The Annals of Applied Probability
2010, Vol. 20, No. 4, 1219–1252
DOI: 10.1214/09-AAP665

CONVERGENCE OF COMPLEX MULTIPLICATIVE CASCADES

BY JULIEN BARRAL, XIONG JIN AND BENOÎT MANDELBROT

Université Paris 13, INRIA and Yale University

The familiar cascade measures are sequences of random positive measures obtained on [0, 1] via b-adic independent cascades. To generalize them, this paper allows the random weights invoked in the cascades to take real or complex values. This yields sequences of random functions whose possible strong or weak limits are natural candidates for modeling multifractal phenomena. Their asymptotic behavior is investigated, yielding a sufficient condition for almost sure uniform convergence to nontrivial statistically self-similar limits. Is the limit function a monofractal function in multifractal time? General sufficient conditions are given under which such is the case, as well as examples for which no natural time change can be used. In most cases when the sufficient condition for convergence does not hold, we show that either the limit is 0 or the sequence diverges almost surely. In the later case, a functional central limit theorem holds, under some conditions. It provides a natural normalization making the sequence converge in law to a standard Brownian motion in multifractal time.

1. Introduction.

1.1. *Foreword about Hölder singularity and multifractal functions.* Multifractal analysis is a natural framework to describe statistically and geometrically the heterogeneity in the distribution at small scales of the Hölder singularities of a given locally bounded function or signal $F : I \mapsto \mathbb{R}$ (or $\mathbb{C}$) where I is a bounded interval. One possible way to define the Hölder singularity at a given point t is by measuring the asymptotic behavior of the oscillation of f around t thanks to the exponent

$$h_F(t) = \liminf_{r\to 0^+} \frac{\log \mathrm{Osc}_F([t-r, t+r])}{\log(r)}$$

or

$$h_F(t) = \liminf_{n\to\infty} \frac{\log_2 \mathrm{Osc}_F(I_n(t))}{-n},$$

Received February 2009; revised November 2009.

AMS 2000 subject classifications. Primary 60F05, 60F15, 60F17, 60G18, 60G42, 60G44; secondary 28A78.

Key words and phrases. Multiplicative cascades, continuous function-valued martingales, functional central limit theorem, laws stable under random weighted mean, multifractals.

where $I_n(t)$ is the dyadic interval of length 2^{-n} containing t and $\mathrm{Osc}_F(J) = \sup_{s,t\in J} |F(t) - F(s)|$. Then the multifractal analysis of F consists in classifying the points with respect to their Hölder singularity. The singularity spectrum of F is the Hausdorff dimension of the Hölder singularities level sets

$$E_F(h) = \{t \in I : h_F(t) = h\}, \qquad h \geq 0.$$

One says that F is monofractal if there exists a unique $h \geq 0$ such that $E_F(h) \neq \varnothing$. Otherwise F is multifractal. Alternatively, one can also compute free energy functions like

$$\tau_F(q) = \liminf_{n\to\infty} \frac{-1}{n} \log_2 \sum_{I\in\mathcal{G}_n, \mathrm{Osc}_F(I)\neq 0} \mathrm{Osc}_F(I)^q \tag{1.1}$$

and say that F is monofractal if τ_F is linear. Let us mention that one always has $\dim E_F(h) \leq \tau_F^*(h) = \inf_{q\in\mathbb{R}} hq - \tau_F(q)$, and one says that the multifractal formalism holds at h if these inequalities are equalities (see [34, 56] for instance). Also, one says that F is monofractal in the strong sense if $\lim_{r\to 0^+} \frac{\log \mathrm{Osc}_F([t-r,t+r])}{\log(r)}$ exists everywhere and is independent of t. Examples of functions that satisfy this property are the Weierstrass functions, Brownian motion and self-affine functions [32, 60].

When F is a continuous multifractal function possessing some scaling invariance property, it may happen that it possesses the remarkable property to be decomposable as a (often strongly) monofractal function B and an increasing multifractal time change G such that $F = B \circ G$. In the known cases, there exists $\beta > 1$ such that $\tau_F(\beta) = 0$, and the function G is such that, roughly speaking, $|G(I)| \approx \mathrm{Osc}_F(I)^\beta$ for I small enough in $\bigcup_{n\geq 1} \mathcal{G}_n$, so that $\tau_G(q) = \tau_F(\beta q)$ ([50, 58]).

1.2. *Some methods that build multifractal processes.* Our goal is to build new multifractal stochastic processes. Since [47–49], one of the main approaches is to construct singular statistically self-similar measures μ on $[0, 1]$ (or $\mathbb{R}_+$). These measures are obtained as almost sure weak limits of absolutely continuous measure-valued martingales $(\mu_n)_{n\geq 1}$ whose densities $(Q_n)_{n\geq 1}$ are martingales generated by multiplicative processes (see [3, 11, 12, 48, 49, 52]). These objects have been used to construct nonmonotonic multifractal stochastic processes as follows: (a) by starting with fractional Brownian motions or stable Lévy processes X, and an independent measure μ, and performing the multifractal time change $X(\mu([0, t]))$ [3, 15, 22, 50, 56]; (b) by integrating the density Q_n with respect to the Brownian motion and letting n tend to ∞ [3, 22]; (c) by using μ to specify the covariance of some Gaussian processes [46]; (d) by considering random wavelet series whose coefficients are built from the multifractal measure μ [2, 14]. Such processes are geared to modeling signals possessing a wildly varying local Hölder regularity as well as scaling invariance properties. Many such signals come from physical or social intermittent phenomena like turbulence [28, 49], spatial rainfall [31], human heart rate [53, 59], Internet traffic [29, 57] and stock exchanges prices [4, 50].

As background, standard statistically self-similar measures consist in the 1-dimensional b-adic canonical cascades constructed on the interval [0, 1] as follows. Fix an integer $b \geq 2$. The b-adic closed subintervals of [0, 1] are naturally encoded by the nodes of the tree $\mathscr{A}^* = \bigcup_{n\geq 0}\{0, \dots, b-1\}^n$ with the convention that $\{0, \dots, b-1\}^0$ contains the root of $\mathscr{A}^*$ denoted $\varnothing$. To each element w of $\mathscr{A}^*$, we associate a nonnegative random weight $W(w)$, these weights being independent and identically distributed with a random variable W such that $\mathbb{E}(W) = 1$. A sequence of random densities $(Q_n)_{n\geq 1}$ is then obtained as follows: Let I be the semi-open to the right b-adic interval encoded by the node $w = w_1 w_2 \cdots w_n$, that is, $I = [\sum_{k=1}^{n} w_k b^{-k}, b^{-n} + \sum_{k=1}^{n} w_k b^{-k}]$. Then

$$Q_n(t) = W(w_1)W(w_1 w_2)\cdots W(w_1 w_2 \cdots w_n) \qquad \text{for } t \in I.$$

Consider the sequence of measures $(\mu_n)_{n\geq 1}$ whose densities with respect to the Lebesgue measure are given by $(Q_n)_{n\geq 1}$. This is the measure-valued martingale (with respect to the natural filtration it generates) introduced in [48, 49].

The study of these martingales and their limits has led to numerous mathematical developments in the theories of probability and geometric measure [1, 5, 6, 16, 23, 25–27, 30, 33, 35–39, 43–45, 54, 55, 61, 62]. These objects, as well as the other statistically self-similar measures mentioned above, are special examples of a general model of positive measure-valued martingale, namely the "T-martingales" developed in [35, 36], which make rigorous the construction and results of the seminal work [47] on log-normal multiplicative chaos.

A natural alternative to the preceding constructions consists of allowing the [0, 1]-martingales $(Q_n)_{n\geq 1}$ to take real or complex values and consider the continuous functions-valued martingale

$$\left(F_n : t \in [0, 1] \mapsto \int_0^t Q_n(s)\,\mathrm{d}s\right)_{n\geq 1}.$$

The companion paper [10] exhibits a class of such [0, 1]-martingales for which we can prove a general uniform convergence theorem to a nontrivial random function. As a consequence, we construct natural extensions of random functions of the now familiar statistically self-similar measures, namely canonical b-adic cascades [48, 49], compound Poisson cascades [8, 11] and infinitely divisible cascades [3, 22].

1.3. *Further results for complex b-adic cascades.* This paper deepens the study of the convergence properties of the complex extension of 1-dimensional canonical b-adic cascades. In particular, we improve the convergence result obtained in [10], and proceed beyond the results obtained in [13] in the special case where W is real-valued and constant in absolute value. In this case, two possibilities exist as n tends to ∞. $(F_n)_{n\geq 1}$ may converge almost surely uniformly to a monofractal process sharing some fractal properties with a fractional Brownian motion of exponent $H \in (1/2, 1)$. If not, F_n is not bounded and the following new functional central limit theorem holds: as n tends to ∞, $F_n/\sqrt{\mathbb{E}(F_n(1)^2)}$ converges in law to the restriction to $[0, 1]$ of the standard Brownian motion. This last result raises a question. Does $F_n/\sqrt{\mathbb{E}(F_n(1)^2)}$ have a weak limit in the general case when $F_n(1)$ is not bounded in L^2 norm? In case of weak convergence, it remains to describe the nature of the limit multifractal process and compare it with other models of random multifractal functions.

On the asymptotic behavior of $(F_n)_{n\geq 1}$, our main results are of the following nature. We obtain a condition on the moments of W that suffices for the uniform convergence of F_n—almost surely and in L^p norm ($p > 1$)—to a nontrivial limit (Theorem 2.1). When this sufficient condition does not hold, we show that—in most of the cases—either F_n converges uniformly to 0, or F_n diverges almost surely in $\mathcal{C}([0, 1])$, namely in the space of complex-valued continuous functions over $[0, 1]$ (Theorems 2.5 and 2.6); it is worth mentioning that the different possible behaviors correspond to the three phases occurring in the mean field theory of directed polymers with random weights in which the free energy is built from a b-adic canonical complex cascade [24] [see Remark 2.3(4)].

Functional central limit theorems (Theorems 2.7 and 2.8) add to the previous results on the almost sure behavior of $(F_n)_{n\geq 1}$. Let W be real valued, and $(F_n)_{n\geq 1}$ converge uniformly almost surely and in L^2 norm to a function F. Then, under weak additional assumptions on the moments of W we prove that $(F_n - F)/\sqrt{\mathbb{E}((F_n - F)(1)^2)}$ converges in law as n tends to ∞. Let F_n be not bounded in L^2. Then, under strong assumptions on the moments of W we prove that $F_n/\sqrt{\mathbb{E}(F_n(1)^2)}$ converges in law as n tends to ∞.

It is remarkable that in both cases the weak limit is standard [50]: it is Brownian motion B in multifractal time $B \circ \widetilde{F}$ where $\widetilde{F}$ is independent of B and is the limit of a positive canonical b-adic cascade. Thus the limit process is one of the processes mentioned above as built from statistically self-similar measures. Among nontrivial functional central limit theorems, these results seem to be the first to exhibit this kind of limit process. They reinforce the importance of multifractal subordination in a monofractal process as a natural operation.

The previous functional central limit theorems raise the following question. Let $(F_n)_{n\geq 1}$ strongly converge to a nontrivial limit F. Can one—as it is the case for some other classes of multifractal functions [3, 50, 51, 58]—decompose F as $\widetilde{B} \circ \widetilde{F}$, where $\widetilde{B}$ is a monofractal process and $\widetilde{F}$ the indefinite integral of a statistically self-similar measure? It turns out that there is a unique natural candidate $\widetilde{F}$ as time change: There exists $\beta > 1$ such that $\mathbb{E}(|W|^\beta) = 1$ and $\widetilde{F}$ is the b-adic canonical cascade generated by the martingale $(|Q_n|^\beta)_{n\geq 1}$. We give sufficient conditions on the moments of W for $F \circ \widetilde{F}^{-1}$ to be indeed a monofractal (in the strong sense specified in Section 1.1) stochastic process of exponent $1/\beta$ (Theorem 2.3). We do not know whether or not they are necessary.

We also consider a more general model of signed multiplicative cascades, namely the signed extension of the most general construction considered in [49]: We only assume that all the vectors of the form $(W(w0), \dots, W(w(b-1)))$ are independent and identically distributed, and $\mathbb{E}(\sum_{i=0}^{b-1} W(i)/b) = 1$. We call this class b-adic independent cascades; such a cascade can be described as a [0, 1]-martingale only if $\mathbb{E}(W(i)) > 0$ for all i (see Remark 2.3 in [10]). All our previous results have an extension to this class. Moreover, one important additional case appears in this class with respect to canonical cascades. It consists of the conservative cascades for which $\sum_{i=0}^{b-1} W(i)/b = 1$ almost surely. We can build examples of such conservative cascades whose limit cannot be naturally decomposed as a monofractal function in a multifractal time [see Theorems 2.2(2) and 2.4 and Remark 2.2(5)].

1.4. *Organization of the paper.* The b-adic independent cascades are constructed in Section 2.1. The results regarding the uniform convergence and the representation of the limit as a monofractal function in multifractal time are given in Section 2.2 while results on degeneracy and divergence as well as central limit theorems are given in Section 2.4. Section 2.5 provides a statement on the multifractal nature of the limit whenever it exists (proof and extensions are given in [9]) as well as the connection of the statistically self-similar processes constructed in this paper with random variables and processes stable under random weighted mean. Sections 3 and 4 are dedicated to the proofs.

1.5. *Definitions.* Given an integer $b \geq 2$, we denote by $\mathscr{A}$ the alphabet $\{0, \dots, b-1\}$ and we define $\mathscr{A}^* = \bigcup_{n \geq 0} \mathscr{A}^n$ (by convention $\mathscr{A}^0$ is the set reduced to the empty word denoted $\varnothing$). The word obtained by concatenation of u and v in $\mathscr{A}^*$ is denoted $u \cdot v$ and sometimes uv. For every $n \geq 0$, the length of an element of $\mathscr{A}^n$ is by definition equal to n and will be denoted by $|w|$. Let $n \geq 1$ and $w = w_1 \cdots w_n \in \mathscr{A}^n$. Then for every $1 \leq k \leq n$, the word $w_1 \cdots w_k$ is denoted $w|k$; if $k = 0$, then $w|0$ stands for $\varnothing$.

For $w \in \mathscr{A}^*$, define $t_w = \sum_{i=1}^{|w|} w_i b^{-i}$ and $I_w = [t_w, t_w + b^{-|w|}]$.

If $f \in \mathcal{C}([0, 1])$ and I is a subinterval of [0, 1], $\Delta f(I)$ denotes by the increment of f over I. Also, $\|f\|_\infty$ denotes the norm $\sup_{t \in [0,1]} |f(t)|$.

Denote by $(\Omega, \mathcal{B}, \mathbb{P})$ the probability space on which the random variables considered in this paper are defined. Write $U \equiv V$ to express that the random variables U and V have the same probability distribution. The probability distribution of a random variable V is denoted by $\mathcal{L}(V)$.

2. Construction of *b*-adic independent cascades and main results.

2.1. *Construction.* Let $W = (W_0, \dots, W_{b-1})$ be a complex vector whose components are integrable and satisfy $\mathbb{E}(\sum_{i=0}^{b-1} W_i) = 1$ (we have modified the normalization with respect to the discussion of Section 1). Then let $(W(w))_{w \in \mathscr{A}^*}$

be a sequence of independent copies of W, and consider the sequence of random functions

$$F_n(t) = F_{W,n}(t) = \int_0^t b^n \prod_{k=1}^{n} W_{u_k}(u|k-1)\,\mathrm{d}u \tag{2.1}$$

(where each non b-adic point u is identified with the word $u_1 \cdots u_n \cdots$ defined by the b-adic expansion $u = \sum_{k\geq 1} u_k b^{-k}$, and we recall that $u|k-1 = u_1 \cdots u_{k-1}$). A special case playing an important role in the sequel is the conservative one, that is, when $\sum_{i=0}^{b-1} W_i = 1$ almost surely. Remarkable functions obtained as the limit of such deterministic sequences F_n are the self-affine functions considered, for instance, in [17, 40, 60] (these functions are called self-affine because their graphs are self-affine sets).

For $p \in \mathbb{R}_+$ let

$$\varphi_W(p) = -\log_b \mathbb{E}\left(\sum_{i=0}^{b-1} |W_i|^p\right). \tag{2.2}$$

Our assumptions imply $-1 \leq \varphi_W(0) \wedge \varphi_W(1) \leq \varphi_W(0) \vee \varphi_W(1) \leq 0$.

2.2. *Strong uniform convergence.* We give sufficient conditions for the almost sure uniform convergence of $(F_n)_{n\geq 1}$. The asymptotic behavior of $(F_n)_{n\geq 1}$ when these conditions do not hold will be examined in Section 2.4.

We distinguish the conservative and nonconservative cases. Our results are illustrated in Figures 1 to 3.

THEOREM 2.1 (Nonconservative case). *Suppose that* $\mathbb{P}(\sum_{i=0}^{b-1} W_i \neq 1) > 0$ *and there exists* $p > 1$ *such that* $\varphi_W(p) > 0$. *Suppose, moreover, that either* $p \in (1, 2]$ *or* $\varphi_W(2) > 0$.

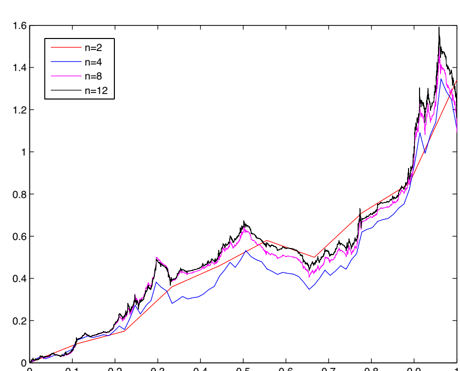

FIG. 1. *Uniform convergence in the nonconservative case*: $F_{W,n}$ *for* $n = 2, 4, 8, 12$ *in the case* $b = 3$ *and* $\varphi_W(\beta) = 0$ *for* $\beta \approx 1.395$.

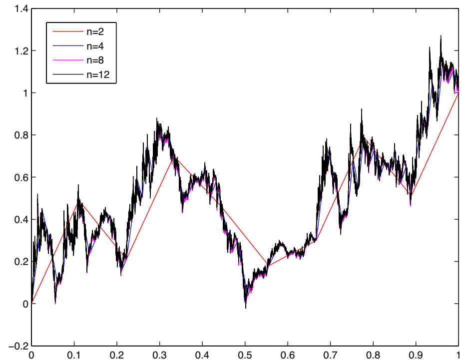

FIG. 2. *Uniform convergence in the noncritical conservative case*: $F_{W,n}$ *for* $n = 2, 4, 8, 12$ *in the case* $b = 3$ *and* $\varphi_W(\beta) = 0$ *for* $\beta \approx 2.172$.

(1) $(F_n)_{n\geq 1}$ *converges uniformly, almost surely and in* L^p *norm, as* n *tends to* ∞, *to a function* $F = F_W$, *which is nondecreasing if* $W \geq 0$. *Moreover, the function* F *is* γ*-Hölder continuous for all* γ *belonging to* $(0, \max_{q\in(1,p]} \varphi_W(q)/q)$.

(2) F *satisfies the statistical scaling invariance property*

$$F = \sum_{i=0}^{b-1} \mathbf{1}_{[i/b,(i+1)/b]}\big(F(i/b) + W_i F_i \circ S_i^{-1}\big), \tag{2.3}$$

where $S_i(t) = (t + i)/b$, *the random objects* $W, F_0, \ldots, F_{b-1}$ *are independent, and the* F_i *are distributed like* F *and the equality holds almost surely.*

THEOREM 2.2 (Conservative case). *Suppose that* $\mathbb{P}(\sum_{i=0}^{b-1} W_i = 1) = 1$.

(1) *If there exists* $p > 1$ *such that* $\varphi_W(p) > 0$, *then the same conclusions as in Theorem* 2.1 *hold.*

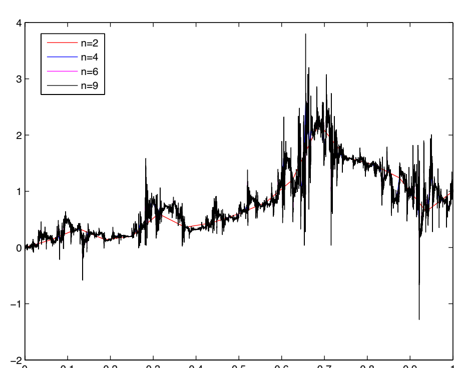

FIG. 3. *Uniform convergence in the critical conservative case*: $F_{W,n}$ *for* $n = 2, 4, 6, 9$ *in the case* $b = 4$ *and* $\varphi_W(p) < 0$ *on* $\mathbb{R}_+$, $\varphi_W(p) \to 0$ *when* $p \to \infty$. *The limit is not uniformly Hölder.*

(2) Critical case. *Suppose that* $\lim_{p\to\infty}\varphi_W(p)=0$ [*in particular* φ_W *is increasing and* $\varphi_W(p)<0$ *for all* $p>1$]. *This is equivalent to the fact that* $\mathbb{P}(\forall 0\le i\le b-1, |W_i|\le 1)=1$ *and* $\sum_{i=0}^{b-1}\mathbb{P}(|W_i|=1)=1$.

Suppose also that $\mathbb{P}(\#\{i:|W_i|=1\}=1)<1$, *and there exists* $\gamma\in(0,1)$ *such that*, *with probability* 1, *one of the two following properties holds for each* $0\le i\le b-1$:

$$\begin{cases} \textit{either } |W_i|\le\gamma, \\ \textit{or } |W_i|=1 \textit{ and } (\sum_{k=0}^{i-1}W_i,\sum_{k=0}^{i}W_i)\in\{(0,1),(1,0)\}. \end{cases} \tag{2.4}$$

Then, *with probability* 1, $(F_n)_{n\ge1}$ *converges almost surely uniformly to a limit* $F=F_W$ *which is not uniformly Hölder and satisfies part* (2) *of Theorem* 2.1.

REMARK 2.1. (1) The sufficient condition for the convergence in L^p of complex-valued martingales like $(F_n(1))_{n\ge1}$ is known in the context of martingales in the branching random walk ([7, 18]); however, the sequence of functions $(F_n)_{n\ge1}$ is not considered in these papers. When W has nonnegative components, it follows from [25] and [39] that this condition is necessary.

(2) Theorem 2.2 goes beyond the construction of deterministic self-affine functions ([17, 60]) which all fall in Theorem 2.2.

(3) The following discussion will be useful for the statement and proof of Theorem 2.5. It is easily seen that F_n vanishes ($F_n=0$) if and only if $\prod_{k=1}^n W_{w_k}(w|k-1)=0$ for all $w\in\mathscr{A}^n$, and in this case, $F_k=0$ for all $k>n$. Thus, if we denote by $\mathscr{V}$ the event $\{\exists n\ge1: F_n=0\}$, we have $\mathscr{V}=\liminf_{n\to\infty}\{F_n=0\}$. Notice that $\mathbb{P}(\mathscr{V})=0$ in the conservative case.

By construction, there are b independent copies $(F_{i,n})_{n\ge1}$ of $(F_n)_{n\ge1}$, independent of W, and converging respectively to F_i almost surely, such that for $n\ge1$ we can write

$$F_n=\sum_{i=0}^{b-1}\mathbf{1}_{[i/b,(i+1)/b]}\big(F_n(i/b)+W_iF_{i,n-1}\circ S_i^{-1}\big).$$

Thus, $\{F_n=0\}=\bigcap_i\{W_iF_{i,n-1}=0\}$. Similarly, $\{F=0\}=\bigcap_i\{W_iF_i=0\}$. It is then not difficult to see that under the assumptions of Theorem 2.1, $\mathbb{P}(\mathscr{V})$ and $\mathbb{P}(F=0)$ are equal to the unique fixed point distinct from 1 of the convex polynomial

$$P(x)=\sum_{k=0}^{b}\mathbb{P}(\#\{0\le i\le b-1:|W_i|>0\}=k)x^k.$$

Consequently, since $\mathscr{V}\subset\{F=0\}$, these events differ from a set of null probability.

We can also interpret the set of words $w\in\mathscr{A}^n$ such that $\prod_{k=1}^n W_{w_k}(w|k-1)=0$ as the nodes of generation n of a Galton–Watson tree whose offspring distribution is given by that of the integer $N=\#\{0\le i\le b-1:|W_i|>0\}$. Then $P(x)=\mathbb{E}(x^N)$.

We end this section by introducing auxiliary nonnegative b-adic independent cascades which will play an important role in the rest of the paper.

DEFINITION 2.1. If $\beta > 0$ and $\varphi_W(\beta) > -\infty$, then for $w \in \mathscr{A}^*$ let

$$W^{(\beta)}(w) = b^{\varphi_W(\beta)}(|W_0(w)|^\beta, \dots, |W_{b-1}(w)|^\beta),$$

and simply denote $W^{(\beta)}(\varnothing)$ by $W^{(\beta)}$. We have $\varphi_{W^{(\beta)}}(p) = \varphi_W(\beta p) - p\varphi_W(\beta)$ for all $p > 0$. In particular, $\varphi_{W^{(\beta)}}(1) = 0$. If $\varphi_{W^{(\beta)}}(p) > 0$ for some $p \in (1, 2)$, we denote by $F_{W^{(\beta)}}$ the nondecreasing function obtained in Theorem 2.1 as the almost sure uniform limit of $F_{W^{(\beta)},n} : t \in [0, 1] \mapsto \int_0^t b^n \prod_{k=1}^n W_{u_k}^{(\beta)}(u|k-1)\,\mathrm{d}u$.

2.3. *Representation as a monofractal function in multifractal time.* As explained in the introduction, in order to qualitatively compare the strong limit F_W obtained in Theorem 2.1 with other models of multifractal processes, it is important to study the possibility to decompose it as a monofractal function in multifractal time. Under the assumptions of Theorems 2.1 and 2.2(1), if we denote by β the smallest solution of $\varphi_W(p) = 0$, the only natural choice at our disposal as time change is the function $F_{W^{(\beta)}}$ introduced in Definition 2.1. In the deterministic case, it is elementary to check that $B_{1/\beta} = F_W \circ F_{W^{(\beta)}}^{-1}$ is monofractal (Section 4.7 in [50]) in the strong sense. In the random case, this also true under strong assumptions on the moments of W as shows Theorem 2.3 which is illustrated in Figures 4 and 5. We do not know whether or not weaker assumptions on W lead to situations in which $B_{1/\beta}$ is not monofractal. However, the functions constructed in Theorem 2.2(2) provide simple examples of statistically self-similar continuous functions for which it seems to be impossible to find a natural decomposition as monofractal functions in multifractal time; at least such a time change cannot be obtained as limit of a positive b-adic independent cascade.

THEOREM 2.3. *Suppose that* $\mathbb{P}(W \in \mathbb{C}^b \setminus \mathbb{R}_+^b) > 0$ *and* $\sum_{i=0}^{b-1} \mathbb{E}(|W_i|^p) < \infty$ *for all* $p \in \mathbb{R}$.

Suppose also that the assumptions of Theorems 2.1 *or* 2.2(1) *hold. Let* β *be the smallest solution of* $\varphi_W(p) = 0$, *and suppose that* $\varphi_W(p) > 0$ *for all* $p > \beta$. *We have* $\beta > 1$, $\varphi_{W^{(\beta)}}(1) = 0$ *and* $\varphi_{W^{(\beta)}}(p) > 0$ *for all* $p > 1$. *Let* $B_{1/\beta} = F_W \circ F_{W^{(\beta)}}^{-1}$.

With probability 1, *the function* $B_{1/\beta}$ *is a monofractal function in the strong sense, with constant pointwise Hölder exponent* $1/\beta$.

THEOREM 2.4. *Suppose that the assumptions of Theorem* 2.2(2) *hold and the components of* W *do not vanish. With probability* 1, *if the function* F_W *can be decomposed as a monofractal function* B *in a multifractal time* G, *then the Hölder exponent of* B *is* 0, *and* G *must be such that* $\dim([0, 1] \setminus \overline{E}_G(\infty)) = 0$, *where*

$$\overline{E}_G(\infty) = \left\{ t \in [0, 1] : \limsup_{r \to 0^+} \frac{\log(\mathrm{Osc}_G([t-r, t+r]))}{\log(r)} = \infty \right\}.$$

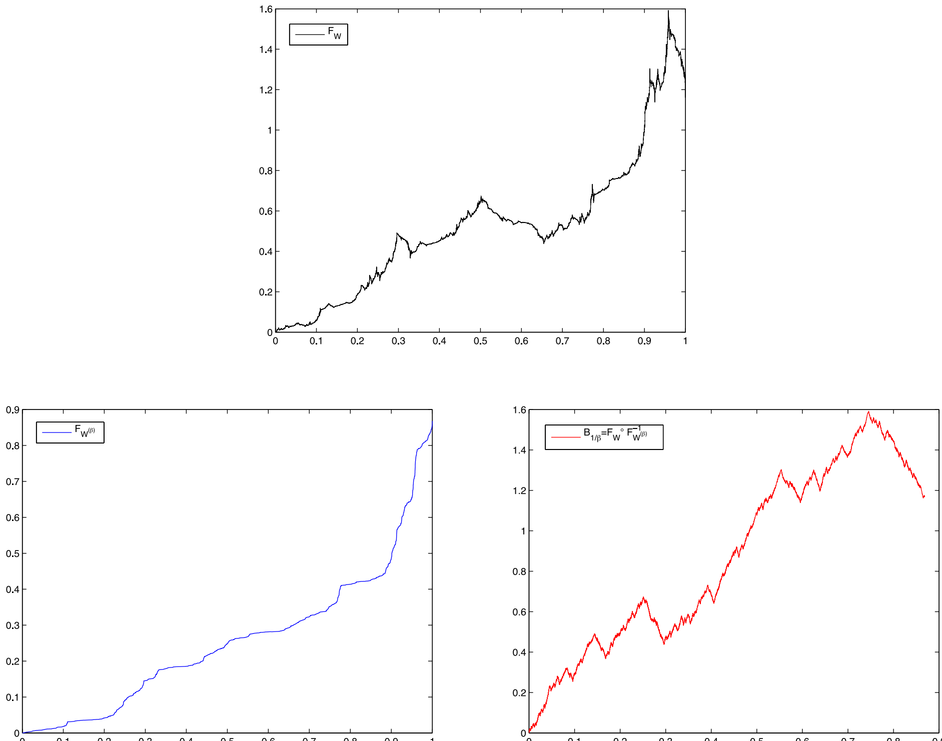

FIG. 4. *The limit function F_W corresponding to the construction of Figure* 1 (*top*) *can be written as the monofractal function* $B_{1/\beta} = F_W \circ F_{W^{(\beta)}}^{-1}$ *of exponent* $H = 1/\beta \approx 0.7168$ (*right-bottom*) *in multifractal time* $F_{W^{(\beta)}}$ (*left-bottom*).

REMARK 2.2. (1) Notice that due to Theorem 2.1, $1/\beta$ must belong to $(1/2, 1)$ when $\mathbb{P}(\sum_{k=0}^{b-1} W_k = 1) < 1$.

(2) When W is deterministic, we are necessarily in the conservative case $\sum_{i=0}^{b-1} W_i = 1$, and Theorem 2.3 is well known (see, for instance, Section 4.7 in [50]). In this case, it is also the simplest illustration of the general result obtained in [58] regarding the representation of multifractal functions as monofractal functions (in the strong sense) in multifractal time (see also [51] for another illustration of this concept).

(3) Under the assumptions of Theorem 2.3, it seems possible to obtain the result by using the general approach developed in [58]. However, this necessitates the use of the extension to the present context of some sophisticated estimates developed for positive cascades in [16]. Thus we will give a short and self-contained proof.

(4) The moments of $\frac{\mathrm{Osc}_{F_W}([0,1])}{F_{W^{(\beta)}}(1)^{1/\beta}}$ are all finite under the strong assumptions of Theorem 2.3. Suppose that we have found sufficient conditions under which there

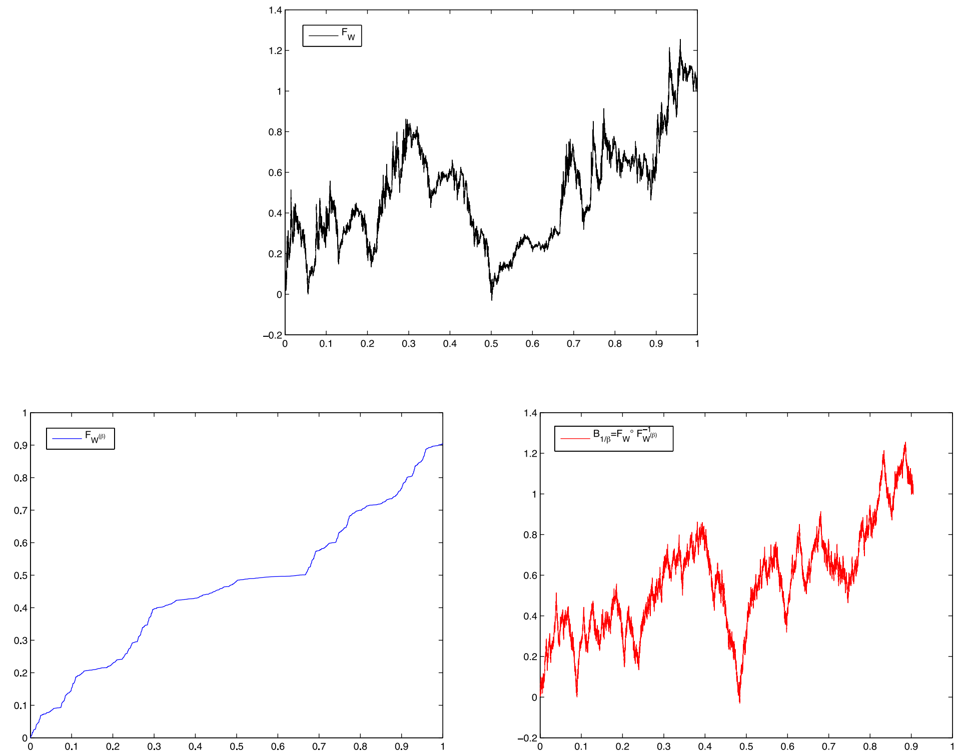

FIG. 5. *The limit function F_W corresponding to the construction of Figure 2 (top) can be written as the monofractal function $B_{1/\beta} = F_W \circ F^{-1}_{W(\beta)}$ of exponent $H = 1/\beta \approx 0.4604$ (right-bottom) in multifractal time $F_{W(\beta)}$ (left-bottom).*

exists $q \neq 0$ such that $\mathbb{E}[(\frac{\mathrm{Osc}_{F_W}([0,1])}{F_{W(\beta)}(1)^{1/\beta}})^q] = \infty$. Then we know how to prove that the function $B_{1/\beta}$ is not monofractal if $q > 0$ and not strongly monofractal if $q < 0$. Thus finding information on the $\frac{\mathrm{Osc}_{F_W}([0,1])}{F_{W(\beta)}(1)^{1/\beta}}$ moments behavior under weak assumptions on W remains an important open question to complete the description of $B_{1/\beta}$.

(5) As a consequence of Theorem 2.4, we get that the time change G cannot be equivalent to the limit of a positive b-adic independent cascade obtained as in Theorems 2.1 or 2.2, in the sense that their derivative in the distribution sense are equivalent positive measures. Indeed, in this case the analysis of such a measure achieved in [5] or [39], implies that there exists $D > 0$ such that $\lim_{n\to 0^+} \frac{\log_b(\mathrm{Osc}_G(I_n(t)))}{-n} = D$ at each point t of a set E of Hausdorff dimension D.

Also, we can notice that if a time change G as described in Theorem 2.4 does exist, from the multifractal analysis point of view, the decomposition would not simplify the study of F_W since it would necessitate having a very fine description of the pointwise divergence of $\frac{\log(\mathrm{Osc}_G([t-r,t+r]))}{\log(r)}$ inside the set $\overline{E}_G(\infty)$.

An example of increasing function G such that $\dim([0,1]\setminus \overline{E}_G(\infty))=0$ is obtained as follows:

$$G(t)=\int_0^t \sum_{n\geq 1}\sum_{0\leq k<b^n} b^{-n^2}\big(b^{n^3}\mathbf{1}_{[kb^{-n},kb^{-n}+b^{-n^3}]}(u)\big)\,\mathrm{d}u.$$

We leave the reader check that t is not a b-adic number and if

$$\delta_t=\limsup_{n\to\infty}\sup_{0\leq k<b^n}\frac{\log(|t-kb^{-n}|)}{-n}<\infty,$$

then $\lim_{r\to 0^+}\frac{\log(\mathrm{Osc}_G([t-r,t+r]))}{\log(r)}=\infty$. Moreover, it is clear that $\dim\{t:\delta_t=\infty\}=0$.

2.4. *Degeneracy, divergence and weak uniform convergence.* Recall that φ_W is concave, $\varphi_W(0)<0$ and $\varphi_W(1)\leq 0$. Let us define

$$p_0=\sup\{p:\varphi_W'(p)\text{ exists and }\varphi_W'(p)p-\varphi_W(p)>0\}. \tag{2.5}$$

In order to simplify the next discussion, we assume that $\varphi_W(p)>-\infty$ for all $p\geq 0$.

Since $\varphi_W(0)<0$, we have $p_0>0$. Also, if $p_0<\infty$ then $\varphi_W(p_0)=0$ if and only if $\varphi_W(p_0)=\varphi_W'(p_0)=0$. If $p_0=\infty$, we define $\varphi_W(p_0)=\lim_{p\to\infty}\varphi_W(p)$.

In the previous section, we have dealt with the convergence of F_n in the following cases that we gather in the condition (C):

(C): One of the following three cases arises:

(1) There exists $p\in(1,2]$ such that $\varphi_W(p)>0$;

(2) $\mathbb{P}(\sum_{k=0}^{b-1}W_k=1)=1$ and there exists $p>1$ such that $\varphi_W(p)>0$;

(3) $\mathbb{P}(\sum_{k=0}^{b-1}W_k=1)=1$, $\mathbb{P}(\forall 0\leq i\leq b-1,\ |W_i|\leq 1)=1$, $\sum_{i=0}^{b-1}\mathbb{P}(|W_i|=1)=1$ and $\mathbb{P}(\#\{i:|W_i|=1\}=1)<1$. Equivalently, $\lim_{p\to\infty}\varphi_W(p)=0$ and $\mathbb{P}(\#\{i:|W_i|=1\}=1)<1$.

Suppose that (C) does not hold. We cannot have simultaneously $p_0\in(1,2]$ and $\varphi_W(p_0)>0$. Also, $\varphi_W(2)\leq 0$. Moreover, if $\mathbb{P}(\sum_{k=0}^{b-1}W_k=1)=1$, then $\varphi_W(p_0)\leq 0$, and if $p_0=\infty$, then $\sum_{i=0}^{b-1}\mathbb{P}(|W_i|=1)>1$ or $\mathbb{P}(\#\{i:|W_i|=1\}=1)=1$ (see also Remark 2.3).

The following results concern the asymptotic behavior of F_n when (C) does not hold. Before stating it, we recall the discussion of Remark 2.1(3).

THEOREM 2.5 (Degeneracy and divergence). *Suppose that* (C) *does not hold and $\varphi_W(p)>-\infty$ for all $p\geq 0$.*

(1) *Suppose that $p_0\in(0,1]$. Then, for all $\alpha\leq\varphi_W(p_0)/p_0$, $b^{n\alpha}F_n$ converges almost surely uniformly to* 0, *and for all $\alpha>\varphi_W(p_0)/p_0$, $b^{n\alpha}F_n$ is unbounded almost surely, conditionally on $\mathscr{V}^c$.*

(2) *Suppose that* $p_0 \in (1,2]$. *We have* $\varphi_W(p_0) \le 0$. *Then, for all* $\alpha > \varphi_W(p_0)/p_0$, $b^{n\alpha} F_n$ *is unbounded almost surely, conditionally on* $\mathscr{V}^c$. *In particular, if* $\varphi_W(p_0) < 0$, *then* F_n *is unbounded almost surely, conditionally on* $\mathscr{V}^c$.

(3) *Suppose that* $p_0 > 2$ *and* $\mathbb{P}(\sum_{k=0}^{b-1} W_k = 1) = 1$. *We have* $\varphi_W(p_0) \le 0$. *If* $p_0 < \infty$, *the same conclusions as in* (2) *hold. If* $p_0 = \infty$, *then* $(F_n)_{n\ge1}$ *diverges in* $\mathcal{C}([0,1])$ *almost surely.*

Moreover, in both cases there is no sequence $(r_n)_{n\ge1}$ *tending to* 0 *or* ∞, *as* $n \to \infty$, *such that* $r_n F_n$ *converges in law to a nontrivial limit in* $\mathcal{C}([0,1])$.

(4) *Suppose that* $p_0 > 2$ *and* $\mathbb{P}(\sum_{k=0}^{b-1} W_k \ne 1) > 0$. *For* $n \ge 1$, *let* $r_n = b^{n\varphi_W(2)/2}$ *if* $\varphi_W(2) < 0$ *and* $r_n = n^{-1/2}$ *if* $\varphi_W(2) = 0$. *The probability distributions of the random functions* $r_n F_n$ *form a tight sequence, and for all* $\alpha > \varphi_W(2)/2$, $b^{n\alpha} F_n$ *is unbounded almost surely, conditionally on* $\mathscr{V}^c$.

REMARK 2.3. (1) Suppose that (C) does not hold and $p_0 < \infty$. Thus $p_0 \in (1,2]$ and $\varphi_W(p_0) = 0$, or when $\mathbb{P}(\sum_{k=0}^{b-1} W_k = 1) = 1$, $p_0 \in (1,\infty)$ and $\varphi_W(p_0) = 0$. What we can only prove is that $\lim_{n\to\infty} \sup_{w\in\mathscr{A}^n} |\Delta F_n(I_w)| = 0$ and $\liminf_{n\to\infty} \frac{\log_b \sup_{w\in\mathscr{A}^n} |\Delta F_n(I_w)|=0}{-n}$. This is not enough to decide whether or not F_n is convergent. It is mainly for the same reason that we cannot deal with the case $\alpha = \varphi_W(p_0)/p_0$ in Theorem 2.5(2) and (3) (when $p_0 < \infty$).

(2) When $p_0 = \infty$, Theorem 2.5 tells nothing about the case where the assumptions of Theorem 2.1(2) hold except that there is no $\gamma \in (0,1)$ such that (2.4) holds.

(3) The results obtained in [25, 30] when $W \ge 0$ show that in this case, when F_n converges almost surely uniformly to 0, there does not exist a sequence $(a_n)_{n\ge1}$ such that F_n/a_n converge in law to a nontrivial process as n tends to ∞ (see the discussion in Section VIII of [30]). Theorem 2.7 shows that allowing the components of W to take values in $\mathbb{R}_-^*$ yields a completely different situation.

(4) One referee brought to our attention reference [24] which studies the possible phases in mean field theory of directed polymers with random complex weights. Specifically, the mathematical question discussed in [24] concerns the asymptotic behavior of $\log|F_n(1)|/n$ in the canonical case, without special assumption on the value of $\mathbb{E}(W_0)$, but under the assumption that the distribution of W_0 is continuous, all its positive moments are finite and $W_0/|W_0|$ is independent of $|W_0|$. The same parameter p_0 as the one defined in (2.5) is considered and three phases are distinguished: phase I corresponds to (C), phase II to parts (1) and (2) of Theorem 2.5 and phase III to part (4) of Theorem 2.5. The results obtained in [24] complete those obtained in the present paper for phases II and III by showing that $\log|F_n(1)|/n$ converges in probability to $-\varphi_W(p_0)/p_0$ in phase II and $-\varphi_W(2)/2$ in phase III. In phase III and under the assumptions of Theorem 2.7, our functional central limit theorem describes the asymptotic behavior of the distribution of $\log|F_n(1)|/n$.

Part (4) of Theorem 2.5 is restated and refined in Theorems 2.6 and 2.7.

We now cease to assume that $\varphi_W(p) > -\infty$ for all $p \geq 0$, but assume that $\varphi_W(2) > -\infty$.

Define

$$\sigma = \begin{cases} \sqrt{\dfrac{\mathbb{E}(|\sum_{i=0}^{b-1} W_i|^2) - 1}{\mathbb{E}(\sum_{i=0}^{b-1} |W_i|^2) - 1}}, & \text{if } \varphi_W(2) < 0, \\ \sqrt{\sum_{i \neq j} \mathbb{E}(W_i \overline{W_j})}, & \text{if } \varphi_W(2) = 0. \end{cases}$$

THEOREM 2.6 [Tightness of $(\mathcal{L}(F_n/\sqrt{\mathbb{E}(F_n(1)^2)})_{n\geq 1})$]. *Suppose that* $\mathbb{P}(\sum_{k=0}^{b-1} W_k \neq 1) > 0$ *and* (C) *does not hold. In particular,* $\varphi_W(2) \leq 0$.

(1) *The sequence* $(F_n(1))_{n\geq 1}$ *is unbounded in* L^2 *norm. Specifically, we have* $\mathbb{E}(|F_n(1)|^2) \sim \sigma^2 b^{-n\varphi_W(2)}$ *if* $\varphi_W(2) < 0$ *and* $\mathbb{E}(|F_n(1)|^2) \sim \sigma^2 n$ *if* $\varphi_W(2) = 0$.

(2) *Suppose that* $p_0 > 2$. *Equivalently,* $\varphi_W(p)/p > \varphi_W(2)/2$ *near* 2^+. *For* $n \geq 1$ *let* $Z_n = F_n/\sqrt{\mathbb{E}(F_n(1)^2)}$.

The sequence $(Z_n(1))_{n\geq 1}$ *is bounded in* L^p *norm for all* p *such that* $\varphi_W(p)/p > \varphi_W(2)/2$.

Moreover, the probability distributions of the random continuous functions $Z_n = F_n/\sqrt{\mathbb{E}(F_n(1)^2)}$ *form a tight sequence.*

(3) *Suppose that* $p_0 > 2$. *We have* $\varphi_{W^{(2)}}(p) > 0$ *near* 1^+ (*remember Definition* 2.1). *Suppose, moreover, that* W *is* $\mathbb{R}^b$*-valued and* $(Z_n)_{n\geq 1}$ *converges in law, as* n *tends to* ∞. *Then the weak limit of* Z_n *is the Brownian motion in multifractal time* $Z = B \circ F_{W^{(2)}}$ *where* B *is a standard Brownian motion independent of* $F_{W^{(2)}}$. *Moreover,* Z *satisfies the statistical scaling invariance property,*

$$Z \equiv \sum_{i=0}^{b-1} \mathbf{1}_{[i/b,(i+1)/b]}\big(Z(i/b) + b^{\varphi_W(2)/2} W_i Z_i \circ S_i^{-1}\big), \tag{2.6}$$

where $S_i(t) = (t+i)/b$, *the random objects* $W, Z_0, \ldots, Z_{b-1}$ *are independent, and the* Z_i *are distributed like* Z.

The following result completes Theorem 2.6 and is illustrated in Figure 6.

THEOREM 2.7 (Functional central limit theorem when F_n is unbounded). *Suppose that* $\mathbb{P}(\sum_{k=0}^{b-1} W_k \neq 1) > 0$ *and* (C) *does not hold. Suppose, moreover, that* W *is* $\mathbb{R}^b$*-valued and* $\varphi_W(p)/p > \varphi_W(2)/2$ *over* $(2, \infty)$ (*equivalently* φ_W *is increasing, or* $|W_k| \leq 1$ *almost surely for all* $0 \leq k \leq b-1$).

Then $(Z_n)_{n\geq 1}$ *converges in law, as* n *tends to* ∞, *to the Brownian motion in multifractal time* Z *described in Theorem* 2.6(3). *Also, the probability distribution of* $Z(1)$ *is determined by its moments.*

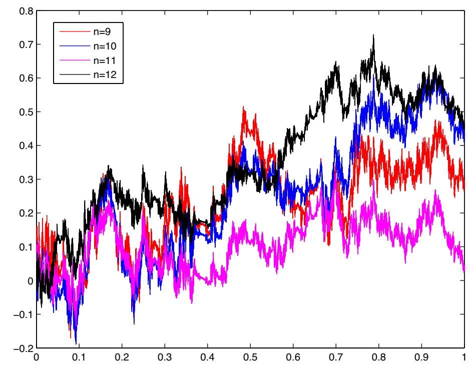

FIG. 6. *Weak convergence of* $F_{W,n}/(\sigma b^{-n\varphi_W(2)/2})$ *to a Brownian motion in multifractal time when* $\varphi_W(2) \le 0$ *and* $\varphi_W(p)/p > \varphi_W(2)/2$ *for all* $p > 2$ *in the nonconservative case.*

Theorem 2.7 has the following natural counterpart when $(F_n)_{n\ge 1}$ converges almost surely.

THEOREM 2.8 (Functional central limit theorem when F_n converges). *Suppose that* $\mathbb{P}(\sum_{k=0}^{b-1} W_k \neq 1) > 0$ *and* (C) *holds. Suppose, moreover, that* W *is* $\mathbb{R}^b$*-valued and* $\varphi_W(p)/p > \varphi_W(2)/2$ *near* 2^+.

Then $(F_n - F)/\sqrt{\mathbb{E}((F_n - F)(1)^2)}$ *converges in law, as n tends to* ∞, *to the Brownian motion in multifractal time described in Theorem* 2.6(3). *Moreover,* $\sqrt{\mathbb{E}((F_n - F)(1)^2)} = b^{-n\varphi_W(2)/2}\sqrt{\mathbb{E}((1 - F_W(1))^2)}$.

REMARK 2.4. (1) The equivalent of $\mathbb{E}(|F_n(1)|^2)$ given in Theorem 2.6(1) is valid in general when $\varphi_W(2) \le 0$.

(2) In Theorem 2.7, if the components of W have the same constant modulus, then the limit process is the standard Brownian motion over [0, 1]. In particular, we recover the result obtained in [13], where the components are i.i.d.

(3) When the components of W are nonnegative, independent, and identically distributed, the scalar central limit theorem deduced for the term $(F_n - F)(1)/\sqrt{\mathbb{E}((F_n - F)(1)^2)}$ from Theorem 2.7 is almost a restatement of the central limit theorem established in Corollary 4.3 in [55].

2.5. *Multifractal nature and scaling invariance properties.*

2.5.1. *Multifractal nature.*

The multifractal analysis of the sample paths of the limit process F obtained in Theorem 2.1 is achieved in [9]. Here we state a simplified version of the result obtained in [9].

THEOREM 2.9. *Under the assumptions of Theorem* 2.1, *assume that* $\mathbb{E}(\sum_{i=0}^{b-1} |W_i|^q)$ *is finite for all* $q \in \mathbb{R}$ *and* $W \neq (1/b, \dots, 1/b)$ (*otherwise* F *is the* *identity function of* $[0,1]$). *Then*, *with probability* 1, *for all* $h \geq 0$, $\dim E_F(h) = \inf_{q\in\mathbb{R}} hq + \log_b \mathbb{E}(\sum_{i=0}^{b-1} |W_i|^q) = \tau_F^*(h)$, *a negative dimension meaning that the corresponding iso-Hölder set is empty. Moreover, the function* F *is monofractal if and only if there exists* $H \in (0,1)$ *such that* $|W_i| = b^{-H}$ *almost surely for all* $\leq i \leq b-1$. *In this case*, *the pointwise Hölder exponent is everywhere equal to* H.

2.5.2. *Link with processes stable under random weighted mean.* The scaling properties of the stochastic processes obtained in Theorems 2.1 and 2.7 are reminiscent of the fact that the distribution of their increments between 0 and 1 is invariant under random weighted mean (using the terminology of [49]), or equivalently that this distribution is a fixed point of a smoothing transformation (using the terminology of [25]). Specifically, this increment $\widetilde{Z}$ satisfies a functional equation of the form

$$\widetilde{Z} \equiv \sum_{i=0}^{b-1} \widetilde{W}_i \widetilde{Z}_i, \tag{2.7}$$

where $\widetilde{Z} \equiv \widetilde{Z}_i$ for all i, and the random variables $\widetilde{W} = (\widetilde{W}_0, \dots, \widetilde{W}_{b-1})$, $\widetilde{Z}_0, \dots, \widetilde{Z}_{b-1}$ are independent.

The square integrable solutions of (2.7) have been studied in [21]. Thanks to [21], we can conclude that: (1) When $p \geq 2$, the limit process F_W obtained in Theorem 2.1 is the unique square integrable continuous stochastic process with expectation the identity function of $[0,1]$ and satisfying (2.3); (2) the limit process $B \circ F_{W^{(2)}}$ obtained in Theorem 2.7 is the unique centered square integrable continuous stochastic process satisfying (2.6).

It is interesting to compare the structure of the process constructed in Theorem 2.1 with other processes naturally associated with (2.7), namely symmetric Lévy processes in multifractal time which also satisfy (2.3) and are obtained as follows: If $\widetilde{W} \geq 0$ and there exists $\beta \in (1,2)$ such that $\varphi_{\widetilde{W}}(\beta) = 0$ and $\varphi'_{\widetilde{W}}(\beta) > 0$, then it is noticed in [25, 30, 42] that a solution of (2.7) is $X_\beta \circ F_{\widetilde{W}^{(\beta)}}(1)$ where $F_{\widetilde{W}^{(\beta)}}$ is the nondecreasing function constructed in Definition 2.1, and X_β is a symmetric stable Lévy process of index β independent of $F_{\widetilde{W}^{(\beta)}}$. The multifractal nature of the jump processes $X_\beta \circ F_{\widetilde{W}^{(\beta)}}$ is given in [15].

3. Proofs of Theorems 2.1, 2.2 and 2.3. We start with a remark. Under the assumptions of Theorems 2.1 and 2.2(1), we have $-1 \leq \varphi_W(0) < \varphi_W(1) \leq 0 < \varphi_W(p)$. Since φ_W is concave this implies that $\varphi_W(q) < q-1$ for all $q \in (1,p]$ except if $\varphi_W(q) = q-1$ for all q. This can happen only if the components of W are positive and equal to $1/b$ almost surely. In this case $F_n(t) = t$ for all $n \geq 1$ and $t \in [0,1]$ and the result obviously holds. We exclude this case in the rest of this section.

For $w \in \mathscr{A}^*$, we denote by $(F_n^{[w]})_{n\geq 1}$ the copy of $(F_n)_{n\geq 1}$ constructed with the random vectors $(W(w \cdot u))_{u\in\mathscr{A}^*}$:

$$F_n^{[w]}(t) = \int_0^t b^n \prod_{k=1}^n W_{u_k}(w \cdot u||w| + k - 1)\,\mathrm{d}u.$$

For $n > |w|$, the increment $\Delta F_n(I_w)$ of F_n over I_w takes the form

$$\Delta F_n(I_w) = Q(w) F_{n-|w|}^{[w]}(1), \tag{3.1}$$

where

$$Q(w) = \prod_{k=0}^{|w|-1} W_{w_{k+1}}(w|k).$$

This implies in particular that for every $n \geq 1$ we have

$$F_n = \sum_{i=0}^{b-1} \mathbf{1}_{[i/b,(i+1)/b]}\big(F_n(i/b) + W_i F_{n-1}^{[i]} \circ S_i^{-1}\big). \tag{3.2}$$

Moreover, $Q(w)$ and $F_{n-|w|}^{[w]}(1)$ are independent, and for each $p \geq 1$, the families $\{F_n^{[w]}\}_{n\geq 1}$, $w \in \mathscr{A}^p$, are independent.

3.1. *Proofs of Theorems* 2.1 *and* 2.2(1). If $\mathbb{P}(\sum_{i=0}^{b-1} W_i = 1) = 1$ then $F_k(1) = 1$ almost surely. If $p \in (1, 2]$, the fact that the martingale $(F_k(1))_{k\geq 1}$ converges almost surely and in L^p norm is a consequence of Theorem 1 in [18], and the case $p > 2$ is a consequence of Theorem 1 in [7] (the positive case is treated in [25] and [39]). Then, equation (3.1) implies the almost sure convergence of the b-adic increments of F_n.

Now we establish the almost sure uniform convergence of F_n. When F_n can be interpreted as a [0, 1]-martingale, that is when the components of W have positive expectations (see [10]), the proof provides a simpler alternative to the general proof given in [10].

Let $q \in (1, p]$ such that $\varphi_W(q) > 0$ and define $M_q = \mathbb{E}(\sup_{k\geq 1} |F_k(1)|^q)$. By using (3.1) as well as the martingale property of $(F_k(1))_{k\geq 1}$ and Doob's inequality we get

$$\begin{aligned}\mathbb{E}\Big(\sup_{n\geq 1} |\Delta F_n(I_w)|^q\Big) &\leq \sum_{n=1}^{|w|} \mathbb{E}(|\Delta F_n(I_w)|^q) + \mathbb{E}(|Q(w)|^q)\mathbb{E}\Big(\sup_{n>|w|} \big|F_{n-|w|}^{[w]}(1)\big|^q\Big) \\ &\leq \sum_{n=1}^{|w|} b^{-(|w|-n)q}\mathbb{E}(|Q(w|n)|^q) + C_q M_q \mathbb{E}(|Q(w)|^q)\end{aligned}$$

for some constant C_q. Consequently, for $\gamma > 0$ and $N \geq 1$ we have

$$\begin{aligned}
&\mathbb{P}\Big(\max_{w\in\mathscr{A}^N}\sup_{n\geq 1}|\Delta F_n(I_w)| > b^{-\gamma N}\Big)\\
&\quad\leq b^{\gamma N q}\sum_{w\in\mathscr{A}^N}\sum_{n=1}^{N} b^{-(N-n)q}\mathbb{E}(|Q(w|n)|^q) + C_q M_q \mathbb{E}(|Q(w)|^q)\\
&\quad= b^{\gamma N q}\Bigg[\sum_{n=1}^{N} b^{-(N-n)(q-1)} b^{-n\varphi_W(q)} + C_q M_q b^{-N\varphi_W(q)}\Bigg]\\
&\quad\leq \bigg[\frac{b^{q-1-\varphi_W(q)}}{b^{q-1-\varphi_W(q)}-1} + C_q M_q\bigg] b^{\gamma N q} b^{-N\varphi(q)},
\end{aligned}$$

where we used the fact that $\varphi_W(q) < q-1$. It follows that if $\gamma < \varphi_W(q)/q$, we have

$$\sum_{N\geq 1}\mathbb{P}\Big(\max_{w\in\mathscr{A}^N}\sup_{n\geq 1}|\Delta F_n(I_w)| > b^{-\gamma N}\Big) < \infty.$$

Due to the Borel–Cantelli lemma, we conclude that, with probability 1,

$$\text{for } N \text{ large enough,}\qquad \max_{w\in\mathscr{A}^N}\sup_{n\geq 1}|\Delta F_n(I_w)| \leq b^{-\gamma N}. \tag{3.3}$$

Next, we use the following classical property: for any continuous complex function f on [0, 1], one has

$$\omega(f,\delta) \leq 2b\sum_{n\geq -\log\delta/\log b}\ \sup_{w\in\mathscr{A}^n}\Delta f(I_w), \tag{3.4}$$

where $\omega(f,\delta)$ stands for the modulus of continuity of f,

$$\omega(f,\delta) = \sup_{\substack{t,s\in[0,1]\\ |t-s|\leq\delta}} |f(t)-f(s)|.$$

Since $F_n(0) = 0$ almost surely for all $n \geq 1$, it follows from (3.3), (3.4) and Ascoli–Arzela's theorem that, with probability 1, the sequence of continuous functions $(F_n)_{n\geq 1}$ is relatively compact, and all the limit of subsequences of F_n are γ-Hölder continuous for all $\gamma < \max_{q\in(1,p]}\varphi(q)/q$. Moreover, by the self-similarity of the construction (3.1) we know that F_n converges almost surely on set of b-adic points. This yields the uniform convergence of F_n and the Hölder regularity property of the limit F.

To see that $(F_n)_{n\geq 1}$ converges in L^p norm, it is enough to prove that the sequence $(\mathbb{E}(\sup_{1\leq k\leq n}\|F_k\|_\infty^p))_{n\geq 1}$ is bounded.

For $n \geq 1$ and $0 \leq i \leq b-1$ define

$$S_n = \sup_{1\leq k\leq n}\|F_k\|_\infty,\qquad S_n(i) = \sup_{1\leq k\leq n}\|F_k^{[i]}\|_\infty$$

and

$$\widetilde{S}_n(i) = \sup_{1\le k\le n} |F_k(ib^{-1})|.$$

Due to (3.2) we have $S_n \le \max_{0\le i\le b-1}[\widetilde{S}_n(i) + |W_i|S_{n-1}(i)]$, so

$$\mathbb{E}(S_n^p) \le \sum_{i=0}^{b-1} \mathbb{E}\big([|W_i|S_{n-1}(i) + \widetilde{S}_n(i)]^p\big).$$

Denote by $\bar{p} \ge 2$ the unique integer such that $\bar{p} - 1 < p \le \bar{p}$. By using the subadditivity of the mapping $x \ge 0 \mapsto x^{p/\bar{p}}$ we get

$$\begin{aligned}
&\mathbb{E}\big((|W_i|S_{n-1}(i) + \widetilde{S}_n(i))^p\big)\\
&\quad \le \mathbb{E}\big(((|W_i|S_{n-1}(i))^{p/\bar{p}} + \widetilde{S}_n(i)^{p/\bar{p}})^{\bar{p}}\big)\\
&\quad \le \mathbb{E}(|W_i|^p)\cdot\mathbb{E}(S_{n-1}(i)^p) + \mathbb{E}(\widetilde{S}_n(i)^p)\\
&\qquad + \sum_{m=1}^{\bar{p}-1}\binom{\bar{p}}{m}\mathbb{E}\big([|W_i|S_{n-1}(i)]^{mp/\bar{p}}[\widetilde{S}_{n-1}(i)]^{(\bar{p}-m)p/\bar{p}}\big).
\end{aligned}$$

Now let us make some remarks:

– The Hölder inequality yields for any pair of nonnegative random variables (U, V) and $m \in [1, \bar{p} - 1]$

$$\mathbb{E}\big(U^{mp/\bar{p}}V^{(\bar{p}-m)p/\bar{p}}\big) \le \mathbb{E}(U^p)^{m/\bar{p}}\mathbb{E}(V^p)^{(\bar{p}-m)/\bar{p}}.$$

– The convergence in L^p norm of $F_n(1)$ implies that $(\widetilde{S}_n(i))_{n\ge1}$ is bounded in L^p.

– The random variables $|W_i|$ and $S_{n-1}(i)$ are independent and $|W_i|$ is in L^p.

– Since the expectation of $F_k(1)$ is equal to 1 for all $k \ge 1$, we have $1 \le \mathbb{E}(S_{n-1}^p)^{m/\bar{p}} \le \mathbb{E}(S_{n-1}^p)^{(\bar{p}-1)/\bar{p}}$ for all $m \in [1, \bar{p} - 1]$.

The previous remarks imply the existence of two constants A and B independent of i such that

$$\begin{aligned}
&\mathbb{E}\big((W_iS_{n-1}(i) + \widetilde{S}_n(i))^p\big)\\
&\quad \le \mathbb{E}(|W_i|^p)\mathbb{E}(S_{n-1}^p) + B + \sum_{m=1}^{\bar{p}-1}\binom{\bar{p}}{m}A^{m/\bar{p}}B^{(\bar{p}-m)/\bar{p}}\mathbb{E}(S_{n-1}^p)^{(\bar{p}-1)/\bar{p}}\\
&\quad = \mathbb{E}(|W_i|^p)\mathbb{E}(S_{n-1}^p) + B + (A+B)^{\bar{p}}\mathbb{E}(S_{n-1}^p)^{(\bar{p}-1)/\bar{p}}.
\end{aligned}$$

Summing this inequality over i we find

$$\mathbb{E}(S_n^p) \le b^{-\varphi_W(p)}\mathbb{E}(S_{n-1}^p) + b(A+B)^{\bar{p}}\mathbb{E}(S_{n-1}^p)^{(\bar{p}-1)/\bar{p}} + b\cdot B.$$

For $x \ge 0$ let $f(x) = b^{-\varphi_W(p)}\cdot x + b(A+B)^{\bar{p}}x^{(\bar{p}-1)/\bar{p}} + b\cdot B$. Since $b^{-\varphi_W(p)} < 1$ and $(\bar{p} - 1)/\bar{p} < 1$, there exists x_0 such that $f(x) < x$ for any $x > x_0$, which implies $\mathbb{E}(S_n^p) \le \max\{x_0, \mathbb{E}(S_{n-1}^p)\}$. This yields the conclusion.

Property (2.3) is a consequence of (3.2).

3.2. *Proof of Theorem* 2.2(2). By construction, if there exists $0 \le i \le b-1$ such that $\mathbb{P}(|W_i| > 1) > 0$, then $\lim_{p\to\infty} \varphi_W(p) = -\infty$. Otherwise, we have $\lim_{p\to\infty} \varphi_W(p) = -\log_b \sum_{i=0}^{b-1} \mathbb{P}(|W_i| = 1)$, and by concavity of φ_W we have $\varphi_W(p) < 0$ for all $p > 0$ and $\lim_{p\to\infty} \varphi_W(p) = 0$ if and only if $\sum_{i=0}^{b-1} \mathbb{P}(|W_i| = 1) = 1$.

For $n \ge 1$ let us define $m_n = \max_{w\in\mathscr{A}^n} |Q(w)|$. Then, due to (3.1), for $p \ge 1$, we have

$$\|F_{n+p} - F_n\|_\infty \le \sup_{w\in\mathscr{A}^n} |Q(w)| \big\|F_p^{[w]} - 1\big\|_\infty \le m_n \sup_{w\in\mathscr{A}^n} \big(1 + \big\|F_p^{[w]}\big\|_\infty\big).$$

We are going to prove that $\lim_{n\to\infty} m_n = 0$, and there exists $C > 0$ such that $\|F_p\|_\infty \le C$ almost surely for all $p \ge 1$. Thus $(F_n)_{n\ge1}$ is almost surely a Cauchy sequence.

We start with the proof of $\lim_{n\to\infty} m_n = 0$. Due to the fact that the components of W are either equal to 1 or less than or equal to γ, the sequence $(m_n)_{n\ge1}$ is nonincreasing and $m_{n+1} = m_n$ or $m_{n+1} \le \gamma m_n$. Thus if $\lim_{n\to\infty} m_n > 0$, we can find an infinite word $w_1 \cdots w_n \cdots$ in $\mathscr{A}^{\mathbb{N}_+}$ and $n_0 \ge 1$ such that or all $n \ge n_0$, $W_{w_{n+1}}(w_1 \cdots w_n) = 1$. By construction, such an infinite word must belong to the boundary of a Galton–Watson tree rooted at $w_1 \cdots w_{n_0}$, and whose offspring distribution generating function is given by $x \mapsto \sum_{k=0}^{b} \mathbb{P}(\#\{i : |W_i| = 1\} = k)x^k$. This tree is subcritical, since we assumed that $\sum_{k=1}^{b} k\mathbb{P}(\#\{i : |W_i| = 1\} = k) = \sum_{i=0}^{b-1} \mathbb{P}(|W_i| = 1) = 1$ and $\mathbb{P}(\#\{i : |W_i| = 1\} = 1) < 1$. Consequently its boundary is empty almost surely, and m_n tends to 0 as $n \to \infty$.

Now we prove by induction that, with probability 1, for all $p \ge 1$ and $w \in \mathscr{A}^p$, we have

$$\max\big(|F_p(t_w)|, |F_p(t_w + b^{-p})|\big) \le 1 + \frac{b+1}{2} \sum_{l=1}^{p-k_w-1} \gamma^l + \frac{b-1}{2}\gamma^{p-k_w}, \tag{3.5}$$

where

$$k_w = \#\{1 \le j \le p : |W_{w_j}(w|j-1)| = 1\}.$$

This will imply that $\|F_p\|_\infty \le 1 + b/(1-\gamma)$ almost surely.

For the case $p = 1$, that $|\sum_{i=0}^{j} W_i| \le 1 + (b-1)\gamma/2$ for all $0 \le j \le b-1$ is a direct consequence of our assumptions $\mathbb{P}(\sum_{i=0}^{b-1} W_i = 1) = 1$, and almost surely either $|W_i| = 1$ and $(\sum_{k=0}^{i-1} W_i, \sum_{k=0}^{i} W_i) \in \{(0,1),(1,0)\}$, or $|W_i| \le \gamma$. Thus $\|F_1\|_\infty \le 1 + (b-1)\gamma/2$.

Now suppose that $p \ge 1$ and (3.5) holds. Let $w \in \mathscr{A}^p$. For every $1 \le j \le b-1$, let $s_j(w) = \sum_{i=0}^{j-1} W_i(w)$. By construction, we have

$$F_{p+1}(t_{wj}) = F_p(t_w) + \big(F_p(t_w + b^{-p}) - F_p(t_w)\big)s_j(w).$$

If $|W_j(w)| = 1$, then by our assumption we have $k_{wj} = k_w + 1$ and $s_j(w) \in \{0, 1\}$, so $F_{p+1}(t_{wj}) \in \{F_p(t_w), F_p(t_w + b^{-p})\}$ and (3.5) holds. Otherwise, $k_{wj} = k_w$, and since $|Q(w)| \leq \gamma^{p-k_w}$ we get

$$\begin{aligned}|F_{p+1}(t_{wj})| &\leq |F_p(t_w)| + \gamma^{p-k_w}|s_j(w)| \\ &\leq 1 + \frac{b+1}{2}\sum_{l=1}^{p-k_w-1}\gamma^l + \frac{b-1}{2}\gamma^{p-k_w} + \gamma^{p-k_w} + \frac{b-1}{2}\gamma^{p+1-k_w} \\ &= 1 + \frac{b+1}{2}\sum_{l=1}^{p-k_{wj}}\gamma^l + \frac{b-1}{2}\gamma^{p+1-k_{wj}}.\end{aligned}$$

Now we prove that the limit F of $(F_n)_{n\geq 1}$ is not uniformly Hölder continuous. To see this, we can consider of any $q > 1$ the positive measure μ_q on $[0, 1]$ obtained almost surely as the derivative in the distribution sense of $F_{W^{(q)}}$ (see Definition 2.1). It follows from the equality $Q(w) = \Delta F(I_w)$ and Theorem IV(i) of [5] that $\lim_{n\to\infty} \log(|\Delta F(I_n(t))|)/-n\log(b) = \varphi'_W(q)$ for μ_q-almost every t where $I_n(t)$ is the semi-open to the right b-adic interval of generation n containing t. Moreover, by construction we have $\lim_{q\to\infty}\varphi'_W(q) = 0$. This implies that, with probability 1, there exists a sequence $(t_k)_{k\geq 1}$ of points in $[0, 1]$ such that $\lim_{k\to\infty} h_F(t_k) = 0$; hence F is not uniformly Hölder.

3.3. *Proof of Theorem* 2.3. The facts that $\beta \in (1, 2)$ when $\mathbb{P}(\sum_{i=0}^{b-1} W_i = 1) < 1$, as well as the properties of $W^{(\beta)}$ are immediate.

We will prove in the end of this section the following properties which hold under the assumptions of Theorem 2.3.

LEMMA 3.1. *Let X and X_β stand respectively for the oscillations of F_W and $F_{W^{(\beta)}}$ over* $[0, 1]$. *For all $q \in \mathbb{R}$ we have $\mathbb{E}(X^q) < \infty$ and $\mathbb{E}(X_\beta^q) < \infty$.*

For every $w \in \mathscr{A}^*$, let $X(w)$ and $X_\beta(w)$ stand respectively for the oscillations of $F_W^{[w]}$ and $F_{W^{(\beta)}}^{[w]}$ over $[0, 1]$. We deduce from the Lemma 3.1 that, with probability 1, for every $\varepsilon > 0$, there exists n_ε such that

$$\forall n \geq n_\varepsilon, \forall w \in \mathscr{A}^n, \forall Y \in \{X, X_\beta\}, \qquad b^{-n\varepsilon} \leq Y(w) \leq b^{n\varepsilon}. \tag{3.6}$$

This implies that

$$\begin{aligned} b^{-n\varepsilon}|Q(w)| &\leq \mathrm{Osc}_{F_W}(I_w) = |Q(w)|X(w) \leq b^{n\varepsilon}|Q(w)|, \\ b^{-n\varepsilon}|Q(w)|^\beta &\leq \mathrm{Osc}_{F_{W^{(\beta)}}}(I_w) = |Q(w)|^\beta X_\beta(w) \leq b^{n\varepsilon}|Q(w)|^\beta. \end{aligned}$$

Consequently,

$$b^{-n(1+1/\beta)\varepsilon} \leq \frac{\mathrm{Osc}_{F_W}(I_w)}{\mathrm{Osc}_{F_{W^{(\beta)}}}(I_w)^{1/\beta}} = \frac{X(w)}{X_\beta(w)^{1/\beta}} \leq b^{n(1+1/\beta)\varepsilon}.$$

Let $B = F_W \circ F_{W^{(\beta)}}^{-1}$. Let $J_w = F_{W^{(\beta)}}(I_w)$. We have $|J_w| = \mathrm{Osc}_{F_{W^{(\beta)}}}(I_w)$, and $\mathrm{Osc}_B(J_w) = \mathrm{Osc}_{F_W}(I_w)$, so the previous inequality is equivalent to

$$b^{-n(1+1/\beta)\varepsilon} \le \frac{\mathrm{Osc}_B(J_w)}{|J_w|^{1/\beta}} = \frac{X(w)}{X_\beta(w)^{1/\beta}} \le b^{n(1+1/\beta)\varepsilon}.$$

Under our assumptions, it is also true that [see Theorem 2.1(1)]

$$\lim_{n\to\infty} \inf_{w\in\mathscr{A}^n} \frac{\log_b |J_w|}{-n} \ge \alpha_0,$$

where $\alpha_0 = \sup_{p>0} \varphi_{W^{(\beta)}}(p)/p > 0$ (in fact the equality holds). Also, we have the following property (we postpone its proof until after that of Lemma 3.1).

LEMMA 3.2. *With probability* 1, *for every* $\varepsilon > 0$, *there exists* n_ε *such that*

$$\forall n \ge n_\varepsilon, \qquad b^{-n\varepsilon} \le \inf_{w\in\mathscr{A}^n} \inf_{0\le i\le b-1} \frac{|J_{wi}|}{|J_w|} \le 1. \tag{3.7}$$

We can also choose the random integer n_ε so that for all $n \ge n_\varepsilon$ (3.7) holds as well as the property $|J_w| \le b^{-n\alpha_0/2}$ for all $w \in \mathscr{A}^n$.

Let $t \in (0,1)$ and $0 < r < \min_{w\in\mathscr{A}^{n_\varepsilon+1}} |J_w|$. Let $w_2 \in \mathscr{A}^*$ such that $|w_2| > n_\varepsilon$, $t \in J_{w_2} \subset [t-r, t+r]$ and $|J_{w_2}|$ is maximal. Then let $(w_1, w_3) \in \mathscr{A}^* \times \mathscr{A}^*$ such that $\min(|w_1|, |w_3|) > n_\varepsilon$, $\min(|J_{w_1}|, |J_{w_3}|) \ge r$, the intervals J_{w_i}, $i \in \{1,2,3\}$, are adjacent, $[t-r, t+r] \subset J_{w_1} \cup J_{w_2} \cup J_{w_3}$, and $|J_{w_1}| + |J_{w_2}| + |J_{w_3}|$ is minimal. This constraint imposes that $|J_w| \le rb^{\varepsilon|w|}$ for $w \in \{w_1, w_3\}$. Otherwise, due to (3.7) we can replace J_w by one of its sons in the covering of $[t-r, t+r]$. Also, we have $2r \ge |J_{w_2}| \ge rb^{-\varepsilon|w_2|}$. Otherwise, since $t \in J_{w_2}$, due to (3.7) we can replace J_{w_2} by its father, hence $|J_{w_2}|$ is not maximal.

Since

$$\mathrm{Osc}_B(J_{w_2}) \le \mathrm{Osc}_B([t-r, t+r]) \le 3 \max_i \mathrm{Osc}_B(J_{w_i}),$$

we have

$$|J_{w_2}|^{1/\beta} b^{-|w_2|(1+1/\beta)\varepsilon} \le \mathrm{Osc}_B([t-r, t+r]) \le 3 \max_i |J_{w_i}|^{1/\beta} b^{|w_i|(1+1/\beta)\varepsilon}.$$

Now we specify $\varepsilon < \alpha_0/4$. We can deduce from the constraints on the length of the intervals J_{w_i} that $b^{\varepsilon|w_i|} \le r^{-4\varepsilon/\alpha_0}$. Consequently, there exists a constant C depending on W only such that for r small enough,

$$r^{1/\beta+C\varepsilon} \le \mathrm{Osc}_B([t-r, t+r]) \le r^{1/\beta-C\varepsilon}.$$

Since this holds for all $0 < \varepsilon < \alpha_0/4$, almost surely for all $t \in (0,1)$, we have in fact that, with probability 1, for all $t \in (0,1)$, $\lim_{r\to 0^+} \frac{\log(\mathrm{Osc}_B([t-r,t+r]))}{\log(r)} = 1/\beta$, hence $h_B(t) = 1/\beta$.

PROOF OF LEMMA 3.1. The part concerning the moments of positive orders is a consequence of Theorem 2.1(1) and the inequality $\mathrm{Osc}_f([0,1]) \le 2\|f\|_\infty$, which holds for any continuous function f on $[0,1]$.

For the moments of negative orders, the case of X_β, which is the increment between 0 and 1 of the increasing function $F_{W^{(\beta)}}$, is treated for instance in any of [5, 42, 54]. For X, we just remark that we have

$$X \ge b^{-1} \sum_{k=0}^{b-1} |W_k| X(k). \tag{3.8}$$

Moreover, the event $\{X = 0\}$ is measurable with respect to $\bigcap_{n\ge 1} \sigma(W(w) : |w| \ge n)$ because the components of W do not vanish. Thus this event has probability 0 or 1. Since the function F_W is not almost surely equal to 0, X is positive with probability 1. We can then use inequality (3.8) in the same way as in [5, 33, 54] when W is positive to prove that all the moments of negative order of X are finite as soon as the same property holds for the random variables $|W_i|$. □

PROOF OF LEMMA 3.2. Let $(\varepsilon_k)_{k\ge 1}$ be a positive sequence converging to 0 at ∞. Since $|J_w| = |Q(w)|^\beta X_\beta(w)$, due to (3.6), for any $k \ge 1$, with probability 1, for n large enough we have $b^{-n\varepsilon_k} \le \inf_{w\in\mathscr{A}^n} \inf_{0\le i\le b-1} |W_i(w)|$. This is an immediate consequence of the fact that all the moments of negative order of the random variables $|W_i|$ are finite. Since the set $\{\varepsilon_k : k \ge 1\}$ is countable, we have the conclusion. □

3.4. *Proof of Theorem* 2.4. Suppose that there exist a continuous and increasing function G defined on $[0,1]$ with $G(0) = 0$ as well as a monofractal continuous function B defined on $[G(0), G(1)]$ such that $F = B \circ G$. We denote by H the Hölder exponent of B (notice that it may be random).

At first, suppose that $H \in (0,1]$. For every $\alpha \in (0, H)$ the function B is uniformly α-Hölder, so there exists $C > 0$ such that $\mathrm{Osc}_B(I) \le C|I|^\alpha$. Consequently, for every $n \in \mathbb{N}_+$ and $q \ge 0$, we have

$$\sum_{w\in\mathscr{A}^n} \mathrm{Osc}_F(I_w)^q \le C^q \sum_{w\in\mathscr{A}^n} |G(I_w)|^{\alpha q}.$$

Since G is increasing, taking $q = 1/\alpha$ yields $\sum_{w\in\mathcal{A}^n} \mathrm{Osc}_F(I_w)^{1/\alpha} \le C^{1/\alpha} G(1)$, hence $\tau_F(1/\alpha) \ge 0$. This is in contradiction to the fact (established in [9]) that $\tau_F = \varphi_W$ over $\mathbb{R}_+$ almost surely.

Now we suppose that $H = 0$. If $\dim([0,1] \setminus \overline{E}_G(\infty)) > 0$, then we have $h_{F_W}(t) = 0 = h_B(G(t))$ at each $t \in [0,1] \setminus \overline{E}_G(\infty)$. But since $\tau_F \ge \varphi_W$ over $\mathbb{R}_+$, we have $\dim E_F(0) \le \tau_F^*(0) \le \inf_{q\ge 0} -\varphi_W(q) = 0$. This yields a contradiction.

4. Proofs of Theorems 2.5, 2.6, 2.7 and 2.8.

4.1. *Proof of Theorem* 2.5. (1) If $\alpha \in \mathbb{R}$ and $n \geq 1$ we define $F_{\alpha,n} = b^{n\alpha} F_n$.

Let $\alpha_0 = \varphi_W(p_0)/p_0 = \varphi_W'(p_0)$, and $W^{(p_0)} = b^{\varphi_W(p_0)}(|W_0|^{p_0}, \dots, |W_{k-1}|^{p_0})$. By construction, we have (see Definition 2.1) $\varphi_{W^{(p_0)}}(1) = \varphi'_{W^{(p_0)}}(1) = 0$. We know from [25, 39] that this implies that $\lim_{n\to\infty} \sum_{w\in\mathscr{A}^n} \Delta F_{W^{(p_0)},n}(I_w) = 0$ almost surely. Consequently, since

$$\|F_{\alpha_0,n}\|_\infty \leq \sum_{w\in\mathscr{A}^n} |\Delta F_{\alpha_0,n}(I_w)| = \sum_{w\in\mathscr{A}^n} \Delta F_{W^{(p_0)},n}(I_w)^{1/p_0}$$

and $1/p_0 \geq 1$, we can conclude that $F_{\alpha,n}$ converges almost surely uniformly to 0 for all $\alpha \leq \alpha_0$.

Now let $\alpha > \alpha_0$ and set $V_\alpha = b^\alpha W$. We have $\varphi_{V_\alpha}(p) = \varphi_W(p) - \alpha p$. Thus, $\varphi'_{V_\alpha}(p_0)p_0 - \varphi_{V_\alpha}(p_0) = \varphi'_W(p_0)p_0 - \varphi_W(p_0) = 0$, $\varphi_{V_\alpha}(p_0) < 0$ and $\varphi'_{V_\alpha}(p_0) < 0$. Consequently, we can find $p \in (0, p_0)$ such that $\varphi'_{V_\alpha}(p) < 0$ and $\varphi'_{V_\alpha}(p)p - \varphi_{V_\alpha}(p) > 0$. Let $V_\alpha^{(p)} = b^{\varphi_{V_\alpha}(p)}(|V_{\alpha,0}|^p, \dots, V_{\alpha,b-1}|^p)$. We have $\varphi_{V_\alpha^{(p)}}(1) = 0$ and $\varphi'_{V_\alpha}(p)p - \varphi_{V_\alpha}(p) > 0$ is equivalent to $\varphi'_{V_\alpha^{(p)}}(1) > 0$, so the nondecreasing function $F_{V_\alpha^{(p)}}$ is well defined and differs from 0 with positive probability by Theorem 2.1. Let us denote by μ the measure $F'_{V_\alpha^{(p)}}$. It follows from Theorem IV(i) in [5] that, with probability 1, conditionally on $\mathscr{V}^c$, we have $\mu \neq 0$ and

$$\lim_{n\to\infty} \frac{\log_b |\Delta F_{\alpha,n}(I_n(t))|}{-n} = -\sum_{k=0}^{b-1} \mathbb{E}\big(V_{\alpha,k}^{(p)} \log_b |V_{\alpha,k}|\big) = \varphi'_{V_\alpha}(p) \qquad \text{for } \mu\text{-a.e. } t,$$

where $I_n(t)$ stands for the semi-open to the right b-adic interval of generation n containing t. Since $\varphi'_{V_\alpha}(p) < 0$, we conclude that $F_{\alpha,n}$ is unbounded.

(2) It is the same proof as in (1).

(3) We recall that in the conservative case we have $\mathbb{P}(\mathscr{V}^c) = 1$. If $p_0 < \infty$, the unboundedness result is proven as in 1. If $p_0 = \infty$ then $\sum_{i=0}^{b-1} \mathbb{P}(|W_i| = 1) > 1$ or $\mathbb{P}(\#\{i : |W_i| = 1\} = 1) = 1$.

We claim that, with probability 1, there exists an infinite word $w_1 \cdots w_n \cdots$ and an integer $n_0 \geq 1$ such that $|\Delta F_{n_0}(I_{w_1\cdots w_{n_0}})| \neq 0$ and $|W_{w_{n+1}}(w_1 \cdots w_n)| = 1$ for all $n \geq n_0$. This implies that the absolute value of the increment of F_n over the interval $I_{w_1\cdots w_n}$ is equal to the constant $|\Delta F_{n_0}(I_{w_1\cdots w_{n_0}})|$ for all $n \geq n_0$, so that $(F_n)_{n\geq 1}$ cannot converge uniformly.

Our claim is immediate if $\mathbb{P}(\#\{i : |W_i| = 1\} = 1) = 1$. Now, suppose that $\sum_{i=0}^{b-1} \mathbb{P}(|W_i| = 1) > 1$. If $n \geq 1$ and $w \in \mathscr{A}^n$, the set of words $v \in \bigcup_{p\geq n+1} \mathscr{A}^p$ with prefix w (i.e., $v|n = w$) such that $|W_{v_{k+1}}(v|k)| = 1$ for all $n \leq k \leq |v| - 1$ for a Galton–Watson tree $T(w)$ whose offspring distribution is given by that of $N_1 = \#\{i : |W_i| = 1\}$. Since $\sum_{i=0}^{b-1} \mathbb{P}(|W_i| = 1) > 1$ this tree is supercritical, so it is finite with a probability $q_1 < 1$. Moreover, the trees $T(w)$, $w \in \mathscr{A}^n$, are independent and independent of $\mathcal{F}_n = \sigma(Q(w) : w \in \mathscr{A}^n)$.

Let E_n be the event $\{\forall w \in \mathscr{A}^n, Q(w) = 0 \text{ or } T(w) \text{ is finite}\}$ and $\mathscr{Z}_n = \{w \in \mathscr{A}^n : Q(w) \neq 0\}$. We have $E_n = \{\forall w \in \mathscr{Z}_n, T(w) \text{ is finite}\}$. Thus

$$\mathbb{P}(E_n) = \mathbb{E}(\mathbb{P}(E_n|\mathcal{F}_n)) = \mathbb{E}(q_1^{\#\mathscr{Z}_n}) = (\mathbb{E}(q_1^N))^n,$$

where $N = \#\{0 \leq i \leq b-1 : |W_i| > 0\}$ [see also Remark 2.1(3)]. Now, since $N \geq 1$ almost surely and $q_1 < 1$, we have $\sum_{n\geq 1} \mathbb{P}(E_n) < \infty$. The conclusion follows from the Borel–Cantelli Lemma.

The result about the nonexistence of a normalization making the sequence weakly convergent is an obvious consequence of the fact that we have $\Delta F_n(I_w) = \Delta F_{|w|}(I_w)$ for all $w \in \mathscr{A}^*$ and $n \geq |w|$.

(4) follows from Theorem 2.6(1) and (2). Indeed, suppose that $\alpha > \varphi_W(2)/2$ and $(b^{n\alpha} F_n)_{n\geq 1}$ is bounded with positive probability. Since the boundedness of this sequence is clearly an event which is measurable with respect to $\bigcap_{p\geq 1} \sigma(W(w), |w| \geq p)$, it occurs with probability 1. In this case, $X_n = r_n F_n$ tends almost surely uniformly to 0, as $n \to \infty$. This contradicts the fact that the L^2 norm of $Z_n(1)$ converges to a positive value σ, and $X_n(1)$ is bounded in L^p norm for p close to 2^+.

4.2. *Proof of Theorem* 2.6. (1) Let $Y_0(w) = 1$ and $Y_n(w) = F_n^{[w]}(1)$ for all $n \geq 1$ and $w \in \mathscr{A}^*$. Also, when $\varphi_W(2) < 0$ let ℓ be the unique solution of $\ell = b^{-\varphi_W(2)}\ell + \sum_{i\neq j} \mathbb{E}(W_i \overline{W_j})$, that is, $\ell = \sum_{i\neq j} \mathbb{E}(W_i \overline{W_j})/(1 - b^{-\varphi_W(2)})$. We can deduce from (3.2) that

$$\mathbb{E}(|Y_n|^2) = b^{-\varphi_W(2)}\mathbb{E}(|Y_{n-1}|^2) + \sum_{i\neq j} \mathbb{E}(W_i \overline{W_j}).$$

Consequently, defining $v_n = \mathbb{E}(|Y_n|^2)$, for $n \geq 0$ we have $v_n = \ell + (v_0 - \ell)b^{-n\varphi_W(2)}$ if $\varphi_W(2) < 0$ and $v_n = v_0 + n\mathbb{E}(W_i \overline{W_j})$ if $\varphi_W(2) = 0$. If $\varphi_W(2) < 0$, we have $1 = v_0 \neq \ell$; otherwise we must have $1 = \sum_{i\neq j} \mathbb{E}(W_i \overline{W_j}) + \mathbb{E}(\sum_{i=0}^{b-1} |W_i|^2) = \mathbb{E}(|\sum_{i=0}^{b-1} W_i|^2)$, and since we have $\mathbb{E}(|\sum_{i=0}^{b-1} W_i|^2)^{1/2} \geq \mathbb{E}(|\sum_{i=0}^{b-1} W_i|) \geq \mathbb{E}(\sum_{i=0}^{b-1} W_i) = 1$, this implies that we are in the conservative case. Also, if $\varphi_W(2) = 0$, then $\sum_{i\neq j} \mathbb{E}(W_i \overline{W_j}) \neq 0$; otherwise $\mathbb{E}(|\sum_{i=0}^{b-1} W_i|^2) = \mathbb{E}(\sum_{i=0}^{b-1} |W_i|^2) = b^{-\varphi_W(2)} = 1$, and we have the same contradiction as in the previous discussion.

Consequently, we have $\mathbb{E}(|Y_n|^2) \sim (1 - \ell)b^{-n\varphi_W(2)} = \sigma^2 b^{-n\varphi_W(2)}$ if $\varphi_W(2) < 0$ and $\mathbb{E}(|Y_n|^2) \sim n \sum_{i\neq j} \mathbb{E}(W_i \overline{W_j}) = \sigma^2 n$ if $\varphi_W(2) = 0$.

(2) We denote by σ_n the equivalent of $\sqrt{\mathbb{E}(|F_n(1)|^2)}$ obtained in (1), that is, $\sigma_n = \sigma b^{-n\varphi_W(2)/2}$ if $\varphi_W(2) < 0$ and $\sigma_n = \sigma\sqrt{n}$ if $\varphi_W(2) = 0$, and we consider $Z_n = F_n/\mathscr{A}^n$ rather than $F_n/\sqrt{\mathbb{E}(|F_n(1)|^2)}$. For $w \in \mathscr{A}^*$, we also denote $F_n^{[w]}/\sigma_n$ by $Z_n^{[w]}$.

We leave the reader to check the following simple properties for $m, n \geq 1$ and $w \in \mathscr{A}^m$: If $n > m$, then

$$\Delta Z_n(I_w) = Q(w) \cdot \begin{cases} b^{m\varphi_W(2)/2} Z_{n-m}^{[w]}(1), & \text{if } \varphi_W(2) < 0, \\ \sqrt{\dfrac{n-m}{n}} Z_{n-m}^{[w]}(1), & \text{if } \varphi_W(2) = 0, \end{cases} \tag{4.1}$$

and if $1 \leq n \leq m$ then

$$\Delta Z_n(I_w) = Q(w|n) b^{n-m} \cdot \begin{cases} b^{n\varphi_W(2)/2}/\sigma, & \text{if } \varphi_W(2) < 0, \\ 1/\sigma\sqrt{n}, & \text{if } \varphi_W(2) = 0. \end{cases} \tag{4.2}$$

To simplify the notations, we denote $Z_n^{[w]}(1)$ by $\widetilde{Z}_n(w)$. Also, we define $\widetilde{W} = b^{\varphi_W(2)/2} W$. By construction, we have

$$\widetilde{Z}_{n+1} = \begin{cases} \displaystyle\sum_{k=0}^{b-1} \widetilde{W}_k \widetilde{Z}_n(k), & \text{if } \varphi_W(2) < 0, \\ \displaystyle\sum_{k=0}^{b-1} \widetilde{W}_k \sqrt{\frac{n}{n+1}} \widetilde{Z}_n(k), & \text{if } \varphi_W(2) = 0. \end{cases} \tag{4.3}$$

This is formally the same [or almost the same when $\varphi_W(2) = 0$] equality as $F_{n+1}(1) = \sum_{k=0}^{b-1} W_k F_n^{[k]}(1)$, with the same properties of independence and equidistribution. Moreover, we have $\varphi_{\widetilde{W}}(p) = \varphi_W(p) - p\varphi_W(2)/2$, and $\widetilde{Z}_n = \widetilde{Z}_n(\varnothing)$ is bounded in L^2 norm. Consequently, when $\varphi_W(p) - p\varphi_W(2)/2 > 0$, the boundedness of $\widetilde{Z}_n$ in L^p is obtained by induction on the integer part of p like for the proof of the boundedness in L^p of $(F_n(1))_{n\geq 1}$ when $\varphi_W(p) > 0$ (see, for instance, [7, 25, 39]).

We now study the tightness of the sequence $(Z_n)_{n\geq 1}$. By Theorem 7.3 of [19], since $Z_n(0) = 0$ almost surely for all $n \geq 1$, it is enough to show that for each positive ε

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}\big(\omega(Z_n, \delta) > \varepsilon\big) = 0 \tag{4.4}$$

(recall that $\omega(f, \cdot)$ stands for the modulus of continuity of f when $f \in \mathcal{C}([0, 1])$).

We fix $p > 2$ such that $\varphi_W(p)/p - \varphi_W(2)/2 > 0$ as well as $\gamma > 0$, and we estimate the L^p norm of the b-adic increments of Z_n by using (4.1) and (4.2).

If $n > m$, then

$$\sum_{w\in\mathscr{A}^m} \|\Delta Z_n(I_w)\|_p^p \leq b^{-pm(\varphi_W(p)/p - \varphi_W(2)/2)} \sup_{k\geq 1} \|Z_k(1)\|_p^p,$$

and if $1 \leq n \leq m$, then

$$\begin{aligned} \sum_{w\in\mathscr{A}^m} \|\Delta Z_n(I_w)\|_p^p &\leq \sum_{w\in\mathscr{A}^m} \mathbb{E}(|Q(w|n)|^p) b^{(n-m)p} b^{np\varphi_W(2)/2}/\sigma^p \\ &= \sum_{v\in\mathscr{A}^n} b^{m-n} \mathbb{E}(|Q(v)|^p) b^{(n-m)p} b^{np\varphi_W(2)/2)}/\sigma^p \\ &= b^{(m-n)(1-p)} b^{-np(\varphi_W(p)/p - \varphi_W(2)/2}/\sigma^p. \end{aligned}$$

Consequently, for all $m_0 \geq 1$ and $n \geq m_0$

$$\begin{aligned}
&\mathbb{P}(\exists m \geq m_0, \exists w \in \mathscr{A}^m : |\Delta Z_n(I_w)| > b^{-\gamma m}) \\
&\quad \leq \sum_{m \geq m_0} b^{p\gamma m} \sum_{w \in \mathscr{A}^m} \|\Delta Z_n(I_w)\|_p^p \\
&\quad \leq \sup_{k \geq 1} \|Z_k(1)\|_p^p \sum_{m=m_0}^{n-1} b^{-pm(\varphi_W(p)/p - \varphi_W(2)/2 - \gamma)} \\
&\qquad + \sigma^{-p} b^{-np(\varphi_W(p)/p - \varphi_W(2)/2 - \gamma)} \sum_{m \geq n} b^{(m-n)(1-p+\gamma p)} \\
&\quad = O\big(b^{-pm_0(\varphi_W(p)/p - \varphi_W(2)/2 - \gamma)}\big),
\end{aligned}$$

if we choose $\gamma \in (0, \min(\varphi_W(p)/p - \varphi_W(2)/2, (p-1)/p))$. We fix such a γ, define $\alpha = \varphi_W(p)/p - \varphi_W(2)/2 - \gamma$ and notice thanks to (3.4) that on the complement of $\{\exists m \geq m_0, \exists w \in \mathscr{A}^m : \Delta Z_n(I_w)| > b^{-\gamma m})\}$ we have $\omega(Z_n, b^{-m_0}) \leq 2b \cdot b^{-\gamma m_0}/(1 - b^{-\gamma})$. Consequently, we have obtained that, for $\delta \in (0, 1)$, and m_0 such that $b^{-m_0-1} < \delta \leq b^{-m_0}$, we have

$$\sup_{n \geq -\log_b(\delta)} \mathbb{P}\bigg(\omega(F_n, \delta) > \frac{2b^{1+\gamma}}{1 - b^{-\gamma}} \delta^\gamma\bigg) = O(\delta^\alpha).$$

(3) The properties of $W^{(2)}$ are obvious. Suppose that $(Z_n)_{n \geq 1}$ converges in distribution to a continuous process Z, as $n \to \infty$. Then the same holds for all the sequences $(Z_n^{[w]})_{n \geq 1}$, and by using (4.3) we see that the limit in distribution of the $Z(1)$ must satisfy

$$Z(1) = \sum_{k=0}^{b-1} \widetilde{W}_k Z^{[k]}(1).$$

Moreover, $\mathbb{E}(Z(1)) = 0$ and $\mathbb{E}(Z^2(1)) = 1$. Consequently, it follows from Theorem 4(ii) in [21] that the characteristic function of $Z(1)$ is given by $\mathbb{E}(\exp(-t^2 \times F_{W^{(2)}}(1)/2))$. It is clear that this is also the characteristic function of $\widetilde{Z} = B \circ F_{W^{(2)}}(1)$.

Now let us prove that for all $p \geq 1$, the vector $V_{p,n} = (\Delta Z_n(I_w))_{w \in \mathscr{A}^p}$ converges in distribution to $(\Delta B \circ F_{W^{(2)}}(I_w))_{w \in \mathscr{A}^p}$, as $n \to \infty$. This will yield the conclusion.

Fix $p \geq 1$ an integer. By definition of the processes $Z_n^{[w]}$, for $n > p$, we have

$$V_{p,n} = \big(r_{p,n} \widetilde{Q}(w) \cdot Z_{n-p}^{[w]}(1)\big)_{w \in \mathscr{A}^p}$$

with $r_{p,n} = 1$ if $\varphi_W(2) < 0$ and $r_{p,n} = \sqrt{(n-p)/n}$ otherwise, and $\widetilde{Q}(w) = \prod_{k=1}^p \widetilde{W}_{w_k}(w|k-1)$. Let $\mathcal{F}_p = \sigma(Q(w), w \in \mathscr{A}^p)$. The random variables $Z_{n-p}^{[w]}(1)$

are independent and independent of $\mathcal{F}_p$. Moreover, they converge in distribution to $\widetilde{Z}$. Thus, if $\xi = (\xi_w)_{w\in\mathscr{A}^p} \in \mathbb{R}^{\mathscr{A}^p}$, we have

$$\lim_{n\to\infty} \mathbb{E}(\exp(i\langle\xi|V_{p,n}\rangle)|\mathcal{F}_p) = \prod_{w\in\mathscr{A}^p} \phi_{\widetilde{Z}}(\xi_w \widetilde{Q}(w)),$$

where $\phi_{\widetilde{Z}}$ is the characteristic function of $\widetilde{Z}$. Consequently, there exists a family $\{F^{[w]}_{W^{(2)}}(1)\}_{w\in\mathscr{A}^p}$ of independent copies of $F_{W^{(2)}}(1)$, this family being also independent of $\mathcal{F}_p$, such that

$$\lim_{n\to\infty} \mathbb{E}(\exp(i\langle\xi|V_{p,n}\rangle)) = \mathbb{E}\biggl(\prod_{w\in\mathscr{A}^p} \exp\bigl(-\xi_w^2 \widetilde{Q}(w)^2 F^{[w]}_{W^{(2)}}(1)/2\bigr)\biggr).$$

To see that the right-hand side is the characteristic function of $U_p = (\Delta B \circ F_{W^{(2)}}(I_w))_{w\in\mathscr{A}^p}$, let us first define $Q^{(2)}(w) = \prod_{k=1}^p W^{(2)}_{w_k}(w|k-1)$. Notice that $Q^{(2)}(w) = \widetilde{Q}(w)^2$. By definition, $B \circ F_{W^{(2)}}(I_w) = B(F_{W^{(2)}}(t_w + b^{-p})) - B(F_{W^{(2)}}(t_w))$. Consequently, if $\xi \in \mathbb{R}^{\mathscr{A}^p}$,

$$\begin{aligned}\mathbb{E}\bigl(\exp(i\langle\xi|U_p\rangle)|\sigma(\mathcal{F}_p : p \ge 1)\bigr) &= \prod_{w\in\mathscr{A}^p} \exp\bigl(-\xi_w^2 \Delta F_{W^{(2)}}(I_w)/2\bigr)\\ &= \prod_{w\in\mathscr{A}^p} \exp\bigl(-\xi_w^2 Q^{(2)}(w) F^{[w]}_{W^{(2)}}(1)/2\bigr).\end{aligned}$$

Since $(\widetilde{Q}(w)^2)_{w\in\mathscr{A}^p} = (Q^{(2)}(w))_{w\in\mathscr{A}^p}$, we have the result.

4.3. *Proof of Theorem* 2.7. The following proposition will be crucial. We postpone its proof until the end of the section.

PROPOSITION 4.1. *The probability distribution of $\widetilde{Z} = B \circ F_{W^{(2)}}(1)$ is symmetric and determined by its moments. Specifically, for every positive integer q define $S_q = \{\beta = (\beta_0,\dots,\beta_{b-1}) \in \mathbb{N}^b : 0 \le \beta_0,\dots,\beta_{b-1} < q, \sum_{k=0}^{b-1}\beta_k = q\}$ and $M^{(q)} = \mathbb{E}(\widetilde{Z}^q)$. Also let $\widetilde{W} = b^{\varphi_W(2)/2}W$. We have $M^{(2)} = 1$ and for every integer $p \ge 2$*

$$M^{(2p)} = \bigl(1 - b^{-\varphi_{\widetilde{W}}(2p)}\bigr)^{-1} \sum_{\substack{\beta\in S_{2p}\\ \beta_k\equiv 0[2]}} \gamma_\beta \mathbb{E}\Biggl(\prod_{k=0}^{b-1} \widetilde{W}_k^{\beta_k}\Biggr) M^{(\beta_k)}, \tag{4.5}$$

where $\gamma_{\beta_0,\dots,\beta_{b-1}} = \frac{q!}{(\beta_0)!\cdots(\beta_{b-1})!}$.

We now prove that the random variables $\widetilde{Z}_n = Z_n(1)$, $n \ge 1$, converge in distribution to $\widetilde{Z}$.

For all positive integers q and n, let $\widetilde{M}_n^{(q)} = \mathbb{E}(\widetilde{Z}_n^q)$, and as in Proposition 4.1 let $S_q = \{\beta = (\beta_0,\dots,\beta_{b-1}) \in \mathbb{N}^b : 0 \le \beta_0,\dots,\beta_{b-1} < q, \sum_{k=0}^{b-1}\beta_k = q\}$.

Due to Proposition 4.1, to get the convergence in distribution of $\widetilde{Z}_n$ to $\widetilde{Z}$, it is enough to show the three following properties:

(1) for every $p \geq 0$ one has the property $(\mathcal{P}_{2p})$: $\widetilde{M}^{(2p)} = \lim_{n\to\infty} \widetilde{M}_n^{(2p)}$ exists. Moreover $\widetilde{M}^{(2)} = 1$;

(2) for every $p \geq 0$ one has the property $(\mathcal{P}_{2p+1})$: $\lim_{n\to\infty} \widetilde{M}_n^{(2p+1)} = 0$;

(3) for $p \geq 2$

$$\widetilde{M}^{(2p)} = \left(1 - b^{-\varphi_{\widetilde{W}}(2p)}\right)^{-1} \sum_{\substack{\beta\in S_{2p}\\ \beta_k\equiv 0[2]}} \gamma_\beta \mathbb{E}\left(\prod_{k=0}^{b-1} \widetilde{W}_k^{\beta_k}\right) \widetilde{M}^{(\beta_k)}.$$

For $n \geq 1$ let $r_n = 1$ if $\varphi_W(2) < 0$ and $r_n = \sqrt{\frac{n}{n+1}}$ otherwise. Let q be an integer ≥ 3. Raising (4.3) to the power q and taking the expectation yields

$$\widetilde{M}_{n+1}^{(q)} = r_n^q b^{-\varphi_{\widetilde{W}}(q)} \widetilde{M}_n^{(q)} + r_n^q \sum_{\beta\in S_q} \gamma_\beta \mathbb{E}\left(\prod_{k=0}^{b-1} \widetilde{W}_k^{\beta_k}\right) \widetilde{M}_n^{(\beta_k)}. \tag{4.6}$$

We show by induction that $((\mathcal{P}_{2p-1}), (\mathcal{P}_{2p}))$ holds for $p \geq 1$, and we deduce the relation (3).

At first, $((\mathcal{P}_1), (\mathcal{P}_2))$ holds by construction. Suppose that $((\mathcal{P}_{2k-1}), (\mathcal{P}_{2k}))$ holds for $1 \leq k \leq p-1$, with $p \geq 2$. In particular, $M_n^{(\beta_k)}$ goes to 0 as n goes to ∞ if β_k is an odd integer belonging to $[1, 2p-3]$. Every element of the set S_{2p-1} must contain an odd component. Due to our induction assumption, this implies that in relation (4.6), the term

$$r_n^{2p-1} b^{-(2p-1)/2} \sum_{\beta\in S_{2p-1}} \gamma_\beta \mathbb{E}\left(\prod_{k=0}^{b-1} \widetilde{W}_k^{\beta_k}\right) \widetilde{M}_n^{(\beta_k)}$$

on the right-hand side goes to 0 at ∞. This yields

$$\widetilde{M}_{n+1}^{(2p-1)} = r_n^{2p-1} b^{-\varphi_{\widetilde{W}}(2p-1)} \widetilde{M}_n^{(2p-1)} + o(1)$$

as $n \to \infty$. Since we have $r_n^{2p-1} b^{-\varphi_{\widetilde{W}}(2p-1)} \leq b^{-\varphi_{\widetilde{W}}(2p-1)} < 1$, this yields $\lim_{n\to\infty} \widetilde{M}_n^{(2p-1)} = 0$, that is to say $(\mathcal{P}_{2p-1})$.

Now, the same argument as above shows that on the right-hand side of $\widetilde{M}_{n+1}^{(2p)}$, we have

$$\lim_{n\to\infty} r_n^{2p} \sum_{\beta\in S_{2p}} \gamma_\beta \mathbb{E}\left(\prod_{k=0}^{b-1} \widetilde{W}_k^{\beta_k}\right) \widetilde{M}_n^{(\beta_k)} = \sum_{\substack{\beta\in S_{2p}\\ \beta_k\equiv 0[2]}} \gamma_\beta \mathbb{E}\left(\prod_{k=0}^{b-1} \widetilde{W}_k^{\beta_k}\right) \widetilde{M}^{(\beta_k)}.$$

Denote by L the right-hand side of the above relation. By using (4.6) we deduce from the previous lines that

$$\widetilde{M}_{n+1}^{(2p)} = r_n^{2p} b^{-\varphi_{\widetilde{W}}(2p)} \widetilde{M}_n^{(2p)} + L + o(1).$$

Consequently, since $\lim_{n\to\infty} r_n = 1$, $\widetilde{M}_n^{(2p)}$ converges to the unique solution of $m = b^{-\varphi_{\widetilde{W}}(2p)} m + L$. This yields both $(\mathcal{P}_{2p})$ and (3).

PROOF OF PROPOSITION 4.1. Due to our assumption $\varphi_{W^{(2)}}(p) > 0$ for all $p > 1$, it follows from Theorem 4.1 in [41] that $\limsup_{k\to\infty} \|F_{W^{(2)}}(1)\|_k/k < \infty$. We also have $\limsup_{k\to\infty} \|B(1)\|_k/k < \infty$. Consequently, since conditionally on $F_{W^{(2)}}$ we have $B \circ F_{W^{(2)}}(1) \equiv F_{W^{(2)}}(1)^{1/2} B(1)$, we have $\limsup_{k\to\infty} \|B \circ F_{W^{(2)}}(1)\|_k/k < \infty$. This ensures (see Proposition 8.49 in [20]) that $B \circ F_{W^{(2)}}(1)$ is determined by its moments.

Now we notice that there exists a vector $(\widetilde{Z}(0), \dots, \widetilde{Z}(b-1))$ independent of $\widetilde{W}$ and whose components are independent copies of $\widetilde{Z}$ such that

$$\widetilde{Z} \equiv \sum_{k=0}^{b-1} \widetilde{W}_k \widetilde{Z}(k). \tag{4.7}$$

This is a consequence of Theorem 4(ii) in [21] as we said in proving Theorem 2.6(2). Since by construction $\mathbb{E}(\widetilde{Z}) = 0$ and $\mathbb{E}(\widetilde{Z}^2) = 1$, we can exploit (4.7) in the same way as (4.3) to get (4.5).

For the sake of completeness, we give a direct argument for (4.7). If we consider the random functions $F_{W^{(2)}}^{[0]}, \dots, F_{W^{(2)}}^{[0]}$ as well as b independent Brownian motions $B_0, \dots, B_{b-1}$, and if we set $\widetilde{Z}(k) = B_k \circ F_{W^{(2)}}^{[k]}(1)$, then for $\xi \in \mathbb{R}$ we have

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(i\left\langle \xi \middle| \sum_{k=0}^{b-1} \widetilde{W}_k \widetilde{Z}(k)\right\rangle\right) \middle| \sigma(\mathcal{F}_p : p \geq 1)\right) \\
&\quad = \prod_{k=0}^{b-1} \exp\bigl(-\xi^2 \widetilde{W}_k^2 F_{W^{(2)}}^{[k]}(1)/2\bigr) \\
&\quad = \prod_{k=0}^{b-1} \exp\bigl(-\xi^2 W_k^{(2)} F_{W^{(2)}}^{[k]}(1)/2\bigr) \\
&\quad = \exp\bigl(-\xi^2 F_{W^{(2)}}(1)/2\bigr).
\end{aligned}$$

We have seen in the previous proof that the expectation of the right-hand side is the characteristic function of $\widetilde{Z}$. This yields (4.7). The previous computation shows that we can obtain Theorem 2.6(4) without using [21]. □

4.4. *Proof of Theorem* 2.8. At first we notice that under our assumptions we have $\varphi_W(2) > 0$. Then, since we can write $(F_n - F)(1) = \sum_{w\in\mathscr{A}^n} Q(w)(1 - F_W^{[w]}(1))$ and the terms $(1 - F_W^{[w]}(1))$ are centered, independent and independent of the $Q(w)$, it is not difficult to see that $\mathbb{E}((F_n - F)(1)^2) = \sigma^2 b^{-n\varphi_W(2)}$ where $\sigma = \sqrt{\mathbb{E}((1 - F_W(1))^2)} > 0$. Also, $\varphi_{W^{(2)}}(p) > 0$ in a neighborhood of 1^+, so $F_{W^{(2)}}$ is nondegenerate.

For $w \in \mathscr{A}^*$ let $R_n^{[w]} = (F_n^{[w]} - F^{[w]})/\sigma b^{-n\varphi_W(2)/2}$. For $m, n \geq 1$ and $w \in \mathscr{A}^m$: If $n > m$, then

$$\Delta R_n(I_w) = |Q(w)| \cdot b^{m\varphi_W(2)/2} R_{n-m}^{[w]}(1), \tag{4.8}$$

and if $1 \leq n \leq m$, then

$$\Delta R_n(I_w) = b^{n-m} b^{n\varphi_W(2)/2} Q(w|n)/\sigma - b^{n\varphi_W(2)/2} Q(w) F^{[w]}(1)/\sigma,$$

and since $\varphi_W(2) > 0$,

$$|\Delta R_n(I_w)| \leq b^{n-m} b^{n\varphi_W(2)/2} |Q(w|n)|/\sigma + b^{m\varphi_W(2)/2} |Q(w)| \big|F^{[w]}(1)\big|/\sigma. \tag{4.9}$$

We deduce from (4.8) that

$$R_n(1) = \sum_{i=0}^{b-1} b^{\varphi_W(2)/2} W_i R_{n-1}^{[i]}(1).$$

Then the same arguments as in the proof of Theorem 2.5(2) show that $R_n(1)$ is bounded in L^p norm for p in a neighborhood of 2^+. Also, (4.8) and (4.9) can be used to prove the tightness of the sequence $(\mathcal{L}(R_n))_{n\geq 1}$ like (4.1) and (4.2) where used to prove the tightness of $(\mathcal{L}(Z_n))_{n\geq 1}$ in the proof of Theorem 2.5(2).

Now let us prove that for all $p \geq 1$, the vector $V_{p,n} = (\Delta R_n(I_w))_{w\in\mathscr{A}^p}$ converges in distribution to $(\Delta B \circ F_{W^{(2)}}(I_w))_{w\in\mathscr{A}^p}$, as $n \to \infty$. This will yield the conclusion. We adopt the same notations as in the proof of Theorem 2.5(2).

By definition of the processes $R_n^{[w]}$, for every $p \geq 1$ and $n > p$, we have

$$V_{p,n} = \big(\widetilde{Q}(w) \cdot R_{n-p}^{[w]}(1)\big)_{w\in\mathscr{A}^p}.$$

Consequently, due to the proof of Theorem 2.5(2), it only remains to prove that $R_p(1)$ converges in distribution to $\widetilde{Z}(1)$. To do this, we remark that it follows from the argument used to prove Proposition 4.1 and its Corollary 4.3 in [55] [which can be directly applied to $R_p(1)$ when W has nonnegative i.i.d. components] that conditionally on $\mathcal{F}_p$, $R_p(1)$ converges in distribution to a centered normal law of standard deviation $\sqrt{F_{W^{(2)}}(1)}$. This implies that $R_p(1)$ converges in law to $\widetilde{Z}(1)$ as $p \to \infty$ [see the expression of the characteristic function of $\widetilde{Z}(1)$ in the proof of Theorem 2.5(3)].

REFERENCES

[1] ARBEITER, M. and PATZSCHKE, N. (1996). Random self-similar multifractals. *Math. Nachr.* **181** 5–42. MR1409071

[2] ARNEODO, A., BACRY, E. and MUZY, J. F. (1998). Random cascades on wavelet dyadic trees. *J. Math. Phys.* **39** 4142–4164. MR1633148

[3] BACRY, E. and MUZY, J. F. (2003). Log-infinitely divisible multifractal processes. *Comm. Math. Phys.* **236** 449–475. MR2021198

[4] BACRY, E., KOZHEMYAK, A. and MUZY, J.-F. (2008). Continuous cascade models for asset returns. *J. Econom. Dynam. Control* **32** 156–199. MR2381693

[5] BARRAL, J. (1999). Moments, continuité, et analyse multifractale des martingales de Mandelbrot. *Probab. Theory Related Fields* **113** 535–569. MR1717530

[6] BARRAL, J. (2000). Continuity of the multifractal spectrum of a random statistically self-similar measure. *J. Theoret. Probab.* **13** 1027–1060. MR1820501

[7] BARRAL, J. (2001). Generalized vector multiplicative cascades. *Adv. in Appl. Probab.* **33** 874–895. MR1875784

[8] BARRAL, J. (2003). Poissonian products of random weights: Uniform convergence and related measures. *Rev. Mat. Iberoamericana* **19** 813–856. MR2053565

[9] BARRAL, J. and JIN, X. (2009). Multifractal analysis of complex random cascades. *Comm. Math. Phys.* To appear.

[10] BARRAL, J., JIN, X. and MANDELBROT, B. (2010). Uniform convergence for complex [0, 1]-martingales. *Ann. Appl. Probab.* **20** 1205–1218.

[11] BARRAL, J. and MANDELBROT, B. B. (2002). Multifractal products of cylindrical pulses. *Probab. Theory Related Fields* **124** 409–430. MR1939653

[12] BARRAL, J. and MANDELBROT, B. B. (2004). Random multiplicative multifractal measures I, II, III. In *Fractal Geometry and Applications*: *A Jubilee of Benoît Mandelbrot* (M. Lapidus and M. V. Frankenkuijsen, eds.). *Proc. Sympos. Pure Math.* **72** 3–90. Amer. Math. Soc., Providence, RI.

[13] BARRAL, J. and MANDELBROT, B. (2009). Fractional multiplicative processes. *Ann. Inst. H. Poincaré Probab. Statist.* **45** 1116–1129. MR2572167

[14] BARRAL, J. and SEURET, S. (2005). From multifractal measures to multifractal wavelet series. *J. Fourier Anal. Appl.* **11** 589–614. MR2182637

[15] BARRAL, J. and SEURET, S. (2007). The singularity spectrum of Lévy processes in multifractal time. *Adv. Math.* **214** 437–468. MR2348038

[16] BARRAL, J. and SEURET, S. (2007). Renewal of singularity sets of random self-similar measures. *Adv. in Appl. Probab.* **39** 162–188. MR2307876

[17] BEDFORD, T. (1989). Hölder exponents and box dimension for self-affine fractal functions. *Constr. Approx.* **5** 33–48. MR982723

[18] BIGGINS, J. D. (1992). Uniform convergence of martingales in the branching random walk. *Ann. Probab.* **20** 137–151. MR1143415

[19] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2nd ed. Wiley, New York. MR1700749

[20] BREIMAN, L. (1992). *Probability*. *Classics in Applied Mathematics* **7**. SIAM, Philadelphia, PA. MR1163370

[21] CALIEBE, A. and RÖSLER, U. (2003). Fixed points with finite variance of a smoothing transformation. *Stochastic Process. Appl.* **107** 105–129. MR1995923

[22] CHAINAIS, P., RIEDI, R. and ABRY, P. (2005). On non-scale-invariant infinitely divisible cascades. *IEEE Trans. Inform. Theory* **51** 1063–1083. MR2237970

[23] COLLET, P. and KOUKIOU, F. (1992). Large deviations for multiplicative chaos. *Comm. Math. Phys.* **147** 329–342. MR1174416

[24] DERRIDA, B., EVANS, M. R. and SPEER, E. R. (1993). Mean field theory of directed polymers with random complex weights. *Comm. Math. Phys.* **156** 221–244. MR1233845

[25] DURRETT, R. and LIGGETT, T. M. (1983). Fixed points of the smoothing transformation. *Z. Wahrsch. Verw. Gebiete* **64** 275–301. MR716487

[26] FALCONER, K. J. (1994). The multifractal spectrum of statistically self-similar measures. *J. Theoret. Probab.* **7** 681–702. MR1284660

[27] FAN, A. H. (2002). On Markov–Mandelbrot martingales. *J. Math. Pures Appl.* (9) **81** 967–982. MR1946911

[28] FRISCH, U. and PARISI, G. (1985). Fully developed turbulence and intermittency in turbulence. *Proc. International Summer School on Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics* (*Proc. Int. Sch. Phys.*, *"Enrico Fermi," Course LXXXVIII*) 84–88. North-Holland, Amsterdam.

[29] GILBERT, A. C., WILLINGER, W. and FELDMANN, A. (1999). Scaling analysis of conservative cascades, with applications to network traffic. *IEEE Trans. Inform. Theory* **45** 971–991. MR1682523

[30] GUIVARC'H, Y. (1990). Sur une extension de la notion de loi semi-stable. *Ann. Inst. H. Poincaré Probab. Statist.* **26** 261–285. MR1063751

[31] GUPTA, V. K. and WAYMIRE, E. C. (1993). A statistical analysis of mesoscale rainfall as a random cascade. *J. Appl. Meteor.* **32** 251–267.

[32] HEURTEAUX, Y. (2003). Weierstrass functions with random phases. *Trans. Amer. Math. Soc.* **355** 3065–3077 (electronic). MR1974675

[33] HOLLEY, R. and WAYMIRE, E. C. (1992). Multifractal dimensions and scaling exponents for strongly bounded random cascades. *Ann. Appl. Probab.* **2** 819–845. MR1189419

[34] JAFFARD, S. (1998). Oscillation spaces: Properties and applications to fractal and multifractral functions. *J. Math. Phys.* **39** 4129–4141. MR1633152

[35] KAHANE, J.-P. (1985). Sur le chaos multiplicatif. *Ann. Sci. Math. Québec* **9** 105–150. MR829798

[36] KAHANE, J.-P. (1987). Positive martingales and random measures. *Chinese Ann. Math. Ser. B* **8** 1–12. MR886744

[37] KAHANE, J.-P. (1987). Multiplications aléatoires et dimensions de Hausdorff. *Ann. Inst. H. Poincaré Probab. Statist.* **23** 289–296. MR898497

[38] KAHANE, J.-P. (1991). Produits de poids aléatoires indépendants et applications. In *Fractal Geometry and Analysis* (*Montreal*, *PQ*, *1989*). *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.* **346** 277–324. Kluwer, Dordrecht. MR1140725

[39] KAHANE, J. P. and PEYRIÈRE, J. (1976). Sur certaines martingales de Benoit Mandelbrot. *Adv. Math.* **22** 131–145. MR0431355

[40] KÔNO, N. (1986). On self-affine functions. *Japan J. Appl. Math.* **3** 259–269. MR899223

[41] LIU, Q. (1996). The growth of an entire characteristic function and the tail probabilities of the limit of a tree martingale. In *Trees* (*Versailles*, *1995*). *Progress in Probability* **40** 51–80. Birkhäuser, Basel. MR1439972

[42] LIU, Q. (2001). Asymptotic properties and absolute continuity of laws stable by random weighted mean. *Stochastic Process. Appl.* **95** 83–107. MR1847093

[43] LIU, Q. (2002). An extension of a functional equation of Poincaré and Mandelbrot. *Asian J. Math.* **6** 145–168. MR1902651

[44] LIU, Q. and ROUAULT, A. (2000). Limit theorems for Mandelbrot's multiplicative cascades. *Ann. Appl. Probab.* **10** 218–239. MR1765209

[45] LIU, Q., RIO, E. and ROUAULT, A. (2003). Limit theorems for multiplicative processes. *J. Theoret. Probab.* **16** 971–1014. MR2033195

[46] LUDEÑA, C. (2008). L^p-variations for multifractal fractional random walks. *Ann. Appl. Probab.* **18** 1138–1163. MR2418240

[47] MANDELBROT, B. (1972). Possible refinement of the lognormal hypothesis concerning the distribution of energy dissipation in intermittent turbulence. In *Statistical Models and Turbulence*. (M. Rosenblatt and C. Van Atta, eds.). *Lecture Notes in Phys.* **12** 333–351. Springer, Berlin. MR0438885

[48] MANDELBROT, B. (1974). Multiplications aléatoires itérées et distributions invariantes par moyennes pondérées. *C. R. Acad. Sci. Paris* **278** 289–292 and 355–358.

[49] MANDELBROT, B. (1974). Intermittent turbulence in self-similar cascades: Divergence of hight moments and dimension of the carrier. *J. Fluid. Mech.* **62** 331–358.

[50] MANDELBROT, B. B. (1997). *Fractals and Scaling in Finance*. Springer, New York. MR1475217

[51] MANDELBROT, B. B. and JAFFARD, S. (1997). Peano–Pólya motions, when time is intrinsic or binomial (uniform or multifractal). *Math. Intelligencer* **19** 21–26. MR1488862

[52] MANNERSALO, P., NORROS, I. and RIEDI, R. H. (2002). Multifractal products of stochastic processes: Construction and some basic properties. *Adv. in Appl. Probab.* **34** 888–903. MR1938947

[53] MEYER, M. and STIEDL, O. (2003). Self-affine fractal variability of human heartbeat interval dynamics in health and disease. *Eur. J. Appl. Physiol.* **90** 305–316.

[54] MOLCHAN, G. M. (1996). Scaling exponents and multifractal dimensions for independent random cascades. *Comm. Math. Phys.* **179** 681–702. MR1400758

[55] OSSIANDER, M. and WAYMIRE, E. C. (2000). Statistical estimation for multiplicative cascades. *Ann. Statist.* **28** 1533–1560. MR1835030

[56] RIEDI, R. H. (2003). Multifractal processes. In *Theory and Applications of Long-Range Dependence* 625–716. Birkhäuser, Boston, MA. MR1957510

[57] RIEDI, R. H. and LÉVY-VÉHEL, J. (1997). TCP Traffic is multifractal: A numerical study. Techical report INRIA-RR-3129, INRIA.

[58] SEURET, S. (2009). On multifractality and time subordination for continuous functions. *Adv. Math.* **220** 936–963. MR2483233

[59] STANLEY, H. E., AMARAL, L. A. N., GOLDBERGER, A. L., HAVLIN, S., IVANOV, P. C. and PENG, C. K. (1999). Statistical physics and physiology: Monofractal and multifractal approaches. *Physica A* **270** 309–324.

[60] URBAŃSKI, M. (1990). The probability distribution and Hausdorff dimension of self-affine functions. *Probab. Theory Related Fields* **84** 377–391. MR1035663

[61] WAYMIRE, E. C. and WILLIAMS, S. C. (1995). Multiplicative cascades: Dimension spectra and dependence. In *Proceedings of the Conference in Honor of Jean-Pierre Kahane* (*Orsay*, *1993*) 589–609. MR1364911

[62] WAYMIRE, E. C. and WILLIAMS, S. C. (1996). A cascade decomposition theory with applications to Markov and exchangeable cascades. *Trans. Amer. Math. Soc.* **348** 585–632. MR1322959

J. BARRAL
LAGA (UMR 7539), INSTITUT GALILÉE
UNIVERSITÉ PARIS 13
99 AVENUE JEAN-BAPTISTE CLÉMENT
93430 VILLETANEUSE
FRANCE
E-MAIL: barral@math.univ-paris13.fr

X. JIN
INRIA
DOMAINE DE VOLUCEAU
78153 LE CHESNAY CEDEX
FRANCE
E-MAIL: Xiong.Jin@inria.fr

B. MANDELBROT
DEPARTMENT OF MATHEMATICS
YALE UNIVERSITY
NEW HAVEN, CONNECTICUT 06520
USA
E-MAIL: Benoit.Mandelbrot@yale.edu