as (5.6) when  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ . After all, (5.6) is a sharp estimate derived from the additional structure resulting from the assumption  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ . Given how little is known about these ruin probabilities in general, (5.6) might be useful as a benchmark against which to compare these more general approximations. We should also mention that in the subexponential case the situation is much better understood. Then Rosiński and Samorodnitsky [27] show

$$\lim_{u \rightarrow \infty} \frac{P(\tau(u) < T)}{\bar{\Pi}_X^+(u)} = \lim_{u \rightarrow \infty} \frac{P(X_T > u)}{\bar{\Pi}_X^+(u)} = T.$$

The first equality is because ruin by time  $T$  is essentially the result of one extremely large claim which greatly exceeds  $u$ . Consequently,  $X$  will not have returned to level  $u$  by time  $T$  on the event  $\tau(u) < T$ . The second equality is a direct consequence of subexponentiality.

**6. Functional limit theorem.** We now address the question of how first passage occurs by time  $T$ , by proving a functional limit theorem for the process conditioned on  $\tau(u) < T$  as  $u \rightarrow \infty$ . We begin by revisiting Theorem 3.1, in the  $\mathcal{S}^{(\alpha)}$  case, with the aid of Proposition 4.3. This allows us to set  $K = \infty$  and take the limit as  $x \rightarrow \infty$  in (3.3).

**THEOREM 6.1.** *Assume  $\overline{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$  and  $H \in \mathcal{H}_T$ . Then*

$$(6.1) \quad \lim_{x \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{E[H(X_{[0, \tau(u-x)]}, X \circ \theta_{\tau(u-x)} - c^u); A(u, x, T)]}{\overline{\Pi}_X^+(u)} = \int_{D \otimes D} H(w, w') \mu^T(dw) \otimes \nu^T(dw').$$

**PROOF.** The limit is finite since, by the same argument as in (3.4) but with  $K = x = \infty$ , we obtain

$$(6.2) \quad \int_{w \in D} \int_{w' \in D} |H(w, w')| \mu^T(dw) \nu^T(dw') \leq C \nu(\mathcal{D}_T) \int_0^T E[e^{\alpha X_{t-}} (1 + e^{-\theta X_{t-}})] dt < \infty,$$

where finiteness follows from Lemma 2.1.

Next, by (3.1), if  $K \geq 0$ , then  $|H|$  is bounded on  $\{(w, w') : w_{\tau_{\Delta}-} \geq K\}$  by some constant  $C$  say. Since  $\tau(u) < 2T$  on  $A(u, x, T)$ , it then follows from (4.9) with  $T$  replaced by  $2T$ , that

$$\lim_{K, x \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{E[H(X_{[0, \tau(u-x)]}, X \circ \theta_{\tau(u-x)} - c^u); X_{\tau(u-x)-} \geq K, A(u, x, T)]}{\overline{\Pi}_X^+(u)} = 0.$$

Thus by (3.3), the limit in (6.1) is given by

$$\begin{aligned} & \lim_{K, x \rightarrow \infty} \int_{D \otimes D} H(w, w') \mu_K^T(dw) \otimes \nu_x^T(dw') \\ &= \lim_{K, x \rightarrow \infty} \int_{\mathcal{D} \otimes \mathcal{D}} H(w, w') I(\phi < K) I(z > -x) \mu^T(dw, dt, d\phi) \\ & \quad \otimes \nu^T(dw', dr, dz). \end{aligned}$$

Now the integrand is trivially dominated by  $|H|$  and

$$\begin{aligned} & \int_{\mathcal{D} \otimes \mathcal{D}} |H(w, w')| \mu^T(dw, dt, d\phi) \otimes v^T(dw', dr, dz) \\ &= \int_{w \in D} \int_{w' \in D} |H(w, w')| \mu^T(dw) v^T(dw') < \infty \end{aligned}$$

by (6.2). Thus by dominated convergence

$$\begin{aligned} & \lim_{K, x \rightarrow \infty} \int_{\mathcal{D} \otimes \mathcal{D}} H(w, w') \mu_K^T(dw) \otimes v_x^T(dw') \\ &= \int_{\mathcal{D} \otimes \mathcal{D}} H(w, w') \mu^T(dw, dt, d\phi) \otimes v^T(dw', dr, dz) \\ &= \int_{\mathcal{D} \otimes \mathcal{D}} H(w, w') \mu^T(dw) \otimes v^T(dw'). \end{aligned} \quad \square$$

To give a clearer understanding of the limit in (6.1), introduce independent  $D$ -valued random variables  $\tilde{Z}$  and  $\tilde{W}$  with distributions given by (2.7) and (2.8), respectively. Clearly  $\tilde{W}$  is the process  $X$  conditioned on  $\tau(0) < T$ , and started with initial distribution

$$(6.3) \quad P(\tilde{W}_0 \in dz) = \frac{1}{v^T(\mathcal{D})} \alpha e^{-\alpha z} P_z(\tau(0) < T) dz.$$

To give a more transparent description of  $\tilde{Z}$ , we first introduce the Esscher transform  $Z$  of  $X$ , defined as follows. Let  $\mathcal{B}([0, s])$  denote the Borel sets in  $\mathbb{R}^{[0, s]}$ . Then for any  $s \geq 0$  and any  $B_s \in \mathcal{B}([0, s])$ ,

$$(6.4) \quad P(\{Z_v : 0 \leq v \leq s\} \in B_s) = e^{-\psi(\alpha)s} E(e^{\alpha X_s}; \{X_v : 0 \leq v \leq s\} \in B_s).$$

Next, recalling (4.3), let  $\tau$  be independent of  $Z$  with distribution

$$(6.5) \quad P(\tau \in dt) = \frac{\mu^T(dt)}{\mu^T(\mathcal{D})} = \frac{I(t < T) e^{\psi(\alpha)t} dt}{\mu^T(\mathcal{D})}.$$

Thus

$$(6.6) \quad P(\tau \in dt) = \frac{\psi(\alpha) e^{\psi(\alpha)t} dt}{e^{\psi(\alpha)T} - 1}, \quad 0 \leq t < T, \text{ if } \psi(\alpha) \neq 0$$

and

$$(6.7) \quad P(\tau \in dt) = \frac{dt}{T}, \quad 0 \leq t < T, \text{ if } \psi(\alpha) = 0.$$

PROPOSITION 6.1. *Assume  $E e^{\alpha X_1} < \infty$ . Then with  $Z$  and  $\tau$  as above,*

$$\{\tilde{Z}_t : t < \tau_\Delta(\tilde{Z})\} \stackrel{d}{=} \{Z_t : t < \tau\}.$$

PROOF. For any  $B_s \in \mathcal{B}([0, s])$

$$\begin{aligned}
 &P(\{\tilde{Z}_v : 0 \leq v \leq s\} \in B_s, s < \tau_\Delta(\tilde{Z})) \\
 &= \frac{1}{\mu^T(\mathcal{D})} \int_{s < t < T} E(e^{\alpha X_{t-}} : \{X_v : 0 \leq v \leq s\} \in B_s) dt \\
 &= \frac{E(e^{\alpha X_s} : \{X_v : 0 \leq v \leq s\} \in B_s)}{\mu^T(\mathcal{D})} \int_{s < t < T} e^{\psi(\alpha)(t-s)} dt \\
 &= P(\{Z_v : 0 \leq v \leq s\} \in B_s) \frac{1}{\mu^T(\mathcal{D})} \int_{s < t < T} e^{\psi(\alpha)t} dt \\
 &= P(\{Z_v : 0 \leq v \leq s\} \in B_s) P(s < \tau) \\
 &= P(\{Z_v : 0 \leq v \leq s\} \in B_s, s < \tau). \quad \square
 \end{aligned}$$

Thus  $\tilde{Z}$  is seen to be the Esscher transform of  $X$  killed at an independent time  $\tau$  with distribution given by (6.5).

With the previous analysis at hand, it is a relatively easy matter to study the process  $X$  conditioned on  $\tau(u) < T$ , when  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ . To do so, first introduce the probability measure

$$P^{(u,T)}(\cdot) = P(\cdot | \tau(u) < T)$$

and let  $E^{(u,T)}$  denote expectation with respect to  $P^{(u,T)}$ . Let  $Z$  and  $\tau$  be distributed as above and let  $(W, \tau)$  be independent of  $Z$  with joint distribution

$$\begin{aligned}
 &P(W \in dw', \tau \in dt) \\
 (6.8) \quad &= \frac{\mu^T(\mathcal{D})}{B(T)} \int_{(-\infty, \infty)} \alpha e^{-\alpha z} P_z(X \in dw', \tau(0) < T - t) dz P(\tau \in dt) \\
 &= \frac{\mu^T(\mathcal{D})}{B(T)} v^{T-t}(dw') P(\tau \in dt),
 \end{aligned}$$

where recall  $B(T)$  is given by (5.7). Observe this is a true probability distribution since, from (2.2),

$$\begin{aligned}
 &\int_{[0,T)} \int_D \int_{(-\infty, \infty)} \alpha e^{-\alpha z} P_z(X \in dw, \tau(0) < T - t) dz P(\tau \in dt) \\
 &= \frac{1}{\mu^T(\mathcal{D})} \int_{[0,T)} E e^{\alpha \bar{X}_{T-t}} \mu(dt) \\
 &= \frac{1}{\mu^T(\mathcal{D})} \int_{[0,T)} e^{\psi(\alpha)t} E e^{\alpha \bar{X}_{T-t}} dt = \frac{B(T)}{\mu^T(\mathcal{D})}.
 \end{aligned}$$

Thus  $W$  is the process  $X$  conditioned on  $\tau(0) < T - \tau$ , and started with initial distribution

$$P(W_0 \in dz) = \frac{\mu^T(\mathcal{D})}{B(T)} \alpha e^{-\alpha z} P_z(\tau(0) < T - \tau) dz.$$

In particular,

$$P(W_0 > 0) = \frac{\mu^T(\mathcal{D})}{B(T)}.$$

LEMMA 6.1. *The joint distribution of  $(Z_{[0,\tau]}, W)$  is given by*

$$(6.9) \quad P(Z_{[0,\tau]} \in dw, W \in dw') = \frac{1}{B(T)} \int_0^\infty \mu^T(dw, dt) v^{T-t}(dw').$$

PROOF. First observe that by (6.4), for  $0 \leq t < T$ ,

$$(6.10) \quad P(Z_{[0,t]} \in dw) = e^{-\psi(\alpha)t} E(e^{\alpha X_t}; X_{[0,t]} \in dw).$$

Thus by (6.5), (6.8), (6.10) and independence

$$\begin{aligned} P(Z_{[0,\tau]} \in dw, W \in dw') &= \int_t P(Z_{[0,t]} \in dw, \tau \in dt, W \in dw') \\ &= \int_t P(Z_{[0,t]} \in dw) P(\tau \in dt, W \in dw') \\ &= \frac{\mu^T(\mathcal{D})}{B(T)} \int_t e^{-\psi(\alpha)t} E(e^{\alpha X_t}; X_{[0,t]} \in dw) v^{T-t}(dw') P(\tau \in dt) \\ &= \frac{1}{B(T)} \int_{[0,T)} E(e^{\alpha X_t}; X_{[0,t]} \in dw) v^{T-t}(dw') dt \\ &= \frac{1}{B(T)} \int_0^\infty \mu^T(dw, dt) v^{T-t}(dw'). \end{aligned} \quad \square$$

We are now ready to state the main result of this section.

THEOREM 6.2 (Functional limit theorem). *Assume that  $\overline{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$  and  $H \in \mathcal{H}_t$  for every  $t \leq T$ . Then*

$$(6.11) \quad \lim_{x \rightarrow \infty} \lim_{u \rightarrow \infty} E^{(u,T)} H(X_{[0,\tau(u-x)]}, X \circ \theta_{\tau(u-x)} - c^u) = EH(Z_{[0,\tau]}, W).$$

PROOF. Set

$$\tilde{H}(w, w') = H(w, w') I(\tau_\Delta(w) + \tau_0(w') < T),$$

and fix  $w_0 \in D$ . Then

$$E_z[\tilde{H}(w_0, X) I(\tau(0) < T); \tau(0) < \infty] = E_z[H(w_0, X); \tau(0) < T - \tau_\Delta(w_0)].$$

Now for every  $w \in D$ , and in particular for  $w = w_0$ ,  $E_z[H(w, X); \tau(0) < T - \tau_\Delta(w_0)]$  is trivially continuous in  $z$  if  $\tau_\Delta(w_0) \geq T$ , while if  $\tau_\Delta(w_0) < T$ , it is continuous a.e. in  $z \in (-\infty, \infty)$  since  $H \in \mathcal{H}_{T-\tau_\Delta(w_0)}$ . Thus  $\tilde{H} \in \mathcal{H}_T$  and so

$$\begin{aligned} & E^{(u,T)} H(X_{[0,\tau(u-x)]}, X \circ \theta_{\tau(u-x)} - c^u) \\ &= \frac{E \tilde{H}(X_{[0,\tau(u-x)]}, X \circ \theta_{\tau(u-x)} - c^u)}{P(\tau(u) < T)} \\ &= \frac{E[\tilde{H}(X_{[0,\tau(u-x)]}, X \circ \theta_{\tau(u-x)} - c^u); A(u, x, T)]}{P(\tau(u) < T)} \\ &\rightarrow \frac{1}{B(T)} \int_{D \otimes D} \tilde{H}(w, w') \mu^T(dw) \otimes v^T(dw') \end{aligned}$$

as  $u \rightarrow \infty$ , then  $x \rightarrow \infty$  by (6.1) and (4.14). This last integral is absolutely convergent by (6.2). Hence, applying (A.12) to the positive and negative parts of  $H$ , this final expression may be rewritten as

$$\begin{aligned} & \frac{1}{B(T)} \int_{D \otimes D} H(w, w') I(\tau_\Delta(w) + \tau_0(w') < T) \mu^T(dw) \otimes v^T(dw') \\ &= \frac{1}{B(T)} \int_{D \otimes D} H(w, w') \int_{[0,\infty)} \mu^T(dw, dt) v^{T-t}(dw') \\ &= EH(Z_{[0,\tau]}, W) \end{aligned}$$

by (6.9).  $\square$

Thus, under  $P^{(u,T)}$ , for large  $u$ , the process  $X$  can be approximated as follows:

- run  $Z$  for times  $0 \leq t < \tau$ ;
- run  $u + W$  from time  $\tau$  on, that is, at time  $\tau + t$ , the value of the process is  $u + W_t$ .

Thus the process behaves like  $Z$  up to an independent time  $\tau$  when it jumps from a neighborhood of 0 to a neighborhood of  $u$ . Its position prior to the jump is  $Z_{\tau-}$  and its position after is  $u + W_0$ . If  $W_0 > 0$  the process  $X - u$  behaves like  $X$  started at  $W_0$ . If  $W_0 \leq 0$ , the process  $X - u$  behaves like  $X$  started at  $W_0$  and conditioned on  $\tau(0) < T - \tau$ .

**7. Conditional distribution of the first passage time and overshoot.** As an illustration of Theorem 6.2, we derive the joint limiting distribution of the first passage time  $\tau(u)$  and the overshoot  $O_u = X_{\tau(u)} - u$ , conditional on  $\tau(u) < T$ . We give two descriptions of the limit, the first in terms of the limiting variables in Theorem 6.2, and the second in terms of fluctuation quantities. This latter description allows us to relate the limiting distribution of the overshoot when  $T < \infty$  to the limiting distribution when  $T = \infty$  (as found in [22]) for the  $Ee^{\alpha X_1} < 1$  case.



**THEOREM 7.1.** *If  $\overline{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ , then for any bounded continuous function  $g : [0, \infty)^2 \rightarrow \mathbb{R}$ ,*

$$\lim_{u \rightarrow \infty} E^{(u,T)} g(O_u, \tau(u)) = E g(W_{\tau_0(W)}, \tau + \tau_0(W)).$$

*Furthermore, the limiting distribution is given by*

$$(7.1) \quad \begin{aligned} &P(W_{\tau_0(W)} \in d\gamma, \tau + \tau_0(W) \in dt) \\ &= \frac{1}{B(T)} \left( \alpha e^{-\alpha\gamma} d\gamma \mu(dt) \right. \\ &\quad \left. + \int_{z \geq 0} \alpha e^{\alpha z} dz \int_{0 \leq r \leq t} \mu(dt - r) P_0(X_{\tau(z)} - z \in d\gamma, \tau(z) \in dr) \right) \end{aligned}$$

*for  $\gamma \geq 0$  and  $0 \leq t < T$ .*

**PROOF.** Set

$$H(w, w') = g(w'_{\tau_0}, \tau_{\Delta}(w) + \tau_0(w')).$$

Then for every  $0 \leq x < u$ ,

$$H(X_{[0, \tau(u-x)]}, X \circ \theta_{\tau(u-x)} - c^u) = g(O_u, \tau(u))$$

if  $\tau(u) < \infty$ . Further,  $H \in \mathcal{H}_t$  for every  $t > 0$ , since  $H$  is of the form (3.12). Thus by (6.11) we obtain

$$\lim_{u \rightarrow \infty} E^{(u,T)} g(O_u, \tau(u)) = E g(W_{\tau_0(W)}, \tau + \tau_0(W)).$$

Now by (6.5) and (6.8),

$$\begin{aligned} &P(W_{\tau_0(W)} \in d\gamma, \tau \in ds, \tau_0(W) \in dr) \\ &= \frac{I(r < T - s)}{B(T)} \int_z \alpha e^{-\alpha z} P_z(X_{\tau(0)} \in d\gamma, \tau(0) \in dr) dz \mu^T(ds). \end{aligned}$$

Hence, letting  $\delta_{\{b\}}$  denote a point mass concentrated at  $b$ , the limiting distribution of  $(O_u, \tau(u))$  is given by

$$\begin{aligned} &P(W_{\tau_0(W)} \in d\gamma, \tau + \tau_0(W) \in dt) \\ &= \frac{1}{B(T)} \int_z \alpha e^{-\alpha z} dz \int_{0 \leq r \leq t} \mu(dt - r) P_z(X_{\tau(0)} \in d\gamma, \tau(0) \in dr) \\ &= \frac{1}{B(T)} \left( \int_{z > 0} \alpha e^{-\alpha z} dz \int_{0 \leq r \leq t} \mu(dt - r) \delta_{\{(z,0)\}}(d\gamma, dr) \right. \\ &\quad \left. + \int_{z \leq 0} \alpha e^{-\alpha z} dz \int_{0 \leq r \leq t} \mu(dt - r) P_z(X_{\tau(0)} \in d\gamma, \tau(0) \in dr) \right) \\ &= \frac{1}{B(T)} \left( \alpha e^{-\alpha\gamma} d\gamma \mu(dt) \right. \\ &\quad \left. + \int_{z \geq 0} \alpha e^{\alpha z} dz \int_{0 \leq r \leq t} \mu(dt - r) P_0(X_{\tau(z)} - z \in d\gamma, \tau(z) \in dr) \right) \end{aligned}$$

for  $\gamma \geq 0$  and  $0 \leq t < T$ .  $\square$

The expression for the limiting distribution in (7.1) may also be written in terms of fluctuation quantities by using the quintuple law of Doney and Kyprianou [12]. In order to do so, we first need to introduce some further notation which is standard in the area; cf. [4, 11, 23]. Recall from Section 2 that  $(L_t)_{t \geq 0}$  is the local time of  $X$  at its maximum. Let  $(L_t^{-1}, H_t)_{t \geq 0}$  be the bivariate ascending ladder process and  $\Pi_{L^{-1}, H}(\cdot, \cdot)$  its Lévy measure. The bivariate renewal function of  $(L^{-1}, H)$  is

$$(7.2) \quad V(s, z) = \int_{t \geq 0} P(L_t^{-1} \leq s, H_t \leq z; t < L_\infty) dt,$$

with associated renewal measure  $V(ds, dz)$ . The Lévy measure of  $H$  will be denoted  $\Pi_H$ , and its Laplace exponent by  $\kappa$  where

$$(7.3) \quad \begin{aligned} \kappa(\beta) &:= -\ln(E(e^{-\beta H_1}; 1 < L_\infty)) \\ &= q + \beta d_H + \int_{y \geq 0} (1 - e^{-\beta y}) \Pi_H(dy). \end{aligned}$$

Here  $d_H \geq 0$  is the drift and  $q \geq 0$  is the killing rate of  $H$  (see, e.g., [23], (6.15) and (6.16)).

Define measures  $\eta_\alpha^V(ds)$  and  $\eta_\alpha^{\Pi_{L^{-1}, H}}(ds)$  on  $[0, \infty)$  by

$$\eta_\alpha^V(ds) = \int_z e^{\alpha z} V(ds, dz), \quad \eta_\alpha^{\Pi_{L^{-1}, H}}(ds) = \int_z (e^{\alpha z} - 1) \Pi_{L^{-1}, H}(ds, dz).$$

If  $Ee^{\alpha X_1} < \infty$  then, as we now show,  $\eta_\alpha^{\Pi_{L^{-1}, H}}$  is a finite measure, while  $\eta_\alpha^V$  is finite on compact sets.

**PROPOSITION 7.1.** *If  $Ee^{\alpha X_1} < \infty$  then  $\int_{z \geq 1} e^{\alpha z} \Pi_H(dz) < \infty$ .*

**PROOF.** By Vigon’s equation amicale inversée [30], for  $z > 0$

$$\Pi_H(dz) = k \int_{y \geq 0} \widehat{V}(dy) \Pi_X(y + dz),$$

where  $\widehat{V}$  is the renewal function of the descending ladder height process, and  $k > 0$  is a constant depending on the normalizations of the local times. Thus

$$\begin{aligned} \int_{z \geq 1} e^{\alpha z} \Pi_H(dz) &= k \int_{z \geq 1} e^{\alpha z} \int_{y \geq 0} \widehat{V}(dy) \Pi_X(y + dz) \\ &= k \int_{y \geq 0} \widehat{V}(dy) \int_{x \geq y+1} e^{\alpha(x-y)} \Pi_X(dx) \\ &\leq k \int_{y \geq 0} e^{-\alpha y} \widehat{V}(dy) \int_{x \geq 1} e^{\alpha x} \Pi_X(dx). \end{aligned}$$

The first integral is finite since for some  $c > 0$ ,  $\widehat{V}([0, y]) \leq cy$  for large  $y$  by [4], Proposition III.1, and the second is finite by [28], Theorem 25.17.  $\square$

Finiteness of  $\eta_\alpha^{\Pi_{L^{-1},H}}$  follows immediately from Proposition 7.1. For  $\eta_\alpha^V$ , Proposition 7.1 and [28], Theorem 25.17, imply that when  $Ee^{\alpha X_1} < \infty$ , we also have  $E(e^{\alpha H_1}; 1 < L_\infty) < \infty$ . Hence, by dominated convergence, for  $a$  sufficiently large,  $E(e^{-aL_1^{-1} + \alpha H_1}; 1 < L_\infty) < 1$ . Thus, for such  $a$ ,

$$\begin{aligned} \int_s e^{-as} \eta_\alpha^V(ds) &= \int_s \int_z e^{-as + \alpha z} \int_{t \geq 0} P(L_t^{-1} \in ds, H_t \in dz, t < L_\infty) dt \\ &= \int_{t \geq 0} (E(e^{-aL_1^{-1} + \alpha H_1}; 1 < L_\infty))^t dt < \infty, \end{aligned}$$

showing that  $\eta_\alpha^V$  is finite on compact sets.

**THEOREM 7.2.** *The limiting distribution in (7.1) may be written alternatively as*

$$\begin{aligned} (7.4) \quad &P(W_{\tau_0(W)} \in d\gamma, \tau + \tau_0(W) \in dt) \\ &= \frac{\alpha}{B(T)} \left( \mu(dt) e^{-\alpha\gamma} d\gamma + d_H(\mu * \eta_\alpha^V)(dt) \delta_{\{0\}}(d\gamma) \right. \\ &\quad \left. + \int_{0 \leq s \leq t} (\mu * \eta_\alpha^V)(dt - s) \int_{y \geq 0} e^{\alpha y} \Pi_{L^{-1},H}(ds, y + d\gamma) dy \right) \end{aligned}$$

for  $\gamma \geq 0$  and  $0 \leq t < T$ .

**PROOF.** By [15], Corollary 3.1, for  $z > 0$  and  $\gamma, r \geq 0$ ,

$$\begin{aligned} (7.5) \quad &P_0(X_{\tau(z)} - z \in d\gamma; \tau(z) \in dr) \\ &= d_H \frac{\partial_-}{\partial_- z} V(dr, z) \delta_{\{0\}}(d\gamma) \\ &\quad + I(\gamma > 0) \int_{0 \leq s \leq r} \int_{0 \leq y \leq z} V(ds, z - dy) \Pi_{L^{-1},H}(dr - s, y + d\gamma), \end{aligned}$$

where  $\partial_-/\partial_-z$  denotes the left derivative. A straightforward calculation involving changes of variable and orders of integration, shows that

$$\begin{aligned} (7.6) \quad &I(\gamma > 0) \int_{z \geq 0} \alpha e^{\alpha z} dz \int_{0 \leq y \leq z} V(ds, z - dy) \Pi_{L^{-1},H}(dr - s, y + d\gamma) \\ &= I(\gamma > 0) \alpha \int_{\zeta \geq 0} e^{\alpha \zeta} V(ds, d\zeta) \int_{y \geq 0} e^{\alpha y} \Pi_{L^{-1},H}(dr - s, y + d\gamma) dy \\ &= \alpha \eta_\alpha^V(ds) \int_{y \geq 0} e^{\alpha y} \Pi_{L^{-1},H}(dr - s, y + d\gamma) dy, \end{aligned}$$

where the  $I(\gamma > 0)$  term may be omitted in the final expression since the measure there assigns no mass to the set  $\{\gamma = 0\}$ . On the other hand, if  $X$  creeps, that is  $d_H > 0$ , then from [15], Theorem 3.1(ii),

$$(7.7) \quad \int_{z \geq 0} \alpha e^{\alpha z} dz \frac{\partial_-}{\partial_- z} V(dr, z) = \alpha \eta_\alpha^V(dr).$$

Thus substituting (7.5), (7.6) and (7.7) into (7.1) gives

$$\begin{aligned} &P(W_{\tau_0(W)} \in d\gamma, \tau + \tau_0(W) \in dt) \\ &= \frac{\alpha}{B(T)} \left( \mu(dt) e^{-\alpha\gamma} d\gamma + d_H(\mu * \eta_\alpha^V)(dt) \delta_{\{0\}}(d\gamma) \right. \\ &\quad \left. + \int_{0 \leq r \leq t} \mu(dt - r) \int_{0 \leq s \leq r} \eta_\alpha^V(ds) \right. \\ &\quad \left. \times \int_{y \geq 0} e^{\alpha y} \Pi_{L^{-1}, H}(dr - s, y + d\gamma) dy \right), \end{aligned}$$

which is the same as (7.4).  $\square$

Each of the three terms in (7.4) has a clear meaning. In order to exit by time  $T$ , the process must take a large jump from a neighborhood of the origin to a neighborhood of the boundary. The first term is a consequence of this jump overshooting the boundary. If the jump undershoots the boundary, then the process crosses the boundary either by creeping, which leads to the second term, or by taking a further (small) jump which results in the final term. From this description we can read off, for example, that the limiting (sub-)distribution of the time at which the conditioned process creeps over the boundary is given by  $B(T)^{-1} \alpha d_H(\mu * \eta_\alpha^V)$ .

The marginal distributions can be obtained from either (7.1) or (7.4). We will focus on the latter, but mention in passing that the expression for the marginal distribution in  $t$  obtained from (7.1), is actually a simple consequence of Theorem 4.1; for  $0 \leq t < T$

$$(7.8) \quad P(\tau + \tau_0(W) \leq t) = \frac{B(t)}{B(T)} = \frac{1}{B(T)} \int_{[0, t]} e^{\psi(\alpha)(t-s)} E e^{\alpha \bar{X}_s} ds.$$

By integrating out  $\gamma$  in (7.4), and noting that

$$(7.9) \quad \alpha \int_{y \geq 0} \int_{\gamma \geq 0} e^{\alpha y} \Pi_{L^{-1}, H}(ds, y + d\gamma) dy = \eta_\alpha^{\Pi_{L^{-1}, H}}(ds),$$

we obtain the alternative characterization

$$(7.10) \quad \begin{aligned} &P(\tau + \tau_0(W) \in dt) \\ &= \frac{1}{B(T)} (\mu(dt) + \alpha d_H(\mu * \eta_\alpha^V)(dt) + (\mu * \eta_\alpha^V * \eta_\alpha^{\Pi_{L^{-1}, H}})(dt)) \end{aligned}$$

for  $0 \leq t < T$ . Similarly, the marginal distribution in  $\gamma$  obtained from (7.4) is

$$\begin{aligned}
 & P(W_{\tau_0(W)} \in d\gamma) \\
 (7.11) \quad &= \frac{\alpha}{B(T)} \left( \mu(T) e^{-\alpha\gamma} d\gamma + d_H(\mu * \eta_\alpha^V)(T) \delta_{\{0\}}(d\gamma) \right. \\
 & \quad \left. + \int_{0 \leq s < T} (\mu * \eta_\alpha^V)(T-s) \int_{y \geq 0} e^{\alpha y} \Pi_{L^{-1}, H}(ds, y + d\gamma) dy \right),
 \end{aligned}$$

where, recall, for any measure  $\eta$  and any  $t$ ,  $\eta(t) = \eta([0, t])$ .

When  $Ee^{\alpha X_1} < 1$  the limiting distribution of the overshoot was found in [22] for the case  $T = \infty$ , that is, conditional on  $\tau(u) < \infty$ . To relate the result in [22] to (7.11), we investigate the limit as  $T \rightarrow \infty$  in (7.11). For this, we recall (2.4), which may be rewritten

$$(7.12) \quad Ee^{\alpha \bar{X}_\infty} = q \eta_\alpha^V(\infty).$$

**THEOREM 7.3.** *If  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$  and  $Ee^{\alpha X_1} < 1$ , then as  $T \rightarrow \infty$ ,*

$$\begin{aligned}
 & P(W_{\tau_0(W)} \in d\gamma) \\
 (7.13) \quad & \rightarrow \frac{\alpha e^{-\alpha\gamma} d\gamma}{Ee^{\alpha \bar{X}_\infty}} + \frac{\alpha}{q} \left( d_H \delta_0(d\gamma) + \int_{y \geq 0} e^{\alpha y} \Pi_H(y + d\gamma) dy \right),
 \end{aligned}$$

where convergence is in the total variation norm.

**PROOF.** Since  $\psi(\alpha) < 0$  when  $Ee^{\alpha X_1} < 1$ , on letting  $T \rightarrow \infty$  we obtain

$$(7.14) \quad \mu(T) \rightarrow \mu(\infty) < \infty,$$

while by Lemma 2.1, (7.12) and monotone convergence

$$(7.15) \quad (\mu * \eta_\alpha^V)(T) = \int_{0 \leq t < T} \eta_\alpha^V(T-t) \mu(dt) \rightarrow \eta_\alpha^V(\infty) \mu(\infty) < \infty$$

and

$$\begin{aligned}
 & \int_{0 \leq s < T} (\mu * \eta_\alpha^V)(T-s) \int_{y \geq 0} e^{\alpha y} \Pi_{L^{-1}, H}(ds, y + d\gamma) dy \\
 (7.16) \quad & \uparrow \eta_\alpha^V(\infty) \mu(\infty) \int_{s \geq 0} \int_{y \geq 0} e^{\alpha y} \Pi_{L^{-1}, H}(ds, y + d\gamma) dy \\
 & = \eta_\alpha^V(\infty) \mu(\infty) \int_{y \geq 0} e^{\alpha y} \Pi_H(y + d\gamma) dy,
 \end{aligned}$$

where convergence is in total variation by monotonicity. Also, again by monotone convergence

$$(7.17) \quad B(T) = \int_{0 \leq t < T} Ee^{\alpha \bar{X}_{T-t}} \mu(dt) \rightarrow Ee^{\alpha \bar{X}_\infty} \mu(\infty) < \infty.$$

Thus (7.13) follows by letting  $T \rightarrow \infty$  in (7.11) and using (7.12).  $\square$

The limiting distribution in (7.13) agrees with the limiting distribution of the overshoot conditional on  $\tau(u) < \infty$  which was found in [22] (see also [12] and [16], (7.5)). Since the only possible atoms in the limiting distributions are at 0, it thus follows that

$$(7.18) \quad \lim_{T \rightarrow \infty} \lim_{u \rightarrow \infty} P^{(u,T)}(O_u < x) = \lim_{u \rightarrow \infty} \lim_{T \rightarrow \infty} P^{(u,T)}(O_u < x)$$

for every  $x \geq 0$ , when  $Ee^{\alpha X_1} < 1$ .

It is interesting to note that if  $Ee^{\alpha X_1} = 1$ , then  $Ee^{\alpha \bar{X}_\infty} = \infty$  [see (7.20) below], and hence, formally the limit in (7.13) becomes

$$(7.19) \quad P(W_{\tau_0(W)} \in d\gamma) \rightarrow \frac{\alpha}{q} \left( d_H \delta_0(d\gamma) + \int_{y \geq 0} e^{\alpha y} \Pi_H(y + d\gamma) dy \right).$$

Under the (minor) additional Cramér–Lundberg assumption (4.18), this again agrees with the limiting distribution of the overshoot conditional on  $\tau(u) < \infty$  (see, e.g., [17]) and so (7.18) continues to hold. However, the argument given above is not rigorous in this case as all the limiting quantities in (7.14)–(7.17) are infinite and hence, cannot be canceled. To prove (7.19), more care needs to be taken with the limiting operations. To this end, we begin by recalling that  $Ee^{\alpha X_1} = 1$  is equivalent to  $\psi(\alpha) = 0$ . Consequently, by the Wiener–Hopf factorization,

$$k\kappa(-\alpha)\widehat{\kappa}(\alpha) = -\psi(\alpha) = 0.$$

Here  $\widehat{\kappa}$  is the Laplace exponent of the descending ladder process  $\widehat{H} \geq 0$  and  $k > 0$  is some constant depending on the normalization of the local times (see, e.g., [23], Theorem 6.16(iv)). Since  $X_t \rightarrow -\infty$  a.s. when  $Ee^{\alpha X_1} = 1$ , it follows that  $\widehat{H}$  is a proper (not killed) subordinator, and hence,  $Ee^{-\alpha \widehat{H}_1} = e^{-\widehat{\kappa}(\alpha)} < 1$ . Thus, if  $\psi(\alpha) = 0$ , then  $\kappa(-\alpha) = 0$ , and so by (7.2) and (7.12),

$$(7.20) \quad Ee^{\alpha \bar{X}_\infty} = q \int_x^\infty e^{\alpha x} \int_{t=0}^\infty P(H_t \in dx, t < L_\infty) dt = \int_0^\infty e^{-\kappa(-\alpha)t} dt = \infty.$$

LEMMA 7.1. Assume  $Ee^{\alpha X_1} = 1$  and (4.18), then

$$(7.21) \quad \lim_{T \rightarrow \infty} \frac{B(T-t)}{B(T)} \rightarrow 1.$$

PROOF. First observe that from (5.7)

$$(7.22) \quad \frac{B(T)}{T} = \frac{1}{T} \int_{0 \leq s < T} Ee^{\alpha \bar{X}_s} ds \rightarrow \infty$$

since  $Ee^{\alpha\bar{X}_\infty} = \infty$ . Thus by (4.19), for fixed  $t$ ,

$$\begin{aligned} \frac{1}{B(T)} \int_{T-t \leq s < T} Ee^{\alpha\bar{X}_s} ds &\leq \frac{1}{B(T)} \int_{T-t \leq s < T} C(1+s) ds \\ &\leq \frac{C(1+T)t}{B(T)} \rightarrow 0 \end{aligned}$$

as  $T \rightarrow \infty$ . Hence, (7.21) holds.  $\square$

LEMMA 7.2. Assume  $\bar{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ ,  $Ee^{\alpha X_1} = 1$  and (4.18), then

$$(7.23) \quad \lim_{T \rightarrow \infty} \frac{\mu(T)}{B(T)} = 0 \quad \text{and} \quad \lim_{T \rightarrow \infty} \frac{(\mu * \eta_\alpha^V)(T)}{B(T)} = \frac{1}{q}.$$

PROOF. The first limit follows immediately from (7.22) since  $\mu(T) = T$ . For the second limit, first observe that by (7.3) and monotone convergence, as  $T \uparrow \infty$ ,

$$(7.24) \quad \alpha d_H + \eta_\alpha^{\Pi_{L^{-1}, H}}(T) \uparrow \alpha d_H + \int_{y \geq 0} (e^{\alpha y} - 1) \Pi_H(dy) = q - \kappa(-\alpha) = q$$

Now by (7.10)

$$\begin{aligned} B_T &= \mu(T) + \alpha d_H (\mu * \eta_\alpha^V)(T) + (\mu * \eta_\alpha^V * \eta_\alpha^{\Pi_{L^{-1}, H}})(T) \\ &= \mu(T) + \int_{t < T} [\alpha d_H + \eta_\alpha^{\Pi_{L^{-1}, H}}(T-t)] (\mu * \eta_\alpha^V)(dt) \\ &\leq \mu(T) + [\alpha d_H + \eta_\alpha^{\Pi_{L^{-1}, H}}(T)] (\mu * \eta_\alpha^V)(T). \end{aligned}$$

Thus by (7.24)

$$\liminf_{T \rightarrow \infty} \frac{(\mu * \eta_\alpha^V)(T)}{B(T)} \geq \frac{1}{q}.$$

For the upper bound, fix  $T_0 > 0$ . Then by (7.10)

$$\begin{aligned} B_{T+T_0} &\geq \mu(T+T_0) + \int_{t < T} [\alpha d_H + \eta_\alpha^{\Pi_{L^{-1}, H}}(T+T_0-t)] (\mu * \eta_\alpha^V)(dt) \\ &\geq [\alpha d_H + \eta_\alpha^{\Pi_{L^{-1}, H}}(T_0)] (\mu * \eta_\alpha^V)(T). \end{aligned}$$

Dividing by  $B(T)$  and using (7.21), gives

$$\limsup_{T \rightarrow \infty} \frac{(\mu * \eta_\alpha^V)(T)}{B(T)} \leq \frac{1}{\alpha d_H + \eta_\alpha^{\Pi_{L^{-1}, H}}(T_0)}.$$

Now let  $T_0 \rightarrow \infty$ .  $\square$

THEOREM 7.4. Assume  $\overline{\Pi}_X^+ \in \mathcal{S}^{(\alpha)}$ ,  $Ee^{\alpha X_1} = 1$  and (4.18), then as  $T \rightarrow \infty$ ,

$$(7.25) \quad P(W_{\tau_0(W)} \in d\gamma) \rightarrow \frac{\alpha}{q} \left( d_H \delta_0(d\gamma) + \int_{y \geq 0} e^{\alpha y} \Pi_H(y + d\gamma) dy \right),$$

where convergence is in the total variation norm.

PROOF. Let

$$f_T(s) = \frac{I(0 \leq s < T)(\mu * \eta_\alpha^V)(T - s)}{B(T)}.$$

Then by (7.21) and (7.23), for fixed  $s$ ,

$$(7.26) \quad f_T(s) \rightarrow \frac{1}{q},$$

while

$$(7.27) \quad \sup_s f_T(s) \leq \frac{(\mu * \eta_\alpha^V)(T)}{B(T)} \rightarrow \frac{1}{q}.$$

Now from (7.9) and (7.11), for any Borel set  $C \subset [0, \infty)$ ,

$$\begin{aligned} & \left| P(W_{\tau_0(W)} \in C) - \frac{\alpha}{q} \left( d_H I_C(0) + \int_{y \geq 0} e^{\alpha y} \Pi_H(y + C) dy \right) \right| \\ & \leq \frac{\mu(T)}{B(T)} + \alpha d_H \left| \frac{(\mu * \eta_\alpha^V)(T)}{B(T)} - q^{-1} \right| \\ & \quad + \int_{s \geq 0} |f_T(s) - q^{-1}| \int_{y \geq 0} \alpha e^{\alpha y} \Pi_{L^{-1}, H}(ds, y + C) dy \\ & \leq \frac{\mu(T)}{B(T)} + \alpha d_H \left| \frac{(\mu * \eta_\alpha^V)(T)}{B(T)} - q^{-1} \right| + \int_{s \geq 0} |f_T(s) - q^{-1}| \eta_\alpha^{\Pi_{L^{-1}, H}}(ds). \end{aligned}$$

Since  $\eta_\alpha^{\Pi_{L^{-1}, H}}(ds)$  is a finite measure, the result follows by taking the supremum over all  $C$  and using (7.23), (7.26), (7.27) and bounded convergence.  $\square$

When  $Ee^{\alpha X_1} > 1$  it is possible that  $q > 0$  or  $q = 0$ . In either case it seems more difficult to obtain an analogue of (7.25), in part because (7.21) no longer holds. One case in which the limit in (7.25) can be found is when  $X$  is a subordinator, and so  $q = 0$ . In this case we may take  $H = X$ , and similar calculations to those above lead to

$$P(W_{\tau_0(W)} \in d\gamma) \rightarrow \frac{\alpha}{\psi(\alpha)} \left( d_H \delta_0(d\gamma) + \int_{y \geq 0} e^{\alpha y} \Pi_H(y + d\gamma) dy \right).$$

On the other hand, since  $X_t \rightarrow \infty$  a.s.,

$$\lim_{T \rightarrow \infty} P^{(u, T)}(O_u \in d\gamma) = P(O_u \in d\gamma),$$



and by standard renewal theory (see, e.g., [23], Theorem 5.7)

$$\lim_{u \rightarrow \infty} P(O_u \in d\gamma) = \frac{1}{m} \left( d_H \delta_0(d\gamma) + \int_{y \geq 0} \Pi_H(y + d\gamma) dy \right),$$

where  $m = EH_1 = EX_1$ . In particular, this shows that (7.18) no longer holds when  $Ee^{\alpha X_1} > 1$ .

A similar discussion applies to the first passage time; if  $Ee^{\alpha X_1} < 1$  then by (4.14), (4.22) and (4.23)

$$(7.28) \quad \lim_{T \rightarrow \infty} \lim_{u \rightarrow \infty} P^{(u,T)}(\tau(u) < t) = \lim_{u \rightarrow \infty} \lim_{T \rightarrow \infty} P^{(u,T)}(\tau(u) < t)$$

for all  $t \geq 0$ . When  $Ee^{\alpha X_1} \geq 1$ , letting  $T \rightarrow \infty$  in (7.8) shows that for all  $t \geq 0$

$$\lim_{T \rightarrow \infty} \lim_{u \rightarrow \infty} P^{(u,T)}(\tau(u) < t) = 0,$$

while by Theorem 4.1 and Remark 4.3

$$\lim_{u \rightarrow \infty} \lim_{T \rightarrow \infty} P^{(u,T)}(\tau(u) < t) = \lim_{u \rightarrow \infty} P(\tau(u) < t | \tau(u) < \infty) = 0.$$

Hence, (7.28) is also valid in this case, but in a degenerate sense.

From the calculations presented in this section, it is hopefully clear that the asymptotic behavior of many other functionals of the path can be investigated in a similar manner.

### APPENDIX

The Appendix gives more details on several formulas involving  $\mu_K$  and  $\nu_x$  where  $K, x \in (-\infty, \infty]$ . They are first defined for product sets  $(A \times B \times C) \in \mathcal{F} \otimes \mathcal{B}([0, \infty)) \otimes \mathcal{B}$  by

$$\mu_K(A \times B \times C) = \int_{t \in B} E(e^{\alpha X_{t-}}; X_{[0,t)} \in A, X_{t-} \in C, X_{t-} < K) dt$$

and

$$\nu_x(A \times B \times C) = \int_{\{z > -x\} \cap C} \alpha e^{-\alpha z} P_z(X \in A, \tau(0) \in B) dz$$

and then extended to measures on  $\mathcal{F} \otimes \mathcal{B}([0, \infty)) \otimes \mathcal{B}$ .

LEMMA A.1. For each  $T \geq 0$  the following equality of measures holds:

$$(A.1) \quad I(t < T) \mu_K(dw, dt, d\phi) = I(\tau_\Delta(w) < T) \mu_K(dw, dt, d\phi)$$

and

$$(A.2) \quad I(r < T) \nu_x(dw', dr, dz) = I(\tau_0(w') < T) \nu_x(dw', dr, dz).$$

PROOF. For (A.1), it suffices to show

$$\int_{A \times B \times C} I(t < T) \mu_K(dw, dt, d\phi) = \int_{A \times B \times C} I(\tau_\Delta(w) < T) \mu_K(dw, dt, d\phi)$$

for every  $(A \times B \times C) \in \mathcal{F} \otimes \mathcal{B}([0, \infty)) \otimes \mathcal{B}$ . Let  $\{w : \tau_\Delta(w) < T\} = A_1 \in \mathcal{F}$ . Observe that

$$X_{[0,t]} \in AA_1 \quad \text{iff } X_{[0,t]} \in A \text{ and } t < T.$$

Thus

$$\begin{aligned} & \int_{A \times B \times C} I(\tau_\Delta(w) < T) \mu_K(dw, dt, d\phi) \\ &= \int_{t \in B} E(e^{\alpha X_{t-}}; X_{[0,t]} \in AA_1, X_{t-} \in C, X_{t-} < K) dt \\ &= \int_{t \in B} I(t < T) E(e^{\alpha X_{t-}}; X_{[0,t]} \in A, X_{t-} \in C, X_{t-} < K) dt \\ &= \mu_K(A \times (B \cap [0, T]) \times C) \\ &= \int_{A \times B \times C} I(t < T) \mu_K(dw, dt, d\phi). \end{aligned}$$

The proof of (A.2) is analogous.  $\square$

LEMMA A.2. For any nonnegative measurable function  $H : D \rightarrow \mathbb{R}$

$$(A.3) \quad \int_D H(w) \mu_K(dw) = \int_0^\infty E(e^{\alpha X_{t-}} H(X_{[0,t]}); X_{t-} < K) dt$$

and

$$(A.4) \quad \int_D H(w') \nu_x(dw') = \int_{z > -x} \alpha e^{-\alpha z} E_z(H(X); \tau(0) < \infty) dz.$$

PROOF. Let  $H = 1_A$  for  $A \in \mathcal{F}$ . Then

$$\begin{aligned} & \int_0^\infty E(e^{\alpha X_{t-}} H(X_{[0,t]}); X_{t-} < K) dt \\ &= \int_0^\infty E(e^{\alpha X_{t-}}; X_{[0,t]} \in A, X_{t-} < K) dt \\ &= \mu_K(A) = \int_D H(w) \mu_K(dw). \end{aligned}$$

Formula (A.3) then follows by standard arguments. The proof of (A.4) is similar.  $\square$

Applying (A.3) to the function  $H(w)f(\tau_\Delta(w))g(w_{\tau_\Delta-}, \bar{w}_{\tau_\Delta-})I(\tau_\Delta(w) < T)$  where  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R}^2 \rightarrow \mathbb{R}$  are nonnegative Borel functions, gives

$$\begin{aligned}
 (A.5) \quad & \int_D H(w)f(\tau_\Delta(w))g(w_{\tau_\Delta-}, \bar{w}_{\tau_\Delta-})\mu_K^T(dw) \\
 &= \int_{[0,T)} f(t)E(e^{\alpha X_{t-}}H(X_{[0,t)})g(X_{t-}, \bar{X}_{t-}); X_{t-} < K) dt.
 \end{aligned}$$

As a special case we obtain

$$\begin{aligned}
 (A.6) \quad & \int_D H(w)f(\tau_\Delta(w))\mu_K^T(dw) \\
 &= \int_{[0,T)} f(t)E(e^{\alpha X_{t-}}H(X_{[0,t)}); X_{t-} < K) dt \\
 &= \int_{[0,T)} f(t) \int_D H(w)E(e^{\alpha X_{t-}}; X_{[0,t)} \in dw, X_{t-} < K) dt \\
 &= \int_{D \otimes [0,\infty)} H(w)f(t)\mu_K^T(dw, dt).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 (A.7) \quad & \int_D H(w')f(\tau_0(w'))v_x^T(dw) \\
 &= \int_{[0,T)} f(r) \int_{z>-x} \alpha e^{-\alpha z} E_z(H(X); \tau(0) \in dr) \\
 &= \int_{D \otimes [0,\infty)} H(w')f(r)v_x^T(dw', dr).
 \end{aligned}$$

LEMMA A.3. *If  $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}$  are nonnegative Borel functions then*

$$\begin{aligned}
 (A.8) \quad & \int_{D \otimes D} f(\tau_\Delta(w), \tau_0(w'))g(w_{\tau_\Delta-}, \bar{w}_{\tau_\Delta-})\mu_K^T(dw) \otimes v_x^T(dw') \\
 &= \int_{[0,T)} \int_{[0,T)} f(t, r)E(e^{\alpha X_{t-}}g(X_{t-}, \bar{X}_{t-}); X_{t-} < K) dt v_x(dr).
 \end{aligned}$$

PROOF. By (A.5) and (A.7), both with  $H \equiv 1$ , (A.8) holds if  $f(t, r) = f_1(t)f_2(r)$ . The general result then follows by standard arguments.  $\square$

Setting  $g \equiv 1$  gives

$$\begin{aligned}
 (A.9) \quad & \int_{D \otimes D} f(\tau_\Delta(w), \tau_0(w'))\mu_K^T(dw) \otimes v_x^T(dw') \\
 &= \int_0^\infty \int_0^\infty f(t, r)\mu_K^T(dt)v_x^T(dr).
 \end{aligned}$$

Taking  $g(y_1, y_2) = 1_{(-\infty, K)}(y_2)$  yields

$$\begin{aligned}
 & \int_{D \otimes D} f(\tau_\Delta(w), \tau_0(w')) I(\bar{w}_{\tau_\Delta-} < K) \mu_K^T(dw) \otimes v_x^T(dw') \\
 \text{(A.10)} \quad &= \int_0^\infty \int_0^\infty f(t, r) I(t < T) E(e^{\alpha X_{t-}}; \bar{X}_{t-} < K) dt v_x^T(dr) \\
 &= \int_0^\infty \int_0^\infty f(t, r) \bar{\mu}_K^T(dt) v_x^T(dr),
 \end{aligned}$$

where

$$\bar{\mu}_K^T(dt) = I(t < T) E(e^{\alpha X_{t-}}; \bar{X}_{t-} < K) dt.$$

LEMMA A.4. *Let  $H : D \otimes D \rightarrow \mathbb{R}$  be nonnegative and measurable, and  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be nonnegative and Borel, then*

$$\begin{aligned}
 & \int_{D \otimes D} H(w, w') f(\tau_\Delta(w), \tau_0(w')) \mu_K^T(dw) \otimes v_x^T(dw') \\
 \text{(A.11)} \quad &= \int_{(D \otimes [0, \infty)) \otimes (D \otimes [0, \infty))} H(w, w') f(t, r) \mu_K^T(dw, dt) \otimes v_x^T(dw', dr).
 \end{aligned}$$

PROOF. If  $H(w, w') = H_1(w)H_2(w')$  and  $f(t, r) = f_1(t)f_2(r)$ , then (A.11) holds by (A.6) and (A.7). The result then follows by standard arguments.  $\square$

As a special case we obtain

$$\begin{aligned}
 & \int_{D \otimes D} H(w, w') I(\tau_\Delta(w) + \tau_0(w') < T) \mu_K^T(dw) \otimes v_x^T(dw') \\
 \text{(A.12)} \quad &= \int_{D \otimes D} H(w, w') \int_{[0, \infty)} \mu_K^T(dw, dt) v_x^{T-t}(dw').
 \end{aligned}$$

**Acknowledgments.** The author would like to thank Ross Maller for his hospitality and willing ear while portions of this work were being carried out. Thanks also to an anonymous referee for some helpful comments.

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