The Annals of Applied Probability
2012, Vol. 22, No. 5, 2133–2164
DOI: 10.1214/11-AAP816

STOCHASTIC APPROXIMATION, COOPERATIVE DYNAMICS AND SUPERMODULAR GAMES[1]

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This paper considers a stochastic approximation algorithm, with decreasing step size and martingale difference noise. Under very mild assumptions, we prove the nonconvergence of this process toward a certain class of repulsive sets for the associated ordinary differential equation (ODE). We then use this result to derive the convergence of the process when the ODE is *cooperative* in the sense of Hirsch [*SIAM J. Math. Anal.* **16** (1985) 423–439]. In particular, this allows us to extend significantly the main result of Hofbauer and Sandholm [*Econometrica* **70** (2002) 2265–2294] on the convergence of *stochastic fictitious play* in *supermodular games*.

1. Introduction. Let $(x_n)_{n\in\mathbb{N}}$ be an $\mathbb{R}^d$-valued *stochastic approximation process* [Robbins and Monro (1951), Kiefer and Wolfowitz (1952)] whose general form can be written as

$$x_{n+1} - x_n = \frac{1}{n+1}\big(F(x_n) + U_{n+1}\big), \tag{1}$$

where $F:\mathbb{R}^d \to \mathbb{R}^d$ is a vector field and (U_n) an $\mathbb{R}^d$-valued martingale differences sequence. Originally, Robbins–Monro algorithms were designed for finding the zeroes of a given deterministic function. They are now widely used in several fields (including signal processing, learning and game theory, optimization, etc.) either as stochastic algorithms or models of learning and evolution.

To obtain information on the behavior of the sample paths (x_n) it is natural to compare them to the trajectories of the ordinary differential equation

$$\dot{x} = F(x). \tag{2}$$

This method—usually referred to as the *ODE method*—was first introduced in Ljung (1977) and has been developed by many authors [including Kushner and Clark (1978), Benveniste, Metivier and Priouret (1990), Duflo (1996), Kushner and Yin (2003)] for very simple dynamics (e.g., linear or gradient-like).

In a series of papers initiated in the late 1990s, the first author and collaborators have formulated an approach to stochastic approximation based on notions

Received December 2010; revised September 2011.

[1]Supported by the Swiss National Science Foundation Grant 200021-103625/1.

MSC2010 subject classifications. Primary 62L20, 37C50; secondary 37C65, 91A12.

Key words and phrases. Stochastic approximation, cooperative systems, stochastic fictitious play, supermodular games.

of dynamical system theory, showing that the asymptotic behavior of (x_n) can be described with a great deal of generality through (2), beyond gradients and simple dynamics. We refer the reader to the survey paper [Pemantle (2007)] for a recent introduction to this approach including several examples, references and discussion of the literature; and to Benaïm (1999) for a comprehensive presentation.

One of the key results is that limit sets of (x_n) are almost surely *internally chain transitive* (ICT) in the sense of Conley (1978). Examples of ICT sets include equilibria, periodic orbits and omega limit sets of (2) but also possibly more complicated sets.

On the other hand, due to the stochastic nature of (1), not every ICT set is likely to be the limit set of (x_n) because the noise may push the process away from certain "unstable" sets. A first result in this direction has been obtained by Pemantle (1990) [see also Brandière and Duflo (1996) and Tarrès (2001)] who proved that, under natural assumptions on (U_n), (x_n) has zero probability to converge toward a linearly unstable equilibrium of (2), provided F is C^2. This later result has been extended to linearly unstable periodic orbits by Benaïm and Hirsch (1995) and to C^2 normally hyperbolic repulsive sets in Benaïm (1999).

The characterization of the limit sets of (1) in terms of ICT sets and the non-convergence results toward unstable equilibria and periodic orbit have been successfully used by a number of authors for analyzing stochastic approximation processes in ecology, game theory, engineering and elsewhere [see, e.g., Section 4 of Pemantle (2007)].

The present paper continues this line of research. Our main motivation is to investigate the behavior of (x_n) when F is *cooperative*; meaning that F is C^k, $k \geq 1$, and

$$\frac{\partial F_i}{\partial x_j} \geq 0 \qquad \text{for } i \neq j.$$

Stochastic approximation processes associated to a cooperative vector field arise as models of learning and evolution in neural networks [Sadeghi (1998)], coordination games [Benaïm and Hirsch (1999a)], supermodular games [Hofbauer and Sandholm (2002)], proportional fair sharing algorithms [Kushner and Whiting (2004)] and Bayesian games [Beggs (2009)].

We will prove (Theorem 4.4 and Corollary 4.6 in Section 4) that, under certain conditions on (U_n):

THEOREM A. *If F is cooperative and irreducible, then (x_n) converges almost surely to the zeroes of F.*

This answers (partially) a conjecture raised in Benaïm (2000). In the context of learning in games, this implies the (almost sure) convergence of the method of *stochastic fictitious play* for supermodular games in full generality, a property conjectured by Hofbauer and Sandholm (2002).

The program leading to the proof of Theorem A began several years ago. The first step has been to identify ICT sets of cooperative vector fields. This has been achieved through the papers Hirsch (1999), Benaïm and Hirsch (1999b) and Benaïm (2000), relying heavily on the seminal work of Hirsch on cooperative dynamics. In brief, an ICT set for a cooperative vector field is either an arc of equilibria (generically an isolated equilibrium) or is contained in a normally repulsive C^1 hypersurface (trajectories are exponentially repelled, in a normal direction).

The second step consists in ruling out such repelling sets as possible limit sets of (1). The main difficulty is that the techniques used in the proof of the nonconvergence results mentioned above require smoother manifold (typically C^2) than can be proved for cooperative vector fields. In Section 3 of this paper we will prove (Theorem 3.12) that, under certain technical assumptions:

THEOREM B. *Let Γ be a compact invariant subset of a C^1 normally repulsive manifold, for a $C^k, k \geq 1$ vector field F. Then, with probability* 1, *the limit set of* (x_n) *cannot be contained in* Γ.

Note that there is no assumption here that F is cooperative. In particular (see Section 3), this later result can be applied to extend the nonconvergence results of Pemantle (1990) and Benaïm and Hirsch (1995) to linearly unstable equilibria (or periodic orbits) when F is merely C^1.

The price to pay is that we require a stronger condition on the noise sequence. The proof is completely different and relies on diffusion approximation techniques rather than on martingale estimates.

The paper is organized as follows. Section 2 sets up the notation and the necessary background on stochastic approximations. Section 3 is devoted to the precise statement and proof of Theorem B. Section 4 proves the main results on cooperative stochastic approximations (Theorem A) and Section 5 applies this to prove the convergence of stochastic fictitious play for supermodular game in full generality. Certain technical results are postponed to the Appendix.

2. Background, notation and hypotheses. Let F denote a locally Lipschitz vector field on $\mathbb{R}^d$. By standard results, the Cauchy problem $\frac{dy}{dt} = F(y)$ with initial condition $y(0) = x$ admits a unique solution $t \to \Phi_t(x)$ defined on an open interval $J_x \subset \mathbb{R}$ containing the origin. For simplicity in the statement of our results we furthermore assume that F is *globally integrable*, meaning that $J_x = \mathbb{R}$ for all $x \in \mathbb{R}^d$. This holds in particular if F is sublinear; that is,

$$\limsup_{\|x\| \to \infty} \frac{\|F(x)\|}{\|x\|} < \infty. \tag{3}$$

We let $\Phi = \{\Phi_t\}_{t \in \mathbb{R}}$ denote the flow induced by F.

A continuous map $\chi : \mathbb{R}_+ \to \mathbb{R}^d$ is called an *asymptotic pseudo-trajectory* (APT) for Φ [Benaïm and Hirsch (1996)] if, for any $T > 0$,

$$\lim_{t \to +\infty} d_\chi(t, T) = 0,$$

where

$$d_\chi(t, T) = \sup_{h \in [0,T]} \|\chi(t+h) - \Phi_h(\chi(t))\|. \tag{4}$$

In other terms, for any $T > 0$, the curve joining $\chi(t)$ to $\chi(t+T)$ shadows the trajectory of the semiflow starting from $\chi(t)$ with arbitrary accuracy, provided t is large enough.

REMARK 2.1. Assume that Φ_1 restricted to $\chi(\mathbb{R}_+)$ is uniformly continuous. This holds in particular if χ or F are bounded maps. Then

$$\lim_{t \to \infty} d_\chi(t, 1) = 0 \quad \Longleftrightarrow \quad \forall T > 0 \qquad \lim_{t \to \infty} d_\chi(t, T) = 0.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with some nondecreasing sequence of σ-algebras $(\mathcal{F}_t)_{t \geq 0}$. Throughout this paper we will consider an $(\mathcal{F}_t)_t$-adapted continuous-time stochastic process $X = (X(t))_{t \geq 0}$ living in $\mathbb{R}^d$ verifying the following condition:

HYPOTHESIS 2.2. *There exists a map $\omega : \mathbb{R}_+^3 \to \mathbb{R}_+$ such that*:

(i) *For any $\delta > 0, T > 0$,*

$$\mathbb{P}\Bigl(\sup_{s \geq t} d_X(s, T) \geq \delta \;\big|\; \mathcal{F}_t\Bigr) \leq \omega(t, \delta, T),$$

(ii) $\lim_{t \to \infty} \omega(t, \delta, T) = 0$.

A sufficient condition ensuring that Hypothesis 2.2 holds is that

$$\mathbb{P}\bigl(d_X(t, T) \geq \delta \mid \mathcal{F}_t\bigr) \leq \int_t^{t+T} r(s, \delta, T)\, ds \tag{5}$$

for some $r : \mathbb{R}^3 \mapsto \mathbb{R}_+$ such that

$$\int_0^\infty r(s, \delta, T)\, ds < \infty.$$

In this case

$$\omega(t, \delta, T) = \int_t^\infty r(s, \delta, T)\, ds.$$

The proof of the following proposition is obvious.

PROPOSITION 2.3. *Under Hypothesis 2.2, X is almost surely an asymptotic trajectory for Φ.*

EXAMPLE 2.4 (Diffusion processes). Let X be solution to the stochastic differential equation

$$dX(t) = F(X(t))\,dt + \sqrt{\gamma(t)}\,dB_t,$$

where F is a globally Lipschitz vector field, (B_t) a standard Brownian motion on $\mathbb{R}^d$ and $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ a decreasing continuous function [or more generally a $(B_t)_t$-adapted decreasing process]. Assume that (almost surely)

$$\int_0^{+\infty} \exp\left(\frac{-c}{\gamma(t)}\right) dt < +\infty$$

for all $c > 0$. Then (5) is satisfied with

$$r(t,\delta,T) = C \exp\left(-\frac{\delta^2 C(T)}{\gamma(t)}\right),$$

where C and $C(T)$ are positive constants. This is proved in Benaïm (1999), Proposition 7.4.

EXAMPLE 2.5 (Robbins–Monro algorithms). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with an increasing sequence of σ-algebra $(\mathcal{F}_n)$. Let (U_n) be an $(\mathcal{F}_n)$-adapted sequence of $\mathbb{R}^d$-valued random variables such that

$$\mathbb{E}(U_{n+1} \mid \mathcal{F}_n) = 0 \quad \text{and} \quad \sup_n \mathbb{E}(\|U_n\|^2) < \infty.$$

Let F be a globally Lipschitz vector field and $(x_n)_n$ be solution to (1) with x_0 measurable with respect to $\mathcal{F}_0$.

Set $\gamma_n = \frac{1}{n}$, $\tau_n = \sum_{i=1}^n \gamma_i$,

$$X(\tau_i + s) := x_i + s\frac{x_{i+1} - x_i}{\gamma_{i+1}} \qquad \text{for } i \in \mathbb{N}, s \in [0, \gamma_{i+1}]$$

and

$$\overline{\gamma}(\tau_i + s) := \gamma_{i+1} \qquad \text{for } s \in [0, \gamma_{i+1}[.$$

The continuous-time process X is almost surely an asymptotic pseudo-trajectory of the flow induced by F [see Benaïm (1999), Proposition 4.2 for a proof]. Additionally, we have the following result [see Benaïm (1999) and more specifically Benaïm (2000)]:

PROPOSITION 2.6. *There exists some constant B such that condition (5) holds with*

$$r(s,\delta,T) = \frac{B\overline{\gamma}(s)}{\delta^2}$$

for any $s \geq \tau_{k_0}$, and $k_0 = [\frac{2}{B\delta^2}] + 1$.

2.1. *The limit set theorem.* A set $L \subset \mathbb{R}^d$ is said to be *invariant* (resp., *positively invariant*) for Φ provided $\Phi_t(L) \subset L$ for all $t \in \mathbb{R}$ (resp., $t \in \mathbb{R}_+$).

Let L be an invariant set for Φ. We let Φ^L denote the restriction of Φ to L. That is, $\Phi_t^L(x) = \Phi_t(x)$ for all $x \in L$ and $t \in \mathbb{R}$. Note that with such a notation, $\Phi = \Phi^{\mathbb{R}^d}$.

An *attractor* for Φ^L is a nonempty compact invariant set $A \subset L$ having a neighborhood U in L such that

$$\lim_{t\to\infty} \operatorname{dist}(\Phi_t^L(x), A) = 0$$

uniformly in $x \in U$. Note that if L is compact, L is always an attractor for Φ^L. An attractor for Φ^L distinct from L is called a *proper attractor*.

The *basin of attraction* of A for Φ^L is the open set (in L) consisting of every $x \in L$ for which $\lim_{t\to\infty} \operatorname{dist}(\Phi_t(x), A) = 0$.

A *global attractor* for Φ is an attractor whose basin is $\mathbb{R}^d$. If such an attractor exists, Φ (resp., F) is called a *dissipative* flow (resp., vector field).

A compact invariant set L is said to be *internally chain-transitive* or *attractor-free* if Φ^L has no proper attractor [see, e.g., Conley (1978)]. Note that such a set is necessarily connected.

A fundamental property of asymptotic pseudo-trajectories is given by the following result due to Benaïm (1996) for stochastic approximation processes and Benaïm and Hirsch (1996) for APT. We refer to Benaïm (1999) for a proof and more details; and also to Pemantle (2007) for a recent overview and some applications.

THEOREM 2.7. *Let χ be a bounded APT; then its limit set*

$$\mathcal{L}(\chi) = \bigcap_{t\geq 0} \overline{\chi([t,\infty[)}$$

is internally chain-transitive.

COROLLARY 2.8. *Under Hypothesis 2.2, the limit set of X is almost surely internally chain-transitive on the event* $\{\sup_{t\geq 0} \|X(t)\| < \infty\}$.

REMARK 2.9. If X is as in Example 2.4 or 2.5, with F locally Lipschitz (instead of globally Lipschitz), then the conclusion of Corollary 2.8 holds true.

3. Nonconvergence toward normally hyperbolic repulsive sets. Throughout this section we assume that F is a C^1 vector field and call Φ the flow induced by F.

A submanifold $S \subset \mathbb{R}^d$ is *locally invariant* if there exists a neighborhood U of Γ in $\mathbb{R}^d$ and a positive time t_0 such that

$$\Phi_t(U \cap S) \subset S$$

for all $|t| \leq t_0$. We let $\mathcal{G}(k,d)$ denote the Grassman manifold of k-dimensional planes in $\mathbb{R}^d$. For $p \in S$, the tangent space of S in p is denoted T_pS.

DEFINITION 3.1. A compact invariant set $\Gamma \subset \mathbb{R}^d$ is called a *normally hyperbolic repulsive set* if $\Gamma \subset S$, where S is a locally invariant C^1, $(d-k)$-dimensional ($k \in \{1,\dots,d\}$) submanifold of $\mathbb{R}^d$ and there exists a continuous map

$$p \in \Gamma \mapsto E_p^u \in \mathcal{G}(k,d),$$

such that:

(i) for any $p \in \Gamma$, we have $\mathbb{R}^d = T_pS \oplus E_p^u$,
(ii) for any $t \in \mathbb{R}$ and any $p \in \Gamma$, $D\Phi_t(p)E_p^u = E^u_{\Phi_t(p)}$,
(iii) there exist positive constants λ and C such that, for any $p \in \Gamma$, $w \in E_p^u$ and $t \geq 0$, we have

$$\|D\Phi_t(p)w\| \geq Ce^{\lambda t}\|w\|.$$

The two basic examples of normally hyperbolic sets are the following. For more details, see Benaïm (1999), Section 9.

EXAMPLE 3.2 (Linearly unstable equilibria). Let p be an equilibrium of (2). That is, $F(p) = 0$. Then $\mathbb{R}^d$ can be written as the direct sum of E_p^s, E_p^c and E_p^u, the generalized eigenspaces corresponding to the eigenvalues of the jacobian matrix $DF(p)$ having, respectively, negative real parts, null real parts and positive real parts. Equilibrium p is said to be *linearly unstable* if $E_p^u \neq \{0\}$. In addition, by classical results in stable manifolds theory, there exists a C^1 (C^k if F is C^k) locally invariant manifold S, tangent to $E_p^s \oplus E_p^c$ at p. Clearly $\mathbb{R}^d = T_pS \oplus E_p^u$ and point (iii) is easily verified, since E_p^u is the direct sum of eigenspaces associated to eigenvalues with positive real parts; see Figure 1.

REMARK 3.3. If $E_p^u = \mathbb{R}^d$, then $S = \{p\}$ is a zero-dimensional manifold.

EXAMPLE 3.4 (Hyperbolic linearly unstable periodic orbit). Let $\Gamma \subset \mathbb{R}^d$ be a periodic orbit of period T. The *Floquet multipliers* of Γ are the eigenvalues of $D\Phi_T(p)$ for any $p \in \Gamma$. The unity is always a Floquet multiplier and Γ is called hyperbolic if the other multipliers all have moduli different from 1. It is called linearly unstable if at least one has modulus strictly greater than 1. If both assumptions are checked, then, for each $p \in \Gamma$, $\mathbb{R}^d$ can be written as the direct sum of three vector spaces E_p^s, E_p^c and E_p^u, invariant under $D\Phi_t(p)$ and such that:

(i) the dimension of E_p^u is at least one and the map $p \mapsto E_p^u$ is C^1 (since F is C^1),
(ii) for any $t \geq 0$, $w \in E_p^u$, we have $\|D\Phi_t(p)w\| \geq Ce^{\lambda t}\|w\|$,

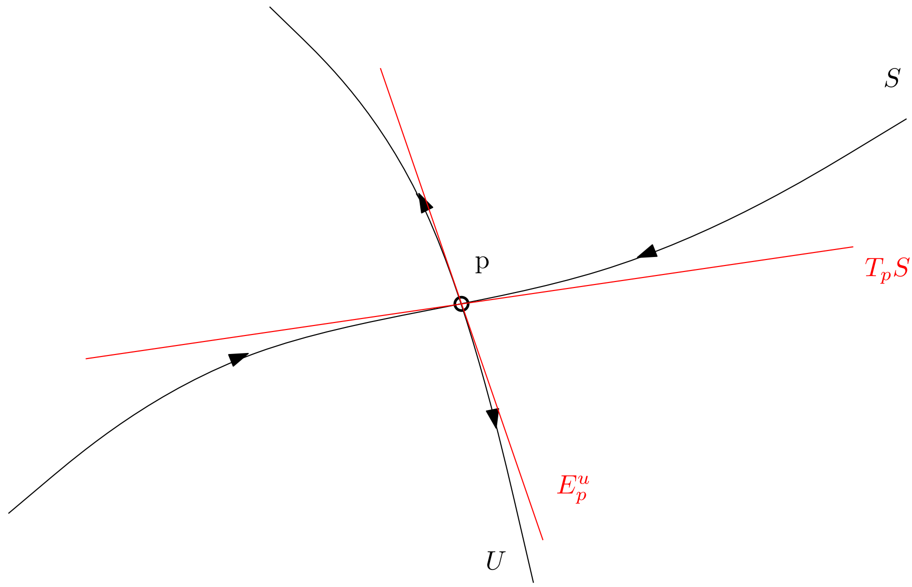

FIG. 1. *Hyperbolic linearly unstable equilibrium*, *with* $\Gamma = \{p\}$, $d = 2$, $k = 1$.

(iii) $E_p^c = \mathrm{Span}(F(p))$,

(iv) the *stable manifold* S defined as the union over $p \in \Gamma$ of the local stable manifolds in p is C^1, locally invariant and $T_pS = E_p^s \oplus E_p^c$; see Figure 2.

As a consequence, Γ is a normally hyperbolic repulsive set.

REMARK 3.5. If $E_p^s = \{0\}$, then $S = \Gamma$.

3.1. *Nonconvergence*: *Sufficient conditions*. We now give a general result, which states that, for a certain class of stochastic processes, convergence to normally hyperbolic repulsive sets occurs with null probability. Namely, let X be a continuous-time $(\mathcal{F}_t)$-adapted process verifying Hypothesis 2.2.

HYPOTHESIS 3.6. *There exists a map* $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\lim_{t\to\infty} \gamma(t) = 0$ *such that*:

(i) *For every ball* $O \subset \mathbb{R}^d$, *there exists* $c > 0$ *such that*

$$\liminf_{t\to\infty} \mathbb{P}\left(\frac{X(t+1) - \Phi_1(X(t))}{\sqrt{\gamma(t)}} \in O \,\middle|\, \mathcal{F}_t\right) > c$$

almost surely.

(ii) *There exists* $a > 0$ *such that*

$$\limsup_{t\to\infty} \omega\bigl(t, a\sqrt{\gamma(t)}, T\bigr) < 1.$$

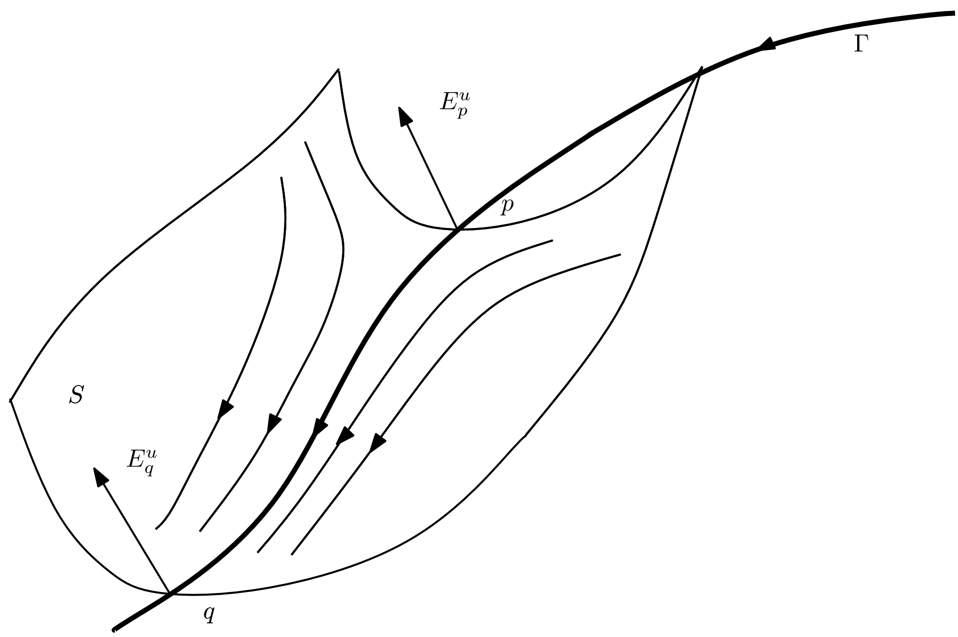

FIG. 2. *Hyperbolic linearly unstable periodic orbit, with $d = 3$, $k = 1$.*

REMARK 3.7. In Hypothesis 3.6 it suffices to assume that condition (i) holds on the event $\{X(\mathbb{R}^+) \subset U\}$ where U is a given neighborhood of Γ.

In order to understand this later hypothesis, let us describe a simple example.

EXAMPLE 3.8 (Linear diffusion processes). Let A be a positive parameter and X be solution to the stochastic differential equation

$$dX(t) = AX(t)\,dt + e^{-t/2}\,dB_t.$$

Notice that this is a particular case of Example 2.4, with $F(x) = Ax$ and $\gamma(t) = e^{-t}$. Hence $X(t)$ satisfies Hypothesis 2.2 with

$$\omega(t, \delta, T) = \int_t^{+\infty} C \exp\biggl(-\frac{\delta^2 C(T)}{e^{-s}}\biggr).$$

Thus

$$\omega\bigl(t, a\sqrt{\gamma(t)}, T\bigr) = \int_t^{\infty} C \exp(-a^2 e^{-t/2} C(T) e^s)\,ds$$

and condition (ii) in Hypothesis 3.6 is clearly satisfied. Let now

$$Y(t+1) := \frac{X(t+1) - \Phi_1(X(t))}{\sqrt{\gamma(t)}} = e^{t/2}\bigl(X(t+1) - e^A X(t)\bigr).$$

A straightforward computation gives

$$X(t) = e^{At} \int_0^t e^{-(A+1/2)s}\, dB_s + X(0).$$

Consequently we have

$$Y(t+1) = e^{A(t+1)+t/2} \int_t^{t+1} e^{-(A+1/2)s}\, dB_s.$$

Hence $Y(t+1)$ is a Gaussian random variable with variance

$$\sigma_A^2 = \frac{e^{2A}}{2A+1}\big(1 - e^{-(2A+1)}\big),$$

which is independent of t and positive. Therefore condition (i) of Hypothesis 3.6 holds.

THEOREM 3.9. *Let X be a continuous $(\mathcal{F}_t)$-adapted process verifying Hypotheses 2.2 and 3.6. Then*

$$\mathbb{P}\big(X(t) \to \Gamma\big) = 0.$$

The proof of Theorem 3.9 is given in the next subsection. We now state two applications of this result, including Example 3.8. The proofs of Theorems 3.10 and 3.12 are technical and are postponed to the Appendix.

THEOREM 3.10. *Let X be as in Example 2.4. Set $l(t) = \log(\gamma(t))$. Assume that*:

(i) *Function l is subadditive*: $l(t+s) \le l(t) + l(s)$. *This holds in particular if l is concave and $l(0) = 0$.*

(ii) *There exist constants $a \ge b > 0$ such that $-a \le \dot{l}(t) \le -b$.*

Then Hypothesis 3.6 holds. In particular, the conclusion of Theorem 3.9 holds.

For the specific case of the Robbins–Monro algorithm, an additional assumption on the noise is needed. Let $\mathcal{S}^+(\mathbb{R}^d)$ denote the set of symmetric definite positive matrices on $\mathbb{R}^d$.

HYPOTHESIS 3.11. *There exist a neighborhood U of Γ and a continuous map*

$$Q : U \to \mathcal{S}^+(\mathbb{R}^d)$$

such that $\mathbb{E}(U_{n+1}U_{n+1}^T \mid \mathcal{F}_n) = Q(x_n)$ on the event $\{x_n \in U\}$.

THEOREM 3.12. *Let $(x_n)_n$ be solution to (1). Assume that $\mathbb{E}(\|U_n\|^{2p} \mid \mathcal{F}_{n-1})$ is almost surely bounded for some $p > 1$, and that Hypothesis 3.11 holds. Then the associated interpolated process $X(t)_{t\ge 0}$ satisfies Hypothesis 3.6 and therefore, the conclusion of Theorem 3.9 holds.*

3.2. *Proof of Theorem* 3.9. For further analysis, it is convenient to extend the map $p \to E^u_p$ to a neighborhood of Γ and to approximate it by a smooth map. More precisely, it is shown in Benaïm [(1999), Section 9.1] that there exists a neighborhood $\mathcal{N}_0 \subset U$ of Γ and a C^1 bundle

$$\tilde{E}^u = \{(p, v) \in S \cap \mathcal{N}_0 \times \mathbb{R}^d : v \in \tilde{E}^u_p\},$$

where $\tilde{E}^u_p \in \mathcal{G}(k, d)$ such that:

(i) for all $p \in S \cap \mathcal{N}_0$, $\mathbb{R}^d = T_p S \oplus \tilde{E}^u_p$;

(ii) the map $H : \tilde{E}^u \mapsto \mathbb{R}^d$ defined by $H(p, v) = p + v$ induces a C^1 diffeomorphism from a neighborhood of the zero section $\{(p, 0) \in \tilde{E}^u\}$ onto $\mathcal{N}_0$.

Let now $V : \mathcal{N}_0 \mapsto \mathbb{R}_+$ be the map defined by $V(x) = \|v\|$ for $H^{-1}(x) = (p, v)$. The form of V implies that there exists $L > 0$ such that

$$d(x, S) \leq V(x) \leq L d(x, S) \tag{6}$$

for all $x \in \mathcal{N}_0$. Then according to Lemma 9.3 in Benaïm (1999) there exist a bounded neighborhood $\mathcal{N}_1 \subset \mathcal{N}_0$ of Γ, and numbers $T > 0$, $\rho > 1$ such that

$$\forall x \in \mathcal{N}_1 \qquad V(\Phi_T(x)) \geq \rho V(x). \tag{7}$$

Given a neighborhood $\mathcal{N} \subset U$ of Γ, we let

$$\mathsf{Out}_\epsilon = \mathsf{Out}_\epsilon(\mathcal{N}, S) := \{x \in \mathcal{N} \mid d(x, S \cap \mathcal{N}) \geq \epsilon\}$$

and

$$\mathsf{In}_\epsilon = \mathsf{In}_\epsilon(\mathcal{N}, S) := \mathcal{N} \setminus \mathsf{Out}_\epsilon.$$

LEMMA 3.13. (i) *There exist a bounded neighborhood $\mathcal{N} \subset U$ of Γ, $T > 0$ and $\rho > 1$ such that, for all $\epsilon > 0$,*

$$\Phi_T(\mathsf{Out}_\epsilon(\mathcal{N}, S)) \cap \mathcal{N} \subset \mathsf{Out}_{\rho\epsilon}(\mathcal{N}, S).$$

In particular, every compact invariant subset contained in $\mathcal{N}$ lies in S.

(ii) *For all $R > 0$, there exist a finite set $\{v_1, \dots, v_n\} \subset \mathbb{R}^d$ and a Borel map $I : \Gamma \mapsto \{1, \dots, n\}$ such that for all $p \in \Gamma$ and $v \in B(v_{I(p)}, 1)$,*

$$p + \epsilon v \in \mathsf{Out}_{R\epsilon}.$$

PROOF. Choose $k \in \mathbb{N}$ such that $\rho^k > L$ and $\mathcal{N} \subset \mathcal{N}_1$ be small enough so that $\Phi_{kT}(\mathcal{N}) \subset \mathcal{N}_1$. Then, using (6) and (7) for all $x \in \mathcal{N}$,

$$d(\Phi_{kT}(x), S) \geq \frac{1}{L} V(\Phi_{kT}(x)) \geq \frac{\rho^k}{L} V(x) \geq \frac{\rho^k}{L} d(x, S).$$

Replacing T by kT and ρ by $\frac{\rho^k}{L}$ gives the result.

We now prove the second assertion. Given $R > 0$, let $f : \Gamma \mapsto \mathbb{R}^d$ be a measurable function such that for all $p \in \Gamma$, $f(p) \in \tilde{E}^u_p$ and $\|f(p)\| = L(R+2)$ where L is the constant appearing in (6). The bundle $\tilde{E}^u$ being locally trivial, it is not hard to construct such a function. By compactness of $\overline{f}(\Gamma)$, there exists a finite set $\{v_1, \dots, v_n\} \subset f(\Gamma)$ such that $f(\Gamma) \subset \bigcup_{i=1}^n B(v_i, 1)$. For $p \in \Gamma$, set

$$I(p) = \min\{i = 1, \dots, n : \|f(p) - v_i\| \le 1\}.$$

Then, for $I(p) = i$ and $v \in B(v_i, 1)$,

$$d\big(p + \epsilon f(p), S\big) \le d(p + \epsilon v, S) + \epsilon \|f(p) - v)\| \le d(p + \epsilon v, S) + 2\epsilon.$$

On the other hand, by (6),

$$d\big(p + \epsilon f(p), S\big) \ge \frac{1}{L} V\big(p + \epsilon f(p)\big) = \frac{\epsilon \|f(p)\|}{L} = \epsilon(R+2).$$

Hence

$$d(p + \epsilon v, S) \ge R\epsilon. \qquad \square$$

COROLLARY 3.14. *Let $\mathcal{N}$, T and ρ be as in Lemma 3.13, and set $\delta = (\rho - 1) > 0$. Let χ be an asymptotic pseudo-trajectory verifying*:

(i) $\chi(0) \in \mathsf{Out}_\epsilon$,
(ii) *for all $t \ge 0$, $d_\chi(t, T) \le \delta\epsilon$.*

Then χ eventually leaves $\mathcal{N}$.

PROOF. Suppose that χ remains in $\mathcal{N}$. We claim that $\chi(kT) \in \mathsf{Out}_\epsilon$ for all $k \in \mathbb{N}$. If $\chi(kT) \in \mathsf{Out}_\epsilon$, then $\Phi_T(\chi(kT)) \in \mathsf{Out}_{\rho\epsilon}$ by Lemma 3.13. Hence $\chi(kT + T) \in \mathsf{Out}_\epsilon$ since $d_\chi(kT, T) \le \delta\epsilon$. This proves the claim by induction on k. Now, by the limit set Theorem 2.7 and Lemma 3.13, the limit set of χ lies in S and we have reached a contradiction. $\square$

Throughout the remainder of the section we let $\mathcal{N}$, T and ρ be as in Lemma 3.13 and $\delta = (\rho - 1) > 0$. Recall that X is a continuous-time $(\mathcal{F}_t)$-adapted process verifying Hypothesis 2.2. We call E_t the event

$$E_t = \{\forall s \ge t : X(s) \in \mathcal{N}\}.$$

LEMMA 3.15. (i) *On the event $\{X(t) \in \mathsf{Out}_\epsilon\}$,*

$$\mathbb{P}(E_t \mid \mathcal{F}_t) \le \omega(t, \delta\epsilon, T).$$

(ii)

$$\mathbb{P}(E_t \mid \mathcal{F}_t) \le 1 - [1 - \omega(t+1, \delta\epsilon, T)]\mathbb{P}\big(X(t+1) \in \mathsf{Out}_\epsilon \mid \mathcal{F}_t\big).$$

PROOF. Since X is an asymptotic pseudo-trajectory which satisfies

$$\mathbb{P}\big(\exists s \geq t : d_X(s, T) \geq \delta\varepsilon \mid \mathcal{F}_t\big) \leq \omega(t, \delta\varepsilon, T),$$

the first inequality follows from Corollary 3.14. Now

$$\begin{aligned}
\mathbb{P}(E_t \mid \mathcal{F}_t) &\leq \mathbb{P}(E_{t+1} \mid \mathcal{F}_t) \\
&= \mathbb{P}\big(E_{t+1}; X(t+1) \in \mathsf{Out}_\epsilon \mid \mathcal{F}_t\big) + \mathbb{P}\big(E_{t+1}; X(t+1) \in \mathsf{In}_\epsilon \mid \mathcal{F}_t\big) \\
&= \mathbb{E}\big(\mathbb{P}(E_{t+1} \mid \mathcal{F}_{t+1})\mathbf{1}_{X(t+1)\in \mathsf{Out}_\epsilon} \mid \mathcal{F}_t\big) + \mathbb{E}\big(\mathbb{P}(E_{t+1} \mid \mathcal{F}_{t+1})\mathbf{1}_{X(t+1)\in \mathsf{In}_\epsilon} \mid \mathcal{F}_t\big) \\
&\leq \omega(t+1, \delta\epsilon, T)\mathbb{P}\big(X(t+1) \in \mathsf{Out}_\epsilon \mid \mathcal{F}_t\big) + \mathbb{P}\big(X(t+1) \in \mathsf{In}_\epsilon \mid \mathcal{F}_t\big). \qquad \square
\end{aligned}$$

LEMMA 3.16. *Assume that there exist a map* $\epsilon : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *with*

$$\lim_{t\to\infty} \epsilon(t) = 0$$

and constants $c > 0$ *and* $c' < 1$ *such that for* t *large enough*,

(i) $\mathbb{P}(X(t+1) \in \mathsf{Out}_{\epsilon(t)} \mid \mathcal{F}_t) \geq c$ *on the event* $\{X(t) \in \mathsf{In}_{\epsilon(t)}\}$.

(ii)

$$\omega\big(t, \delta\epsilon(t), T\big) < c'.$$

Then

$$\mathbb{P}\big(X(t) \to \Gamma\big) = 0.$$

PROOF. One has

$$\{X(t) \to \Gamma\} \subset \bigcup_{n\in\mathbb{N}} E_n$$

and it suffices to prove that $\mathbb{P}(E_n) = 0$ for all $n \in \mathbb{N}$.

For all $t \geq n$, $E_n \subset E_t$. Thus

$$\mathbb{P}(E_n \mid \mathcal{F}_t) \leq \mathbb{P}(E_t \mid \mathcal{F}_t) \leq \max\big(c', 1 - (1 - c')c\big),$$

where the last inequality follows from the assumptions and Lemma 3.15. Now, by a classical Martingale result,

$$1 > \max\big(c', 1 - (1 - c')c\big) \geq \lim_{t\to\infty} \mathbb{P}(E_n \mid \mathcal{F}_t) \to \mathbf{1}_{E_n}$$

almost surely. Hence the result. $\square$

We are now ready to prove Theorem 3.9. We shall prove that the assumptions of Lemma 3.16 are fulfilled with $\epsilon(t) = \frac{\sqrt{\gamma(t)}}{\alpha}$, where $\alpha = \frac{\delta}{a}$ and a is given by Hypothesis 3.6(ii). Condition (ii) of the lemma is clearly verified.

To check condition (i) we assume that $X(t) \in \mathsf{In}_{\epsilon(t)}$. Hence (for t large enough), $\Phi_1(X(t))$ lies in $\mathcal{N}_0 \subset \mathcal{N}$ and we can write

$$\Phi_1(X(t)) = p(t) + v(t)$$

with $(p(t), v(t)) \in \tilde{E}^u_{p(t)}$ (see the beginning of the section). Then, by the triangle inequality,

$$d\big(X(t+1), S\big) \geq d(p(t) + \alpha\epsilon(t)Y(t+1), S) - \|v(t)\|$$

with

$$Y(t+1) = \frac{X(t+1) - \Phi_1(X(t))}{\alpha\epsilon(t)}.$$

Now

$$\|v(t)\| = V(\Phi_1(X(t))) \leq Ld(\Phi_1(X(t), S)) \leq M\epsilon(t),$$

where the first inequality follows from the Lipschitz continuity of the map V [see (6)], and the second from the Lipschitz continuity of Φ_1 and invariance of S. Thus

$$\frac{d(X(t+1), S)}{\epsilon(t)} \geq U_t - M,$$

where

$$U_t = \frac{d(p(t) + \alpha\epsilon(t)Y(t+1), S)}{\epsilon(t)}.$$

Let $R = \frac{1+M}{\alpha}$ and call $(v_i)_{i=1,\dots,n}$ the finite set of vectors given by Lemma 3.13(ii). Then, by Hypothesis 3.6(ii), there exists $c > 0$ such that

$$\mathbb{P}\big(Y(t+1) \in B(v_i, 1) \mid \mathcal{F}_t\big) \geq c \qquad \forall i$$

almost surely. Consequently,

$$\begin{aligned}\mathbb{P}\big(U_t \geq (1+M) \mid \mathcal{F}_t\big) &= \mathbb{P}\big(p(t) + \alpha\epsilon(t)Y(t+1) \in \mathsf{Out}_{R\alpha\epsilon(t)} \mid \mathcal{F}_t\big) \\ &\geq \mathbb{P}\big(Y(t+1) \in B\big(v_{I(p(t))}, 1\big) \mid \mathcal{F}_t\big) \geq c.\end{aligned}$$

This proves that condition (i) of the lemma is verified.

4. Application to cooperative dynamics. Throughout this section we assume that F is C^1 and that for all $x \in \mathbb{R}^d$ the Jacobian matrix $DF(x) = (\frac{\partial F_i}{\partial x_j}(x))$ has nonnegative off-diagonal entries and is irreducible. Such a vector field F is said to be *cooperative* and *irreducible*; see Hirsch (1985). We refer the reader to Hirsch and Smith (2006) for a recent survey on the subject. We furthermore assume that F is *dissipative*, meaning that it admits a global attractor.

For $x, y \in \mathbb{R}^d$, $x \geq y$ means that $x_j \geq y_j$ for all j. If, additionally, $x \neq y$, we write $x > y$. If $x_j > y_j$ for all j, it is denoted $x \gg y$. Given two sets $A, B \subset \mathbb{R}^d$, we write $A \leq B$ provided $x \leq y$ for all $x \in A$ and $y \in B$. Set A is called *unordered* if for all $x, y \in A$, $x \leq y \Rightarrow x = y$.

The vector field F being cooperative and irreducible, its flow has positive derivatives; see Hirsch (1985), Hirsch and Smith (2006). That is, $D\Phi_t(x) \gg 0$ for $x \in \mathbb{R}^d$ and $t > 0$. This implies that it is *strongly monotonic* in the sense that $\phi_t(x) \gg \phi_t(y)$ for all $x > y$ and $t > 0$.

We let $\mathcal{E}$ denote the equilibria set of F. A point $p \in \mathcal{E}$ is called *linearly unstable* if the Jacobian matrix $DF(p)$ has at least one eigenvalue with positive real part. We let $\mathcal{E}^+$ denote the set of such equilibria and $\mathcal{E}^- = \mathcal{E} \setminus \mathcal{E}^+$.

An equilibrium point $p \in \mathcal{E}$ is said to be *asymptotically stable from below* if there exists $x < p$ such that $\phi_t(x) \to p$. The subset of equilibria which satisfy this property is denoted $\mathcal{E}_{\text{asb}}$. Note that if $p \in \mathcal{E}_{\text{asb}}$, then there exists a nonempty open set of initial conditions from which the solution trajectories converge to p. In particular, $\mathcal{E}_{\text{asb}}$ is countable. Given $p \in \mathcal{E}_{\text{asb}}$, we introduce the set of points whose limit set dominates p:

$$V(p) := \{x \mid \omega(x) \geq p\},$$

and we let S_p denote its boundary: $S_p := \partial V(p)$. The following proposition is basically due to Hirsch [(1988), Theorem 2.1], but the C^1 regularity was proved by Terescak (1996). Our statement follows from Proposition 3.2 in Benaïm (2000), where more details can be found.

PROPOSITION 4.1. *There exists a unique equilibrium $p^* \in \mathcal{E}_{\text{asb}}$ such that $V(p^*) = \mathbb{R}^d$. For any other $p \in \mathcal{E}_{\text{asb}} \setminus \{p^*\}$, S_p is a C^1 unordered invariant hypersurface, diffeomorphic to $\mathbb{R}^{d-1}$.*

For $p \in \mathcal{E}_{\text{asb}} \setminus \{p^*\}$, we let $\mathcal{R}(\Phi^{S_p})$ denote the chain-recurrent set of Φ restricted to S_p; or equivalently, the union of all internally chain-transitive sets contained in S_p. We also set

$$\mathcal{R}'_p = \mathcal{R}(\Phi^{S_p}) \setminus \{\mathcal{E}^- \cap S_p\}.$$

The first part of the next theorem is proved in Benaïm (2000) (see the proof of Proposition 3.2) and the second part restates Theorem 3.3 in the same paper [relying heavily on Hirsch (1999)].

THEOREM 4.2. *For any $p \in \mathcal{E}_{\text{asb}} \setminus \{p^*\}$, the set $\mathcal{R}'_p$ is a repulsive normally hyperbolic set (in the sense of Section 3). Any internally chain-transitive set is either an ordered arc included in $\mathcal{E}^-$ or is contained in $\mathcal{R}'_p$ for some $p \in \mathcal{E}_{\text{asb}} \setminus \{p^*\}$.*

REMARK 4.3. By a result of Jiang (1991), if F is real analytic, it cannot have a nondegenerate ordered arc of equilibria.

As a consequence of these results we get the following:

THEOREM 4.4. *Let X be a continuous $(\mathcal{F}_t)$-adapted stochastic process verifying Hypotheses 2.2 and 3.6. Then, on the event $\{\sup_t \|X(t)\| < \infty\}$, the limit set of X is almost surely an ordered arc contained in $\mathcal{E}^-$. In case F is real analytic, $X(t)$ converges almost surely to an equilibrium $p \in \mathcal{E}^-$.*

PROOF. Follows from Theorems 2.7, 4.2 and 3.9. □

COROLLARY 4.5. *Let X be the process given in Example 2.4 with $-a \leq \frac{\dot\gamma(t)}{\gamma(t)} \leq -b$ and $a \geq b > 0$. Then the conclusions of Theorem 4.4 hold.*

Let Λ denote the global attractor of F.

COROLLARY 4.6. *Let (x_n) be the Robbins–Monro algorithm given in Example 2.5. Assume that Hypothesis 3.11 holds where U is a neighborhood of Λ. Then the conclusions of Theorem 4.4 hold.*

5. Stochastic fictitious play in supermodular games.

5.1. *General settings.* Let us consider an N-persons game in normal form. Player i's action set is finite and denoted A^i; Δ^i is the mixed strategies set:

$$\Delta^i := \left\{x^i = (x^i(\alpha))_{\alpha \in A^i} \,\middle|\, x^i(\alpha) \geq 0, \sum_{\alpha \in A^i} x^i(\alpha) = 1\right\}$$

and $u^i : A^i \mapsto \mathbb{R}$ his utility function. The set of action profiles (resp., mixed strategy profiles) is denoted $A := \bigtimes_{i=1}^N A^i$ (resp., $\Delta := \bigtimes_{i=1}^N \Delta^i$). The utility functions $(u^i)_{i=1,\dots,N}$ are defined on A but linearly extended to Δ:

$$x = (x^1, \dots, x^N) \in \Delta \mapsto u^i(x) := \sum_{a=(a^1,\dots,a^N)\in A} u^i(a) x^1(a^1) \cdots x^N(a^N).$$

We call $G(N, A, u)$ the game induced by these parameters.

Standing notation. As usual in game theory, we let $a^{-i} = (a^j)_{j \neq i}$, $x^{-i} = (x^j)_{j\neq i}$, $A^{-i} = \bigtimes_{j \neq i} A^j$, etc. We may write (a^i, a^{-i}) for $a = (a^1, \dots, a^N)$ and so on.

5.2. *Perturbed best response dynamic.* To shorten notation let us take the point of view of player 1. A *choice function* for player 1 is a continuously differentiable map $C : \mathbb{R}^{A^1} \mapsto \Delta^1$.

We say that C is a *stochastic choice function* if there exists a positive probability density $f:\mathbb{R}^{A^1}\mapsto\mathbb{R}^+$ such that for all $\Pi\in\mathbb{R}^{A^1}$, $C(\Pi)$ is the law of the random variable

$$\arg\max_{\beta\in A^1}\big(\Pi(\beta)+\varepsilon(\beta)\big),$$

where $\varepsilon\in\mathbb{R}^{A^1}$ is a random variable having distribution $f(x)\,dx$. A classical example of stochastic choice function is the Logit map:

$$L(\Pi)(\alpha)=\frac{\exp(\eta^{-1}\Pi(\alpha))}{\sum_{\beta\in A^1}\exp(\eta^{-1}\Pi(\beta))}.$$

It is induced by an extreme value density [see Fudenberg and Levine (1998) and Hofbauer and Sandholm (2002)].

Given a choice function C, the *smooth* (or *perturbed*) *best response* associated to C is the map $\mathbf{br}^1:\Delta^{-1}\mapsto\Delta^1$ defined by

$$\mathbf{br}^1(y)=C(u^1(\cdot,y)).$$

In the remainder of the section, an N-tuple of perturbed best response maps is given and we let $\mathbf{br}:\Delta\mapsto\Delta$ denote the map defined by

$$\mathbf{br}(x):=(\mathbf{br}^1(x^{-1}),\dots,\mathbf{br}^N(x^{-N})).$$

Let $T\Delta$ be the tangent space to Δ. The *perturbed best response* vector field is the smooth vector field $F:\Delta\mapsto T\Delta$, defined as

$$F(x)=-x+\mathbf{br}(x). \tag{8}$$

REMARK 5.1. By construction, F can be defined as a vector field on $\mathbb{R}^A$ which satisfies condition (3) and is dissipative with a global attractor contained in Δ.

The set of *perturbed Nash equilibria* is the set of $x\in\Delta$ such that $F(x)=0$. It will be referred to as PNE.

5.3. *Stochastic fictitious play.* Let $a_n=(a_n^1,\dots,a_n^N)\in A$ denote the action profile realized at stage n and $\overline{x}_n\in\Delta$ be the empirical distribution of moves up to time n:

$$\overline{x}_n:=\left(\frac{1}{n}\sum_{m=1}^n\delta_{a_m^1},\dots,\frac{1}{n}\sum_{m=1}^n\delta_{a_m^N}\right).$$

From now on, we assume that agents play repeatedly and independently. That is,

$$\mathbb{P}\big(a_{n+1}=(a^1,\dots,a^N)\mid\mathcal{F}_n\big)=\prod_{i=1}^N\mathbb{P}(a_{n+1}^i=a^i\mid\mathcal{F}_n),$$

where $(\mathcal{F}_n)_n = \sigma(a_1, \dots, a_n)$ (or any other σ-field representing the history up to time n). We furthermore assume that

$$\mathbb{P}(a^i_{n+1} = \cdot \mid \mathcal{F}_n) = \mathbf{br}^i(\overline{x}_n^{-i}), \tag{9}$$

where $\overline{x}_n^{-i}$ are the empirical moves of player i opponents up to time n.

This type of adaptive behavior is called *Stochastic Fictitious Play* (SFP) and was originally introduced in Fudenberg and Kreps (1993). The concept behind is that players use fictitious play strategies in a game where payoff functions are perturbed by some random variables in the spirit of Harsanyi (1973). We refer the reader to Fudenberg and Levine (1998) for more details.

A simple computation gives

$$\overline{x}_{n+1} = \overline{x}_n + \frac{1}{n+1}\big(F(\overline{x}_n) + U_{n+1}\big), \tag{10}$$

where F is the perturbed best response vector field (8) and U_{n+1} is a bounded martingale difference given by

$$U_{n+1} := (\delta_{a^1_{n+1}}, \dots, \delta_{a^N_{n+1}}) - \mathbf{br}(\overline{x}_n).$$

Equation (10) in connection with stochastic approximation theory has been used by many authors for analyzing the behavior of SFP for different classes of games, including 2×2 games [Fudenberg and Kreps (1993), Benaïm and Hirsch (1999a)] and Potential and zero-sum games [Hofbauer and Sandholm (2002)]. One of the main questions is to prove whether or not $\overline{x}_n$ converges to the set of PNE. We will address this question below for the so-called class of supermodular games.

By an obvious abuse of language, we will say that an $m \times m$ matrix A is positive definite if, for any $\zeta \in T\Delta$, we have

$$\zeta \neq 0 \quad \Longrightarrow \quad \zeta^T A \zeta > 0.$$

In the following, the set of matrices which are positive definite in this sense is denoted $\mathcal{S}^+(T\Delta)$.

LEMMA 5.2. *Assume that for each i, the choice function of player i takes values into the interior of Δ^i (notice that this property is always satisfied for stochastic choice functions). Then there exists a continuous function $Q : \Delta \to \mathcal{S}^+(T\Delta)$ such that*

$$\mathbb{E}(U_{n+1}U^T_{n+1} \mid \mathcal{F}_n) = Q(\overline{x}_n).$$

PROOF. Let, for $x \in \Delta$ and $i \in \{1, \dots, N\}$, $Q^i(x)$ denote the quadratic form on $T\Delta^i$ defined by

$$Q^i(x)(\zeta^i) = \sum_{\alpha \in A^i} \langle \delta_\alpha - \mathbf{br}^i(x^{-i}), \zeta^i \rangle^2 \mathbf{br}^i(x^{-i})_\alpha.$$

Equivalently, $Q^i(x)(\zeta^i)$ is the variance of $\alpha \mapsto \langle \delta_\alpha, \zeta^i \rangle$ under the law $\mathbf{br}^i(x^{-i})$. Let $Q(x)$ denote the quadratic form on $T\Delta$ defined by

$$Q(x)(\zeta) = \sum_{i=1}^{N} Q^i(x)(\zeta^i).$$

Since $\mathbf{br}^i(x^{-i})_\alpha > 0$ and $\{\delta_\alpha - \mathbf{br}^i(x^{-i}) : \alpha \in A^i\}$ spans $T\Delta^i$, $Q^i(x)$ is nondegenerate for all i. Hence $Q(x)$ is nondegenerate. □

5.4. *Supermodular games.* We assume here that for each $i = 1, \ldots, N$ the action set A^i is equipped with a total ordering denoted $\leq$; and we focus our attention on games such that, for a given player, the reward he obtains by switching to a higher action increases when his opponents choose higher strategies. Such games are called *supermodular* and arise in many economic applications; see, for example, Topkis (1979) or Milgrom and Roberts (1990).

DEFINITION 5.3. We say that the game $G(N, A, u)$ is (strictly) *supermodular* if, for any pair of distinct players (i, j) and any action profiles $a = (a^1, \ldots, a^N)$ and $b = (b^1, \ldots, b^N)$ such that $a^i > b^i$ and $a^{-i} = b^{-i}$, the quantity $u^i(a) - u^i(b)$ is (strictly) increasing in $a^j = b^j$, for $j \neq i$.

REMARK 5.4. In the particular case where each action set A^i is equal to the couple $\{0, 1\}$, the state space is the hypercube $[0, 1]^N$ and these games have been defined as *coordination games* in Benaïm and Hirsch (1999a).

In the remainder of this section we set $A^i = \{1, \ldots, m^i\}$. Let $B^i : \mathbb{R}^{m^i} \mapsto \mathbb{R}^{m^i-1}$ be the operator defined by

$$B^i_j(u) = \sum_{k=j+1}^{m^i} u_k, \qquad j = 1, \ldots, m^i - 1.$$

Similarly let $B : \bigtimes_{i=1}^N \mathbb{R}^{m^i} \mapsto \bigtimes_{i=1}^N \mathbb{R}^{m^i-1}$ be defined by $B(x^1, \ldots, x^N) = (B^1(x^1), \ldots, B^N(x^N))$. Note that B induces a one-to-one map from Δ onto $B(\Delta)$. By abuse of notation we write B^{-1} its inverse. The following result is proved in Hofbauer and Sandholm (2002).

THEOREM 5.5 [Hofbauer and Sandholm (2002)]. *Assume that $G(N, A, u)$ is strictly supermodular and that for each i, $\mathbf{br}^i$ is associated to a stochastic choice function. Then, the vector field $G : B(\Delta) \mapsto B(T\Delta)$, defined by*

$$G(y) = BF(B^{-1}(y)) = -y + B\mathbf{br}(B^{-1}(y))$$

is cooperative and irreducible.

Hofbauer and Sandholm then used this theorem combined with results from Benaïm (2000) to describe the limit set of stochastic fictitious play for supermodular game. In view of the new results obtained in this paper and specifically in Section 4, we are now able to improve notably their results and to prove the convergence of stochastic fictitious play for supermodular games in full generality.

THEOREM 5.6. *Assume that the assumptions of Theorem 5.5 are satisfied. Then the limit set of $(\overline{x}_n)_n$ is almost surely an ordered arc of PNE that is not linearly unstable. If we furthermore assume that the choice functions are real analytic (e.g., in the logit case), then $(\overline{x}_n)_n$ almost surely converges toward a PNE that is not linearly unstable.*

PROOF. Set $\overline{y}_n = B\overline{x}_n$. Then

$$\overline{y}_{n+1} - \overline{y}_n = \frac{1}{n+1}\big(G(\overline{y}_n) + V_{n+1}\big)$$

with $V_n = BU_n$. Then

$$\mathbb{E}(V_{n+1}^t V_{n+1} \mid \mathcal{F}_n) = BQ(B^{-1}(\overline{y}_n))^t B.$$

By Lemma 5.2 and Theorem 5.5, the conditions to apply Corollary 4.6 are met. □

APPENDIX

A.1. Proof of Theorem 3.10. The assumptions on γ easily imply that

$$\frac{\gamma(t)}{\gamma(s+t)} \geq \frac{1}{\gamma(s)} \geq e^{bs}.$$

Thus

$$\omega\big(t, a\sqrt{\gamma(t)}, T\big) \leq C \int_0^\infty \exp\left(-a^2 e^{bs} C(T)\right)$$

and condition (iii) of Hypothesis 3.6 holds. Let

$$A_s^t = \left[DF(\Phi_s(X_t)) - \frac{1}{2}\frac{\dot{\gamma}(t+s)}{\gamma(t+s)}\right]$$

and let $\{Y_s^t, s \geq 0\}$ be solution to

$$dY_s^t = A_s^t Y_s^t + dB_{t+s}$$

with initial condition $Y_0^t = 0$. Condition (i) of Hypothesis 3.6 follows from the following lemma.

LEMMA A.1.

$$\lim_{t\to\infty} \mathbb{P}\Big(\sup_{0\le s\le 1}\Big\| Y_s^t - \frac{X_{t+s}-\Phi_s(X_t)}{\sqrt{\gamma(t+s)}}\Big\| \ge \epsilon \;\Big|\; \mathcal{F}_t\Big) = 0.$$

In particular, Hypothesis 3.6(i) *holds with* $Y(t) = Y_1^{t-1}$ *for all* $t \ge 1$.

PROOF. Set $\alpha(s) = 1/\sqrt{\gamma(s)}$, $Z_s^t = X_{t+s} - \Phi_s(X_t)$ and $\hat{Y}_s^t = \alpha(t+s)Z_s^t$. Then

$$\begin{aligned} dZ_s^t &= \big(F(X_{t+s}) - F(\Phi_s(X_t))\big)\,ds + \sqrt{\gamma(t+s)}\,dB_{t+s} \\ &= [DF(\Phi_s(X_t))Z_s^t + o(\|Z_s^t\|)]\,ds + \sqrt{\gamma(t+s)}\,dB_{t+s}. \end{aligned}$$

Hence

$$d\hat{Y}_s^t = \Big[DF(\Phi_s(X_t)) + \frac{\dot{\alpha}(t+s)}{\alpha(t+s)}\Big]\hat{Y}_s^t + dB_{t+s} + \alpha(t+s)o(\|Z_s^t\|),$$

where $o(z) = z\eta(z)$ and $\lim_{z\to 0}\eta(z) = \eta(0) = 0$. Then

$$Y_s^t - \hat{Y}_s^t = \int_0^s A_u^t(Y_u^t - \hat{Y}_u^t)\,du + \int_0^s \alpha(t+u)o(\|Z_u^t\|)\,du.$$

Thus, by Gronwall's inequality,

$$\sup_{0\le s\le 1}\|Y_s^t - \hat{Y}_s^t\| \le e^K R_t$$

with

$$R_t = \sup_{0\le s\le 1}\alpha(t+s)o(\|Z_s^t\|)$$

and

$$K = \sup_{s,t}\|A_s^t\| \le \|DF\| + \frac{a}{2}. \tag{11}$$

To conclude the proof it remains to show that

$$\mathbb{P}(R_t \ge \delta \mid \mathcal{F}_t) \to 0$$

as $t \to \infty$.

It follows from the estimate given in Example 2.4 that

$$\mathbb{P}\Big(\sup_{0\le s\le 1}\|Z_s^t\| \ge \delta \;\Big|\; \mathcal{F}_t\Big) \le \int_t^{t+1} C\exp\Big(\frac{-\delta^2 C(1)}{\gamma(s)}\Big)\,ds \le C\exp\Big(-\frac{\delta^2 C(1)}{\gamma(t+1)}\Big).$$

Thus

$$\begin{aligned} \mathbb{P}\Big(\sup_{0\le s\le 1}\alpha(t+s)\|Z_s^t\| \ge R \;\Big|\; \mathcal{F}_t\Big) &\le \mathbb{P}\Big(\|Z_s^t\| \ge \frac{R}{\alpha(t+1)} \;\Big|\; \mathcal{F}_t\Big) \\ &\le C\exp(-R^2 C(1)). \end{aligned}$$

Now,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{0\le s\le 1} \alpha(t+s)\|Z_s^t\|\eta(\|Z_s^t\|) \ge \delta \,\Big|\, \mathcal{F}_t\Big)\\
&\qquad \le \mathbb{P}\Big(\sup_{0\le s\le 1} \alpha(t+s)\|Z_s^t\| \ge R \,\Big|\, \mathcal{F}_t\Big) + \mathbb{P}\Big(\sup_{0\le s\le 1} \eta(\|Z_s^t\|) \ge \frac{\delta}{R} \,\Big|\, \mathcal{F}_t\Big)\\
&\qquad \le C\exp(-R^2C(1)) + \mathbb{P}\Big(\sup_{0\le s\le 1} \eta(\|Z_s^t\|) \ge \frac{\delta}{R} \,\Big|\, \mathcal{F}_t\Big).
\end{aligned}$$

Since $\lim_{z\to 0}\eta(z) = 0$,

$$\limsup_{t\to\infty} \mathbb{P}\Big(\sup_{0\le s\le 1} \alpha(t+s)\|Z_s^t\|\eta(\|Z_s^t\|) \ge \delta \,\Big|\, \mathcal{F}_t\Big) \le C\exp(-R^2C(1))$$

and since R is arbitrary, this proves the result. $\square$

It remains to prove that condition (ii) of Hypothesis 3.6 holds.

LEMMA A.2. *Let Σ be an $n\times n$ self-adjoint positive definite matrix and*

$$f_\Sigma(x) = \frac{\exp(-(1/2)\langle\Sigma^{-1}x, x\rangle)}{\sqrt{\det(\Sigma)(2\pi)^n}}$$

the density of a centered Gaussian vector with covariance Σ. Let $0<\alpha\le\beta$, respectively, denote the smallest and largest eigenvalues of Σ. Then

$$f_\Sigma(x) \ge \left(\frac{\alpha}{\beta}\right)^{n/2} f_{\alpha Id}(x).$$

PROOF. Follows from the estimates $\det(\Sigma)\le\beta^n$ and $\langle\Sigma^{-1}x, x\rangle \le \frac{\|x\|^2}{\alpha}$. $\square$

Since Y_s^t is a linear function of $\{B_{t+u}, 0\le u\le s\}$, it is a Gaussian vector under the conditional probability $\mathbb{P}(\cdot\,|\mathcal{F}_t)$. By Itô's formulas, its covariance matrix is solution to

$$\frac{d\Sigma_s^t}{ds} = A_s^t\Sigma_s^t + \Sigma_s^t A_s^{t*} + Id$$

with initial condition $\Sigma_0^t = 0$, where A_s^{t*} stands for the transpose of A_s^t. It is then easy to check that

$$\Sigma_s^t = \int_0^s U^t(u)U^{t*}(u)\,du,$$

where $U^t(s)$ is the solution to

$$\frac{dU}{ds} = A_s^t U, \qquad U(0) = Id. \tag{12}$$

Using (12) we see that $U^t(s)$ is invertible and that its inverse $(U^t(s))^{-1}$ solves

$$\frac{dV}{ds} = -VA_s^t, \qquad V(0) = Id.$$

Using again (12) combined with the estimate (11) and Gronwall's lemma, we get

$$\|U^t(s)\| \le e^{Ks}.$$

Similarly,

$$\|(U^t(s))^{-1}\| \le e^{Ks}.$$

It follows that for all vector h,

$$e^{-Ks}\|h\| \le \|U^t(s)h\| \le e^{Ks}\|h\|.$$

Hence

$$a\|h\|^2 \le \langle \Sigma_1^t h, h\rangle \le b\|h\|^2,$$

where $a = \int_0^1 e^{-2Ku}\,du$ and $b = \int_0^1 e^{2Ku}\,du$. The result then follows from Lemma A.2.

A.2. Proof of Theorem 3.12. Recall that $(\mathcal{F}_n)_n$ is a given filtration to which the stochastic process $(x_n)_n$ is adapted. Let $m_n := \sup\{k \in \mathbb{N} \mid \tau_k \le n\}$ and call $(\mathcal{G}_n)_n$ the sigma algebra $(\mathcal{F}_{m_n})_n$. Let $n \ge 1$ and $k_n := m_{n+1} - m_n$. We denote by t_j^n the quantity $\tau_{m_n+j} - \tau_{m_n}$ $(j = 0, \dots, k_n)$ and $t_n := t_{k_n}^n$. Notice that $|t_n - 1| \le \gamma_{m_n}$.

For the continuous-time interpolated process induced by a discrete process $(x_n)_n$, Hypothesis 3.6 is satisfied if there exists a vanishing positive sequence $(\gamma(n))_n$ and a $\mathcal{G}_n$-adapted random sequence $(Y_n)_n$ such that:

(i) for any $\alpha > 0$,

$$\lim_{n\to+\infty} \mathbb{P}\left(\left\|\frac{x_{m_{n+1}} - \Phi_{t_n}(x_{m_n})}{\sqrt{\gamma(n)}} - Y_{n+1}\right\| > \alpha \,\Big|\, \mathcal{G}_n\right) = 0,$$

(ii) for any open ball $O \subset \mathbb{R}^d$, there exists a positive number δ such that

$$\liminf_{n\to+\infty} \mathbb{P}(Y_{n+1} \in O \mid \mathcal{G}_n) > \delta \qquad \text{almost surely,}$$

(iii) there exists $a > 0$ such that

$$\limsup_{n\to+\infty} \omega\big(n, a\sqrt{\gamma(n)}, T\big) < 1.$$

Let $\gamma(n) := \sum_{k=1}^{k_n} \gamma_{m_n+k}^2$. First, by Proposition 2.6, the map ω corresponding to the process $(x_n)_n$ is given by

$$\omega(n, \delta, T) = \frac{B\int_n^{+\infty} \overline{\gamma}(u)\,du}{\delta^2}.$$

Hence,

$$\omega\bigl(n, a\sqrt{\gamma(n)}, T\bigr) \le \frac{B}{a^2} \frac{\sum_{m_n}^{+\infty} \gamma_i^2}{\sum_{m_n+1}^{m_{n+1}} \gamma_i^2}.$$

Since

$$\limsup_n \frac{\sum_{m_n}^{+\infty} \gamma_i^2}{\sum_{m_n+1}^{m_{n+1}} \gamma_i^2} < +\infty,$$

the quantity $\omega(n, a\sqrt{\gamma(n)}, T)$ is smaller than 1, for a large enough. The next lemma corresponds to Lemma A.1.

LEMMA A.3. *Point* (i) *is satisfied for this choice of* $(\gamma(n))_n$ *and the random sequence* $(Y_n)_n$ *given by*

$$\frac{1}{\sqrt{\gamma(n-1)}} \sum_{j=1}^{k_{n-1}} \gamma_{m_{n-1}+j} \left(\prod_{k=j+1}^{k_{n-1}} \bigl(I_d + \gamma_{m_{n-1}+k} DF(\phi_{t_{k-1}^{n-1}}(x_{m_{n-1}}))\bigr) \right) U_{m_{n-1}+j}.$$

PROOF. Set $\hat{Y}_{n+1} := \frac{x_{m_{n+1}} - \phi_{t_n}(x_{m_n})}{\sqrt{\gamma(n)}}$. We have, for $j = 0, \dots, k_n - 1$,

$$\phi_{t_{j+1}^n}(x_{m_n}) - \phi_{t_j^n}(x_{m_n}) = \gamma_{m_n+j+1} F(\phi_{t_j^n}(x_{m_n})) + \mathcal{O}(\gamma_{m_n+j}^2).$$

Then, denoting

$$\hat{Y}_j^n := \frac{1}{\sqrt{\gamma(n)}} \bigl(x_{m_n+j} - \phi_{t_j^n}(x_{m_n})\bigr) \qquad (j = 0, \dots, k_n),$$

we have

$$\hat{Y}_{j+1}^n - \hat{Y}_j^n = \frac{\gamma_{m_n+j+1}}{\sqrt{\gamma(n)}} [F(x_{m_n+j}) - F(\phi_{t_j^n}(x_{m_n})) + U_{m_n+j+1}] + \mathcal{O}\left(\frac{\gamma_{m_n+j+1}^2}{\sqrt{\gamma(n)}}\right).$$

Consequently,

$$\hat{Y}_{j+1}^n - \hat{Y}_j^n = \gamma_{m_n+j+1} \left(DF(\phi_{t_j^n}(x_{m_n})) \hat{Y}_j^n + \frac{R^n(j)}{\sqrt{\gamma(n)}} + \frac{U_{m_n+j+1}}{\sqrt{\gamma(n)}} \right) + \mathcal{O}\left(\frac{\gamma_{m_n+j+1}^2}{\sqrt{\gamma(n)}}\right) \qquad (j = 0, \dots, k_n - 1),$$

where

$$R^n(j) := F(x_{m_n+j}) - F(\phi_{t_j^n}(x_{m_n})) - DF(\phi_{t_j^n}(x_{m_n})) \cdot \bigl(x_{m_n+j} - \phi_{t_j^n}(x_{m_n})\bigr).$$

By a recursive argument,

$$\begin{aligned}\hat{Y}_{n+1} - Y_{n+1} &= \hat{Y}^n_{k_n} - Y_{n+1}\\ &= \frac{1}{\sqrt{\gamma(n)}} \sum_{j=1}^{k_n} \gamma_{m_n+j} \left(\prod_{k=j+1}^{k_n} \left(I_d + \gamma_{m_n+k} DF(\phi_{t^n_{k-1}}(x_{m_n}))\right) \right) R^n(j)\\ &\quad + \mathcal{O}(e^{-n/2}),\end{aligned}$$

since $\hat{Y}^n_0 = 0$ and $\sum_{j=0}^{k_n-1} \frac{\gamma_{m_n+j+1}}{\sqrt{\gamma(n)}} = \sqrt{\gamma(n)} = \mathcal{O}(e^{-n/2})$.

Recall that $\sum_{j=1}^{k_n} \gamma_{m_n+j} \le 1 + \gamma_{m_n+1}$ and DF is bounded. Consequently, there exists a real number K such that for n large enough,

$$\begin{aligned}&\frac{1}{\sqrt{\gamma(n)}} \left\| \sum_{j=1}^{k_n} \gamma_{m_n+j} \left(\prod_{k=j+1}^{k_n} \left(I_d + \gamma_{m_n+k} DF(\phi_{t^n_{k-1}}(x_{m_n}))\right) \right) R^n(j) \right\|\\ &\quad \le e^K \frac{1}{\sqrt{\gamma(n)}} \sup_{j=1,\dots,k_n} R^n(j) = e^K R_n,\end{aligned}$$

where $R_n := \frac{1}{\sqrt{\gamma(n)}} \sup_{j=1,\dots,k_n} R^n(j)$. By an application of results due to Benaim [see Benaïm (1999), Proposition 4.1, formula (11) and identity (13) with $q = 2$], we have

$$\mathbb{E}\Big(\sup_{j=0,\dots,k_n-1} \|x_{m_n+j} - \phi_{t^n_j}(x_{m_n})\|^2 \bigm| \mathcal{G}_n \Big) \le C\gamma(n),$$

where C is some positive constant. Additionally, by definition of DF,

$$R^n(j)^2 \le h\big(\|x_{m_n+j} - \phi_{t^n_j}(x_{m_n})\|^2\big)$$

for some function $h : \mathbb{R}^*_+ \to \mathbb{R}^*_+$, strictly increasing and such that $h(x)/x \to_{x\to 0^+} 0^+$. An immediate consequence is that

$$\begin{aligned}&\mathbb{P}(R_n \ge \alpha \mid \mathcal{G}_n)\\ &\quad \le \mathbb{P}\Big(\sup_{j=0,\dots,k_n-1} h\big(\|x_{m_n+j} - \phi_{t^n_j}(x_{m_n})\|^2\big) \ge \alpha^2 \gamma(n) \bigm| \mathcal{G}_n \Big)\\ &\quad \le \mathbb{P}\Big(\sup_{j=0,\dots,k_n-1} \|x_{m_n+j} - \phi_{t^n_j}(x_{m_n})\|^2 \ge h^{-1}(\alpha^2\gamma(n)) \bigm| \mathcal{G}_n \Big)\\ &\quad \le \frac{C\gamma(n)}{h^{-1}(\alpha^2\gamma(n))} \to_{n\to+\infty} 0,\end{aligned}$$

which proves the result. $\square$

To simplify notation, we call E the Euclidean space $\mathbb{R}^d$. Given $n \in \mathbb{N}$, the random variable x_n can be written $h_n(U_1, \dots, U_n)$, where $h_n : (E^n, (\mathcal{B}_E)^n) \to$

$(E, \mathcal{B}_E)$ is a measurable function. We denote by $\mathcal{P}_U$ the probability distribution induced by the measurable process $U = (U_n)_n : (\Omega, \mathcal{F}) \to (E^{\mathbb{N}}, (\mathcal{B}_E)^{\mathbb{N}})$. We keep the notation $\mathcal{F}_n$ for the sigma field $(\mathcal{B}_E)^n \times E^{\mathbb{N}}$ when it does not imply any ambiguity.

PROPOSITION A.4. *There exists a function $P_n : (\mathcal{B}_E)^{\mathbb{N}} \times E^{\mathbb{N}} \to [0, 1]$ called a* regular conditional distribution of U given $\mathcal{F}_n$ *in the sense that, for any $u \in E^{\mathbb{N}}$, $P_n(\cdot, u)$ is a probability measure on $(E^{\mathbb{N}}, (\mathcal{B}_E)^{\mathbb{N}})$ and that, for any $B \in (\mathcal{B}_E)^{\mathbb{N}}$, the random variable $P_n(B, \cdot)$ is $\mathcal{F}_n$-measurable with*

$$\mathbb{P}_n(B, \cdot) = \mathbb{P}_U(B \mid \mathcal{F}_n)(\cdot) \qquad \mathbb{P}_U\text{-almost surely.}$$

For convenience, given $u \in E^{\mathbb{N}}$, we denote by $\mathbb{P}_n^u$ the probability measure $\mathbb{P}_n(\cdot, u)$ and $\mathbb{E}_n^u$ the corresponding expectation. Given a measurable function $y : (E^{\mathbb{N}}, (\mathcal{B}_E)^{\mathbb{N}}) \to (E, \mathcal{B}_E)$, we have

$$\mathbb{E}_n^{\cdot}(y) = \mathbb{E}_U(y \mid \mathcal{F}_n) = \mathbb{E}(y(U) \mid \mathcal{F}_n)(\cdot) \qquad \mathbb{P}_U\text{-a.s.}$$

LEMMA A.5. *Let $k < i$ be two natural numbers and $y : (E^{\mathbb{N}}, (\mathcal{B}_E)^{\mathbb{N}}) \to (E, \mathcal{B}_E)$ be a measurable function. There exists a subset $\Omega_0(y) \subset E^{\mathbb{N}}$ such that $\mathbb{P}_U(\Omega_0(y)) = 1$ and, for any $u_0 \in \Omega_0(y)$, $\mathbb{E}_k^{u_0}(y \mid \mathcal{F}_i)$ and $\mathbb{E}_U(y \mid \mathcal{F}_i)$ are $\mathbb{P}_U$-almost surely equal.*

PROOF. The random variable $z := \mathbb{E}_U(y \mid \mathcal{F}_i)$ is $\mathcal{F}_i$-measurable. Pick a countable π-class $\mathcal{D}$ such that $\sigma(\mathcal{D}) = \mathcal{F}_k$. Given $A \in \mathcal{D}$, we claim that there exists a set $\Omega_0(y, A)$ such that $\mathbb{P}_U(\Omega_0(y, A)) = 1$ and, for any $u_0 \in \Omega_0(y, A)$, we have:

(1) $\mathbb{E}_k^{u_0}(\mathbb{E}(\mathbb{I}_A y \mid \mathcal{F}_i)) = \mathbb{E}_U(\mathbb{E}(\mathbb{I}_A y \mid \mathcal{F}_i) \mid \mathcal{F}_k)(u_0)$,
(2) $\mathbb{E}_k^{u_0}(\mathbb{I}_A y) = \mathbb{E}_U(\mathbb{I}_A y \mid \mathcal{F}_k)(u_0)$,
(3) $\mathbb{I}_A \mathbb{E}_U(y \mid \mathcal{F}_i) = \mathbb{E}_U(\mathbb{I}_A y \mid \mathcal{F}_i)$ $\mathbb{P}_k^{u_0}$-a.s.

Let us construct $\Omega_0(y, A)$. First, there exist two sets $\Omega_0^1(y, A)$ and $\Omega_0^2(y, A)$ on which, respectively, points (1) and (2) are satisfied and such that $\mathbb{P}_U(\Omega_0^j(y, A)) = 1$, $j = 1, 2$. Now for the last point, one must first consider a set $\Omega^3(y, A)$ such that $\mathbb{P}_U(\Omega^3(y, A)) = 1$ and, for any $u \in \Omega^3(y, A)$,

$$\mathbb{I}_A(u)\mathbb{E}_U(y \mid \mathcal{F}_i)(u) = \mathbb{E}_U(\mathbb{I}_A y \mid \mathcal{F}_i)(u).$$

Then, by definition of $\mathbb{P}_k^{u_0}$, there exists a set $\Omega_0^3(y, A)$ [which depends on $\Omega^3(y, A)$] such that $\mathbb{P}_U(\Omega_0^3(y, A)) = 1$ and, for any $u_0 \in \Omega_0^3(y, A)$,

$$\mathbb{P}_k^{u_0}(\Omega^3(y, A)) = \mathbb{P}_U(\Omega^3(y, A) \mid \mathcal{F}_k)(u_0) = 1.$$

Finally, pick $\Omega_0(y, A) := \Omega_0^1(y, A) \cap \Omega_0^2(y, A) \cap \Omega_0^3(y, A)$.

Now take

$$\Omega_0(y) := \bigcap_{A \in \mathcal{D}} \Omega(y, A).$$

By countability of $\mathcal{D}$, we have $\mathbb{P}_U(\Omega_0(y)) = 1$. There remains to prove that, for any $u_0 \in \Omega_0(y)$,

$$\begin{aligned}
\int_A z \, d\mathbb{P}_k^{u_0} &= \int_A y \, d\mathbb{P}_k^{u_0} \qquad \text{for any } A \in \mathcal{D}, \\
\mathbb{E}_k^{u_0}(\mathbb{I}_A z) &= \mathbb{E}_k^{u_0}(\mathbb{I}_A \mathbb{E}_U(y \mid \mathcal{F}_i)) \\
&= \mathbb{E}_k^{u_0}(\mathbb{E}_U(\mathbb{I}_A y \mid \mathcal{F}_i)) \\
&= \mathbb{E}_U(\mathbb{E}_U(\mathbb{I}_A y \mid \mathcal{F}_i) \mid \mathcal{F}_k)(u_0) \\
&= \mathbb{E}_U(\mathbb{I}_A y \mid \mathcal{F}_k)(u_0) \\
&= \mathbb{E}_k^{u_0}(\mathbb{I}_A y).
\end{aligned}$$

The second equality follows from point (3), the third from point (1) and the fifth from point (2). The lemma is proved. □

The following result is due to Hall and Heyde (1980) [see Theorems 3.4 or 2, page 351, in Chow and Teicher (1998) for a version adapted to our situation]. It is a central limit result for double arrays. We apply it to prove point (ii).

THEOREM A.6 (Hall and Heyde). *For any $n \geq 1$, let k_n be a positive integer and $(\Omega_n, \mathcal{F}^n, \mathbb{P}_n)$ a probability space. Consider $\mathcal{F}_1^n \subset \mathcal{F}_2^n \subset \cdots \subset \mathcal{F}_{k_n}^n \subset \mathcal{F}^n$ an increasing family of sigma fields and $(y_j^n)_{j=1,\dots,k_n}$ a $(\mathcal{F}_j^n)_{j=1,\dots,k_n}$-adapted family of random variables. Assume that*:

* *for $j = 1, \dots, k_n$,*

$$\mathbb{E}_n(y_j^n \mid \mathcal{F}_{j-1}^n) = 0,$$

* *we have*

$$\sum_{j=1}^{k_n} \mathbb{E}_n\big(\|Y_j^n\|^2 \mathbb{I}_{\|Y_j^n\| > \varepsilon} \mid \mathcal{F}_{j-1}^n\big) \xrightarrow[n \to +\infty]{dist.} 0,$$

* *there exists a positive, $\mathcal{F}_1^n$-adapted random sequence $(w_n)_n$ such that*

$$\sum_{i=1}^{k_n} \mathbb{E}_n(y_j^n (y_j^n)^T \mid \mathcal{F}_{j-1}^n) - w_n \xrightarrow[n \to +\infty]{dist.} 0,$$

* *there exists a positive random matrix η, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which satisfies*

$$\sum_{j=1}^{k_n} \mathbb{E}_n(y_j^n (y_j^n)^T \mid \mathcal{F}_{j-1}^n) \xrightarrow[n \to +\infty]{dist.} \eta.$$

Then, denoting $y_{n+1} := \sum_{j=1}^{k_n} y_j^n$, *the sequence* $(y_n)_n$ *converges in distribution to some random variable* y *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *and whose characteristic function is given by* $\mathbb{E}(e^{-(1/2)t^T \eta t})$. *In particular,*

$$\lim_{n\to+\infty} \mathbb{E}_n\big(e^{i\langle t, y_{n+1}\rangle}\big) = \mathbb{E}\big(e^{-(1/2)t^T \eta t}\big).$$

Let us get back to our settings. Let $n \in \mathbb{N}$ and $j \in \{1, \dots, k_n\}$. Consider the measurable functions $y_j^n : (E^{\mathbb{N}}, (\mathcal{B}_E)^{\mathbb{N}}) \to (E, \mathcal{B}_E)$, given by

$$y_j^n(u) := \frac{\gamma_{m_n+j}}{\sqrt{\gamma(n)}} \left(\prod_{k=j+1}^{k_n} \big(I_d + \gamma_{m_n+k} DF(\phi_{t_{k-1}^n}(x_{m_n}))\big)\right) u_{m_n+j},$$

where $x_n = h_n(u_1, \dots, u_n)$. Finally, call $y_n := \sum_{j=1}^{k_n} y_j^n$.

COROLLARY A.7. *Given an open ball* O *in* E, *there exist* $\delta > 0$ *and a set* Ω_0 *such that* $\mathbb{P}_U(\Omega_0) = 1$ *and*, *for any* $u_0 \in \Omega_0$,

$$\liminf_n \mathbb{P}_U(y_{n+1} \in O \mid \mathcal{G}_n)(u_0) > \delta.$$

PROOF. Let Ω_0 be the set

$$\bigcap_{n\in\mathbb{N}, j=1,\dots,k_n, r\in\mathbb{Q}} \Omega_0\big(y_j^n, \|y_j^n\|^2 \mathbb{I}_{\|y_j^n\|>r}, y_j^n (y_j^n)^T, \mathbb{I}_{\|x_{m_n+j} - \Phi_{t_j^n}(x_{m_n})\|>r}, \mathbb{I}_{y_n \in O}\big).$$

By countability, $\mathbb{P}(\Omega_0) = 1$. Pick $u_0 \in \Omega_0$. We apply Theorem A.6 to $(\Omega_n, \mathcal{F}^n, \mathbb{P}_n) := (E^{\mathbb{N}}, (\mathcal{B}_E)^{\mathbb{N}}, \mathbb{P}_{m_n}^{u_0})$, $\mathcal{F}_j^n = \mathcal{F}_{m_n+j}$ and the double array of random variables $(y_j^n)_{n,j}$.

We now verify that the assumptions required to apply Theorem A.6 hold. First of all,

$$\mathbb{E}_{m_n}^{u_0}(y_j^n \mid \mathcal{F}_{j-1}^n) = \mathbb{E}_U(y_j^n \mid \mathcal{F}_{j-1}^n) = 0 \qquad \text{a.s.}$$

Second, let

$$\Pi_{n,j} := \prod_{k=j+1}^{k_n} \big(I_d + \gamma_{m_n+k} DF(\phi_{t_{k-1}^n}(x_{m_n}))\big).$$

A simple computation gives

$$e^{-2\|DF\|_\infty} \le \|\Pi_{n,j}\| \le e^{\|DF\|_\infty}.$$

Recall that there exists $p > 1$ such that the sequence of random variables $(\mathbb{E}_U(\|u_n\|^{2p} \mid \mathcal{F}_{n-1}))_n$ is almost surely bounded. Hence, taking q such that

$1/p + 1/q = 1$ and choosing $r \in \mathbb{Q}$,

$$\begin{aligned}
\mathbb{E}_U\big(\|y_j^n\|^2 \mathbb{I}_{\|y_j^n\|>r} \mid \mathcal{F}_{j-1}^n\big) &\le \mathbb{E}_U(\|y_j^n\|^{2p} \mid \mathcal{F}_{j-1}^n)^{1/p} \mathbb{P}_U(\|y_j^n\|^{2p} > r^{2p} \mid \mathcal{F}_{j-1}^n)^{1/q} \\
&\le \frac{1}{r^{2p/q}} \mathbb{E}_U(\|y_j^n\|^{2p} \mid \mathcal{F}_{j-1}^n) \\
&\le \frac{1}{r^{2p/q}} \frac{\gamma_{m_n+j}^{2p}}{\gamma(n)} e^{\|DF\|_\infty} \mathbb{E}_U(\|u_{m_n+j}\|^{2p} \mid \mathcal{F}_{j-1}^n) \\
&\le C(r) \frac{\gamma_{m_n+j}^{2p}}{\gamma(n)} \mathbb{E}_U(\|u_{m_n+j}\|^{2p} \mid \mathcal{F}_{j-1}^n).
\end{aligned}$$

Consequently,

$$\sum_{j=1}^{k_n} \mathbb{E}_U\big(\|y_j^n\|^2 \mathbb{I}_{\|y_j^n\|>r} \mid \mathcal{F}_{j-1}^n\big) \le C(r) \sup_j \gamma_{m_n+j}^{2(p-1)} \sup_j \mathbb{E}_U(\|u_{m_n+j}\|^{2p} \mid \mathcal{F}_{j-1}^n),$$

which converges to 0 almost surely. Since u_0 belongs to the set $\Omega_0(\|y_j^n\|^2 \mathbb{I}_{\|y_j^n\|>r})$, for any $j = 1, \dots, k_n$,

$$\sum_{j=1}^{k_n} \mathbb{E}_{m_n}^{u_0}\big(\|y_j^n\|^2 \mathbb{I}_{\|y_j^n\|>r} \mid \mathcal{F}_{j-1}^n\big) = \sum_{j=1}^{k_n} \mathbb{E}_U\big(\|y_j^n\|^2 \mathbb{I}_{\|y_j^n\|>r} \mid \mathcal{F}_{j-1}^n\big) \qquad \mathbb{P}_U\text{-a.s.}$$

and the second point holds.

From now on, we call

$$W_n := \sum_{j=1}^{k_n} \mathbb{E}_U((y_j^n)(y_j^n)^T \mid \mathcal{F}_{j-1}^n).$$

We have

$$\begin{aligned}
&\mathbb{E}_U((y_j^n)(y_j^n)^T \mid \mathcal{F}_{j-1}^n) \\
&\quad = \frac{1}{\gamma(n)} \gamma_{m_n+j}^2 \Pi_{n,j} \mathbb{E}_U(u_{m_n+j} u_{m_n+j}^T \mid \mathcal{F}_{j-1}^n) \Pi_{n,j}^T \\
&\quad = \frac{1}{\gamma(n)} \gamma_{m_n+j}^2 \Pi_{n,j} Q(x_{m_n+j-1}) \Pi_{n,j}^T.
\end{aligned}$$

Consequently,

$$W_n = \frac{1}{\gamma(n)} \sum_{j=1}^{k_n} \gamma_{m_n+j}^2 \Pi_{n,j} Q(x_{m_n+j-1}) \Pi_{n,j}^T.$$

Let w_n be the $\mathcal{F}_{n,1}$-measurable random variable defined by

$$w_n := \frac{1}{\gamma(n)} \sum_{j=1}^{k_n} \gamma_{m_n+j}^2 \Pi_{n,j} Q(\phi_{t_{j-1}^n}(x_{m_n})) \Pi_{n,j}^T.$$

Pick $r \in \mathbb{Q}$. By definition of Ω_0 and Assumption 2.2(i),

$$\begin{aligned}\mathbb{P}^{u_0}_{m_n}\Bigl(\sup_{j=1,\dots,k_n} \|\phi_{t^n_{j-1}}(x_{m_n}) - x_{m_n+j-1}\| > r\Bigr) \\ &= \mathbb{P}\Bigl(\sup_{j=1,\dots,k_n} \|\phi_{t^n_{j-1}}(x_{m_n}) - x_{m_n+j-1}\| > r \bigm| \mathcal{G}_n\Bigr) \\ &\le \omega(n, r, 1) \to 0,\end{aligned}$$

which implies that

$$W_n - w_n \xrightarrow[n\to+\infty]{\text{dist.}} 0.$$

Since the application Q takes values in $[\Lambda^- I_d, \Lambda^+ I_d]$ and $\|\Pi_{n,j}\|$ is bounded above and away from zero, we have

$$0 < a^- \le \Pi_{n,j} Q(x_{m_n+j-1})\Pi^T_{n,j} \le a^+ < +\infty.$$

W_n is a convex combination of such quantities, therefore is bounded. Pick some increasing sequence of integers $(n_p)_p$. Without loss of generality, we may assume that $(W_{n_p})_p$ converges in distribution to some random variable η^{u_0}, defined on the probability space induced by U and which takes values in $\mathcal{S}^+(E) \cap [a^- I_d, a^+ I_d]$.

Now by Theorem A.6,

$$y_{n_p} \xrightarrow[p\to+\infty]{\mathcal{L}} y^{u_0}$$

with $\mathbb{E}_U(e^{i\langle t, y^{u_0}\rangle}) = \mathbb{E}(e^{-(1/2)t^T \eta^{u_0} t})$. In particular, by definition of Ω_0,

$$\lim_p \mathbb{P}_U(y_{n_p+1} \in O \mid \mathcal{G}_{n_p})(u_0) = \lim_p \mathbb{P}^{u_0}_{n_p}(y_{n_p+1} \in O) = \mathbb{P}(y^{u_0} \in O) > \delta,$$

where δ depends on the parameters a^- and a^+ but not on $u_0 \in \Omega_0$ and $(n_p)_p$. The proof is complete. $\square$

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