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OPTIMAL INVESTMENT POLICY AND DIVIDEND PAYMENT STRATEGY IN AN INSURANCE COMPANY

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We consider in this paper the optimal dividend problem for an insurance company whose uncontrolled reserve process evolves as a classical Cramér–Lundberg process. The firm has the option of investing part of the surplus in a Black–Scholes financial market. The objective is to find a strategy consisting of both investment and dividend payment policies which maximizes the cumulative expected discounted dividend pay-outs until the time of bankruptcy. We show that the optimal value function is the smallest viscosity solution of the associated second-order integro-differential Hamilton–Jacobi–Bellman equation. We study the regularity of the optimal value function. We show that the optimal dividend payment strategy has a band structure. We find a method to construct a candidate solution and obtain a verification result to check optimality. Finally, we give an example where the optimal dividend strategy is not barrier and the optimal value function is not twice continuously differentiable.

1. Introduction. A classical problem in actuarial mathematics is to maximize the cumulative expected discounted dividend pay-outs. In the Cramér–Lundberg setting, this optimization problem was introduced by De Finetti (1957); Gerber (1969) proved the existence of an optimal dividend payment strategy and showed that it has a band structure. The cumulative expected discounted dividend pay-outs is a way to value a company as it can be seen, for instance, in the classical paper by Miller and Modigliani (1961) for the deterministic case and more recently in Sethi, Derzko and Lehoczky (1984a, 1984b) and Sethi (1996) for the stochastic case.

In this paper we consider this optimization problem in the classical Cramér–Lundberg setting, but we allow the management the possibility of controlling the stream of dividend pay-outs and of investing part of the surplus in a Black and Scholes financial market. We impose a borrowing constraint: short-selling of stocks or to borrow money to buy stocks is not allowed. Technically, the unconstrained optimization problem is simpler.

Azcue and Muler (2005) consider the problem of maximizing the cumulative expected discounted dividend pay-outs of an insurance company when the management has the possibility of controlling the risk exposure by reinsurance. In this

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case, the optimal value function was characterized as the smallest viscosity solution of the first-order integro-differential Hamilton–Jacobi–Bellman equation, and the optimal dividend payment strategy was found.

In this paper, the optimization problem is more complex than the one we treated before. One difference is that the associated Hamilton–Jacobi–Bellman equation is a nonlinear degenerate second-order integro-differential equation subject to a differential constraint. The possibility that the ellipticity of the second-order operator involved in this equation can degenerate at any point together with the fact that there is an integral term, makes it more difficult to prove the existence and regularity of solutions. However, when we obtain the solution of this operator in Section 6, we see that the ellipticity only degenerates at zero and so the degeneracy is not as serious as it could be (the solution turns out to be twice continuous differentiable). Another difference is that, since in this case the controlled surplus involves a Brownian motion, there is not an optimal strategy. Nevertheless, we prove that the optimal value function can be written explicitly as a limit of value functions of strategies. So, we introduce the notion of limit dividend strategies and prove that the optimal limit strategy has a *band structure*.

In a diffusion setting, which means that the surplus is modeled as a Brownian motion, different cases were studied; we can mention, for instance, Asmussen and Taksar (1997) for the problem of dividend optimization and Højgaard and Taksar (2004) for the case of dividend, reinsurance and portfolio optimization. The main difference between the two settings is that the HJB equation in the diffusion case is a differential equation and not an integro-differential one. Other differences are that in the diffusion setting the optimal strategies are always barrier strategies, that there is a natural boundary condition at zero for the associated HJB equation and that this equation has always classical concave solutions; these properties might not occur in the Cramér–Lundberg setting.

Avram, Palmowski and Pistorius (2007) study the problem of maximizing the discounted dividend pay-outs when the uncontrolled surplus of the company follows a general spectrally negative Lévy process in absence of investment. The HJB equation associated with this optimization problem is also a second-order integro-differential equation but its ellipticity does not degenerate.

In both, Højgaard and Taksar (2004) and Avram, Palmowski and Pistorius (2007), the corresponding HJB equations are second-order equations whose ellipticity does not degenerate at zero, so to characterize the optimal value function among the solutions of the HJB equation they use the natural boundary condition at zero. In this paper, we do not have a natural condition at zero but we do not need this boundary condition because the ellipticity of the HJB degenerates at this point. The lack of a boundary condition at zero makes more difficult to obtain a numerical scheme.

The main results of this paper are the following:

In the first part of the paper, we obtain the optimal value function as the smallest viscosity solutions of the associated HJB equation, and we prove a verification theorem that allows us, since the optimal value function has not a natural boundary condition at zero, to recognize the optimal value function among the many viscosity solutions of the associated HJB equation.

From Section 6 on, we assume that the claim-size distribution has a bounded density; this allows us to show that the optimal value function is twice continuously differentiable except possibly for some points. We find the optimal value function for small surpluses, and we prove that the optimal strategy is *stationary*, that is, the decision of what proportion of the surplus is invested in the risky asset, and how much to pay out as dividends at any time depends only on the current surplus. We also prove that the optimal dividend payment policy has a band structure. In particular, the optimal dividend payment policy for large surpluses is to pay out immediately the surplus exceeding certain level as dividends. We also obtain the best barrier strategy and show both an example where the optimal dividend payment policy is barrier as well as an example where it is not. The second example shows that, even for claim-size distributions with bounded density, the optimal value function could be neither concave nor twice continuously differentiable.

This paper is organized as follows. In Section 2, we state the optimization problem and prove some properties about the regularity and growth of the optimal value function. In Section 3, we state the dynamic programming principle and show that the optimal value function is a viscosity solution of the HJB equation associated with the optimization problem. In Section 4, we prove the uniqueness of viscosity solutions of the HJB equation with a boundary condition at zero. In Section 5, we prove that the optimal value function is the smallest supersolution of the HJB equation and give a verification theorem that states that a supersolution which can be obtained as a limit of value functions of admissible strategies is the optimal value function. In Section 6, we construct via a fixed-point operator a classical solution of the second-order integro-differential equation involved in the HJB equation. In Section 7, we use the solution obtained in Section 6 to obtain the value function of the optimal barrier strategy. In Section 8, we find the optimal value function for small surpluses, show that the optimal strategy is stationary and prove that the optimal dividend payment policy has a band structure. In Section 9, we show some numerical examples. We have placed some technical lemmas in the Appendix to improve the readability of the main text.

2. The stochastic control problem. We assume that the surplus of an insurance company in the absence of control of dividends payment and investment follows the classical Cramér–Lundberg process; that is, the surplus X_t of the company is described by

$$X_t = x + pt - \sum_{i=1}^{N_t} U_i, \tag{2.1}$$

where x is the initial surplus, p is the premium rate, N_t is a Poisson process with claim arrival intensity $\beta > 0$ and the claim sizes U_i are i.i.d. random variables with distribution F. We assume that the distribution F has finite expectation μ and satisfies $F(0)=0$.

We consider that the financial market is described as a classical Black–Scholes model where we have a risk-free asset with price process B_t and a risky asset with price process S_t satisfying

$$\begin{cases} dB_t = r_0 B_t\, dt, \\ dS_t = r S_t\, dt + \sigma S_t\, dW_t, \end{cases}$$

where W_t is a standard Brownian motion independent to the process X_t. We consider for simplicity $r_0 = 0$.

We define Ω as the set of paths with left and right limits and $(\Omega, \mathcal{F}, P)$ as the complete probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the processes X_t and W_t. A *control strategy* is a process $\pi = (\gamma_t, L_t)$ where $\gamma_t \in [0, 1]$ is the proportion of the surplus invested in stocks at time t, and L_t is the cumulative dividends the company has paid out until time t. The control strategy (γ_t, L_t) is *admissible* if the process γ_t is predictable and the process L_t is predictable, nondecreasing and càglàd (left continuous with right limits).

We are considering the case where $\gamma_t \in [0, 1]$ because we are allowing neither short-selling of stocks nor borrowing money from other sources to buy stocks.

Denote by Π_x the set of all the admissible control strategies with initial surplus x. For any $\pi \in \Pi_x$, the controlled risk process X_t^π can be written as

$$X_t^\pi = x + pt + r\int_0^t X_s^\pi \gamma_s\, ds + \sigma \int_0^t X_s^\pi \gamma_s\, dW_s - \sum_{i=1}^{N_t} U_i - L_t. \tag{2.2}$$

All the jumps of the process X_t^π are downward, $X_{t-}^\pi - X_t^\pi > 0$ if there is a claim at time t and $X_t^\pi - X_{t+}^\pi > 0$ only at the discontinuities of L_t. We also ask $\Delta L_t := L_{t^+} - L_t \leq X_t^\pi$ for any $t \geq 0$; this means that the company cannot pay immediately an amount of dividends exceeding the surplus.

Given an admissible strategy $\pi \in \Pi_x$, let $\tau^\pi = \inf\{t \geq 0 : X_t^\pi < 0\}$ be the *ruin time* of the company, note that it can only occur at the arrival of a claim. We define the value function of π by

$$V_\pi(x) = E_x\left(\int_0^{\tau^\pi} e^{-cs}\, dL_s\right), \tag{2.3}$$

where c is the discount factor. The integral is interpreted pathwise in a Lebesgue–Stieltjes sense.

We consider the following optimization problem:

$$V(x) = \sup\{V_\pi(x) \text{ with } \pi \in \Pi_x\} \qquad \text{for } x \geq 0. \tag{2.4}$$

For technical reasons, we define $V(x) = 0$ for $x < 0$. We restrict ourselves to the case $c > r > 0$; we will see in Remark 2.4 that in the case $c < r$, the optimal value function is infinite.

To show that the optimal value function V is well defined and to describe some of its basic properties, we first state some results of the related controlled risk process without claims and without paying dividends.

LEMMA 2.1. *Given $x \geq 0$ and any admissible investment strategy $\gamma_t \in [0, 1]$ consider the process*,

$$Y_t = x + mt + r \int_0^t Y_s \gamma_s \, ds + \sigma \int_0^t Y_s \gamma_s \, dW_s.$$

(a) *If $m \geq 0$, then $E_x(Y_t e^{-ct}) \leq e^{-(c-r)t}(x + m(1 - e^{-rt})/r)$.*
(b) *If $x > 0$ and $\tilde{\tau} = \inf\{t : Y_t < 0\}$, then $\lim_{h \to 0} P(\tilde{\tau} < h) = 0$.*
(c) *If $\gamma_t \equiv 1$, then $E_x(Y_t e^{-ct}) = e^{-(c-r)t}(x + m(1 - e^{-rt})/r)$ for any $m \in \mathbf{R}$.*

PROOF. We can write $Y_t = xU_t + U_t \int_0^t m U_s^{-1} \, ds$ where

$$U_t = e^{\int_0^t (r\gamma_s - \sigma^2/2\gamma_s^2) \, ds + \int_0^t \sigma \gamma_s \, dW_s}. \tag{2.5}$$

The process $e^{-\int_0^t r\gamma_s \, ds} U_t$ is a martingale [see, for instance, Karatzas and Shreve (1991)]. Then the results follow using elementary computations for linear diffusion processes. □

In the next two propositions, we prove that V has linear growth, and we give bounds on the increments of V using the value functions of some simple admissible strategies.

PROPOSITION 2.2. *The optimal value function V is well defined and satisfies*

$$x + p/(\beta + c) \leq V(x) \leq rx/(c - r) + p/(c - r) \qquad \text{for } x \geq 0.$$

PROOF. Consider an initial surplus $x \geq 0$. Given any $\pi = (\gamma_s, L_s) \in \Pi_x$, consider the controlled process X_t^π for $t \geq 0$, and define $X_t^\pi = 0$ for $t < 0$. Then

$$\begin{aligned} \tilde{L}_s &= L_s - \sigma \int_0^s X_u^\pi \gamma_u \, dW_u \\ &\leq x + ps + r \int_0^s X_u^\pi \gamma_u \, du - \sum_{i=1}^{N_s} U_i \\ &\leq x + ps + r \int_0^s X_u^\pi \gamma_u \, du. \end{aligned}$$

Consider the process Y_t defined as in Lemma 2.1 with $m = p$ and the investment strategy γ_s corresponding to π. Since $X_t^\pi \leq Y_t$, we obtain from Lemma 2.1(a) that

$E_x(X_t^\pi e^{-ct}) \le e^{-(c-r)t}(x + p(1-e^{-rt})/r)$. Since $r < c$ and e^{-cs} is a positive and decreasing function, we have that

$$\begin{aligned} V_\pi(x) &= E_x\left(\int_0^\tau e^{-cs}\,dL_s\right) = E_x\left(\int_0^\tau e^{-cs}\,d\widetilde{L}_s\right) \\ &\le E_x\left(\int_0^\infty e^{-cs}\,d\left(x + ps + r\int_0^s X_u^\pi \gamma_u\,du\right)\right) \\ &\le \int_0^\infty e^{-cs} p\,ds + r\int_0^\infty E_x(e^{-cs}X_s^\pi)\,ds \\ &\le rx/(c-r) + p/(c-r). \end{aligned}$$

So $V(x) = \sup_{\pi\in\Pi_x} V_\pi(x)$ is well defined and satisfies the second inequality.

Let us prove now the first inequality. Given an initial surplus $x \ge 0$, consider the admissible strategy π_0 which pays immediately the whole surplus x and then pays the incoming premium p as dividends until the first claim which in this strategy means ruin. Define τ_1 as the time arrival of the first claim; we have

$$V_{\pi_0}(x) = x + pE_x\left(\int_0^{\tau_1} e^{-ct}\,dt\right) = x + p/(\beta + c),$$

but by definition $V(x) \ge V_{\pi_0}(x)$, so we get the result. □

PROPOSITION 2.3. *If $y > x \ge 0$, the function V satisfies*:
(a) $V(y) - V(x) \ge y - x$;
(b) $V(y) - V(x) \le (e^{(c+\beta)(y-x)/p} - 1)V(x)$.

PROOF. (a) Given $\varepsilon > 0$, consider an admissible strategy $\pi \in \Pi_x$ with $V_\pi(x) \ge V(x) - \varepsilon$. We define a new strategy $\overline{\pi} \in \Pi_y$ in the following way, pay immediately $y - x$ as dividends and then follow the strategy $\pi \in \Pi_x$; this new strategy is admissible. We have that

$$V(y) \ge V_{\overline{\pi}}(y) = V_\pi(x) + (y - x) \ge V(x) - \varepsilon + (y - x)$$

and the result follows.

(b) Given $\varepsilon > 0$, take an admissible strategy $\pi \in \Pi_y$ such that $V_\pi(y) \ge V(y) - \varepsilon$. Let us define the strategy $\overline{\pi} \in \Pi_x$ that starting at x, pay no dividends and invest all the surplus in bonds if $X_t^{\overline{\pi}} < y$ and follow strategy π when the current surplus reaches y. This strategy is admissible. If there is no claim up to time $t_0 = (y - x)/p$, the surplus $X_{t_0}^{\overline{\pi}} = y$. The probability of reaching y before the first claim is $e^{-\beta t_0}$, so we obtain

$$V(x) \ge V_{\overline{\pi}}(x) \ge V_\pi(y)e^{-(c+\beta)t_0} \ge \bigl(V(y) - \varepsilon\bigr)e^{-(c+\beta)(y-x)/p}$$

and we get the result. □

As a direct consequence of the previous proposition we have that V is increasing and locally Lipschitz in $[0,+\infty)$, this implies that V is absolutely continuous, that $V'(x)$ exists a.e. and that $1 \leq V'(x) \leq V(x)(c+\beta)/p$ at the points where the derivative exists. We will prove later in this paper that V is continuously differentiable with bounded derivative and that the linear growth condition given by Proposition 2.2 can be improved to $V(x) \leq x + p/c$ for $x \geq 0$.

REMARK 2.4. The value function V is infinite in the case that $c < r$. To see this, let us consider the worst possible case, that is $p \leq \beta\mu$. We can assume that $x > x_0 := (\beta\mu - p + 1)/r > 0$ because, if the initial surplus x is smaller than x_0 there is a positive probability that the surplus surpass the level x_0 [take, for instance, the strategy which pays no dividends and keeps all the surplus in bonds up to time $T = (x_0 - x)/p + 1$]. Given $t_0 > 0$, consider the following admissible strategy $\pi_{t_0} \in \Pi_x$: divide the company in two departments, one of them deals only with the investment and the payment of dividends and the other with the insurance business. The investment department starts with capital x, invest all the surplus on risky assets and diverts to the insurance department a constant flow $p_0 = \beta\mu - p + 1$ up to time $t_0 \wedge \tilde{\tau}_1$ when the whole surplus is paid as dividends. Here $\tilde{\tau}_1$ is the first time the surplus of the investment department reaches zero. Let $X_t^{(1)}$ be the surplus process of the investment department, we have that $X^{(1)}_{t\wedge t_0 \wedge \tilde{\tau}_1} \geq Y_{t\wedge t_0}$ where Y_t is the process described in Lemma 2.1(c) with $m = -p_0$. The insurance department starts with no surplus, pays no dividends and receives a constant flow $p_0 + p$ that is larger than $\beta\mu$ up to time $t_0 \wedge \tilde{\tau}_1 \wedge \tilde{\tau}_2$, where $\tilde{\tau}_2$ is the ruin time of the insurance department (assuming that the insurance department keeps always receiving the constant flow $p_0 + p$). The stopping time $\tilde{\tau}_2$ is independent of both $\tilde{\tau}_1$ and the process Y_t. Call $\tau = t_0 \wedge \tilde{\tau}_1 \wedge \tilde{\tau}_2$, the value function of this admissible strategy satisfies

$$\begin{aligned} V_{\pi_{t_0}}(x) &\geq E_x\big(X_\tau^{(1)} e^{-c\tau} \chi_{\{\tilde{\tau}_1 \geq t_0, \tilde{\tau}_2 \geq t_0\}}\big) \geq E_x\big(Y_{t_0} e^{-ct_0} \chi_{\{\tilde{\tau}_1 \geq t_0, \tilde{\tau}_2 \geq t_0\}}\big) \\ &= E_x\big(Y_{t_0} e^{-ct_0} \chi_{\{\tilde{\tau}_1 \geq t_0\}}\big) P(\{\tilde{\tau}_2 \geq t_0\}) \geq E_x(Y_{t_0} e^{-ct_0}) P(\{\tilde{\tau}_2 = \infty\}), \end{aligned}$$

because $Y_{t_0} < 0$ for $t_0 > \tilde{\tau}_1$. We can compute the survival probability of the insurance department [see, for instance, Teugels (2003)] as $P(\{\tilde{\tau}_2 = \infty\}) = 1 - \beta\mu/(p_0 + p) > 0$. So, from Lemma 2.1(c), we conclude that $V(x) \geq \lim_{t_0\to\infty} V_{\pi_{t_0}}(x) = \infty$.

3. The Hamilton–Jacobi–Bellman equation. In this section we associate a Hamilton–Jacobi–Bellman equation to the optimization problem (2.4) and we prove that the optimal value function V is a viscosity solution of this equation.

The notion of viscosity solution was introduced by Crandall and Lions (1983) for first order Hamilton–Jacobi equations and by Lions (1983) for second-order partial differential equations. Nowadays, it is a standard tool for studying HJB equations [see, for instance, Fleming and Soner (1993) and Bardi and Capuzzo-Dolcetta (1997)].

We first state the *dynamic programming principle*; the proof is similar to the one in Azcue and Muler (2005).

PROPOSITION 3.1. *For any $x \geq 0$ and any stopping time τ, we can write*

$$V(x) = \sup_{\pi=(\gamma_t, L_t)\in\Pi_x} E_x\left(\int_0^{\tau\wedge\tau^\pi} e^{-cs}\, dL_s + e^{-c(\tau\wedge\tau^\pi)} V(X^\pi_{\tau\wedge\tau^\pi})\right).$$

The HJB equation associated to the optimization problem (2.4) is the following fully nonlinear second-order degenerate integro-differential equation with derivative constraint:

$$\max\{1 - u'(x), \mathcal{L}^*(u)(x)\} = 0, \tag{3.1}$$

where

$$\mathcal{L}^*(u)(x) = \sup_{\gamma\in[0,1]} \mathcal{L}_\gamma(u)(x) \tag{3.2}$$

and

$$\begin{aligned}\mathcal{L}_\gamma(u)(x) &= \sigma^2\gamma^2x^2u''(x)/2 + (p + r\gamma x)u'(x)\\ &\quad - (c+\beta)u(x) + \beta\int_0^x u(x-\alpha)\, dF(\alpha).\end{aligned} \tag{3.3}$$

This equation is obtained assuming that the optimal value function V is twice continuously differentiable. We will show in Section 9 that this is not always the case, so we consider *viscosity solutions* of this equation.

DEFINITION 3.2. A continuous function $\underline{u} : [0, \infty) \to \mathbf{R}$ is a viscosity subsolution of (3.1) at $x \in (0, \infty)$ if any twice continuously differentiable function ψ defined in $(0, \infty)$ with $\psi(x) = \underline{u}(x)$ such that $\underline{u} - \psi$ reaches the maximum at x satisfies $\max\{1 - \psi'(x), \mathcal{L}^*(\psi)(x)\} \geq 0$, and a continuous function $\overline{u} : [0, \infty) \to \mathbf{R}$ is a viscosity supersolution of (3.1) at $x \in (0, \infty)$ if any twice continuously differentiable function φ defined in $(0, \infty)$ with $\varphi(x) = \overline{u}(x)$ such that $\overline{u} - \varphi$ reaches the minimum at x satisfies $\max\{1 - \varphi'(x), \mathcal{L}^*(\varphi)(x)\} \leq 0$.

Finally, a continuous function $u : [0, \infty) \to \mathbf{R}$ is a viscosity solution of (3.1) if it is both a viscosity subsolution and a viscosity supersolution at any $x \in (0, \infty)$.

In addition to Definition 3.2, there are two other equivalent formulations of viscosity solutions. The proof of the equivalence of these definitions is standard [see, for instance, Benth, Karlsen and Reikvam (2002)]. We use the three definitions indistinctly.

DEFINITION 3.3. Given a twice continuously differentiable function f and a continuous function u, let us define the operator,

$$\begin{aligned}\mathcal{L}_\gamma(u, f)(x) &= \sigma^2\gamma^2x^2 f''(x)/2 + (p + r\gamma x) f'(x) \\ &\quad - (c+\beta)u(x) + \beta\int_0^x u(x-\alpha)\,dF(\alpha).\end{aligned} \tag{3.4}$$

A continuous function $\underline{u}:[0,\infty) \to \mathbf{R}$ is a viscosity subsolution of (3.1) at $x \in (0,\infty)$ if any twice continuously differentiable function ψ defined in $(0,\infty)$ such that $\underline{u} - \psi$ reaches the maximum at x satisfies $\max\{1 - \psi'(x), \sup_{\gamma\in[0,1]} \mathcal{L}_\gamma(\underline{u}, \psi)(x)\} \ge 0$, and a twice continuous function $\overline{u}:[0,\infty) \to \mathbf{R}$ is a viscosity supersolution of (3.1) at $x \in (0,\infty)$ if any twice continuously differentiable function φ defined in $(0,\infty)$ such that $\overline{u} - \varphi$ reaches the minimum at x satisfies $\max\{1 - \varphi'(x), \sup_{\gamma\in[0,1]} \mathcal{L}_\gamma(\overline{u}, \varphi)(x)\} \le 0$.

DEFINITION 3.4. Given any continuous function $u:[0,\infty) \to \mathbf{R}$ and any $x > 0$, the set of second superdifferentials of u at x is defined as

$$D^+u(x) = \left\{(d,q) \text{ such that } \limsup_{h\to 0} \frac{u(x+h) - u(x) - hd - h^2q/2}{h^2} \le 0\right\}$$

and the set of second subdifferentials of u at x is defined as

$$D^-u(x) = \left\{(d,q) \text{ such that } \liminf_{h\to 0} \frac{u(x+h) - u(x) - hd - h^2q/2}{h^2} \ge 0\right\}.$$

Let us call

$$\begin{aligned}\mathcal{L}_\gamma(u, d, q)(x) &= \sigma^2\gamma^2x^2 q/2 + (p + r\gamma x)d - (c+\beta)u(x) \\ &\quad + \beta\int_0^x u(x-\alpha)\,dF(\alpha).\end{aligned} \tag{3.5}$$

A continuous function $\underline{u}:[0,\infty) \to \mathbf{R}$ is a viscosity subsolution of (3.1) at $x \in (0,\infty)$ if $\max\{1 - d, \sup_{\gamma\in[0,1]} \mathcal{L}_\gamma(\underline{u}, d, q)(x)\} \ge 0$ for all $(d,q) \in D^+\underline{u}(x)$ and $\overline{u}:[0,\infty) \to \mathbf{R}$ is a viscosity supersolution of (3.1) at $x \in (0,\infty)$ if $\max\{1 - d, \sup_{\gamma\in[0,1]} \mathcal{L}_\gamma(\overline{u}, d, q)(x)\} \le 0$ for all $(d,q) \in D^-\overline{u}(x)$.

The next proposition states the semiconcavity of the viscosity solutions of the HJB equation.

PROPOSITION 3.5. *Any absolutely continuous and nondecreasing supersolution of* $\mathcal{L}^*(u) = 0$ *in* $(0,\infty)$ *is semiconcave in any interval* $[x_0, x_1] \subset (0,\infty)$.

PROOF. It is enough to prove that there exists a constant K and a sequence of semiconcave functions v_n in $[0, x_1]$ such that $v_n'' \le K$ a.e. and $v_n \to \overline{u}$ uniformly in $[0, x_1]$.

Since $\overline{u}$ is an absolutely continuous function, there exists $k_0 \geq 1$ such that $|\overline{u}(x) - \overline{u}(y)| \leq k_0|x-y|$ for all $x, y \in [0, x_1]$. Let us define, for any $x \in [0, x_1]$,

$$v_n(x) = \inf_{y \in [0,x_1]} \{\overline{u}(y) + n^2(x-y)^2/2\}. \tag{3.6}$$

It can be proved, as in Lemma 5.1 of Fleming and Soner (1993), that v_n is semiconcave and the inequality $0 \leq \overline{u}(x) - v_n(x) \leq 2k_0^2/n^2$ holds for all $x \in [0, x_1]$, so $v_n \to \overline{u}$ uniformly. We have that if $x + h \leq x_1$, then $v_n(x+h) - v_n(x) \leq k_0 h$ for $h \leq x_1 - x$. In effect, take $y_0 \in [0, x_1]$ such that $v_n(x) = \overline{u}(y_0) + n^2(x-y_0)^2/2$, we have

$$\begin{aligned} v_n(x+h) - v_n(x) &\leq \big(\overline{u}(y_0+h) + n^2(x-y_0)^2/2\big) - \big(\overline{u}(y_0) + n^2(x-y_0)^2/2\big) \\ &= \overline{u}(y_0+h) - \overline{u}(y_0) \\ &\leq k_0 h. \end{aligned}$$

Since v_n is semiconcave, the set

$$A = \{x \in [0, x_1] : v_n'(x) \text{ and } v_n''(x) \text{ exist for all } n \in \mathbf{N} \text{ and } F(x) = F(x^-)\}$$

has full measure.

We want to prove that

$$v_n''(x) \leq 8(c+\beta)\overline{u}(x_1)/(\sigma^2 x_0^2) \qquad \text{in } [x_0, x_1] \cap A. \tag{3.7}$$

Take $\overline{x} \in [x_0, x_1] \cap A$, and consider $\overline{y}_n \in [0, x_1]$ such that

$$v_n(\overline{x}) = \overline{u}(\overline{y}_n) + n^2(\overline{x} - \overline{y}_n)^2/2. \tag{3.8}$$

It can be proved that

$$x_0/2 \leq \overline{y}_n \leq \overline{x} \quad \text{and} \quad \overline{x} - \overline{y}_n \leq 2k_0/n^2. \tag{3.9}$$

By (3.6), we have

$$v_n(\overline{x} + h) \leq \overline{u}(\overline{y}_n + h) + n^2(\overline{x} - \overline{y}_n)^2/2,$$

so we obtain from (3.8) that

$$\begin{aligned} &\liminf_{h \to 0} \frac{\overline{u}(\overline{y}_n + h) - \overline{u}(\overline{y}_n) - h v_n'(\overline{x}) - h^2 v_n''(\overline{x})/2}{h^2} \\ &\qquad \geq \liminf_{h \to 0} \frac{v_n(\overline{x}+h) - v_n(\overline{x}) - h v_n'(\overline{x}) - h^2 v_n''(\overline{x})/2}{h^2} = 0. \end{aligned}$$

Then we have that $(v_n'(\overline{x}), v_n''(\overline{x})) \in D^-\overline{u}(\overline{y}_n)$.

Since $\overline{u}$ is a viscosity supersolution of (3.1) at $\overline{y}_n$, we have from Definition 3.4 that

$$\mathcal{L}_1(\overline{u}, v_n'(\overline{x}), v_n''(\overline{x}))(\overline{y}_n) \leq 0. \tag{3.10}$$

If $v_n''(\overline{x}) \leq 0$, inequality (3.7) holds, and if $v_n''(\overline{x}) > 0$, from (3.9) and (3.10) we get that

$$\sigma^2 x_0^2 v_n''(\overline{x})/8 \leq \sigma^2 \overline{y}^2 v_n''(\overline{x})/2 \leq (c+\beta)\overline{u}(\overline{y}) \leq (c+\beta)\overline{u}(x_1)$$

and so we have (3.7). □

The next proposition states that the optimal value function of our control problem is a viscosity solution of equation (3.1). We will show in the next section that this result is not enough to characterize univocally the optimal value function.

PROPOSITION 3.6. *The optimal value function V is a viscosity solution of* (3.1) *in* $(0, \infty)$.

PROOF. We prove first that V is a viscosity supersolution. Let us call τ_1 and U_1 the time and the size of the first claim. For fixed $l_0 \geq 0$ and $\gamma_0 \in [0, 1]$, consider the admissible strategy $\pi_0 = (\gamma_0, tl_0) \in \Pi_x$.

Assume first that $l_0 > p$. Given any $h > 0$, consider the process Y_t defined in Lemma 2.1 with $m = p - l_0$ and $\gamma_t = \gamma_0$. Let us consider $\widetilde{\tau} = \inf\{t : Y_t < 0\}$. Using Proposition 3.1 with $\tau = \tau_1 \wedge h$, we obtain that

$$V(x) \geq E_x\biggl(\int_0^{\tau\wedge\tau^{\pi_0}} e^{-cs} l_0\, ds + e^{-c(\tau\wedge\tau^{\pi_0})} V(X^{\pi_0}_{\tau\wedge\tau^{\pi_0}})\biggr). \tag{3.11}$$

Note that $\tau \wedge \tau^{\pi_0} = \tau \wedge \widetilde{\tau}$, so we have

$$\begin{aligned} E_x\biggl(\int_0^{\tau\wedge\widetilde{\tau}} e^{-cs} l_0\, ds\biggr) &\geq E_x\biggl(\chi_{\{\tau\leq\widetilde{\tau}\}} \int_0^{\tau} e^{-cs} l_0\, ds\biggr) \\ &= E_x\biggl(\int_0^{\tau} e^{-cs} l_0\, ds\biggr) - E_x\biggl(\chi_{\{\widetilde{\tau}<\tau\}} \int_0^{\tau} e^{-cs} l_0\, ds\biggr) \\ &\geq E_x\biggl(\int_0^{\tau} e^{-cs} l_0\, ds\biggr) - hl_0 P(\widetilde{\tau} < h) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &E_x\bigl(e^{-c(\tau\wedge\widetilde{\tau})} V(X^{\pi_0}_{\tau\wedge\widetilde{\tau}})\bigr) \\ &\quad = E_x\bigl(\chi_{\{\tau\leq\widetilde{\tau}\}} e^{-c\tau} V(X^{\pi_0}_{\tau})\bigr) + E_x\bigl(\chi_{\{\widetilde{\tau}<\tau\}} e^{-c\widetilde{\tau}} V(0)\bigr) \\ &\quad = E_x\bigl(\chi_{\{\tau_1\neq\tau\leq\widetilde{\tau}\}} e^{-ch} V(Y_h)\bigr) + E_x\bigl(\chi_{\{\tau_1=\tau\leq\widetilde{\tau}\}} e^{-c\tau_1} V(Y_{\tau_1} - U_1)\bigr) \\ &\qquad + E_x\bigl(\chi_{\{\widetilde{\tau}<\tau\}} e^{-c\widetilde{\tau}} V(0)\bigr) \\ &\quad \geq E_x\bigl(\chi_{\{\tau_1\neq\tau\}} e^{-ch} V(Y_h)\bigr) - E_x\bigl(\chi_{\{\tau_1\neq\tau>\widetilde{\tau}\}} e^{-ch} V(Y_h)\bigr) \\ &\qquad + E_x\bigl(\chi_{\{\tau_1=\tau\}} e^{-c\tau_1} V(Y_{\tau_1} - U_1)\bigr) \\ &\qquad - E_x\bigl(\chi_{\{\tau_1=\tau>\widetilde{\tau}\}} e^{-c\tau_1} V(Y_{\tau_1} - U_1)\bigr) \\ &\quad \geq E_x\bigl(\chi_{\{\tau_1\neq\tau\}} e^{-ch} V(Y_h)\bigr) + E_x\bigl(\chi_{\{\tau_1=\tau\}} e^{-c\tau_1} V(Y_{\tau_1} - U_1)\bigr), \end{aligned} \tag{3.13}$$

because $V(y)=0$ for $y<0$ and $Y_\tau - U_1 < Y_\tau < 0$ for $\tau > \widetilde{\tau}$. Then, from (3.11), (3.12) and (3.13), we get that

$$\begin{aligned} V(x) \geq{}& l_0\bigl(1-e^{-h(c+\beta)}\bigr)/(c+\beta) - hl_0 P(\widetilde{\tau}<h) + e^{-(c+\beta)h}E_x(V(Y_h)) \\ &+ \beta\int_0^h\biggl(\int_0^\infty E_x\bigl(V_x(Y_s-\alpha)\bigr)\,dF(\alpha)\biggr)e^{-(c+\beta)s}\,ds. \end{aligned} \tag{3.14}$$

Assume now that $l_0 \leq p$, we obtain with a simpler argument that

$$\begin{aligned} V(x) \geq{}& l_0\bigl(1-e^{-h(c+\beta)}\bigr)/(c+\beta) + e^{-(c+\beta)h}E_x(V(Y_h)) \\ &+ \beta\int_0^h\biggl(\int_0^\infty E_x\bigl(V(Y_s-\alpha)\bigr)\,dF(\alpha)\biggr)e^{-(c+\beta)s}\,ds. \end{aligned} \tag{3.15}$$

Dividing by h, we get from (3.14) and (3.15) that

$$\begin{aligned} 0 \geq{}& l_0\bigl(1-e^{-h(c+\beta)}\bigr)/\bigl((c+\beta)h\bigr) + e^{-h(\beta+c)}\bigl(E_x(V(Y_h))-V(x)\bigr)/h \\ &+ \bigl(e^{-h(c+\beta)}-1\bigr)V(x)/h \\ &+ (\beta/h)\int_0^h\biggl(\int_0^\infty E_x\bigl(V(Y_s-\alpha)-V(x)\bigr)\,dF(\alpha)\biggr)e^{-(c+\beta)s}\,ds \\ &+ V(x)(\beta/h)\int_0^h e^{-(c+\beta)s}\,ds - l_0P(\widetilde{\tau}<h) \end{aligned}$$

and so

$$\begin{aligned} 0 \geq{}& \bigl(1-e^{-h(c+\beta)}\bigr)l_0/\bigl((c+\beta)h\bigr) + e^{-h(\beta+c)}\bigl(E_x\bigl(V(Y_h)-V(x)\bigr)\bigr)/h \\ &+ c\bigl(e^{-h(c+\beta)}-1\bigr)V(x)/\bigl((c+\beta)h\bigr) \\ &+ (\beta/h)\int_0^h\biggl(\int_0^\infty E_x\bigl(V(Y_s-\alpha)-V(x)\bigr)\,dF(\alpha)\biggr)e^{-(c+\beta)s}\,ds \\ &- l_0P(\widetilde{\tau}<h). \end{aligned} \tag{3.16}$$

Let $\varphi:(0,\infty)\to\mathbf{R}$ be a twice continuously differentiable function such that $V-\varphi$ reaches the minimum in $(0,\infty)$ at x with $\varphi(x)=V(x)$. Since $x>0$, we can assume without loss of generality that φ is defined in $\mathbf{R}$ and that $\varphi(y)\leq 0$ for $y<0$. From (3.16) we get

$$\begin{aligned} 0 \geq{}& \bigl(1-e^{-h(c+\beta)}\bigr)l_0/\bigl((c+\beta)h\bigr) + E_x\bigl(\varphi(Y_h)-\varphi(x)\bigr)e^{-h(\beta+c)}/h \\ &+ c\bigl(e^{-h(c+\beta)}-1\bigr)V(x)/\bigl((c+\beta)h\bigr) \\ &+ (\beta/h)\int_0^h\biggl(\int_0^\infty E_x\bigl(V(Y_s-\alpha)-V(x)\bigr)\,dF(\alpha)\biggr)e^{-(c+\beta)s}\,ds \\ &- l_0P(\widetilde{\tau}<h). \end{aligned} \tag{3.17}$$

But, since φ is twice continuously, we get, from Itô's formula,

$$
\begin{aligned}
\varphi(Y_h) - \varphi(x) &= \int_0^h \varphi'(Y_s)\, dY_s + (\sigma^2\gamma_0^2/2) \int_0^h \varphi''(Y_s) Y_s^2\, ds \\
&= \int_0^h \bigl(\varphi'(Y_s)\bigl((p - l_0) + r\gamma_0 Y_s\bigr) + \varphi''(Y_s) Y_s^2 \sigma^2 \gamma_0^2/2\bigr)\, ds \qquad (3.18)\\
&\quad + \int_0^h \varphi'(Y_s)\sigma\gamma_0 Y_s\, dW_s.
\end{aligned}
$$

Note that the last term of (3.18) is a martingale. Letting h go to 0^+ in (3.17), we obtain from Lemma 2.1(b) and (3.18) that

$$
\begin{aligned}
0 \geq\; & l_0\bigl(1 - \varphi'(x)\bigr) \\
&+ \biggl(\sigma^2\gamma_0^2 x^2 \varphi''(x)/2 + (p + r\gamma_0 x)\varphi'(x) \\
&\quad - (\beta + c)V(x) + \beta \int_0^\infty V(x - \alpha)\, dF(\alpha)\biggr).
\end{aligned}
$$

Since this inequality holds for all $l_0 \geq 0$, we have that $\varphi'(x) \geq 1$, and taking $l_0 = 0$ we get $\mathcal{L}_{\gamma_0}(V, \varphi)(x) \leq 0$, so

$$
\max\Bigl\{1 - \varphi'(x), \sup_{\gamma \in [0,1]} \mathcal{L}_\gamma(V, \varphi)(x)\Bigr\} \leq 0
$$

and we have the result.

The proof that V is a viscosity subsolution at any $x > 0$ is similar to the one of Proposition 3.8 of Azcue and Muler (2005), but in this case we should also consider a martingale that involves the Brownian motion W_t. □

From Propositions 3.5 and 3.6 we get the following corollary.

COROLLARY 3.7. *The optimal value function V is semiconcave in any interval $[x_0, x_1] \subset (0, \infty)$ and so V'' exists a.e.*

4. Comparison principle for viscosity solutions. We prove in this section a *comparison principle* for viscosity solutions of (3.1), and as a consequence we obtain the uniqueness with the boundary condition $u(0)$ among all the functions u which satisfy the following regularity and growth assumptions:

(A.1) $u : [0, \infty) \to \mathbf{R}$ is locally Lipschitz.

(A.2) If $0 \leq x < y$, then $u(y) - u(x) \geq y - x$.

(A.3) There exists a constant $k > 0$ such that $u(x) \leq x + k$ for all $x \in [0, \infty)$.

PROPOSITION 4.1. *If $\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution of (3.1) in $(0, \infty)$ with $\underline{u}(0) \leq \overline{u}(0)$ and they satisfy the conditions* (A.1), (A.2) *and* (A.3), *then $\underline{u} \leq \overline{u}$ in $(0, \infty)$.*

PROOF. The first part of this proof is similar to the proof of Proposition 4.2 of Azcue and Muler (2005) although in this case we should also use the tools provided by Crandall, Ishii and Lions (1992) to prove comparison principles for second-order differential equations and adapt them to integro-differential equations.

Assume that $\underline{u}(x_0) - \overline{u}(x_0) > 0$ for some point $x_0 > 0$. It is straightforward to show that the functions $\overline{u}^s(x) = s\overline{u}(x)$ with $s > 1$ are also a supersolution and satisfy $\overline{u}^s(0) \geq \underline{u}(0)$. If φ is a continuously differentiable function such that the minimum of $\overline{u}^s - \varphi$ is attained at x_0 then $1 - \varphi'(x_0) \leq 1 - s < 0$. Let us take $s_0 > 1$ with $\underline{u}(x_0) - \overline{u}^{s_0}(x_0) > 0$ and define

$$M = \sup_{x \geq 0}\bigl(\underline{u}(x) - \overline{u}^{s_0}(x)\bigr). \tag{4.1}$$

From assumptions (A.2) and (A.3) we obtain, as in Proposition 4.2 of Azcue and Muler (2005), that

$$0 < \underline{u}(x_0) - \overline{u}^{s_0}(x_0) \leq M = \max_{x \in [0,b]}\bigl(\underline{u}(x) - \overline{u}^{s_0}(x)\bigr), \tag{4.2}$$

where $b = k/(s_0 - 1)$. Call $x^* = \arg\max_{x \in [0,b]}(\underline{u}(x) - \overline{u}^{s_0}(x))$. Since $\underline{u}(x)$ and $\overline{u}^{s_0}(x)$ satisfy assumption (A.1), there exists a constant $m > 0$ such that

$$\frac{\underline{u}(x_1) - \underline{u}(x_2)}{x_1 - x_2} \leq m, \qquad \frac{\overline{u}^{s_0}(x_1) - \overline{u}^{s_0}(x_2)}{x_1 - x_2} \leq m \tag{4.3}$$

for $0 \leq x_2 \leq x_1 \leq b$.

Let us consider

$$A = \{(x, y) : 0 \leq y \leq b, 0 \leq x \leq y\}$$

and for any $\lambda > 0$ the functions

$$\Phi^\lambda(x, y) = \lambda(x - y)^2/2 + 2m/\bigl(\lambda^2(y - x) + \lambda\bigr), \tag{4.4}$$

$$\Sigma^\lambda(x, y) = \underline{u}(x) - \overline{u}^{s_0}(y) - \Phi^\lambda(x, y). \tag{4.5}$$

Calling $M_\lambda = \max_A \Sigma^\lambda$ and $(x_\lambda, y_\lambda) = \arg\max_A \Sigma^\lambda$, we obtain that $M_\lambda \geq \Sigma^\lambda(x^*, x^*) = M - 2m/\lambda$ and so, from (4.2) we get that $M_\lambda > 0$ for $\lambda \geq 4m/M$ and

$$\liminf_{\lambda \to \infty} M_\lambda \geq M. \tag{4.6}$$

Since $(x_\lambda, y_\lambda) \in A$, we have that

$$y_\lambda \geq x_\lambda. \tag{4.7}$$

As in Proposition 4.2 of Azcue and Muler (2005), we can show that for any $\lambda \geq \lambda_0 = \max\{2m/\delta, 4m/M\}$ the point $(x_\lambda, y_\lambda) \notin \partial A$.

Using Theorem 3.2 of Crandall, Ishii and Lions (1992), it can be proved that for any $\delta > 0$, there exist real numbers A_δ and B_δ such that

$$(\Phi_x^\lambda(x_\lambda, y_\lambda), A_\delta) \in D^+\underline{u}(x_\lambda) \tag{4.8}$$

and

$$(-\Phi_y^\lambda(x_\lambda, y_\lambda), B_\delta) \in D^- \overline{u}^s(y_\lambda) \tag{4.9}$$

with

$$D^2\Phi^\lambda(x_\lambda, y_\lambda) + \delta(D^2\Phi^\lambda(x_\lambda, y_\lambda))^2 - \begin{pmatrix} A_\delta & 0 \\ 0 & -B_\delta \end{pmatrix} \geq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.10}$$

where $D^2\Phi^\lambda$ corresponds to the matrix of second derivatives of Φ^λ, and D^+ and D^- are defined in Definition 3.4. The inequality in (4.10) means that the matrix on the left-hand side is positive-semidefinite. So, we obtain from (4.8) and (4.9) that

$$\max\Big\{1 - \Phi_x^\lambda(x_\lambda, y_\lambda), \sup_{\gamma\in[0,1]} \mathcal{L}_\gamma(\underline{u}, \Phi_x^\lambda(x_\lambda, y_\lambda), A_\delta)(x_\lambda)\Big\} \geq 0 \tag{4.11}$$

and

$$\max\Big\{1 + \Phi_y^\lambda(x_\lambda, y_\lambda), \sup_{\gamma\in[0,1]} \mathcal{L}_\gamma(\overline{u}^{s_0}, -\Phi_y^\lambda(x_\lambda, y_\lambda), B_\delta)(y_\lambda)\Big\} \leq 0. \tag{4.12}$$

From (4.10), we obtain that

$$\begin{aligned} &A_\delta x_\lambda^2 - B_\delta y_\lambda^2 \\ &\quad \leq \big((\lambda + (4m\lambda)/\big(\lambda(y_\lambda - x_\lambda) + 1\big)^3\big) \\ &\qquad + 2\delta\big(\lambda + (4m\lambda)/\big(\lambda(y_\lambda - x_\lambda) + 1\big)^3\big)^2\big)(x_\lambda - y_\lambda)^2. \end{aligned} \tag{4.13}$$

We also have from (4.4) that

$$\Phi_x^\lambda(x_\lambda, y_\lambda) + \Phi_y^\lambda(x_\lambda, y_\lambda) = 0 \tag{4.14}$$

and

$$\begin{aligned} &x_\lambda\Phi_x^\lambda(x_\lambda, y_\lambda) + y_\lambda\Phi_y^\lambda(x_\lambda, y_\lambda) \\ &\quad = \lambda(x_\lambda - y_\lambda)^2 + 2m(x_\lambda - y_\lambda)/\big(\lambda(y_\lambda - x_\lambda) + 1\big)^2. \end{aligned} \tag{4.15}$$

But $(-\Phi_y^\lambda(x_\lambda, y_\lambda), B_\delta) \in D^-\overline{u}^s(y_\lambda)$, so we obtain that $-\Phi_y^\lambda(x_\lambda, y_\lambda) \geq s_0 > 1$, and so we conclude from (4.11) and (4.14) that

$$\sup_{\gamma\in[0,1]} \mathcal{L}_\gamma(\underline{u}, \Phi_x^\lambda(x_\lambda, y_\lambda), A_\delta)(x_\lambda) \geq 0. \tag{4.16}$$

Therefore, taking $\gamma_\lambda = \arg\max \mathcal{L}_\gamma(\underline{u}, \Phi_x^\lambda(x_\lambda, y_\lambda), A_\delta)(x_\lambda)$ we get from (4.12) and (4.16) that

$$0 \leq \mathcal{L}_{\gamma_\lambda}(\underline{u}, \Phi_x^\lambda(x_\lambda, y_\lambda), A_\delta)(x_\lambda) - \mathcal{L}_{\gamma_\lambda}(\overline{u}^{s_0}, -\Phi_y^\lambda(x_\lambda, y_\lambda), B_\delta)(y_\lambda)$$

and so

$$
\begin{aligned}
&(c+\beta)\big(\underline{u}(x_\lambda)-\overline{u}^{s_0}(y_\lambda)\big)\\
&\quad\le \sigma^2\gamma_\lambda^2(A_\delta x_\lambda^2 - B_\delta y_\lambda^2)/2\\
&\qquad + p\big(\Phi_x^\lambda(x_\lambda,y_\lambda)+\Phi_y^\lambda(x_\lambda,y_\lambda)\big)\\
&\qquad + r\gamma_\lambda\big(\Phi_x^\lambda(x_\lambda,y_\lambda)x_\lambda+\Phi_y^\lambda(x_\lambda,y_\lambda)y_\lambda\big)\\
&\qquad + \beta\bigg(\int_0^{x_\lambda}\underline{u}(x_\lambda-\alpha)\,dF(\alpha)-\int_0^{y_\lambda}\overline{u}^{s_0}(y_\lambda-\alpha)\,dF(\alpha)\bigg).
\end{aligned}\tag{4.17}
$$

Using the inequality

$$\Sigma^\lambda(x_\lambda,x_\lambda)+\Sigma^\lambda(y_\lambda,y_\lambda)\le 2\Sigma^\lambda(x_\lambda,y_\lambda),$$

we obtain that

$$\lambda(x_\lambda-y_\lambda)^2\le \underline{u}(x_\lambda)-\underline{u}(y_\lambda)+\overline{u}^{s_0}(x_\lambda)-\overline{u}^{s_0}(y_\lambda)+4m(y_\lambda-x_\lambda);$$

then we have from (4.3) that

$$\lambda(x_\lambda-y_\lambda)^2\le 6m|x_\lambda-y_\lambda|.\tag{4.18}$$

We can find a sequence $\lambda_n\to\infty$ such that $(x_{\lambda_n},y_{\lambda_n})\to(\overline{x},\overline{y})\in A$. From (4.18), we get that $|x_{\lambda_n}-y_{\lambda_n}|\le 6m/\lambda_n$ and this gives $\overline{x}=\overline{y}$ and so $\lim_{n\to\infty}\lambda_n(x_{\lambda_n}-y_{\lambda_n})^2=0$. Taking $\delta=1/\lambda$, we get using that $y_{\lambda_n}\ge x_{\lambda_n}$ for all n, (4.13), (4.14), (4.15) and (4.17)

$$(c+\beta)\big(\underline{u}(\overline{x})-\overline{u}^{s_0}(\overline{x})\big)\le\beta\int_0^C\big(\underline{u}(\overline{x}-\alpha)-\overline{u}^{s_0}(\overline{x}-\alpha)\big)\,dF(\alpha),\tag{4.19}$$

where C can be equal to either $\overline{x}$ or $\overline{x}^-$.

From (4.6) and (4.19) we obtain $M\le\beta M/(c+\beta)$. This is a contradiction because $M>0$ and $\beta/(c+\beta)<1$. □

From the previous proposition, we conclude the following corollary.

COROLLARY 4.2. *For any* $u_0>0$, *there is at most one viscosity solution of* (3.1) *in* $(0,+\infty)$ *satisfying assumptions* (A.1), (A.2) *and* (A.3) *with the boundary condition* $u(0)=u_0$.

5. Characterization of V as the smallest supersolution and a verification result. In Sections 2 and 3, we have proved that the optimal value function V is well defined and that it is a viscosity solution of (3.1). In Section 4, we have proved that (3.1) has a comparison principle that gives us uniqueness of viscosity solutions with a given boundary condition. As it can be seen in the next remark there are infinitely many classical solutions of the HJB equation satisfying (A.1), (A.2) and (A.3).

REMARK 5.1. Note that $u(x) = k + x$ is a viscosity solution of (3.1) in $[0, \infty)$ for any $k \geq p/c$ because $u' = 1$ and $\mathcal{L}^*(u) \leq 0$.

Our main goal in this section is to characterize V among all the viscosity solutions of (3.1). We show that the optimal value function V is the smallest of the absolutely continuous supersolutions of the HJB equation. We use this result to prove a *verification theorem* that states that if a supersolution of the HJB equation is obtained, either as a value function of an admissible strategy, or as a limit of value functions of admissible strategies, then this supersolution should be the optimal value function.

Later in this section, using the Corollary 4.2, we also characterize V as the viscosity solution of the HJB equation with the smallest possible boundary condition at zero.

To prove Proposition 5.3 we need the following technical lemma.

LEMMA 5.2. *Let $\overline{u}$ be an absolutely continuous nonnegative supersolution of* (3.1) *in* $(0, +\infty)$. *Given any pair of real numbers $x_1 > x_0 > 0$, we can find a sequence of nonnegative functions $u_n : \mathbf{R} \to \mathbf{R}$ such that*:

(a) *u_n is twice continuously differentiable*,

(b) *u_n converges uniformly to $\overline{u}$ in* $[0, x_1]$,

(c) $u_n' \geq 1$ *in* $[x_0, x_1]$,

(d) $\limsup_{n\to\infty} \mathcal{L}^*(u_n)(x) \leq \beta\overline{u}(0)(F(x) - F(x^-))$ *for* $x \in [x_0, x_1]$.

PROOF. Let us consider an even and twice-continuously differentiable function ϕ with support included in $(-1, 1)$, with integral one, such that $\phi' \geq 0$ in $(-1, 0)$ and $\phi' \leq 0$ in $(0, 1)$. Consider $\phi_n(x) = n\phi(n(x - 1/n))$ and define u_n as the left-sided convolution $u_n(x) = (\overline{u} * \overline{\phi}_n)(x)$. The results (a) and (b) follow using standard techniques [see, for instance, Wheeden and Zygmund (1977)]; (c) follows because $\overline{u}' \geq 1$ a.e.

Let us prove (d). By Proposition 3.5, $\overline{u}$ is semiconcave and so $\overline{u}''$ exists a.e., and the possible jumps of $\overline{u}$ are downward. So, the left-sided convolution u_n satisfies $u_n''(x) \leq (\overline{u}'' * \overline{\phi}_n)(x)$. The result (d) follows because $\mathcal{L}_\gamma(\overline{u})(x) \leq 0$ a.e. for any $\gamma \in [0, 1]$, and it can be shown that

$$\limsup_{n\to\infty}\bigl(\mathcal{L}_\gamma(u_n)(x) - \bigl(\mathcal{L}_\gamma(\overline{u}) * \overline{\phi}_n\bigr)(x)\bigr) \leq \beta\overline{u}(0)\bigl(F(x) - F(x^-)\bigr)$$

for all $x \in [x_0, x_1]$. □

PROPOSITION 5.3. *Let $\overline{u}$ be an absolutely continuous nonnegative supersolution of* (3.1) *in* $(0, +\infty)$, *then* $\overline{u} \geq V$ *in* $[0, +\infty)$.

PROOF. Let us define S as the set of discontinuity points of the claim-size distribution F. Since F is increasing S is a countable set. Take $x > 0$, by Lemmas A.1

and A.2 (included in the Appendix), it is enough to prove that for any pair (x_0, x_1) such that $0 < x_0 \le x \le x_1$, we have

$$\sup_{\pi \in \Pi_x^{[x_0,x_1]} \cap \Pi_x(S)} V_\pi(x) \le \overline{u}(x),$$

where $\Pi_x^{[x_0,x_1]} = \{\pi \in \Pi_x : x_0 \le X_t^\pi \le x_1, t \ge 0\}$ and $\Pi_x(S)$ is the set of all the admissible strategies $\pi \in \Pi_x$ such that the measure of

$$\{(\omega, t) \in \Omega \times [0, \infty) : X_t^\pi(\omega) \in S\}$$

is zero.

Take $\pi = (\gamma_t, L_t) \in \Pi_x^{[x_0,x_1]} \cap \Pi_x(S)$. Consider the functions u_n defined in Lemma 5.2; since they are twice continuously differentiable, we can write

$$\begin{aligned} &u_n(X_{\tau^\pi})e^{-c\tau^\pi} - u_n(x) \\ &\quad = \int_0^{\tau^\pi} u_n'(X_s)e^{-cs}\,dX_s - c\int_0^{\tau^\pi} u_n(X_s)e^{-cs}\,ds \\ &\qquad + (\sigma^2/2)\int_0^{\tau^\pi} u_n''(X_s)\gamma_s^2 X_s^2 e^{-cs}\,ds \end{aligned} \tag{5.1}$$

for any $t \ge 0$.

Note that, since L_t is nondecreasing and left-continuous, it can be written as

$$L_t = \int_0^t dL_s^c + \sum_{X_{s^+} \ne X_s, s<t} (L_{s^+} - L_s), \tag{5.2}$$

where L_s^c is a continuous and nondecreasing function. Hence, using expressions (2.2) and (5.2), we get

$$\begin{aligned} &\int_0^{\tau^\pi} u_n'(X_s)e^{-cs}\,dX_s \\ &\quad = \int_0^{\tau^\pi} u_n'(X_s)e^{-cs}p\,ds + \int_0^{\tau^\pi} u_n'(X_s)e^{-cs} r X_s\gamma_s\,ds \\ &\qquad + \int_0^{\tau^\pi} u_n'(X_s)e^{-cs}\sigma X_s\gamma_s\,dW_s \\ &\qquad - \int_0^{\tau^\pi} u_n'(X_s)e^{-cs}\,dL^c(s) \\ &\qquad + \sum_{X_{s^-} \ne X_s, s\le\tau^\pi} \big(u_n(X_s) - u_n(X_{s^-})\big)e^{-cs} \\ &\qquad + \sum_{X_{s^+} \ne X_s, s<\tau^\pi} \big(u_n(X_{s^+}) - u_n(X_s)\big)e^{-cs}. \end{aligned} \tag{5.3}$$

We have that $X_{s^+} \neq X_s$ only at the discontinuities of L_s, so $X_{s^+} - X_s = -(L_{s^+} - L_s)$ and

$$\sum_{X_{s^+} \neq X_s, s < \tau^\pi} \big(u_n(X_{s^+}) - u_n(X_s)\big)e^{-cs} = -\sum_{L_{s^+} \neq L_s, s < \tau^\pi} \left(\int_0^{L_{s^+}-L_s} u_n'(X_s - \alpha)\, d\alpha\right) e^{-cs}.$$

From Lemma 5.2(c), $u_n' \geq 1$, so we obtain

$$\begin{aligned} & -\int_0^{\tau^\pi} u_n'(X_s)e^{-cs}\, dL_s^c \\ & \quad + \sum_{X_{s^+} \neq X_s, s < \tau^\pi} \big(u_n(X_{s^+}) - u_n(X_s)\big)e^{-cs} \\ & \quad \leq -\left(\int_0^{t\wedge\tau^\pi} e^{-cs}\, dL_s^c + \sum_{L_{s^+} \neq L_s, s < \tau^\pi} \left(\int_0^{L_{s^+}-L_s} d\alpha\right) e^{-cs}\right) \\ & \quad = -\int_0^{\tau^\pi} e^{-cs}\, dL_s. \end{aligned} \tag{5.4}$$

Since $X_s \neq X_{s^-}$ only at the arrival of a claim, the process

$$\begin{aligned} M_t^{(1)} = & \sum_{X_{s^-} \neq X_s, s \leq t} \big(u_n(X_s) - u_n(X_{s^-})\big)e^{-cs} \\ & - \beta \int_0^t e^{-cs} \int_0^\infty \big(u_n(X_{s^-} - \alpha) - u_n(X_{s^-})\big)\, dF(\alpha)\, ds \end{aligned} \tag{5.5}$$

is a martingale with zero-expectation.

From (5.1), (5.3), (5.4) and (5.5), we obtain

$$\begin{aligned} & u_n(X_{\tau^\pi})e^{-c\tau^\pi} - u_n(x) \\ & \quad \leq \int_0^{\tau^\pi} \mathcal{L}_{\gamma_s}(u_n)(X_{s^-})e^{-cs}\, ds - \int_0^{\tau^\pi} e^{-cs}\, dL_s + M_{\tau^\pi}^{(1)} + M_{\tau^\pi}^{(2)}, \end{aligned} \tag{5.6}$$

where

$$M_t^{(2)} = \int_0^t u_n'(X_s)e^{-cs}\sigma X_s \gamma_s\, dW_s$$

is a martingale with zero-expectation.

We have $E_x(\int_0^{\tau^\pi} e^{-cs}\, dL_s) = V_\pi(x)$, $E_x(u_n(X_{\tau^\pi})e^{-c\tau^\pi}) \geq 0$ and from Lemma 5.2(d), since $\pi \in \Pi_x(S)$, we have that

$$\lim_{n\to\infty} E_x\left(\int_0^{t\wedge\tau^\pi} \mathcal{L}_{\gamma_s}(u_n)(X_{s^-})e^{-cs}\, ds\right) \leq 0$$

for all t. Then, from Lemma 5.2(b), we obtain $\overline{u}(x) = \lim_{n\to\infty} u_n(x) \geq V_\pi(x)$. □

In order to state the *verification theorem*, we need to extend the concept of strategies by the following definition.

DEFINITION 5.4. (a) Fix $x \geq 0$, let us define the map $\mathcal{V}_x : \Pi_x \to [0, \infty)$ as $\mathcal{V}_x(\pi) = V_\pi(x)$. We give to Π_x the initial topology of $\mathcal{V}_x$ and define $\widetilde{\Pi}_x$ as the completion of Π_x under this topology [see, for instance, Kelley (1955)]. We say that the elements of $\widetilde{\Pi}_x$ are limit strategies.

(b) Given $\widetilde{\pi} \in \widetilde{\Pi}_x$, there exists a sequence $\pi^k \in \Pi_x$ such that $\lim_{k\to\infty} \pi^k = \widetilde{\pi}$, we define $V_{\widetilde{\pi}}(x) = \lim_{k\to\infty} V_{\pi^k}(x)$.

From Proposition 5.3, we get the following *verification theorem*.

THEOREM 5.5. *Let π be a limit strategy such that the corresponding value function $V_{\widetilde{\pi}}$ is an absolutely continuous supersolution of* (3.1) *in* $(0, \infty)$, *then* $V_{\widetilde{\pi}} = V$.

We conclude from Remark 5.1 and Proposition 5.3 that the optimal value function V satisfies

$$V(x) \leq x + p/c \qquad \text{for } x \geq 0, \tag{5.7}$$

and so it satisfies (A.3). By Propositions 2.2 and 2.3, the optimal value function V also satisfies (A.1) and (A.2). Therefore, from Corollary 4.2 and Proposition 5.3 we get the following corollary.

COROLLARY 5.6. *The function V can be also characterized as the unique viscosity solution of* (3.1) *satisfying assumptions* (A.1), (A.2) *and* (A.3) *with the boundary condition*,

$$V(0) = \inf\{u(0) : u \textit{ viscosity supersolution of } (3.1) \textit{ satisfying } (A.3)\}.$$

6. Solutions of the second-order differential equation. In the previous sections we have characterized the optimal value function V without assuming any regularity conditions on the claim-size distribution function F. To find the optimal value function V and the value function of barrier strategies, we need some technical results about the solutions of

$$\mathcal{L}^*(W) = 0 \tag{6.1}$$

on open sets. In order to have classical solutions of this equation, we assume, from this section on, that the claim-size distribution function F has a bounded density.

If we do not assume this, we would have to deal with viscosity solutions of (6.1) and this adds some technical problems.

Equation (6.1) is similar to the HJB equation that arises in the problem of maximizing the survival probability of an insurance company whose uncontrolled reserve follows the classical Cramér–Lundberg process and where the management has the possibility of investing in the financial market. Azcue and Muler (2009) considered this problem and showed that the optimal survival probability function δ is a classical solution of $\mathcal{L}^*(\delta) = 0$ in $(0, \infty)$, but with parameter c equal to zero.

The existence and uniqueness of classical solutions of (6.1) is not straightforward since the ellipticity of $\mathcal{L}^*$ degenerates at 0 and could degenerate at any positive point. However, we prove in this section that the optimal γ in (6.1) is not zero for positive points. On the other hand, the degeneracy of the ellipticity of the operator at zero gives the uniqueness of twice continuously differentiable solutions of (6.1) in $(0, \infty)$ with only one boundary condition at zero.

In the next proposition we construct, via a fixed-point argument, the unique twice continuously differentiable solution of (6.1) in $(0, \infty)$ with the boundary condition $W(0) = 1$.

PROPOSITION 6.1. (a) *There exists a unique increasing classical solution* W *of* (6.1) *in* $(0, \infty)$ *with the boundary condition*, $W(0) = 1$ *and* $\mathcal{L}^*(W) = \mathcal{L}_{\widetilde{\gamma}(W)}(W) = 0$, *where*

$$\widetilde{\gamma}(W)(x) = \begin{cases} -rW'(x)/(\sigma^2 x W''(x)), \\ \qquad \text{if } 0 < -rW'(x)/(\sigma^2 x W''(x)) \le 1, \\ 1, \qquad \text{otherwise}. \end{cases} \tag{6.2}$$

(b) *The function* W *can be written as* $W(x) = 1 + \int_0^x w(s)\,ds$, *where* w *is the unique nonnegative fixed point of the operator*,

$$T(w)(x) = \inf_{\Gamma \in \mathcal{G}} \frac{2\int_0^x A_\Gamma(s) M(W)(s)\,ds}{\sigma^2 x^2 \Gamma(x)^2 A_\Gamma(x)}. \tag{6.3}$$

Here

$$\mathcal{G} = \{\Gamma : [0, \infty) \to (0, 1] \\ \text{piecewise continuous with } \inf(\Gamma) > 0\}, \tag{6.4}$$

$$M(W)(x) = (c + \beta)W(x) - \beta \int_0^x W(x - \alpha)\,dF(\alpha)$$

and

$$A_\Gamma(x) = e^{\int_1^x 2(p + r\Gamma(s)s)/(\sigma^2 \Gamma(s)^2 s^2)\,ds} / (\Gamma(x)^2 x^2).$$

PROOF. We give here a sketch of the proof and refer to Sections 3 and 4 in Azcue and Muler (2009) for details since the proof is similar.

It can be proved that if U is any classical increasing solution of (6.1) with $U(0) = 1$, then $u = U'$ is a fixed-point of (6.3) and also that there is a unique continuous nonnegative fixed-point w of (6.3). It can be proved that w is locally Lipschitz, and so $W(x) = 1 + \int_0^x w(s)\,ds$ is semiconcave in any compact set included in $(0, \infty)$. The next step consists of proving that W is twice continuously differentiable and so it is a classical solution of (6.1). To do that, we construct twice continuously differentiable increasing solutions of the second-order integro-differential equations, $\mathcal{L}_1(W_1) = 0$ and $\sup_{\gamma\in\mathbf{R}} \mathcal{L}_\gamma(W_2) = 0$, and show then that W coincides locally with one or the other and also that W is obtained by gluing smoothly solutions of these equations. Hipp and Plum (2000) and Schmidli (2002) studied and found classical solutions of the equation $\sup_{\gamma\in\mathbf{R}} \mathcal{L}_\gamma(W) = 0$ with $c = 0$ for the problem of minimizing the ruin probability of an insurance company without borrowing constraints.

Finally, since $\mathcal{L}_\gamma(W)(x)$ is a quadratic function on γ and W is increasing, the maximum is attained at $\gamma = 1$ or at the vertex $\gamma = -rW'(x)/(\sigma^2 x W''(x))$. It can be shown that the vertex cannot be zero. □

REMARK 6.2. Given $\Gamma \in \mathcal{G}$, consider the related problem of finding the survival probability $S(x)$ of an insurance company with initial surplus x, whose uncontrolled reserve follows the classical Cramér–Lundberg process and where the management invests a proportion Γ of the current surplus in the financial market. The function S can be called the scale function of the surplus process as in Revuz and Yor (1999), and it is a solution of $\mathcal{L}_\Gamma(S) = 0$ but with parameter c equal to zero [see Azcue and Muler (2009)]. So, as in Proposition 6.1, we have

$$S'(x) = \frac{2\int_0^x A_\Gamma(s)\beta(S(s) - \int_0^s S(s-\alpha)\,dF(\alpha))\,ds}{\sigma^2 x^2 \Gamma(x)^2 A_\Gamma(x)}.$$

PROPOSITION 6.3. (a) *The function* $\widetilde{\gamma}(W)$ *defined in* (6.2) *can be written as*

$$\widetilde{\gamma}(W)(x) = \min\{1, 2\big(M(W)(x) - pW'(x)\big)/(rxW'(x))\}.$$

(b) *There exists* $\varepsilon > 0$ *such that* $\widetilde{\gamma}(W)(x) = 1$ *for* $x \in [0, \varepsilon)$.

(c) $W'(0^+) = (c+\beta)/p$ *and* $W''(0^+) = (c+\beta-r)(c+\beta)/p^2 - F'(0)\beta/p$.

PROOF. We obtain (a) by replacing the value of $W''(x)$ obtained from the equation $\mathcal{L}_{\widetilde{\gamma}(W)}(W) = 0$ in the definition (6.2). To prove (b) and (c) consider W_1 the unique increasing twice continuously differentiable solution of $\mathcal{L}_1(W_1) = 0$. It can be proved using L'Hôpital's rule that

$$W_1'(0^+) = (c+\beta)/p \quad \text{and} \quad W_1''(0^+) = (c+\beta-r)(c+\beta)/p^2 - F'(0)\beta/p,$$

and so

$$\lim_{x\to 0^+} |rW_1'(x)/(\sigma^2 x W_1''(x))| = +\infty.$$

We conclude that there exists $\varepsilon > 0$ such that $\widetilde{\gamma}(W_1) = 1$ in $[0, \varepsilon)$. Therefore, using that $\mathcal{L}^*(W_1) = \mathcal{L}_{\widetilde{\gamma}(W_1)}(W_1)$, we obtain that W_1 satisfies $\mathcal{L}^*(W_1) = 0$ in $[0, \varepsilon)$ and so $W(x) = W_1(x)$ for small values of x. $\square$

In an analogous way, given a positive x_0 and an increasing continuous function W_0 defined in $[0, x_0]$ such that W_0 is differentiable at x_0, we can construct the unique twice continuously differentiable solution of

$$\mathcal{L}^*(U, W_0)(x) = 0 \qquad \text{for } x > x_0 \tag{6.5}$$

with boundary conditions $U(x_0) = W_0(x_0)$ and $U'(x_0) = W_0'(x_0)$ where

$$\mathcal{L}^*(U, W_0) = \sup_{\gamma \in [0,1]} \mathcal{L}_\gamma(U, W_0), \tag{6.6}$$

$$\mathcal{L}_\gamma(U, W_0)(x) = \sigma^2\gamma^2x^2U''(x)/2 + (p + r\gamma x)U'(x) - M(U, W_0)(x) \tag{6.7}$$

and

$$\begin{aligned} M(U, W_0)(x) = (c + \beta)U(x) - \beta \int_0^{x-x_0} U(x - \alpha)\, dF(\alpha) \\ - \beta \int_{x-x_0}^{x} W_0(x - \alpha)\, dF(\alpha). \end{aligned} \tag{6.8}$$

The next proposition is analogous to Propositions 6.1 and 6.3(a); the proof follows by using a fixed-point argument similar to the one used in Proposition 6.1.

PROPOSITION 6.4. *Assume that W_0 is a continuous, positive and increasing function in $[0, x_0]$ and that W_0 is differentiable at x_0.*

(a) *There exists a unique twice continuously differentiable solution U of (6.5) in (x_0, ∞) with $U(x_0) = W_0(x_0)$ and $U'(x_0) = W_0'(x_0)$.*

(b) *If we define*

$$\widetilde{\gamma}(U, W_0)(x) = \min\{1, 2\big(M(U, W_0)(x) - pU'(x)\big)/(rxU'(x))\}$$

we have that

$$\mathcal{L}^*(U, W_0) = \mathcal{L}_{\widetilde{\gamma}(U,W_0)}(U, W_0) = 0.$$

7. Barrier strategies. A dividend payment policy is called *barrier with level* y when all excess surplus above y is paid out immediately as dividends, but there is no dividends payment when surplus is less than y. In this section we would like to obtain the *optimal barrier strategy*, that is, the admissible strategy that maximizes the cumulative expected discounted dividends among all the strategies whose dividend policies are barrier. We would also like to prove that the optimal barrier strategy is *stationary*, in the sense that the decision on how much dividend to pay and how to invest at any time depends only on the current surplus. Note that a stationary strategy π determines an admissible strategy $\pi_x \in \Pi_x$ for each initial surplus x.

In the classical Cramér–Lundberg model without the possibility of investment, there exists an optimal barrier strategy. Let y^* be the optimal level. It has been proved [for instance, in Azcue and Muler (2005)] that the optimal policy for current surplus y^* is to pay all the incoming premium as dividends in order to maintain the surplus at level y^* until the arrival of the next claim.

In the model with investment, it is possible to define similar barrier strategies for any level y (if the current surplus is y, pay all the incoming premium as dividends and keep all the surplus in bonds), but these barrier strategies are never optimal. In fact there is not a stationary barrier strategy which is optimal, since it is not possible to determine the dividends payment policy when the current surplus coincides with the threshold. We construct in this section a candidate of optimal barrier strategy as an explicit limit of stationary admissible barrier strategies and find its value function. In the next sections we will prove that this strategy is indeed the optimal barrier strategy, also we will show that the optimal strategy in (2.4) could be nonbarrier, but this optimal strategy and the optimal barrier strategy coincide for small surpluses.

First in this section we use the function W, constructed in Section 6, to obtain the value function of a limit barrier strategy with a given level y and the best investment policy. Later we find the optimal level of these strategies. In all the cases, the optimal investment policy is stationary in the sense that the decision on how to invest depends only on the current surplus.

DEFINITION 7.1. Given a predictable process $\gamma_t \in [0, 1]$ and points $0 < z < y$, we define recursively for initial surplus $x \geq 0$, the admissible strategy $\pi_x^{(\gamma_t,z,y)} \in \Pi_x$ as:

1. If $x > y$, pay immediately the surplus $x - y$ as dividends and follow the strategy $\pi_y^{(\gamma_t,z,y)} \in \Pi_y$.
2. If $x \leq y$, follow the admissible strategy $(\gamma_t, 0)$ up to the exit time $\tau^* = \min\{\tau_y, \tau^\pi\}$ where

$$\tau_y = \min\{t : X_t^{\pi_x^{(\gamma_t,z,y)}} = y\}$$

   and τ^π is the ruin time. When $\tau^* = \tau_y$, pay immediately $y - z$ as dividends and follow the strategy $\pi_z^{(\gamma_t,z,y)}$ with initial surplus z.

Let us call $\overline{\Pi}_x^{z,y}$ the set of all these strategies, and let us consider for all $x \in [0, y]$ the function

$$W_{z,y}(x) = \sup_{\pi \in \overline{\Pi}_x^{z,y}} V_\pi(x). \tag{7.1}$$

We define $W_{z,y}(x) = 0$ for $x < 0$. We first state some basic properties of the function $W_{z,y}$. The proof of the next proposition is similar to the proof of Propositions 2.2 and 2.3.

PROPOSITION 7.2. *We have that*:
(a) *The value function $W_{z,y}$ is well defined.*
(b) *If $y \geq x_2 > x_1$, then*

$$W_{z,y}(x_2) - W_{z,y}(x_1) \leq \left(e^{(c+\beta)(x_2 - x_1)/p} - 1\right) W_{z,y}(x_1).$$

(c) $W_{z,y}(y) = W_{z,y}(z) + (y - z)$.
(d) *$W_{z,y}$ is increasing in* $[0, y]$.
(e) *$W_{z,y}$ is absolutely continuous in* $[0, y]$.

Let us state now a dynamic programming principle for these value functions.

PROPOSITION 7.3. *Given $x \in [0, y]$ and any stopping time τ, we have that*

$$W_{z,y}(x) = \sup_{(\gamma_t)\text{ admissible}} E_x\left(e^{-c(\tau \wedge \tau^*)} W_{z,y}\left(X^{(\gamma_t, 0)}_{\tau \wedge \tau^*}\right)\right),$$

where τ^* is the stopping time defined in Definition 7.1.

In the next proposition, we show that all the functions $W_{z,y}$ are multiples of the function W obtained in Proposition 6.1; this allows us to describe the optimal investment policy for (7.1).

PROPOSITION 7.4. (a) *We have that*

$$W_{z,y}(x) = \begin{cases} \dfrac{W(x)}{(W(y) - W(z))/(y - z)}, & \text{if } 0 \leq x < y, \\[2ex] \dfrac{W(y)}{(W(y) - W(z))/(y - z)} + (x - y), & \text{if } x \geq y, \end{cases}$$

where W is the function obtained in Proposition 6.1.

(b) *$W_{z,y}(x)$ is the value function of the admissible stationary strategy $\pi_x \in \overline{\Pi}^{z,y}_x$, the optimal investment policy depends only on the current surplus $X^{\pi}_{t^-}$ and it is given by $\overline{\gamma}_t = \widetilde{\gamma}(W)(X^{\pi}_{t^-})$ where the function $\widetilde{\gamma}(W)$ is defined in Proposition* 6.1.

PROOF. We extend the definition of W as $W(x) = 0$ for $x < 0$. Let us take any admissible strategy $\pi = (\gamma_t, L_t) \in \overline{\Pi}^{z,y}_x$ and consider the stopping times τ_y and τ^* defined in Definition 7.1. Up to time τ^*, the dividend payment policy L_t is zero, so the strategy π only depends on the investment policy $\gamma = (\gamma_t)$. To simplify notation, we denote X^{γ}_t the corresponding controlled risk process starting at x. This process satisfies up to τ^* the following stochastic differential equation:

$$dX^{\gamma}_s = (p + rX^{\gamma}_s \gamma_s)\,ds + \sigma X^{\gamma}_s \gamma_s\,dW_s - d\left(\sum_{i=1}^{N_s} U_i\right). \tag{7.2}$$

Since the function $W(x)$ is twice continuously differentiable, using the expressions (7.2) and the Itô's formula for semimartingales [see Protter (1992)], it can be shown with arguments similar to the proof of Proposition 5.3 that

$$W(X^{\gamma}_{\tau^*})e^{-c\tau^*} - W(x) = \int_0^{\tau^*} \mathcal{L}_{\gamma_s}(W)(X^{\gamma}_{s-})e^{-cs}\,ds + M^{(1)}_{\tau^*} + M^{(2)}_{\tau^*}, \tag{7.3}$$

where

$$\begin{aligned} M^{(1)}_t = {} & \sum_{X_{s-}\neq X, s\le t} \big(W(X^{\gamma}_s) - W(X^{\gamma}_{s-})\big)e^{-cs} \\ & - \beta \int_0^t e^{-cs} \int_0^{\infty} \big(W(X^{\gamma}_{s-} - \alpha) - W(X^{\gamma}_{s-})\big)\,dF(\alpha)\,ds \end{aligned} \tag{7.4}$$

and

$$M^{(2)}_t = \int_0^t W'(X^{\gamma}_s)e^{-cs}\sigma X^{\gamma}_s \gamma_s\,dW_s \tag{7.5}$$

are martingales with zero-expectation.

Note that we have

$$E_x(W(X^{\gamma}_{\tau^*})e^{-c\tau^*}) = E_x\big(W(X^{\gamma}_{\tau^*})e^{-c\tau^*}\chi_{\{\tau^*=\tau_y\}}\big) = E_x\big(W(y)e^{-c\tau^*}\chi_{\{\tau^*=\tau_y\}}\big).$$

From (7.4), (7.5) and (7.3), by Proposition 6.1, we get that

$$\sup_{\gamma \text{ admissible}} E_x(W(X^{\gamma}_{\tau^*})e^{-c\tau^*}) = E_x(W(X^{\overline{\gamma}}_{\tau^*})e^{-c\tau^*}) = W(x)$$

and so $\sup_{\gamma \text{ admissible}} E_x(e^{-c\tau^*}\chi_{\{\tau^*=\tau_y\}}) = W(x)/W(y)$. The supremum is reached at the process $\overline{\gamma} = (\overline{\gamma}_t)$. On the other hand, from Proposition 7.3, we obtain that

$$\sup_{\gamma \text{ admissible}} E\big(e^{-c\tau^*}\chi_{\{\tau^*=\tau_y\}}\big) = W_{z,y}(x)/W_{z,y}(y),$$

and the result follows from $W_{z,y}(y) = W_{z,y}(z) + (y - z)$. □

Note that the optimal investment policy of all the strategies defined above does not depend on the value of z. The corresponding controlled risk process with initial surplus $x \le y$ never exceeds the threshold y. In the next definition we define the limit dividend barrier strategies $\widetilde{\pi}^y_x$ for any $x \in [0, y)$.

DEFINITION 7.5. Given a sequence $z_n \nearrow y$ and any current surplus $x \in [0, y)$, take $\pi_x^{(\overline{\gamma}_t, z_n, y)} \in \overline{\Pi}_x^{z,y}$. We define $\widetilde{\pi}^y_x = \lim_{n\to\infty} \pi_x^{(\overline{\gamma}_t, z_n, y)}$.

In the next proposition we obtain the expression for the limit value function; the proof follows immediately from Proposition 7.4.

PROPOSITION 7.6. *We have that*

$$V_{\widetilde{\pi}_x^y}(x) = \lim_{n\to\infty} W_{z_n,y}(x) = \begin{cases} W(x)/W'(y), & \text{if } 0 \le x < y, \\ W(y)/W'(y) + (x-y), & \text{if } x \ge y. \end{cases}$$

Note that the function $V_{\widetilde{\pi}_x^y}$ is twice continuously differentiable in $(0, y) \cup (y, \infty)$ and differentiable at y. We show now that W' reaches the minimum.

PROPOSITION 7.7. *Consider the function W defined in Proposition 6.1, then*

$$w_1 = \inf W' = W'(x) > 0$$

for some $x \ge 0$. Call $x_* = \min\{x \ge 0 : W'(x) = w_1\}$.

PROOF. Define for $u \ge 0$, the function $G(u) = \inf_{x\in[0,u]} W'(x)$. Since W' is a continuous positive function, then G is continuous, nonincreasing and positive. We want to prove that there exists u_0 such that $G(u)$ is constant for $u \ge u_0$. Suppose that this is not the case, then there exists $u_2 > u_1 > p/(c-r)$ such that $G(u_2) < G(u_1) < G(p/(c-r))$. Consider

$$x_1 = \min\{x : W'(x) = G(u_1)\}, \qquad x_2 = \min\{x : W'(x) = G(u_2)\}.$$

Note that $x_2 > u_1 \ge x_1 > p/(c-r)$. Let us consider the value functions of the limit barrier strategies,

$$\mathcal{U}_{x_i}(x) = \begin{cases} W(x)/W'(x_i), & \text{if } x < x_i, \\ W(x_i)/W'(x_i) + (x - x_i), & \text{if } x \ge x_i, \end{cases}$$

for $i = 1, 2$.

We prove now that $\mathcal{U}_{x_i}$ is a supersolution of (3.1) in $x > 0$. Since W is a solution of (6.1), $W'(x_i) \le W'(x)$ for $x \in (0, x_i]$ and $\mathcal{U}_{x_i}' = 1$ in (x_i, ∞) we only need to show that $\mathcal{U}_{x_i}$ is a supersolution of (6.1) in $[x_i, \infty)$. Let us show first that $\mathcal{U}_{x_i}$ is a supersolution at $x > x_i$, take any $\gamma \in [0, 1]$, since $\mathcal{U}_{x_i}$ is increasing and $\mathcal{U}_{x_i} \ge x_i$, we have that $\mathcal{L}_\gamma(\mathcal{U}_{x_i}) < 0$. Let us show now that $\mathcal{U}_{x_i}$ is a supersolution at x_i. We have that $\mathcal{U}_{x_i}'(x_i) = 1$, take q such that

$$\begin{aligned} q/2 &\le \liminf_{h\to 0} \frac{(\mathcal{U}_{x_i}(x_i+h) - \mathcal{U}_{x_i}(x_i))/h - 1}{h} \\ &\le \lim_{h\to 0^+} \frac{(\mathcal{U}_{x_i}(x_i+h) - \mathcal{U}_{x_i}(x_i))/h - 1}{h} = 0. \end{aligned}$$

Since $\mathcal{U}_{x_i}$ is a supersolution for $x > x_i$ and $\sup_{\gamma\in[0,1]} \mathcal{L}_\gamma(\mathcal{U}_{x_i}, 1, q)(x)$ is continuous for $x \ge x_i$ we have that $\sup_{\gamma\in[0,1]} \mathcal{L}_\gamma(\mathcal{U}_{x_i}, 1, q)(x) \le 0$.

Since $\mathcal{U}_{x_i}$ is the value function of a limit strategy we have that $\mathcal{U}_{x_i} \le V$, and since $\mathcal{U}_{x_i}$ is a supersolution of (3.1), we have that $\mathcal{U}_{x_i} \ge V$. Then $\mathcal{U}_{x_i} = V$ for $i = 1, 2$, and this is a contradiction since $\mathcal{U}_{x_1} \ne \mathcal{U}_{x_2}$. □

In the next proposition we see that the value x_* defined in Proposition 7.7 is the optimal threshold of the dividend barrier strategies given in Definition 7.5. We also give a test to see whether the value function of the limit barrier strategy $\tilde{\pi}_x^{x_*}$ is the optimal value function V at x. The proof follows directly from Proposition 7.7 and Theorem 5.5.

PROPOSITION 7.8. *Define $V_1(x)$ as the value function of the limit barrier strategy obtained in Proposition 7.6 with barrier $x_* := \arg\min W'$. Then*:

(a) $V_1(x) = \max_{y \geq 0} V_{\tilde{\pi}_x^y}(x)$ *for all $x \geq 0$ and the function V_1 is twice continuously differentiable*.

(b) *If V_1 is a viscosity supersolution of* (3.1), *then V_1 coincides with the optimal value function V*.

In Remark 8.7 of the next section, we will see that the limit stationary barrier strategy $\tilde{\pi}^{x_*}$ defined as $\tilde{\pi}_x^{x_*} \in \Pi_x$ for any initial surplus $x \geq 0$ is the optimal barrier strategy. Note that the investment policy corresponding to this strategy is stationary and it is given by

$$\gamma^*(u) = \min\{1, 2(M(W)(u) - pW'(u))/(ruW'(u))\}$$

for any current surplus $u \in [0, x_*]$. Also note that, by Proposition 6.3(b), $\gamma^* = 1$ for small surpluses. This means that the whole surplus should be invested in stocks. In the unconstrained case where it is allowed to borrow money to buy risky assets, it can be seen that optimal investment policy tends to infinite as the surplus goes to zero, that is, for small surpluses the company should always borrow money to buy stocks.

8. Band structure of the optimal dividend strategy. We will show in Section 9 that the optimal value function V is not always the value function of a limit barrier strategy. Nevertheless, we prove in this section that the optimal dividend payment policy has a band structure. As in the case of the optimal barrier strategy, V is not the value function of a stationary admissible strategy, but it can be written explicitly as a limit of value functions of admissible stationary strategies.

We have shown in Section 3 that V is a viscosity solution of equation (3.1). In this section we see that V can be obtained by gluing, in a smooth way, classical solutions of $\mathcal{L}^*(V) = 0$ on an open set $\mathcal{C}_0$ with solutions of $V' = 1$ on a set $\mathcal{B}_0$. The set $\mathcal{B}_0$ is a disjoint union of left-open, right-closed intervals. These sets will be defined in Proposition 8.4.

When the current surplus x is in the set $\mathcal{B}_0$, the optimal dividend payment policy should be to pay out immediately a positive sum of dividends, and when the current surplus x is in the set $\mathcal{C}_0$, the optimal strategy should be to pay no dividends and to follow the investment policy $\gamma(x) = \arg\max_{\gamma \in [0,1]} \mathcal{L}_\gamma(V)(x)$ which depends only on the current surplus x. In the simplest case, when the optimal value function V is the solution of $\mathcal{L}^*(V) = 0$ in $\mathcal{C}_0 = (0, y^*)$ and $V' = 1$ in $\mathcal{B}_0 = (y^*, \infty)$, the optimal dividend payment policy is barrier.

We see that V is continuously differentiable; it is twice continuously differentiable in $\mathcal{B}_0$ and $\mathcal{C}_0$, but at some points outside $\mathcal{B}_0 \cup \mathcal{C}_0$, the second derivative could not exist. So we still need the notion of viscosity solutions to characterize V as a solution of the associated HJB equation.

We also prove in this section that, for small surpluses, the optimal strategy coincides with the optimal barrier obtained in Section 7, and for large surpluses, the optimal strategy is to pay out as dividends the surplus exceeding some level.

In the next proposition, we give conditions under which the optimal value function V is the supremum of the value functions corresponding to admissible strategies with surplus not exceeding $\widehat{x}$.

PROPOSITION 8.1. *Assume there exists $\widehat{x} > 0$ with $V'(\widehat{x}) = 1$; then*

$$V(x) = \sup_{\pi \in \Pi_x^{\widehat{x}}} V_\pi(x) \qquad \text{for all } x \le \widehat{x}.$$

PROOF. Given any $\varepsilon > 0$, let us consider the twice continuously differentiable solution g of the equation $\mathcal{L}^*(g) = 0$ for the special case $\beta = 0$. From Proposition 7.7, we get that $\inf_{x \ge 0} g'(x) = g'(x_*) > 0$ for some $x_* \ge 0$. So $\lim_{x\to\infty} g(x) = \infty$ and we can find a number D such that $g(D) \ge 2g(\widehat{x})V(\widehat{x})/\varepsilon$. Consider $x_n = \widehat{x} - D/n$, and define $h_n = (V(x_n) - V(\widehat{x}))/(x_n - \widehat{x}) - 1$. Since $V'(\widehat{x}) = 1$, we have that h_n goes to 0 as n goes to infinity, and so we can find an integer n_0 large enough such that $h_{n_0} < \varepsilon/(8D)$.

We can find points $0 = y_0 < y_1 < \cdots < y_M = \widehat{x}$ such that $V(y_{j+1}) - V(y_j) \le \varepsilon/(16n_0)$ and admissible strategies $\pi_{y_j} \in \Pi_{y_j}$ such that $V(y_j) - V_{\pi_{y_j}}(y_j) \le \varepsilon/(16n_0)$. Consider, for any $x \in [0, \widehat{x}]$, the point $y(x) = \max\{y_j : y_j \le x\}$ and the strategy $\pi_x \in \Pi_x$ which pays out immediately $x - y(x)$ as dividends and then follows the strategy $\pi_{y(x)} \in \Pi_{y(x)}$. We obtain that $V(x) - V_{\pi_x}(x) \le \varepsilon/(8n_0)$ for any $x \in [0, \widehat{x}]$.

For any $x \in [0, \widehat{x}]$, we define recursively strategies $\pi_x^k \in \Pi_x$ as follows. For $k = 0$, take $\pi_0 = \pi_x$. For $k > 0$ and for the initial surplus $x \le x_{n_0}$, follow the strategy π_x while $X_t^\pi < \widehat{x}$, when the surplus X_t^π reaches $\widehat{x}$, pay out immediately the difference $\widehat{x} - x_{n_0}$ as dividend and then follow the strategy $\pi_{x_{n_0}}^{k-1} \in \Pi_{x_{n_0}}$. For $k > 0$ and for the initial surplus $x \in (x_{n_0}, \widehat{x}]$, pay out immediately the difference $x - x_{n_0}$ as dividend and then follow the strategy $\pi_{x_{n_0}}^{k-1} \in \Pi_{x_{n_0}}$.

With arguments similar to Lemma A.5 in Azcue and Muler (2005) it can be seen that, for any $x \in [0, \widehat{x}]$ and $k \ge 0$ the strategy $\pi_x^k \in \Pi_x$ is admissible and

$$V(x) - V_{\pi_x^{n_0}}(x) < \varepsilon/2 \qquad \text{for all } x \in [0, \widehat{x}]. \tag{8.1}$$

Let us prove now that, for any $x \in [0, \widehat{x}]$, there exists an admissible strategy $\widetilde{\pi} \in \Pi_x^{\widehat{x}}$ such that

$$V_{\pi_x^{n_0}}(x) - V_{\widetilde{\pi}}(x) < \varepsilon/2 \qquad \text{for all } x \in [0, \widehat{x}]. \tag{8.2}$$

Let us define $\widehat{\tau} = \inf\{t > 0 : X_t^{\pi_x^{n_0}} > \widehat{x}\}$. Consider the process $Y_t^{\pi_x^{n_0}}$ defined in Lemma 2.1, as the process corresponding to $X_t^{\pi_x^{n_0}}$ without claims and without paying dividends, but starting at $Y_0^{\pi_x^{n_0}} = x$. Since the process $X_t^{\pi_x^{n_0}}$ should pass at least n_0 times through the interval $[x_{n_0}, \widehat{x}]$ before surpassing $\widehat{x}$, we obtain that

$$\widehat{\tau} \geq \tau_{n_0}^Y := \inf\{t > 0 : Y_t^{\pi_x^{n_0}} > x_{n_0} + n_0(\widehat{x} - x_{n_0})\}. \tag{8.3}$$

To prove this, consider $X_t^{\widetilde{\pi}_x^{n_0}}$ the corresponding process without the dividends payment $\widehat{x} - x_{n_0}$ in each step, then

$$\begin{aligned} &\inf\left\{t > 0 : X_t^{\pi_x^{n_0}} > \widehat{x} = x_{n_0}\left(1 + \frac{\widehat{x} - x_{n_0}}{x_{n_0}}\right)\right\} \\ &\quad = \inf\left\{t > 0 : X_t^{\widetilde{\pi}_x^{n_0}} > x_{n_0}\left(\frac{\widehat{x}}{x_{n_0}}\right)^{n_0}\right\}, \end{aligned}$$

and since $x_{n_0}(\widehat{x}/x_{n_0})^{n_0} \geq x_{n_0} + n_0(\widehat{x} - x_{n_0})$ and $Y_t^{\pi_x^{n_0}} \geq X_t^{\widetilde{\pi}_x^{n_0}}$, we obtain that $\widehat{\tau} \geq \tau_{n_0}^Y$.

Since $\mathcal{L}^*(g) = 0$ and $Y_{\tau_{n_0}^Y}^{\pi} = x_{n_0} + n_0(\widehat{x} - x_{n_0})$ we have, using Itô's formula, that

$$g\big(x_{n_0} + n_0(\widehat{x} - x_{n_0})\big) E(e^{-c\tau_{n_0}^Y}) \leq g(x).$$

So, we have from the fact that g is increasing and (8.3) that

$$\begin{aligned} E(e^{-c\widehat{\tau}}) \leq E(e^{-c\tau_{n_0}^Y}) &\leq \frac{g(x)}{g(x_{n_0} + n_0(\widehat{x} - x_{n_0}))} \\ &\leq \frac{g(\widehat{x})}{g(D)} \leq \frac{\varepsilon}{2V(\widehat{x})}. \end{aligned} \tag{8.4}$$

Again, with arguments similar to Lemma A.5 in Azcue and Muler (2005), we obtain

$$V_{\pi_{n_0}}(x) - V_{\widetilde{\pi}}(x) \leq E(e^{-c\widehat{\tau}})\big(V(\widehat{x}) - \widehat{x}\big).$$

So using (8.4), we conclude (8.2). From (8.1) and (8.2) we get the result. □

We have to introduce some auxiliary sets to define precisely the sets $\mathcal{B}_0$ and $\mathcal{C}_0$ mentioned above.

DEFINITION 8.2. Let us define the continuous function

$$\Lambda(x) = (p + rx) - M(V)(x), \tag{8.5}$$

where the operator M is defined in (6.4), and the sets:

- $\mathcal{A} = \{x \in [0, \infty)$ such that $V'(x^+) = 1$ and $\Lambda(x) = 0\}$,
- $\mathcal{B} = \{x \in (0, \infty)$ such that $V'(x) = 1$ and $\Lambda(x) < 0\}$,
- $\mathcal{C} = [0, \infty) - (\mathcal{A} \cup \mathcal{B})$.

LEMMA 8.3. *The following situations are not possible*:

1. $V'(x^+) = 1$ *and* $\Lambda(x) > 0$.
2. $1 = V'(x^+) < V'(x^-)$ *and* $\Lambda(x) = 0$.

So, we conclude that

$$\mathcal{A} = \{x \in [0, \infty) \textit{ such that } V'(x) = 1 \textit{ and } \Lambda(x) = 0\},$$

$$\mathcal{B} = \{x \in (0, \infty) \textit{ such that } V'(x) = 1 \textit{ and } \Lambda(x) < 0\},$$

$$\mathcal{C} = \{x \in (0, \infty) \textit{ such that } V'(x^+) > 1\}$$
$$\cup \{x \in (0, \infty) \textit{ such that } V'(x^-) > V'(x^+) = 1 \textit{ and } \Lambda(x) < 0\}.$$

PROOF. Let us prove first that given $x \geq 0$, if $V'(x^+) = 1$ then $\Lambda(x) \leq 0$. Assume that $\Lambda(x) > 0$, then we can find $\delta > 0$ such that $\Lambda(y) > 0$ for all $y \in [x, x + \delta)$. Let us define D as the set of points in $(x, x + \delta)$ where V' and V'' exist, since V is semiconcave the set D has full measure. The function V is a supersolution of (3.1), then for any $y \in D$ we have

$$0 \geq \mathcal{L}^*(V)(y) \geq \sigma^2 y^2 V''(y)/2 + \Lambda(y)$$

and so $V''(y) \leq -2\Lambda(y)/(\sigma^2 y^2) < 0$. Then, since V is semiconcave, we have that for any $y \in D$

$$V'(y) - 1 = V'(y) - V'(x^+) \leq \int_x^y V''(s)\, ds < 0$$

and this is a contradiction because $V'(y) \geq 1$.

Let us prove now that if $x \in \mathcal{A}$ and $x > 0$, then V is differentiable at x and $V'(x) = 1$. If we have that $1 = V'(x^+) < V'(x^-)$, take any $d \in (V'(x^+), V'(x^-))$, then

$$\limsup_{h \to 0} \frac{(V(x + h) - V(x))/h - d}{h} = -\infty$$

and so, for any q, we have that

$$\max\Big\{1 - d, \max_{\gamma \in [0,1]} \big(\sigma^2 x^2 \gamma^2 q/2 + (p + rx\gamma)d - M(V)(x)\big)\Big\} \geq 0$$

and then, since $d > 1$, so

$$\max_{\gamma \in [0,1]} \big(\sigma^2 x^2 \gamma^2 q/2 + (p + rx\gamma)d - M(V)(x)\big) \geq 0.$$

Since this holds for any q, taking a sequence $q_n \to -\infty$, we obtain that $pd - M(V)(x) \geq 0$ for any $d \in (1, V'(x^-))$. This implies that $p - M(V)(x) \geq 0$ and so $\Lambda(x) > 0$, which is a contradiction. □

DEFINITION 8.4. We define the sets $\mathcal{A}_0$, $\mathcal{B}_0$ and $\mathcal{C}_0$ as:

- $\mathcal{B}_0 = \mathcal{B} \cup \{a \in \mathcal{A} : (a - \vartheta, a) \subset \mathcal{A} \cup \mathcal{B}$ for some $\vartheta > 0\}$,
- $\mathcal{C}_0 = \mathcal{C} \cup \{a \in \mathcal{A} : (a - \vartheta, a) \cup (a, a + \vartheta) \subset \mathcal{C}$ for some $\vartheta > 0\}$,
- $\mathcal{A}_0 = [0, \infty) - (\mathcal{C}_0 \cup \mathcal{B}_0)$.

PROPOSITION 8.5. *The sets introduced in Definition 8.4 satisfy the following properties*:

(a) $\mathcal{B}_0$ *is a disjoint union of intervals that are left-open and right-closed.*

(b) *If* $(x_0, \widehat{x}] \subset \mathcal{B}_0$ *and* $x_0 \notin \mathcal{B}_0$, *then* $x_0 \in \mathcal{A}_0$.

(c) *There exists* $x^* \geq 0$ *such that* $(x^*, \infty) \subset \mathcal{B}_0$.

(d) $\mathcal{C}_0$ *is an open set in* $[0, \infty)$, *that is, if* $0 \in \mathcal{C}_0$, *there exists* $\delta > 0$ *such that* $[0, \delta) \subset \mathcal{C}_0$ *and if a positive* $x \in \mathcal{C}_0$ *there exists* $\delta > 0$ *such that* $(x - \delta, x + \delta) \subset \mathcal{C}_0$.

(e) *Both* $\mathcal{A}_0$ *and* $\mathcal{B}_0$ *are nonempty.*

PROOF. The proof follows immediately from Definition 8.4 and Lemmas A.5 and A.6 included in the Appendix. □

From the previous proposition we can conclude that the upper boundary of any connected component of $\mathcal{C}_0$ belongs to $\mathcal{A}_0$ and also that the the lower boundary of any connected component of $\mathcal{B}_0$ belongs to $\mathcal{A}_0$.

The next proposition describes the optimal value function V for small initial surpluses.

PROPOSITION 8.6. *Consider the function* W *defined in Proposition 6.1 and the values* w_1 *and* x_* *defined in Proposition 7.7, then the optimal value function* $V(x)$ *coincides with* $W(x)/w_1$ *for all* $x \in [0, x_*]$. *In particular,* V *is twice continuously differentiable in* $[0, x_*]$.

PROOF. By Lemma A.6(b) included in the Appendix, $\mathcal{A}$ is left closed, so there exists $m = \min \mathcal{A}$. Note that, by Proposition 7.7, w_1 and x_* are well defined. Consider V_1 the value function of the limit strategy $\widetilde{\pi}_x^{x_*}$ obtained in Proposition 7.6. From (2.4), we have that $V_1(x) \leq V(x)$.

If $m > x_*$, we have from Proposition 5.3 that $V(x) \leq W(x)/W'(x_*)$ in $[0, \infty)$ because $W(x)/W'(x_*)$ is a supersolution of (3.1). So $V(x) = W(x)/W'(x_*) = V_1(x)$ in $[0, x_*]$. Then $V'(x_*) = 1$ and this implies that $x_* \in \mathcal{A} \cup \mathcal{B}$; this is a contradiction since in both cases there would exist a point in $\mathcal{A}$ smaller that m. In particular, if $x_* = 0$, then $m = 0$.

If $0 < m < x_*$, since $V_1' \geq 1$ and $\mathcal{L}^*(V_1) = 0$ in $(0, m)$, we have that V_1 is a supersolution of (3.1) in $(0, m)$ and since $V'(m) = 1$, by Proposition 8.1,

$W(x)/W'(x_*) = V(x)$ in $[0, m]$, but then $1 = V'(m) = W'(m)/W'(x_*)$ and this is a again a contradiction because by definition of x_*, $W'(m)/W'(x_*) > 1$.

Finally, in the case that $m = 0$, since $0 \in \mathcal{A}$ we have from (8.5) that $V(0) = (c + \beta)/p$, but from Proposition 6.3(c) we have that $W'(0) = (c + \beta)/p$, and so we get $V(0) = W(0)/W'(0)$. This implies that $x_* = 0$ because if x_* were positive, we would obtain

$$V(0) = W(0)/W'(0) < W(0)/W'(x_*) \le V(0).$$

Therefore, $m = x_*$ and $V = V_1$ in $[0, x_*]$. □

The previous proposition allows us to obtain V for small surpluses using only the function W. In the case that $x_* = 0$, we only obtain from this proposition the value at zero, $V(0) = (c+\beta)/p$. Hence, using Corollary 4.2, we can conclude that V is the unique viscosity solution of (3.1) with the boundary condition $V(0) = W(0)/w_1$.

REMARK 8.7. The limit stationary strategy $\widetilde{\pi}^{x_*}$ defined in Section 7 is the optimal barrier strategy. In effect, the optimal barrier strategy is the one with maximum value function al 0 and, by Proposition 8.6, the value function of this limit stationary strategy is $V_1(0) = W(0)/w_1 = V(0)$.

Let us show now that V is a classical solution of $\mathcal{L}^*(V) = 0$ in $\mathcal{C}_0$.

PROPOSITION 8.8. (a) *Let* (x_1, x_2) *with* $x_1 > 0$ *be a connected component of* C_0. *Consider* U *the unique classical solution of*

$$\mathcal{L}^*(U, V)(x) = 0 \tag{8.6}$$

in (x_1, ∞) *with* $U(x_1) = V(x_1)$ *and* $U'(x_1) = V'(x_1) = 1$. *Then* $V = U$ *in* $[x_1, x_2]$.

(b) *The optimal value function* V *is a classical solution of* $\mathcal{L}^*(V) = 0$ *in the open set* $\mathcal{C}_0$.

PROOF. Using Lemma A.8 included in the Appendix, it only remains to prove that V is twice continuously differentiable at the points $a \in \mathcal{A}$ such that, there exists $\delta > 0$ with $(a - \delta, a) \cup (a, a + \delta) \subset \mathcal{C}$. The number

$$\gamma^*(a) = \min\{1, 2(M(V)(a) - pV'(a))/(raV'(a))\}$$

is positive because $M(V)(a) - pV'(a) > \Lambda(a) = 0$ and $V'(a) = 1$. Take any sequence $u_n \to a$ with $u_n \in \mathcal{C}$; we have from Propositions 6.1 and 6.4 that

$$V''(u_n) = 2((p + ru_n\gamma_n)V'(u_n) - M(V)(u_n))/(\sigma^2 u_n^2 \gamma_n^2),$$

where

$$\gamma_n = \min\{1, 2(M(V)(u_n) - pV'(u_n))/(ru_nV'(u_n))\}.$$

Since V is semiconcave we get that

$$\lim_{n\to\infty} V''(u_n) = 2\big(\big(p + ra\gamma^*(a)\big)V'(a) - M(V)(a)\big)/(\sigma^2 a^2 \gamma^*(a)^2),$$

so V is twice continuously differentiable at a. □

REMARK 8.9. The optimal value function V is continuously differentiable at $(0,\infty)$ because it is continuously differentiable both in $\mathcal{C}_0$ and in the interior of $\mathcal{B}_0$. At any other point x we have that V' is continuously differentiable since

$$\lim_{y\in\mathcal{B}_0, y\to x} V'(y) = \lim_{y\in\mathcal{C}_0, y\to x} V'(y) = 1 = V'(x).$$

We prove now that V can be written as a limit of value functions of admissible stationary strategies. All of these admissible strategies coincide on $\mathcal{B}_0$ and $\mathcal{C}_0$. If the current surplus is in $\mathcal{B}_0$, the optimal strategy is to pay out as dividends the amount exceeding the lower boundary of the connected component of $\mathcal{B}_0$. If the current surplus is $x \in \mathcal{C}_0$, the optimal strategy is to pay no dividends and to invest $\gamma(x) = \arg\max_{\gamma\in[0,1]} \mathcal{L}_\gamma(V)(x)$. Finally, if the current surplus is in $\mathcal{A}_0$, we need to consider a limit of admissible strategies similar to the one we used to obtain barrier strategies in Section 7.

We define admissible stationary strategies π based upon the sets $\mathcal{A}_0$, $\mathcal{B}_0$ and $\mathcal{C}_0$ introduced in Definition 8.4. Since these strategies are stationary, for any $x \geq 0$ we can denote $\pi(x) \in \Pi_x$ the corresponding strategy with initial surplus x.

DEFINITION 8.10. Given a finite subset $\mathcal{A}' \subset \mathcal{A}_0$ and a number $u > 0$ satisfying the following conditions:

1. if $\min \mathcal{A}_0 = 0$ then $0 \in \mathcal{A}'$,
2. $c_a = a/e^u \in \mathcal{C}_0$ for all positive $a \in \mathcal{A}'$,

we define recursively the admissible stationary strategy π in the following way:

- If the current surplus $x \in \mathcal{C}_0$, pay no dividends and take

$$\gamma^*(x) = \min\{1, 2\big(M(V)(x) - pV'(x)\big)/(rxV'(x))\}$$

up to the exit time τ of $\mathcal{C}_0$. Then follow the strategy $\pi(x_1) \in \Pi_{x_1}$ where $x_1 = X_\tau^{\pi(x)} \in \mathcal{A}_0 \cup \mathcal{B}_0$.
- If the current surplus $x \in \mathcal{B}_0$, by Proposition 8.5(a) and (b), there exists $a \in \mathcal{A}_0$ such that $(a, x] \subset \mathcal{B}_0$. In this case pay out immediately $x - a$ as dividends, and follow the strategy $\pi(a) \in \Pi_a$ described below.
- If the current surplus $x \in \mathcal{A}_0 \setminus \mathcal{A}'$, pay out immediately $x - a$ as dividends where a is the maximum element of $\mathcal{A}'$ smaller than x, and then follow the strategy $\pi(a) \in \Pi_a$.
- If the current surplus is $a \in \mathcal{A}'$, pay out immediately $a - c_a$ as dividends and then follow the strategy $\pi(c_a) \in \Pi_{c_a}$.

- In the case that the current surplus is $0 \in \mathcal{A}'$, pay out all the incoming premium as dividends up to the ruin time.

In the case that $\mathcal{A}_0$ is finite, V can be written as the limit (with u going to zero) of the value functions of the admissible strategies defined above taking $\mathcal{A}' = \mathcal{A}_0$; but in the case that $\mathcal{A}_0$ is infinite, we have to consider finite subsets $\mathcal{A}' \subset \mathcal{A}_0$. This result is proved in the next theorem.

THEOREM 8.11. *Given $\varepsilon > 0$, we can find a finite set $\mathcal{A}' \subset \mathcal{A}_0$ and a number $u > 0$ such that the admissible stationary strategy introduced in Definition 8.10 satisfies $V(x) - V_{\pi(x)}(x) < \varepsilon$ for all $x \geq 0$. In the case that $\mathcal{A}_0$ is finite, we can take $\mathcal{A}' = \mathcal{A}_0$.*

PROOF. We assume that $\min \mathcal{A}_0 > 0$, in the case $\min \mathcal{A}_0 = 0$ the proof is similar. Let us consider $\widehat{x} = \max \mathcal{A}_0$ and the twice continuously differentiable solution g of the equation $\mathcal{L}^*(g) = 0$ for the special case $\beta = 0$. From Proposition 7.7, we get that $\inf_{x \geq 0} g'(x) = g'(x_*) > 0$ for some $x_* \geq 0$. Since $\lim_{x\to\infty} g(x) = \infty$, we can find a number M such that $g(1)/g(e^M) \leq \varepsilon/(4V(\widehat{x}))$.

We can find $\delta > 0$ such that, if $h \leq \delta$ then

$$0 \leq \big(V(a+h) - V(a)\big)/h - 1 \leq \varepsilon/(4\widehat{x}). \tag{8.7}$$

In effect, V' is absolutely continuous in $[0, \widehat{x}]$, $V'(a) = 1$ for all $a \in \mathcal{A}_0 \cup \mathcal{B}_0$, and from Proposition 8.5(b) and (c) we have that $[\widehat{x}, \infty) \subseteq \mathcal{A}_0 \cup \mathcal{B}_0$.

Given δ, take the finite set $\mathcal{A}_\delta$ and the number $\varsigma > 0$ given by Lemma A.9 included in the Appendix, and take $u > 0$ such that

$$u \leq \delta/(2\widehat{x}), \qquad a - \varsigma < a/e^u \tag{8.8}$$

and

$$0 \leq \frac{V(a) - V(a/e^u)}{a - a/e^u} - 1 \leq \varepsilon/\big(8(M+2)\widehat{x}\big) \tag{8.9}$$

for all $a \in \mathcal{A}_\delta$. Take $N = \#\mathcal{A}_\delta$,

$$k_0 = [M/u + N] + 1 \tag{8.10}$$

and admissible strategies $\overline{\pi}(a) \in \Pi_a$ with $a \in \mathcal{A}_\delta$ such that

$$V(a) - V_{\overline{\pi}(a)}(a) \leq \varepsilon/\big(4(2k_0+3)\big) \qquad \text{for all } a \in \mathcal{A}_\delta. \tag{8.11}$$

Let us define $c_a = a/e^u$ for all $a \in \mathcal{A}_\delta$, then, by (8.8), $c_a \in \mathcal{C}_0$. Take the admissible stationary strategy π associated with $u > 0$ and the finite set $\mathcal{A}_\delta$ given by Definition 8.10.

We define recursively a family of admissible strategies $\overline{\pi}_k(x) \in \Pi_x$ for all $x \geq 0$ and $k \geq 0$, in the following way:

- Take $\overline{\pi}_0(a)$ as the admissible strategy $\overline{\pi}(a)$ defined in (8.11) for all $a \in \mathcal{A}_\delta$.
- If the surplus $x \in \mathcal{C}_0$, pay no dividends and take
$$\gamma^*(x) = \min\{1, 2(M(V)(x) - pV'(x))/(rxV'(x))\}$$
up to the exit time τ_0 of $\mathcal{C}_0$. Then follow the strategy $\overline{\pi}_k(x_1) \in \Pi_{x_1}$ starting at x_1 where $x_1 = X_{\tau_0}^{\overline{\pi}_k(x)} \in \mathcal{A}_0 \cup \mathcal{B}_0$.
- If the surplus $x \in \mathcal{B}_0$, by Proposition 8.5(a) and (b), there exists $a \in \mathcal{A}_0$ such that $(a, x] \subset \mathcal{B}_0$. In this case, pay out immediately $x - a$ as dividends and follow the strategy $\overline{\pi}_k(a) \in \Pi_a$ described below.
- If the surplus $x \in \mathcal{A}_0 \setminus \mathcal{A}_\delta$, pay out immediately $x - a$ as dividends where a is the maximum element of $\mathcal{A}_\delta$ smaller than x, and then follow the strategy $\overline{\pi}_k(a) \in \Pi_a$.
- If the surplus is $a \in \mathcal{A}_\delta$ with $a > 0$, pay out immediately $a - c_a$ as dividends and then follow the strategy $\overline{\pi}_{k-1}(c_a) \in \Pi_{c_a}$.

To simplify notation we write $V_{\overline{\pi}_k}(x)$ instead of $V_{\overline{\pi}_k(x)}(x)$. Let us prove first that

$$\max_{x \ge 0}(V(x) - V_{\overline{\pi}_k}(x)) \le 3\varepsilon/4. \tag{8.12}$$

Given any initial surplus $x \ge 0$, note that all the processes $X_t^{\overline{\pi}_k}$ with $k \ge 0$ coincide for $t \le \tau \wedge \widehat{\tau}$ where τ is the time of arriving to $\mathcal{A}_\delta$ and $\widehat{\tau}$ the ruin time. So, using the dynamic programing principle, we have that

$$\begin{aligned}
&|V_{\overline{\pi}_{k_0}}(x) - V_{\overline{\pi}_0}(x)| \\
&\quad = |E_x(e^{-c(\tau \wedge \widehat{\tau})}(V_{\overline{\pi}_{k_0}}(X_{\tau \wedge \widehat{\tau}}^{\overline{\pi}_{k_0}}) - V_{\overline{\pi}_0}(X_{\tau \wedge \widehat{\tau}}^{\overline{\pi}_{k_0}})))| \\
&\quad \le E_x(|e^{-c(\tau \wedge \widehat{\tau})}(V_{\overline{\pi}_{k_0}}(X_{\tau \wedge \widehat{\tau}}^{\overline{\pi}_{k_0}}) - V_{\overline{\pi}_0}(X_{\tau \wedge \widehat{\tau}}^{\overline{\pi}_{k_0}}))\chi_{\{\tau < \widehat{\tau}\}}|) \\
&\quad \le \max_{a \in \mathcal{A}_\delta} |V_{\overline{\pi}_{k_0}}(a) - V_{\overline{\pi}_0}(a)|.
\end{aligned} \tag{8.13}$$

Consider $a \in \mathcal{A}_\delta$, the processes $X_t^{\overline{\pi}_k}$ starting at a and $\widehat{\tau}$ the ruin time, we define as usual $X_t^{\overline{\pi}_k} = X_{\widehat{\tau}}^{\overline{\pi}_k}$ for $t \ge \widehat{\tau}$. Let τ_k be the kth time that $X_t^{\overline{\pi}_{k_0+1}}$ reaches $\mathcal{A}_\delta$ and let

$$K = \{k \ge 0 \text{ such that } \tau_{k+1} < \widehat{\tau} \text{ and } X_{\tau_k}^{\overline{\pi}_{k_0}} = X_{\tau_{k+1}}^{\overline{\pi}_{k_0}}\}.$$

Since the processes $X_t^{\overline{\pi}_k}$ and $X_t^{\overline{\pi}_{k-1}}$ coincide until $\tau_{k-1} \wedge \widehat{\tau}$, we have using (8.11) and (8.9) that

$$\begin{aligned}
&|V_{\overline{\pi}_{k_0}}(a) - V_{\overline{\pi}_0}(a)| \\
&\quad \le \sum_{k=0}^{k_0-1} |V_{\overline{\pi}_{k+1}}(a) - V_{\overline{\pi}_k}(a)| \\
&\quad = E_a\left(\sum_{k=0}^{k_0-1} (e^{-c\tau_k}|V_{\overline{\pi}_1}(X_{\tau_k}^{\overline{\pi}_{k_0}}) - V_{\overline{\pi}_0}(X_{\tau_k}^{\overline{\pi}_{k_0}})|)\chi_{\{\tau_k < \widehat{\tau}\}}\right).
\end{aligned} \tag{8.14}$$

We denote $a_0 = X_{\tau_k}^{\overline{\pi}_{k_0}} \in \mathcal{A}_\delta$. We define $\widetilde{\tau}_k$ as the first time that $X_t^{\overline{\pi}_{k_0}}$ leaves $\mathcal{C}_0$ after τ_k, and we denote $a_2 = X_{\widetilde{\tau}_k}^{\overline{\pi}_{k_0}}$. We obtain, using Itô's formula, Proposition 8.8(b) and the definition of $\overline{\pi}_0$,

$$
\begin{aligned}
&|V_{\overline{\pi}_1}(a_0) - V_{\overline{\pi}_0}(a_0)| \\
&\quad = |V_{\overline{\pi}_0}(c_{a_0}) + (a_0 - c_{a_0}) - V_{\overline{\pi}_0}(a_0)| \\
&\quad = \big|E\big(\big(V_{\overline{\pi}_0}(a_2) - V(a_2)\big)e^{-c(\widetilde{\tau}_k - \tau_k)}|\mathcal{F}_{\tau_k}\big) \\
&\qquad + a_0 - c_{a_0} - V(a_0) + V(c_{a_0}) + V(a_0) - V_{\overline{\pi}_0}(a_0)\big| \\
&\quad \le E\big(\big(V(a_2) - V_{\overline{\pi}_0}(a_2)\big)e^{-c(\widetilde{\tau}_k - \tau_k)}|\mathcal{F}_{\tau_k}\big) + \big(V(a_0) - V_{\overline{\pi}_0}(a_0)\big) \\
&\qquad + (a_0 - c_{a_0})\left(\frac{V(a_0) - V(c_{a_0})}{a_0 - c_{a_0}} - 1\right).
\end{aligned} \tag{8.15}
$$

From (8.11), (8.8), (8.9) and using that $e^{-u} \ge 1 - u$, we obtain that

$$
\begin{aligned}
&\big(V(a_0) - V_{\overline{\pi}_0}(a_0)\big) + (a_0 - c_{a_0})\left(\frac{V(a_0) - V(c_{a_0})}{a_0 - c_{a_0}} - 1\right) \\
&\qquad \le \frac{\varepsilon}{4(2k_0+3)} + \frac{\varepsilon}{8(M+2)\widehat{x}}(a_0 - c_{a_0}) \\
&\qquad \le \frac{\varepsilon}{4(2k_0+3)} + \frac{\varepsilon}{8(M+2)}u.
\end{aligned} \tag{8.16}
$$

If $k \notin K$ and $a_2 \ge 0$ denote $a_1 = X_{\tau_{k+1}}^{\overline{\pi}_{k_0}} \in \mathcal{A}_\delta$. We obtain that $a_0 > a_1$, and by (8.11),

$$
\begin{aligned}
&E\big(\big(V(a_2) - V_{\overline{\pi}_0}(a_2)\big)e^{-c(\widetilde{\tau}_k - \tau_k)}|\mathcal{F}_{\tau_k}\big) \\
&\quad = E\big(\big(V(a_2) - V_{\overline{\pi}_0}(a_2)\big)e^{-c(\widetilde{\tau}_k - \tau_k)}\chi_{\{\widetilde{\tau}_k < \widehat{\tau}\}}|\mathcal{F}_{\tau_k}\big) \\
&\quad \le E\big(\big(V(a_2) - V_{\overline{\pi}_0}(a_2)\big)\chi_{\{\widetilde{\tau}_k < \widehat{\tau}\}}|\mathcal{F}_{\tau_k}\big) \\
&\quad = E\big(\big(V(a_0) - V_{\overline{\pi}_0}(a_0)\big)\chi_{\{k \in K\}}|\mathcal{F}_{\tau_k}\big) \\
&\qquad + E\big(\big(V(a_2) - V_{\overline{\pi}_0}(a_2)\big)\chi_{\{k \notin K\}}\chi_{\{\widetilde{\tau}_k < \widehat{\tau}\}}|\mathcal{F}_{\tau_k}\big) \\
&\quad \le \frac{\varepsilon}{4(2k_0+3)} + E\big(\big(V(a_2) - V(a_1) - (a_2 - a_1) \\
&\qquad\qquad + V(a_1) - V_{\overline{\pi}_0}(a_1)\big)\chi_{\{k \notin K\}}\chi_{\{\widetilde{\tau}_k < \widehat{\tau}\}}|\mathcal{F}_{\tau_k}\big) \\
&\quad \le \frac{2\varepsilon}{4(2k_0+3)} + E\big(\big(V(a_2) - V(a_1) - (a_2 - a_1)\big)\chi_{\{k \notin K\}}\chi_{\{\widetilde{\tau}_k < \widehat{\tau}\}}|\mathcal{F}_{\tau_k}\big).
\end{aligned} \tag{8.17}
$$

Note that $(a_1, a_2) \cap \mathcal{A}_\delta = \phi$, and so there is no connected component (r_1, r_2) of $\mathcal{C}_0$ included in $[a_1, a_2]$ with length greater than δ. In effect, if such component exists, then $r_2 \in \mathcal{A}_0 \setminus \mathcal{A}_\delta$, and this contradicts Lemma A.9(b) included in the Appendix.

Then we can find $a_1 = x_1 \le x_2 \le \cdots \le x_n = a_2$ such that $x_i \in \mathcal{A}_0 \cup \mathcal{B}_0$ and $x_{i+1} - x_i < \delta$. So we get, by (8.7),

$$\begin{aligned}
&E\big(\big(V(a_2) - V(a_1) - (a_2 - a_1)\big)\chi_{\{k\notin K\}}|\mathcal{F}_{\tau_k}\big)\\
&\quad = E\left(\left(\sum_{i=1}^{n} V(x_{i+1}) - V(x_i) - (x_{i+1} - x_i)\right)\chi_{\{k\notin K\}}\chi_{\{\tilde{\tau}_k<\hat{\tau}\}}|\mathcal{F}_{\tau_k}\right)\\
&\quad \le \frac{\varepsilon}{4\hat{x}} E\left(\sum_{i=1}^{n}(x_{i+1} - x_i)\chi_{\{k\notin K\}}\chi_{\{\tilde{\tau}_k<\hat{\tau}\}}|\mathcal{F}_{\tau_k}\right)\\
&\quad \le \frac{\varepsilon}{4\hat{x}} E\big((X_{\tau_k}^{\overline{\pi}_{k_0}} - X_{\tau_{k+1}}^{\overline{\pi}_{k_0}})\chi_{\{\tilde{\tau}_k<\hat{\tau}\}}|\mathcal{F}_{\tau_k}\big).
\end{aligned} \tag{8.18}$$

From (8.15)–(8.18) and from (8.11), (8.8) and (8.9) we obtain that

$$\begin{aligned}
&|V_{\overline{\pi}_1}(X_{\tau_k}^{\overline{\pi}_{k_0}}) - V_{\overline{\pi}_0}(X_{\tau_k}^{\overline{\pi}_{k_0}})|\\
&\quad \le \frac{3\varepsilon}{4(2k_0+3)} + \frac{\varepsilon}{8(M+2)}u + \frac{\varepsilon}{4\hat{x}} E\big((X_{\tau_k}^{\overline{\pi}_{k_0}} - X_{\tau_{k+1}}^{\overline{\pi}_{k_0}})\chi_{\{\tilde{\tau}_k<\hat{\tau}\}}|\mathcal{F}_{\tau_k}\big).
\end{aligned} \tag{8.19}$$

And so from (8.14) and (8.19), we have using (8.10) and Lemma A.9(c) included in the Appendix, that

$$\begin{aligned}
|V_{\overline{\pi}_{k_0}}(a) - V_{\overline{\pi}_0}(a)| &\le E_a\left(\sum_{k=0}^{k_0-1} |V_{\overline{\pi}_1}(X_{\tau_k}^{\overline{\pi}_{k_0}}) - V_{\overline{\pi}_0}(X_{\tau_k}^{\overline{\pi}_{k_0}})|\right)\\
&\le k_0\left(\frac{3\varepsilon}{4(2k_0+3)} + \frac{\varepsilon}{8(M+2)}u\right)\\
&\quad + \frac{\varepsilon}{4\hat{x}} E_a\big((X_{\tau_k}^{\overline{\pi}_{k_0}} - X_{\tau_{k+1}}^{\overline{\pi}_{k_0}})\chi_{\{\tilde{\tau}_k<\hat{\tau}\}}\big).
\end{aligned}$$

So we have proved (8.12).

Let us prove now that

$$\max_{x\ge 0}|V_\pi(x) - V_{\overline{\pi}_{k_0}}(x)| \le \varepsilon/4. \tag{8.20}$$

Given any initial surplus $x \ge 0$, consider the process $X_t^{\overline{\pi}_{k_0}}$ with initial value x. Since the processes $X_t^{\overline{\pi}_{k_0}}$ and X_t^{π} coincide up to $\tau_{k_0} \wedge \hat{\tau}$,

$$\begin{aligned}
&\max_{x\ge 0}|V_\pi(x) - V_{\overline{\pi}_{k_0}}(x)|\\
&\quad \le E_x\big(e^{-c(\tau_{k_0}\wedge\hat{\tau})}|V_\pi(X_{\tau_{k_0}\wedge\hat{\tau}}^{\pi}) - V_{\overline{\pi}_0}(X_{\tau_{k_0}\wedge\hat{\tau}}^{\pi})|\big)\\
&\quad = E_x\big(e^{-c(\tau_{k_0}\wedge\hat{\tau})}|V_\pi(X_{\tau_{k_0}\wedge\hat{\tau}}^{\pi}) - V_{\overline{\pi}_0}(X_{\tau_{k_0}\wedge\hat{\tau}}^{\pi})|\chi_{\{\tau_{k_0}<\hat{\tau}\}}\big)\\
&\quad = E_x(e^{-c\tau_{k_0}})V(\hat{x}).
\end{aligned} \tag{8.21}$$

Consider the process Y_t^π defined in Lemma 2.1, as the process corresponding to X_t^π without claims and without paying dividends, but starting at $Y_0^\pi = 1$. When the process X_t^π arrives the k_0th time to $\mathcal{A}_\delta$, it should have already passed $k_0 - N$ times through intervals of the form (c_a, a) with $a \in \mathcal{A}_\delta$. So

$$\tau_{k_0} \geq T_{k_0-N} := \min\{t : Y_t^\pi \geq e^{(k_0-N)u}\}.$$

Let γ_t be the investment policy corresponding to the strategy π. We have using Itô's formula that

$$\begin{aligned} g\big(e^{(k_0-N)u}\big)&E(e^{-cT_{k_0-N}}) - g(1) \\ &= E\biggl(\int_0^{T_{k_0-N}} e^{-cs}\bigl(\sigma^2\gamma_s^2(Y_s^\pi)^2 g''(Y_s^\pi)/2 \\ &\qquad\qquad + (p + r\gamma_s Y_s^\pi)g'(Y_s^\pi) - cg(Y_s^\pi)\bigr)\,ds\biggr) \leq 0, \end{aligned}$$

so we get

$$E(e^{-c\tau_{k_0}}) \leq E(e^{-cT_{k_0-N}}) \leq g(1)/g\big(e^{(k_0-N)u}\big) \leq g(1)/g(e^M) \leq \varepsilon/(4V(\widehat{x}))$$

and from (8.21) we obtain (8.20).

We get the result combining (8.12) and (8.20). □

REMARK 8.12. From Propositions 8.5 and 8.11, we conclude that the optimal strategy for large surpluses is to pay out as dividends the amount exceeding $a^* = \max \mathcal{A}_0$.

REMARK 8.13. Propositions 8.6 and 8.8, Remark 8.9, and the fact that the derivative of the optimal value function should be one in $\mathcal{A}_0 \cup \mathcal{B}_0$, suggest a method to construct the optimal value function in the case that the optimal dividend payment policy has the structure of a finite band: we could construct the value function of the best one-band strategy, the best two-band strategy, etc. as candidates of the optimal value function. If any of these candidates is a viscosity solution of (3.1), it should be V. We use this method to find the optimal value function in the examples of the next section.

9. Numerical examples and final remarks. In this section we present numerical approximations of the optimal value function V. In order to do this, we obtain as a first step an approximation of the function W using the fixed-point operator defined in Proposition 6.1; it is not possible to use an standard approximation scheme because of the lack of both the ellipticity of the equation (6.1) and the boundary condition at zero.

We construct two examples of optimal value functions. In one example the optimal dividend payment policy is barrier and in the other it is not.

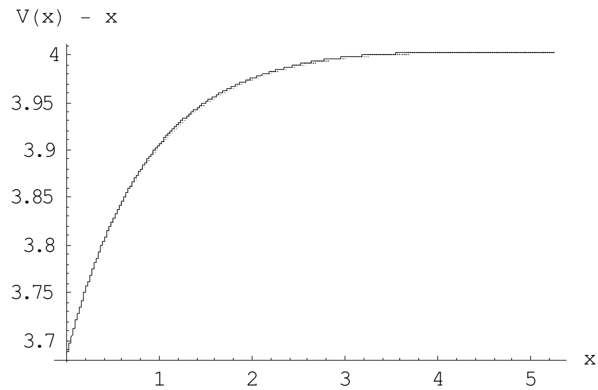

FIG. 1. *$V(x) - x$ for an exponential distribution.*

EXAMPLE 9.1. We consider the exponential distribution $F(x) = 1 - e^{-x}$ and parameters $p = 4$, $\beta = 1$, $c = 0.5$, $r = 0.3$, $\sigma = 2$. We first obtain numerically, using Proposition 6.1, the function W and we get that the derivative reaches the minimum at $y = 4.846$. Then, we prove that the value function V_1 of the optimal barrier strategy is a solution of (3.1) and so, by Proposition 7.8, $V = V_1$ and V is twice continuously differentiable.

We show in Figure 1 the function $V(x) - x$ and in Figure 2 the optimal investment policy $\gamma^*(x)$ for $x \in [0, y]$. Note that, according to Proposition 6.3(b), $\gamma^* = 1$ for small surpluses.

EXAMPLE 9.2. We consider the following claim distribution:

$$F(x) = \begin{cases} 0, & \text{if } x \in [0, 7/10], \\ (10/3)(x - 7/10), & \text{if } x \in (7/10, 1], \\ 1, & \text{if } x > 1, \end{cases}$$

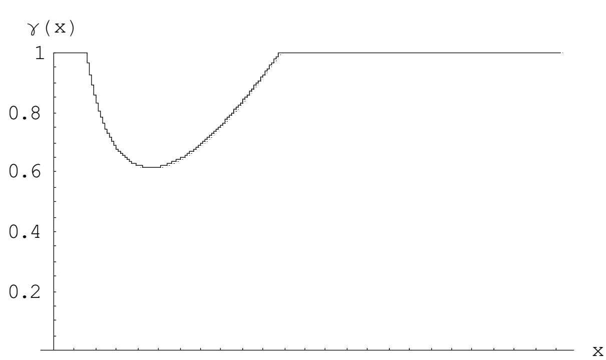

FIG. 2. *γ for an exponential distribution.*

and parameters $p = 1.6$, $\beta = 1$, $c = 0.3$, $r = 0.2$, $\sigma = 1$.

We prove that the derivative of the function W of Proposition 6.1 reaches the minimum at zero, so the value function of the optimal barrier strategy is $V_1(x) = x + (c + \beta)/p$, but in this case V_1 is not a supersolution of (3.1).

We now look for the best two-band strategy. First we obtain numerically, using Proposition 6.4, the function

$$W_y(x) = \begin{cases} x + (c + \beta)/p, & \text{if } x \le y, \\ U_1(x), & \text{if } y > x, \end{cases}$$

for each $y > 0$, where U_1 is the unique solution of $\mathcal{L}^*(U_1, W_y) = 0$ in (y, ∞) with boundary conditions $U_1(y) = W_y(y)$ and $U_1'(y) = 1$. Take

$$y_1 = \min\{y : \text{there exists } z > y \text{ with } V_y'(z) = 1\}$$

and z_1 with $V_{y_1}'(z_1) = 1$. We get $y_1 = 0.291$, $z_1 = 2.926$ and we can prove that

$$V_{y_1}(x) = \begin{cases} W_{y_1}(x), & \text{if } x \le z_1, \\ W_{y_1}(z_1) + (x - z_1), & \text{if } y > z_1, \end{cases}$$

is a viscosity solution of (3.1). Hence $V = V_{y_1}$ because V_{y_1} is the value function of a limit strategy corresponding to the sets $\mathcal{A}_0 = \{0, z_1\}$, $\mathcal{B}_0 = (0, y_1] \cup (z_1, \infty)$ and $\mathcal{C}_0 = (y_1, z_1)$.

We show in Figure 3 the function $V(x) - x$, in Figure 4 the derivative of V and in Figure 5 the optimal investment policy $\gamma^*(x)$ for $x \in (y_1, z_1)$. It can be seen in Figure 4 that V is not twice continuously differentiable at y_1.

Let us finally note that in the setting of diffusion approximation [see, for instance, Højgaard and Taksar (2004)], the optimal value function V is always twice continuously differentiable, concave and comes from an optimal barrier strategy. We see in the last example that this is not always the case in the Cramér–Lundberg setting.

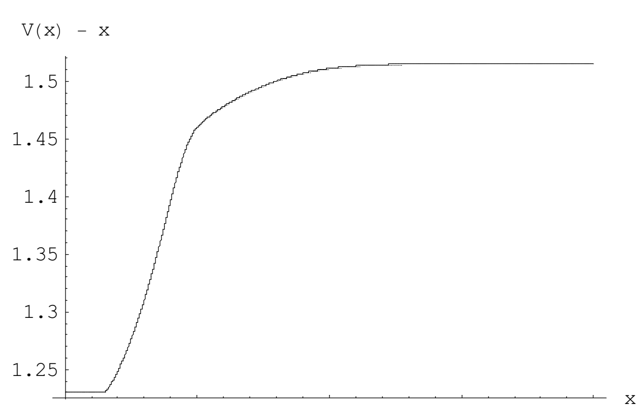

FIG. 3. $V(x) - x$ *for a non-monotone density distribution.*

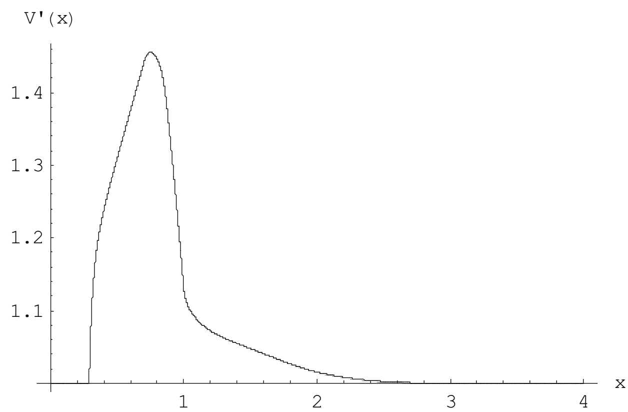

FIG. 4. *V' for a non-monotone density distribution.*

APPENDIX: TECHNICAL LEMMAS

Lemmas for Proposition 5.3. In the following two lemmas we show that, in order to define V as a supremum of value function of admissible strategies, we can discard the strategies where the surplus stays a positive time at the points of a given countable set, and also that V can be written as a limit of value functions of strategies whose surpluses are confined in compact subsets of $(0, \infty)$.

LEMMA A.1. (a) *Given $x \geq 0$ and $x_1 > x$, let us define $\Pi_x^{x_1}$ as the set of $\pi \in \Pi_x$ such that $X_t^\pi \leq x_1$ for all $t \geq 0$ and $\mathcal{V}^{x_1}(x) = \sup\{V_\pi(x)$ with $\pi \in \Pi_x^{x_1}\}$, then*

$$\lim_{x_1 \to \infty} \mathcal{V}^{x_1}(x) = V(x).$$

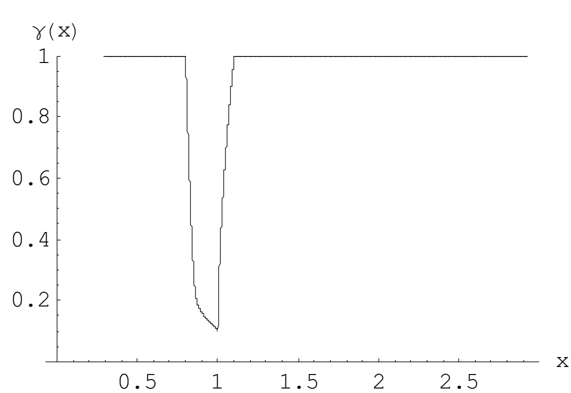

FIG. 5. *γ for a non-monotone density distribution.*

(b) *Given* $x_0 \geq x \geq 0$, *let us define* $\Pi_x^{[x_0,\infty)}$ *as the set of* $\pi \in \Pi_x$ *such that* $X_t^\pi \geq x_0$ *for all* $t \geq 0$ *and* $\mathcal{V}_{x_0}(x) = \sup\{V_\pi(x)$ *with* $\pi \in \Pi_x^{[x_0,\infty)}\}$, *then*

$$\lim_{x_0 \searrow 0} \mathcal{V}_{x_0}(x) = V(x).$$

PROOF. (a) Given $\varepsilon > 0$ consider $\pi \in \Pi_x$ such that $V(x) < V_\pi(x) + \varepsilon$ and consider for any $x_1 > 0$ the admissible strategy $\pi_{x_1} \in \Pi_x^{x_1}$ which coincides with the strategy π while the surplus is less than x_1, and pay out x_1 as dividends at the moment τ_{x_1} when the surplus reaches x_1. Since

$$\lim_{x_1 \to \infty} E_x\left(\int_0^{\tau_{x_1} \wedge \tau^\pi} e^{-cs}\, dL_s^\pi\right) = V_\pi(x),$$

there exists x_1 large enough such that $V(x) - V_{\pi_{x_1}}(x) < 2\varepsilon$.

(b) Take $x_0 \in (0, x)$. We can find an admissible strategy $\pi \in \Pi_{x-x_0}$ such that $V(x - x_0) < V_\pi(x - x_0) + x_0$. Define the admissible strategy $\pi_0 \in \Pi_x$ which invests x_0 in bonds and then follows the strategy π corresponding to initial surplus $x - x_0$ up to the time $\tau_{x_0} = \inf\{t : X_t^{\pi_0} < x_0\}$. Then we have that $\mathcal{V}_{x_0}(x) \geq V_{\pi_0}(x) = V_\pi(x - x_0)$ and the result follows from the continuity of V at x. □

LEMMA A.2. *Given* $x \geq 0$ *and a countable set* $S \subset [0, \infty)$, *let* $\Pi_x(S)$ *be the set of all the admissible strategies* $\pi \in \Pi_x$ *such that the set*

$$\{(\omega, t) \in \Omega \times [\mathbf{0}, \infty) : X_t^\pi(\omega) \in S\},$$

has zero measure. Then $V(x) = \sup_{\pi \in \Pi_x(S)} V_\pi(x)$.

PROOF. Given $\varepsilon > 0$, take $\pi = (\gamma_t, L_t) \in \Pi_x$ such that $V(x) - V_\pi(x) < \varepsilon/2$. Given any $a \in (0, \varepsilon/2)$, consider the stopping times $\tau_a = \inf\{t : L_t \geq a\}$ and $\tau_0 = \inf\{t : L_t \geq 0\}$ and the admissible strategy $\pi_a = (\gamma_t^a, L_t^a)$ such that the dividend policy consists in paying no dividends up to time τ_a and following the dividend policy $L_t - a$ afterward, and such that the amount of the surplus invested in stocks coincides with the amount of the surplus invested in stocks in the original strategy. We have that $X_t^{\pi_a}$ coincides with X_t^π for $t \in [0, \tau_0 \wedge \tau^\pi]$, that $X_t^{\pi_a} - X_t^\pi \in (0, a)$ if $t \in (\tau_0, \tau_a \wedge \tau^\pi)$ and that $X_t^{\pi_a} - X_t^\pi = a$ if $t \in [\tau_a, \tau^\pi]$. We obtain that $\tau^{\pi_a} \geq \tau^\pi$ and that $V_\pi(x) - V_{\pi_a}(x) \leq a < \varepsilon/2$, and so $V(x) - V_{\pi_a}(x) \leq \varepsilon$ for all $a \in (0, \varepsilon/2)$. Note that, fixing $x_i \in S$ we have that

$$1 \geq P\left(\bigcup_{a \in (0,\varepsilon/2)} \{(\omega, t) : X_t^\pi = x_i - a\}\right) \geq P\left(\bigcup_{a \in (0,\varepsilon/2)} \{(\omega, t) : X_t^{\pi_a} = x_i, \tau_a \leq t\}\right),$$

and the last union is disjoint. Then the set of $a \in (0, \varepsilon/2)$ such that

$$P\big(\{(\omega, t) : X_t^{\pi_a} = x_i, \tau_a \leq t\}\big) > 0$$

is countable. So, since S is countable, there exists $a_0 \in (0, \varepsilon/2)$ such that

$$P(\{(\omega, t) : X_t^{\pi_a} \in S, \tau_{a_0} \le t\}) = 0.$$

If $t < \tau_{a_0}$, then $L_t^{a_0} = 0$ and $X_t^{\pi_{a_0}} = X_t^{(\gamma_t, 0)}$ which does not depend on a_0. Define $\tau^0 = 0$ and call τ^i the time of the ith claim, we obtain that

$$\{(\omega, t) : X_t^{\pi_{a_0}} \in S, t \le \tau_a\} = \bigcup_{i=0}^{\infty} \{(\omega, t) : X_t^{\pi_{a_0}} \in S, t \in [\tau^i \wedge \tau_{a_0}, \tau^{i+1} \wedge \tau_{a_0})\},$$

but if $t \in [\tau^i \wedge \tau_{a_0}, \tau^{i+1} \wedge \tau_{a_0})$, we have that $X_t^{(\gamma_t, 0)}$ is a linear diffusion [see, for instance, Borodin and Salminen (2002)], and so

$$P(\{(\omega, t) : X_t^{\pi_{a_0}} \in S, t \in [\tau^i \wedge \tau_{a_0}, \tau^{i+1} \wedge \tau_{a_0})\}) = 0.$$

We conclude that $P((\omega, t) : X_t^{\pi_{a_0}} \in S) = 0$. □

Lemmas for Proposition 8.5. We need the following result in order to prove Lemma A.4.

LEMMA A.3. *Assume that* $V'(\widehat{x}) = 1$ *for some* $\widehat{x} > 0$ *and* $\overline{u}$ *is an absolutely continuous supersolution of* (3.1) *in* $(0, \widehat{x})$, *then* $\overline{u} \ge V$ *in* $[0, \widehat{x}]$.

PROOF. The argument coincides with the one used to prove Proposition 5.3, but taking admissible strategies π such that the corresponding controlled risk process X_t satisfies $X_t \le \widehat{x}$. □

The following lemma gives conditions under which the optimal value function V is linear in some interval.

LEMMA A.4. *Given any* $y > 0$, *we define*

$$\mathcal{U}_y(x) = \begin{cases} V(x), & \text{if } x \le y, \\ V(y) - y + x, & \text{if } x > y. \end{cases} \tag{A.1}$$

(a) *If* $\mathcal{U}_y$ *is supersolution of* (3.1) *in* (y, ∞), *then* $\mathcal{U}_y = V$ *in* $[0, \infty)$.

(b) *Assume that* $V'(\widehat{x}) = 1$ *for some* $\widehat{x} > 0$ *and there exists* $y < \widehat{x}$ *such that* $\mathcal{U}_y$ *is supersolution of* (3.1) *in* $(y, \widehat{x}]$ *then* $\mathcal{U}_y = V$ *in* $[0, \widehat{x}]$.

PROOF. (a) Let us prove first that $\mathcal{U}_y$ is a supersolution of (3.1). We only need to check it at y. In the case that $\mathcal{U}_y'(y^-) = V'(y^-) > 1 = \mathcal{U}_y'(y^+)$, there is no test for viscosity supersolution at y and in the case that $\mathcal{U}_y'(y) = 1$. Take q such that

$$\begin{aligned} q/2 &\le \liminf_{h \to 0} \frac{(\mathcal{U}_y(y+h) - \mathcal{U}_y(y))/h - 1}{h} \\ &\le \lim_{h \to 0^+} \frac{(\mathcal{U}_y(y+h) - \mathcal{U}_y(y))/h - 1}{h} = 0. \end{aligned}$$

Since $\mathcal{U}_y$ is a supersolution for $x > y$ and $\sup_{\gamma\in[0,1]} \mathcal{L}_\gamma(\mathcal{U}_y, 1, q)(x)$ is right continuous for $x \geq y$ we have that $\sup_{\gamma\in[0,1]} \mathcal{L}_\gamma(\mathcal{U}_y, 1, q)(y) \leq 0$.

From Proposition 5.3 we get that $\mathcal{U}_y \geq V$. Let us prove now that $\mathcal{U}_y(x) \leq V(x)$ for all $x > y$. Given any $\varepsilon > 0$, take an admissible strategy $\pi \in \Pi_y$ such that $V_\pi(y) \geq V(y) - \varepsilon$. For any initial surplus $x \geq y$, we define a new strategy $\pi_x \in \Pi_x$ as follows: pay out immediately the excedent $x - y$ as dividend, and then use the strategy π. Since π is admissible, π_x is also admissible. We get that, for all $x > y$ and $\varepsilon > 0$,

$$\mathcal{U}_y(x) - \varepsilon = x - y + V(y) - \varepsilon \leq x - y + V_\pi(y) = V_{\pi_x}(x) \leq V(x),$$

and so we get the result.

The proof of (b) is analogous to the proof of (a) using Lemma A.3. $\square$

LEMMA A.5. *If $\Lambda(x_0) < 0$, there exists $h_0 > 0$, such that the function $\mathcal{U}_{x_0-h_0}$ defined in (A.1) is a supersolution of (3.1) in $(x_0 - h_0, x_0 + h_0)$.*

PROOF. Since V is locally Lipschitz, for a small $h > 0$ and $x \in (x_0 - h_0, x_0 + h_0)$, there exists $K > 1$ such that $V(x) - V(x_0 - h) \leq K(x - x_0 + h)$. By definition $\mathcal{U}_{x_0-h}(x) = V(x_0 - h) + x - x_0 + h$, and so $V(x) - \mathcal{U}_{x_0-h}(x) \leq (K - 1)(x - x_0 + h)$. Then we obtain that $|\Lambda(x) - \mathcal{L}^*(\mathcal{U}_{x_0-h})(x)| \leq (c + 2\beta)(K - 1)(x - x_0 + h)$. By assumption, $\Lambda(x_0) < 0$, since Λ is continuous for h small enough and $x \in (x_0 - h_0, x_0 + h_0)$ we have that $\Lambda(x) < 0$. Therefore, there exists h_0 small enough such that $\mathcal{L}^*(\mathcal{U}_{x_0-h_0})(x) < 0$ for $x \in (x_0 - h_0, x_0 + h_0)$, and so we have the result. $\square$

LEMMA A.6. *The sets introduced in Definition 8.2 satisfy the following properties*:

(a) *$\mathcal{B}$ is a left-open set*, *that is if $x \in \mathcal{B}$ there exists $\delta > 0$ such that $(x - \delta, x] \subset \mathcal{B}$.*

(b) *$\mathcal{A}$ is a left closed set*, *that is if $x_n \in \mathcal{A}$ and $x_n \searrow x$ then $x \in \mathcal{A}$.*

(c) *If $(x_0, \widehat{x}] \subset \mathcal{B}$ and $x_0 \notin \mathcal{B}$ then $x_0 \in \mathcal{A}$.*

(d) *There is a x^* such that $(x^*, \infty) \subset \mathcal{B}$.*

(e) *$\mathcal{C}$ is an open set in $[0, \infty)$, that is if $0 \in \mathcal{C}$, there exists $\delta > 0$ such that $(0, \delta) \subset \mathcal{C}$ and if a positive $x \in \mathcal{C}$ there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subset \mathcal{C}$.*

(f) *Both $\mathcal{A}$ and $\mathcal{B}$ are nonempty.*

PROOF. (a) Assume that $x_0 \in \mathcal{B}$. By Lemma A.5, we can find $h_0 > 0$, such that the function $U_{x_0-h_0}$ defined in (A.1) is a supersolution of (3.1) in $(x_0 - h_0, x_0]$, and then, by Lemma A.4(b), since $V'(x_0) = 1$, we have $\mathcal{U}_{x_0-h_0} = V$ at $[0, x_0)$ and so $(x_0 - h_0, x_0] \subset \mathcal{B}$.

(b) It follows from the right continuity of the function $\Lambda(x)$ and $V'(x^+)$.

(c) Since $\Lambda(x)$ is continuous and $V'(x^+)$ is right continuous, we have that $V'(x_0^+) = 1$ and $\Lambda(x_0) \leq 0$. But $x_0 \notin \mathcal{B}$, so either $\Lambda(x_0) = 0$ and $V'(x_0^+) = 1$

or $V'(x_0^-) > V'(x_0^+) = 1$ and $\Lambda(x_0) < 0$. In the first case $x_0 \in \mathcal{A}$, let us see that the second case is not possible. Since $\Lambda(x_0) < 0$, by Lemma A.5, we can find $h_0 > 0$, such that the function $U_{x_0-h_0}$ defined in (A.1) is a supersolution of (3.1) in $(x_0 - h_0, x_0 + h_0)$ and $x_0 + h_0 \in \mathcal{B}$. Since $V'(x_0 + h_0) = 1$, we have from Lemma A.4(b) that $\mathcal{U}_{x_0-h_0} = V$ at $[0, x_0 + h_0)$ and so $x_0 \in (x_0 - h_0, x_0 + h_0] \subset \mathcal{B}$; this is a contradiction.

(d) For each $y > 0$ let us consider the functions $\mathcal{U}_y$ defined in (A.1). We will show that, if $y \geq p/(c-r)$, then $\mathcal{U}_y$ is a viscosity supersolution of (3.1) for all $x \in (y, \infty)$, and the result follows from Lemma A.4(a). Since $\mathcal{U}'_y = 1$ in (y, ∞) we only need to show that $\mathcal{L}^*(\mathcal{U}_y) \leq 0$ in (y, ∞). Take any $\gamma \in [0, 1]$, since $\mathcal{U}_y$ is increasing, we have that $\mathcal{L}_\gamma(\mathcal{U}_y)(x) \leq p + (r-c)y$. Hence, the result follows with $x^* = p/(c-r)$.

(e) Take $x \in \mathcal{C}$, if there is no $\delta > 0$ such that $[x, x+\delta) \subset \mathcal{C}$, then we can find a sequence $x_n \in \mathcal{A} \cup \mathcal{B}$ such than $x_n \searrow x$. If there is a subsequence $x_{n_k} \in \mathcal{A}$, then by (b) we get that $x \in \mathcal{A}$, and if a subsequence $x_{n_k} \in \mathcal{B}$, by (c) we can find a sequence $y_k \in \mathcal{A}$ with $x < y_k < x_{n_k}$; then again by (b) we get that $x \in \mathcal{A}$. Take a positive $x \in \mathcal{C}$. If there is no $\delta > 0$ such that $[x - \delta, x) \subset \mathcal{C}$, then we can find a sequence $x_n \in \mathcal{A} \cup \mathcal{B}$ such than $x_n \nearrow x$. Then, $V'(x^-) = 1$ and then $V'(x) = 1$. Then, since $x \in \mathcal{C}$, $\Lambda(x) > 0$ but since Λ is continuous $\Lambda(x) = \lim_{n\to\infty} \Lambda(x_n) \leq 0$, and this is a contradiction.

(f) It follows from (c) and (d). $\square$

Lemmas for Proposition 8.8.

LEMMA A.7. *The optimal value function V is a viscosity solution of $\mathcal{L}^*(V) = 0$ on the open set $\mathcal{C}$.*

PROOF. It follows from (3.1) that V is a viscosity supersolution of $\mathcal{L}^*(V) = 0$. Let us prove that it is a viscosity subsolution of $\mathcal{L}^*(V) = 0$ in $\mathcal{C}$. First consider $x \in \mathcal{C}$ with $1 \leq V'(x^+) < V'(x^-)$. Take any $d \in (V'(x^+), V'(x^-))$; we have that

$$\limsup_{h\to 0} \frac{(V(x+h) - V(x))/h - d}{h} = -\infty$$

and then, for any q,

$$\max\Big\{1 - d, \max_{\gamma\in[0,1]} \big(\sigma^2 x^2 \gamma^2 q/2 + (p + rx\gamma)d - M(V)(x)\big)\Big\} \geq 0,$$

so, since $d > 1$, we have that

$$\max_{\gamma\in[0,1]} \big(\sigma^2 x^2 \gamma^2 q/2 + (p + rx\gamma)d - M(V)(x)\big) \geq 0. \tag{A.2}$$

Since this holds for any q, taking a sequence $q_n \to -\infty$,

$$pd - M(V)(x) \geq 0 \qquad \text{for any } d \in (V'(x^+), V'(x^-))$$

that implies $pV'(x^+) - M(V)(x) \geq 0$, and so (A.2) holds for any $d \in [V'(x^+), V'(x^-)]$ and any q. So V is a viscosity subsolution of $\mathcal{L}^*(V) = 0$ at x.

Next consider $x \in \mathcal{C}$ such that V is differentiable with $1 < V'(x)$. We have $d = V'(x) > 1$, and then

$$\max\left\{1 - d, \max_{\gamma \in [0,1]} \big(\sigma^2 x^2 \gamma^2 q/2 + (p + rx\gamma)d - M(V)(x)\big)\right\} \geq 0$$

implies that

$$\max_{\gamma \in [0,1]} \big(\sigma^2 x^2 \gamma^2 q/2 + (p + rx\gamma)d - M(V)(x)\big) \geq 0$$

and so V is a viscosity subsolution of $\mathcal{L}^*(V) = 0$ at x.

Finally, the case in which $1 = V'(x)$ and $\Lambda(x) > 0$ cannot happen by Lemma 8.3. □

LEMMA A.8. (a) *Given* $x_1 > 0$, *there exists a unique absolutely continuous, increasing viscosity solution of*

$$\mathcal{L}^*(U, V) = 0 \tag{A.3}$$

in (x_1, ∞) *that is differentiable at* x_1, *with boundary conditions* $U(x_1) = V(x_1)$ *and* $U'(x_1) = V'(x_1) = 1$.

(b) *Let* (x_1, x_2) *with* $x_1 > 0$ *be a connected component of* $\mathcal{C}$, *the function* U *defined in* (a) *coincides with* V *in* $[x_1, x_2]$.

(c) *The optimal value function* V *is a classical solution of* $\mathcal{L}^*(V) = 0$ *on the open set* $\mathcal{C}$.

PROOF. (a) The existence of U follows from Proposition 6.4. Let us prove the uniqueness. Given an interval (x_1, y), with arguments similar to the ones used in the proof of Proposition 4.1, it can be proved that, if a supersolution of (8.6) is greater than a subsolution of (8.6) in the boundaries of the interval, it is also greater in the interior. From this result we conclude that, if $\overline{u}$ and $\underline{u}$ are supersolution and subsolution of (8.6) with $\overline{u}(x_1) = \underline{u}(x_1)$, then

$$\max_{x \in [x_1, y]} \{\underline{u}(x) - \overline{u}(x)\} \leq \max\{0, \underline{u}(y) - \overline{u}(y)\}.$$

Let us take $\overline{w}$ and $\underline{w}$ supersolution and subsolution of (8.6), respectively, with $\overline{w}(x_1) = \underline{w}(x_1) = V(x_1)$ and $\overline{w}'(x_1) = \underline{w}'(x_1) = V'(x_1)$, and define $\underline{w}_\varepsilon(x) = \underline{w}(x) + \varepsilon(e^{(c+\beta)/p(x-x_1)} - 1)$. Since $\mathcal{L}_\gamma(e^{(c+\beta)/p(x-x_1)} - 1) \geq 0$, we obtain that $\underline{w}_\varepsilon$ is also a subsolution with $\underline{w}_\varepsilon(x_1) = V(x_1)$ and $\underline{w}_\varepsilon'(x_1) = V'(x_1) + \varepsilon(c + \beta)/p > V'(x_1)$. Then, since $\underline{w}_\varepsilon(x) - \overline{u}(x)$ is positive for $x \in (x_1, x_1 + \delta)$ for some positive δ, we have

$$\max_{x \in [x_1, y]} \{\underline{w}_\varepsilon(x) - \overline{u}(x)\} \leq \max\{0, \underline{w}_\varepsilon(y) - \overline{u}(y)\} = \underline{w}_\varepsilon(y) - \overline{u}(y),$$

so we obtain that $\max_{x\in[x_1,y]}\{\underline{w}_\varepsilon(x)-\overline{u}(x)\}=\underline{w}_\varepsilon(y)-\overline{u}(y)$, and so $\underline{w}_\varepsilon(x)-\overline{w}(x)$ is increasing and positive for all $x>x_1$ and $\varepsilon>0$. Then $\underline{w}(x)\geq\overline{w}(x)$ for all $x>x_1$.

[(b) and (c)] We showed in Lemma A.7 that V is a viscosity solution of $\mathcal{L}^*(V)=0$ on the open set $\mathcal{C}$, so let us show now that V is twice continuously differentiable. In the case $x_1=0$, the result follows from Proposition 8.6, and in the case $x_1>0$, we have that $V'(x_1)=1$, and the result follows from Proposition 6.4(a) and (b). □

Lemma for Theorem 8.11.

LEMMA A.9. *Given $\delta>0$, we can find a finite set $\mathcal{A}_\delta\subset\mathcal{A}_0$ and a number $\varsigma>0$ satisfying*:

(a) $(a-\varsigma,a)\subset\mathcal{C}_0$ *for all* $a\in\mathcal{A}_\delta$.

(b) $\{a\in\mathcal{A}_0 : a-\max(\mathcal{A}_0\cap[0,a))\geq\delta\}\subset\mathcal{A}_\delta$.

(c) $\#\mathcal{A}_\delta\leq 2\widehat{x}/\delta$.

PROOF. Consider $\widehat{\mathcal{A}}=\{a\in\mathcal{A}_0:$ there exists $c_a<a$ with $(c_a,a)\subset\mathcal{C}_0\}$ and

$$\mathcal{D}=\{a\in\mathcal{A}:(a-\vartheta,a)\subset\mathcal{A}\cup\mathcal{B}\text{ for some }\vartheta>0\}\subset\mathcal{B}_0.$$

Let us prove first that if $(x_0,x_1)\cap\mathcal{A}_0\neq\phi$, then $(x_0,x_1)\cap\widehat{\mathcal{A}}\neq\phi$. In the case that $(x_0,x_1)\cap\mathcal{C}_0=\phi$, then $(x_0,x_1)\subset\mathcal{B}_0$ and this is is a contradiction. In the case that $(x_0,x_1)\cap\mathcal{C}_0\neq\phi$, since $\mathcal{C}_0$ is open, there exists $c\in(r_1,r_2)\subset\mathcal{C}_0$ with $r_1,r_2\notin\mathcal{C}_0$; if $r_1\leq x_0<x_1\leq r_2$, we have a contradiction because $(x_0,x_1)\subset\mathcal{C}_0$; if $r_2<x_1$ and $r_2\in\mathcal{A}_0$, we have that $r_2\in\widehat{\mathcal{A}}$; and if $x_0<r_1<x_1\leq r_2$, the interval (x_0,r_1) cannot be included in $\mathcal{B}_0$ because we would have that $(x_0,x_1)\subset\mathcal{C}_0\cup\mathcal{B}_0$, so there exists $c\in\mathcal{C}_0\cap(x_0,r_1)$, take $a=\sup(\mathcal{C}_0\cap(x_0,r_1))$ then $a\in(x_0,x_1)\cap\widehat{\mathcal{A}}$.

Let us prove now that $(\mathcal{A}_0\cup\mathcal{D})\subset(\bigcup_{a\in\widehat{\mathcal{A}}}(c_a,a+\delta))\cup(\bigcup_{d\in\mathcal{D}}(d-\delta,d+\delta))$. In effect, given $a_0\in\mathcal{A}_0\setminus\widehat{\mathcal{A}}$, we have that $(a_0-\delta,a_0)$ is not included in $\mathcal{C}_0$. Then $(a_0-\delta,a_0)\cap\mathcal{A}_0\neq\phi$, because if $(a_0-\delta,a_0)\subset\mathcal{B}_0$ then $a_0\in\mathcal{B}_0$ and if $c\in\mathcal{C}_0\cap(a_0-\delta,a_0)\neq\phi$, the right boundary of the connected component of $\mathcal{C}_0$ containing c belongs to $\mathcal{A}_0$. Hence, $(a_0-\delta,a_0)\cap\mathcal{A}_0\neq\phi$, and then $(a_0-\delta,a_0)\cap\widehat{\mathcal{A}}\neq\phi$. Take $\overline{a}\in(a_0-\delta,a_0)\cap\widehat{\mathcal{A}}$, and we have that $a_0\in(c_{\overline{a}},\overline{a}+\delta)$.

Since $\mathcal{A}_0\cup\mathcal{D}$ is a compact set, we can find finite sets $\mathcal{A}'_\delta\subset\mathcal{A}_0$ and $\mathcal{B}_\delta\subset\mathcal{D}$ such that $(\mathcal{A}_0\cup\mathcal{D})\subset(\bigcup_{a\in\mathcal{A}'_\delta}(c_a,a+\delta))\cup(\bigcup_{d\in\mathcal{B}_\delta}(d-\delta,d+\delta))$. Finally consider the set $\mathcal{A}_\delta$ obtained from $\mathcal{A}'_\delta$ removing some points in such a way that the distance between two consecutive points is larger than $\delta/2$ and adding the set $\{a\in\mathcal{A}_0:a-\max(\mathcal{A}_0\cap[0,a))\geq\delta\}$. Take $\varsigma=\min_{a\in\mathcal{A}_\delta}(a-c_a)$. □

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