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# A functional limit theorem for irregular SDEs

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**Abstract.** Let $X_1, X_2, \ldots$ be a sequence of i.i.d. real-valued random variables with mean zero, and consider the scaled random walk of the form $Y_{k+1}^N = Y_k^N + a_N(Y_k^N)X_{k+1}$, where $a_N : \mathbb{R} \to \mathbb{R}_+$. We show, under mild assumptions on the law of $X_i$, that one can choose the scale factor $a_N$ in such a way that the process $(Y_{\lfloor Nt \rfloor}^N)_{t\in\mathbb{R}_+}$ converges in distribution to a given diffusion $(M_t)_{t\in\mathbb{R}_+}$ solving a stochastic differential equation with possibly irregular coefficients, as $N \to \infty$. To this end we embed the scaled random walks into the diffusion $M$ with a sequence of stopping times with expected time step $1/N$.

**Résumé.** Soit $X_1, X_2, \ldots$ une suite de variables aléatoires indépendantes avec espérance $E(X_i) = 0$, et $Y_{k+1}^N = Y_k^N + a_N(Y_k^N)X_{k+1}$ une marche aléatoire renormalisée avec une fonction $a_N : \mathbb{R} \to \mathbb{R}_+$. On montre, sous certaines conditions légères sur la loi de $X_i$, que l'on peut choisir le facteur $a_N$ d'une facon que $(Y_{\lfloor Nt \rfloor}^N)_{t\in\mathbb{R}_+}$ converge en loi, quand $N$ tend vers l'infini, vers une diffusion $(M_t)_{t\in\mathbb{R}_+}$ étant la solution d'une équation differentielle stochastique avec des coefficients irréguliers. À cet effet, nous plongeons la marche aléatoire renormalisée dans la diffusion $M$ par une suite de temps d'arrêt ayant un pas de temps avec espérance $1/N$.

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## Introduction

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. integrable random variables with $E(X_i) = 0$. Let $a_N : \mathbb{R} \to \mathbb{R}_+$ be a function depending on $N \in \mathbb{N}$, and let $(Y_k^N)_{k\in\mathbb{N}_0}$ be the process satisfying $Y_0^N = m \in \mathbb{R}$ and

$$Y_{k+1}^N = Y_k^N + a_N\bigl(Y_k^N\bigr)X_{k+1}, \quad k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}. \tag{1}$$

We extend $Y^N$ to a continuous time processes by defining $Y_t^N = Y_{\lfloor t \rfloor}^N + (t - \lfloor t \rfloor)(Y_{\lfloor t \rfloor+1}^N - Y_{\lfloor t \rfloor}^N)$.

Consider the particular case where $E(X_i^2) = 1$ and $a_N$ is constant equal to $\frac{1}{\sqrt{N}}$. Then $(Y_k^N)_{k\in\mathbb{N}_0}$ is the random walk generated by $(X_k)_{k\in\mathbb{N}_0}$, scaled by the constant $\frac{1}{\sqrt{N}}$, and Donsker's theorem implies that the continuous-time process $(Y_{Nt}^N)_{t\in\mathbb{R}_+}$ converges in distribution to a Brownian motion as $N \to \infty$ (see e.g. [3,14] or Section 8.6 in [4]).

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In this paper we address the question of whether one can choose the scale factor $a_N$ in such a way that the scaled random walk $(Y^N_{Nt})_{t\in\mathbb{R}_+}$ converges in distribution to a time homogeneous diffusion $M$ satisfying the stochastic differential equation (SDE)

$$dM_t = \eta(M_t)\,dW_t, \qquad M_0 = m, \tag{2}$$

where $W$ is a Brownian motion, and $\eta\colon\mathbb{R}\to\mathbb{R}$ is a Borel-measurable function that satisfies the Engelbert–Schmidt conditions (see [5]) in some interval $(l,r)$, $-\infty\le l<r\le\infty$, and vanishes outside $(l,r)$.

If convergence takes place, then one can use the limiting process $M$ as a proxy for the scaled random walk $Y^N$ for large $N$; or vice versa, $Y^N$ can be used for approximating the SDE $M$. One can thus profit from tools for continuous-time *and* discrete-time processes for analyzing both processes $M$ and $Y^N$.

If $\eta$ is Lipschitz continuous, then a natural choice for the scale factor is $a_N(y)=\frac{1}{\sqrt{N}}\eta(y)$. Then $(Y^N_k)_{k\in\mathbb{N}_0}$ can be interpreted as the Euler approximation of $M$, and it is known that it converges in distribution to $M$ (see e.g. [10]). For *arbitrary* diffusion coefficients $\eta$ satisfying the Engelbert–Schmidt conditions the question of whether there exist scale factors such $Y^N$ converges to $M$ has not been solved. If the diffusion coefficient is very irregular, then the diffusion intensity $\eta(x_0)$ at a fixed state point $x_0$ can not be used as an approximation of the diffusion coefficient in the neighborhood of $x_0$. Therefore, in order to have convergence, the scaling factors $a_N$ need to take into account the global structure of $\eta$.

Recall that Skorokhod proves Donsker's theorem by embedding in law the random walk scaled by the constant $\frac{1}{\sqrt{N}}$ into the Brownian motion with a sequence of stopping times (see [15]). We take on Skorokhod's idea and show, under some nice conditions on the distribution of $X_i$, that there exists a scale factor $a_N\colon(l,r)\to(0,\infty)$ such that $(Y^N_k)_{k\in\mathbb{N}_0}$ can be embedded into the diffusion $M$ with a sequence of stopping times with expected time step $1/N$.

Loosely speaking, the embedding works as follows. We first choose $a_N(m)$ (recall that $m$ is the starting point in (2)) and an $(\mathcal{F}_t)$-stopping time $\rho_1$ such that $E(\rho_1)=1/N$ and $M_{\rho_1}\stackrel{d}{=}Y^N_1$, where $(\mathcal{F}_t)$ denotes the underlying filtration. Conditionally on $\{M_{\rho_1}=y\}$ we choose $a_N(y)$ and an $(\mathcal{F}_{\rho_1+t})$-stopping time $\rho_2$ such that $E(\rho_2)=1/N$ and $M_{\rho_1+\rho_2}\stackrel{d}{=}y+a_N(y)X_2$. By proceeding like this we obtain a sequence of $(\mathcal{F}_t)$-stopping times $\tau_k=\rho_1+\dots+\rho_k$ such that $(M_{\tau_k})_{k\in\mathbb{N}_0}$ has the same distribution as the scaled random walk $(Y^N_k)_{k\in\mathbb{N}_0}$.

The times $\rho_k$ turn out to be pairwise uncorrelated and we can check that they satisfy a certain uniform integrability property (see Lemma 3.4). Under such a uniform integrability property we prove a version of the weak law of large numbers for uncorrelated arrays, which is also interesting in itself because we do not require finiteness of the second moments (see Theorem 3.2). This weak law of large numbers entails that for all $t\in\mathbb{R}_+$ we have $\tau_{\lfloor Nt\rfloor}\to t$ in probability, as $N\to\infty$. From this, one can deduce that $(M_{\tau_{\lfloor Nt\rfloor}})_{t\in\mathbb{R}_+}$ converges in probability to $M$ uniformly on compact time intervals. Therefore, $(Y^N_{Nt})_{t\in\mathbb{R}_+}$ converges in distribution to $M$.

For our approach to work one needs to make sure that for every $N\in\mathbb{N}$ and $y\in(l,r)$ there exists a scale factor $a_N(y)$ such that the distribution of $y+a_N(y)X_i$ can be embedded into the diffusion $M$, conditioned to $M_0=y$, with a stopping time with expectation $1/N$. The collection of distributions that can be embedded into $M$ with integrable stopping times is fully described in [1]. Moreover, there is a closed form integral expression for the minimal expectation of an embedding stopping time (see Theorem 3 in [1]). This allows us to derive weak sufficient conditions (see Section 2) for the existence of a scale factor $a_N\colon(l,r)\to(0,\infty)$ such that $(Y^N_k)_{k\in\mathbb{N}_0}$ can be embedded into $M$ with stopping times having expectation $1/N$.

Our approach to generalize Donsker's theorem is essentially different from the one pioneered by Stone in [16] (also see [2] for a recent generalization to tree-valued processes). In that approach the approximating processes are *continuous-time* Markov processes that *do not jump over points in their state spaces* (that is, they can be e.g. diffusions or birth and death processes). On the contrary, in this paper we approximate $M$ via *discrete-time* Markov chains. Another conceptual difference is that we develop our theory without requiring that the approximating Markov chains do not jump over points. On an informal level, one might view conditions (18)–(19) and (26)–(29) at which we arrive in Section 2 as an indication of what comes out when we want to allow overjumping.

The paper is organized in the following way. In Section 1, we recall a necessary and sufficient condition, derived in [1], for a distribution to be embeddable into the diffusion $M$ with an *integrable* stopping time. Moreover, we slightly generalize an integral formula for the minimal expectation of an embedding stopping time. In Section 2, we characterize families of scaled random walks whose laws can be embedded into $M$ via a sequence of increasing stopping times such that the expected distance between two consecutive stopping times is equal to $1/N$, for $N \in \mathbb{N}$. In Section 3, we provide sufficient conditions for a sequence of scaled random walks, embeddable into $M$, to converge in distribution to $M$.

## 1. Embedding distributions in integrable time

In this section we recall a necessary and sufficient condition from [1] for a centered distribution to be embeddable in a diffusion via an integrable stopping time.

Let $I = (l, r)$ with $l \in [-\infty, \infty)$ and $r \in (-\infty, \infty]$. As usual we denote by $\bar{I}$ the closure of $I$ in $\mathbb{R}$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a Borel-measurable function satisfying

$$\eta(x) \neq 0 \quad \text{for all } x \in I, \tag{3}$$

$$\frac{1}{\eta^2} \in L^1_{\mathrm{loc}}(I), \tag{4}$$

$$\eta(x) = 0 \quad \text{for all } x \in \mathbb{R} \setminus I, \tag{5}$$

where $L^1_{\mathrm{loc}}(I)$ denotes the set of functions that are locally integrable on $I$.

Consider the SDE

$$dM_t = \eta(M_t)\, dW_t, \qquad M_0 \sim \gamma, \tag{6}$$

where $\gamma$ is a probability measure on $I$. The assumptions (3)–(5) imply that (6) possesses a weak solution that is unique in law (see e.g. [5] or Theorem 5.5.7 in [9]). This means that there exists a pair of processes $(M, W)$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, with $(\mathcal{F}_t)$ satisfying the usual conditions, such that $W$ is an $(\mathcal{F}_t)$-Brownian motion, $M_0$ is an $\mathcal{F}_0$-measurable random variable with distribution $\gamma$ and $(M, W)$ satisfies the SDE (6). Let us note that $M$ stays in $l$ (resp. $r$) once it hits $l$ (resp. $r$).

For all $y \in I$ and $x \in \mathbb{R}$ we define

$$q(y, x) = \int_y^x \int_y^u \frac{2}{\eta^2(z)}\, dz\, du.$$

Notice that Itô's formula implies that the process $(q(M_0, M_t) - t)$ is a local martingale starting in 0. The assumptions (3)–(5) imply that for all $y \in I$ the nonnegative function $q(y, \cdot)$ is finite on $I$ and equal to $\infty$ on $\mathbb{R} \setminus [l, r]$. Besides, $q(y, \cdot)$ is strictly convex on $I$, strictly decreasing to zero on $(l, y)$ and strictly increasing from zero on $(y, r)$. Moreover, for all $y, \bar{y} \in I$ and $x \in \mathbb{R}$ we have

$$q(y, x) = q(\bar{y}, x) - q(\bar{y}, y) - q_x(\bar{y}, y)(x - y), \tag{7}$$

where $q_x$ denotes the partial derivative of $q$ with respect to the second argument.

Recall that by Feller's test for explosions we have $q(y, l+) = \infty$ if and only if the probability for the process $M$ to attain $l$ in finite time is equal to zero. Notice that the non-explosion condition $q(y, l+) = \infty$ does *not* depend on $y$. Moreover, if $l = -\infty$, then $q(y, l+) = \infty$, and hence any solution of (6) does not attain $-\infty$ in finite time. Similar statements hold true for the right-hand side boundary $r$.

We next recall a result from [1] providing a necessary and sufficient condition for a distribution to be embeddable in $M$ with an integrable stopping time. Let $\mu$ be a centered probability measure on $\mathbb{R}$, i.e. $\int |x| \mu(dx) < \infty$ and $\int x \mu(dx) = 0$. Moreover, we assume that $\mu \neq \delta_0$. Let $K(y, a, \cdot)$, $y \in \bar{I}$, $a \in \mathbb{R}_+$, be the location-scale family of probability distributions defined by

$$K(y, a, B) = \mu\left(\frac{B - y}{a}\right), \quad B \in \mathcal{B}(\mathbb{R}), \tag{8}$$

whenever $a > 0$; and $K(y, 0, \cdot) = \delta_y$. Consider the problem of finding a stopping time $\tau$ such that

$$\mathrm{Law}(M_\tau | \mathcal{F}_0) = K\big(M_0, a(M_0), \cdot\big), \tag{9}$$

where $\mathrm{Law}(M_\tau|\mathcal{F}_0)$ denotes the regular conditional distribution of $M_\tau$ with respect to $\mathcal{F}_0$ and $a\colon \bar{I} \to \mathbb{R}_+$ is a given Borel function. The unconditional version of this problem is usually referred to as the *Skorokhod embedding problem* or the *SEP*, see [7] or [13] for an overview. In the subsequent sections we need embedding stopping times that satisfy $E[\tau|\mathcal{F}_0] < \infty$ a.s. Notice that the conditional expectation is always defined because $\tau$ is nonnegative. For all $y \in I$ we define

$$Q(y) = \int_{\mathbb{R}} q(y,x) K\big(y, a(y), dx\big). \tag{10}$$

One can show that $Q(y)$ is the minimal expected time required for embedding $K(y, a(y), \cdot)$ into $M$, conditioned to $M_0 = y$. To provide an intuition, suppose that the starting in 0 local martingale $(q(M_0, M_t) - t)$ is a true martingale and $\tau$ is a solution of the embedding problem (9). If the optional sampling theorem applies, then $E[\tau|\mathcal{F}_0] = E[q(M_0, M_\tau)|\mathcal{F}_0] = Q(M_0)$. More formally, we have the following result, which is a straightforward generalization of Theorem 3 and Proposition 4 in [1]:

**Theorem 1.1.**

(i) *Any $(\mathcal{F}_t)$-stopping time $\tau$ solving (9) satisfies $E[\tau|\mathcal{F}_0] \geq Q(M_0)$ a.s.*
(ii) *There exists a solution $\tau$ of the embedding problem (9) satisfying the property $E[\tau|\mathcal{F}_0] < \infty$ a.s. if and only if*

$$Q(M_0) < \infty \quad a.s. \tag{11}$$

*In this case, there exists an embedding stopping time $\tau$ with*

$$E[\tau|\mathcal{F}_0] = Q(M_0) \quad a.s. \tag{12}$$

For the proof of the main results of Section 3 it turns out to be helpful to work with the specific solution of the embedding problem (9) provided in [1]. For the reader's convenience we briefly explain the solution method in the Appendix.

## 2. Embedding scaled random walks

Let $(M, W)$ be a weak solution of

$$dM_t = \eta(M_t)\,dW_t, \qquad M_0 = m, \tag{13}$$

with $m \in I$. Moreover, let $X_1, X_2, \ldots$ be a sequence of i.i.d. real-valued integrable random variables with $E(X_i) = 0$. We denote the distribution of $X_i$ by $\mu$. Throughout we assume that $\mu \neq \delta_0$.

***Definition 2.1.*** *Let $a : \mathbb{R} \to \mathbb{R}_+$ be a Borel function. The process $Y = (Y_k)_{k\in\mathbb{N}_0}$, defined by $Y_0 = m$ and*

$$Y_{k+1} = Y_k + a(Y_k)X_{k+1}, \quad k \geq 0, \tag{14}$$

*is called random walk generated by $(X_k)_{k\in\mathbb{N}_0}$ with scale factor $a$ and starting point $m$.*

*We say that $Y = (Y_k)_{k\in\mathbb{N}_0}$ is a scaled random walk if there exists a scale factor $a$ such that $Y$ is the random walk generated by $(X_k)_{k\in\mathbb{N}_0}$ with scale factor $a$.*

In this section we aim at constructing scale factors such that $(Y_k)_{k\in\mathbb{N}_0}$ can be embedded in distribution into $M$ with a sequence of stopping times $(\tau_k)_{k\in\mathbb{N}_0}$ such that $(M_{\tau_k})_{k\in\mathbb{N}_0} \stackrel{d}{=} (Y_k)_{k\in\mathbb{N}_0}$, that is, both discrete time processes have the same law. More precisely, we solve the following problem.

***Problem (P).*** *Let $N \in \mathbb{N}$. Does there exist a scale factor $a_N$ such that the associated scaled random walk $(Y_k^N)_{k\in\mathbb{N}_0}$ with $Y_0^N = m$ can be embedded in distribution into $M$ with a nondecreasing sequence of $(\mathcal{F}_t)$-stopping times $(\tau_k^N)_{k\in\mathbb{N}_0}$*

*with*

$$\tau_0^N = 0 \quad \textit{and} \quad E\big[\tau_{k+1}^N | \mathcal{F}_{\tau_k^N}\big] = \tau_k^N + \frac{1}{N}, \tag{15}$$

*for all* $k \geq 0$?

In order to determine the scale factor solving Problem (P), we introduce, for all $y \in I$, the mapping $G_y : [0, \infty) \to [0, \infty]$ defined via

$$G_y(a) = \int_{\mathbb{R}} q(y, x) K(y, a, dx) = \int_{\mathbb{R}} q(y, y + ax) \mu(dx). \tag{16}$$

Recall that $G_y(a)$ is the minimal expected time needed for embedding $K(y, a, dx)$ into $M$ (cf. Theorem 1.1 and the discussion following (10)). For $y \in I$ let

$$a_{\inf}(y) := \inf\big\{a \in [0, \infty) : G_y(a) = \infty\big\}, \quad \inf \varnothing := \infty.$$

Notice that, for all $y \in I$, the map $G_y(\cdot)$ is increasing on $[0, \infty)$ and strictly increasing on $[0, a_{\inf}(y)]$ with $G_y(0) = 0$, left-continuous by the monotone convergence theorem, and continuous on $[0, \infty) \setminus \{a_{\inf}(y)\}$ by the dominated convergence theorem.

We now provide sufficient conditions guaranteeing that a solution of Problem (P) exists. We need to distinguish four cases.

### 2.1. *Case* 1: $l = -\infty$ *and* $r = \infty$

In this subsection we make the following assumption.

(A1) There exists $y \in I$ such that $G_y(a) < \infty$ for all $a > 0$.

**Lemma 2.2.** (A1) *is equivalent to the condition that for all* $y \in I$ *and* $a > 0$ *we have* $G_y(a) < \infty$.

**Proof.** Let $\bar{y} \in I$ and suppose that $G_{\bar{y}}(a) < \infty$ for all $a > 0$. Let $y \in I$ and notice that $q(y, x) = q(\bar{y}, x) - q(\bar{y}, y) - q_x(\bar{y}, y)(x - y)$. Since $\mu$ is centered, we have

$$G_y(a) = \int_{\mathbb{R}} q(y, y + ax) \mu(dx) = \int_{\mathbb{R}} q(\bar{y}, y + ax) \mu(dx) - q(\bar{y}, y).$$

For all $a > 0$ and $x \in \mathbb{R}$ with $|x| \geq \frac{|y - \bar{y}|}{a}$ we have

$$q(\bar{y}, y + ax) \leq q(\bar{y}, \bar{y} + 2ax).$$

From this we obtain $G_y(a) < \infty$. □

The following theorem provides a solution to Problem (P) in Case 1.

**Theorem 2.3.** *If* (A1) *is satisfied, then for all* $N \in \mathbb{N}$ *there exists a unique scale factor* $a_N$ *satisfying*

$$G_y\big(a_N(y)\big) = \frac{1}{N}, \quad y \in I. \tag{17}$$

*Moreover, the random walk* $(Y_k^N)_{k \in \mathbb{N}_0}$ *generated by* $(X_k)_{k \in \mathbb{N}_0}$ *with scale factor* $a_N$ *and starting point* $m$ *can be embedded into* $M$ *with a sequence of stopping times satisfying* (15).

***Remark 2.4.*** *It is worth noting that* (A1) *is satisfied whenever* $\mu$ *has compact support.*

For the proof of the theorem we need the following auxiliary result.

**Lemma 2.5.** *If* (A1) *is satisfied, then $G_y$ is a bijective mapping from* $[0, \infty)$ *to* $[0, \infty)$, *for all $y \in I$.*

**Proof.** Notice that $\lim_{x \to \pm\infty} q(y, x + y) = \infty$. Moreover, if $0 \le a < b$ and $x \ne 0$, then $q(y, y + ax) < q(y, y + bx)$. Therefore, $G_y$ is strictly increasing and by monotone convergence, $\lim_{a \to \infty} G_y(a) = \infty$. Condition (A1), Lemma 2.2 and a dominated convergence argument show that $G_y$ is continuous, and consequently, bijective. □

**Proof of Theorem 2.3.** Let $N \in \mathbb{N}$. Lemma 2.5 implies that for all $y \in I$ there exists a scale factor $a_N(y)$ that satisfies (17).

We next define a sequence of stopping times $(\tau(k))_{k \in \mathbb{N}_0}$ that embeds the transition probabilities into the diffusion $M$. First define $\tau(0) = 0$. Suppose that $\tau(k)$ is already defined. Set $\tilde{M}_t = M_{t+\tau(k)}$, for $t \ge 0$, and observe that

$$d\tilde{M}_t = \eta(\tilde{M}_t)\, d(W_{t+\tau(k)} - W_{\tau(k)}), \qquad \tilde{M}_0 = M_{\tau(k)}.$$

Theorem 1.1 implies that there exists an $(\mathcal{F}_{t+\tau(k)})$-stopping time $\rho(k+1)$ with

$$E\big[\rho(k+1) | \mathcal{F}_{\tau(k)}\big] = Q(M_{\tau(k)}) = G_{M_{\tau(k)}}\big(a_N(M_{\tau(k)})\big) = \frac{1}{N}$$

such that $\tilde{M}_{\rho(k+1)} \stackrel{d}{=} Y_k^N + a_N(Y_k^N) X_{k+1}$. Now define $\tau(k+1) = \tau(k) + \rho(k+1)$. By construction, the sequence $(M_{\tau(k)})_{k \in \mathbb{N}_0}$ has the same distribution as $(Y_k^N)_{k \in \mathbb{N}_0}$. □

The next example shows that a scale factor satisfying (17) does not necessarily exist if (A1) does not hold true.

***Example 2.6.*** *Let $\mu$ be the probability measure with density $f(x) = c\frac{e^{-|x|}}{1+x^2}$, where $c = (\int \frac{e^{-|x|}}{1+x^2}\, dx)^{-1}$. Moreover let $m = 0$ and $\eta(x) = e^{-x/2}$, $x \in \mathbb{R}$. Then we have $q(y, x) = 2e^y(e^{x-y} - (x - y) - 1)$. A straightforward calculation shows that $G_y(a) = \infty$ for $a > 1$. Therefore Condition* (A1) *is not satisfied. Moreover, for $a = 1$ we have that*

$$G_y(1) = 2e^y c \int_{\mathbb{R}} \big(e^x - x - 1\big) \frac{e^{-|x|}}{1 + x^2}\, dx < \infty.$$

*By considering the limit $y \to -\infty$ we see that for every $N \in \mathbb{N}$ there exists $y \in \mathbb{R}$ such that $G_y(1) < 1/N$. In particular, there exists no solution to* (17).

### 2.2. *Case* 2: $l > -\infty$ *and* $r = \infty$

Here we impose the following assumption.

(A2) $\inf \operatorname{supp} \mu > -\infty$ and there exists $y \in I$ such that the integral over the positive real line $\int_{\mathbb{R}_+} q(y, y + ax)\mu(dx) < \infty$ for all $a > 0$.

For every $y > l$ we set $\bar{a}(y) = \frac{l-y}{\inf \operatorname{supp} \mu}$. Note that for all $a \le \bar{a}(y)$ we have $a \operatorname{supp} \mu \subset [l - y, \infty)$.
We now present a solution to Problem (P) in Case 2.

**Theorem 2.7.** *Suppose that* (A2) *is satisfied and additionally that the following implications hold true*:

$$\text{if } q(m, l+) < \infty, \quad \text{then } \mu\big(\{\inf \operatorname{supp} \mu\}\big) > 0, \tag{18}$$

$$\text{if } q(m, l+) = \infty, \quad \text{then } \liminf_{y \to \infty} G_y\big(\bar{a}(y)\big) > 0 \text{ and } \liminf_{y \searrow l} G_y\big(\bar{a}(y)\big) > 0. \tag{19}$$

*Then there exists $N_0 \in \mathbb{N}$ such that for all $N \ge N_0$ there exists a unique scale factor $a_N$ satisfying*

$$a_N(y) = \sup\left\{a \in [0, \infty) : G_y(a) \le \frac{1}{N}\right\}, \qquad y \in I, \tag{20}$$

*and $a_N(l)=0$. Moreover, the random walk $(Y_k^N)_{k\in\mathbb{N}_0}$, generated by $(X_k)_{k\in\mathbb{N}_0}$ with scale factor $a_N$ and starting in $m$, can be embedded in $M$ with stopping times satisfying* (15).

*In the case $q(m,l+)<\infty$, we can take $N_0=1$; while in the case $q(m,l+)=\infty$, the scale factor $a_N$ of* (20) *satisfies* (17) *for all $y\in I$ and $N\geq N_0$ (here $N_0\geq 1$ can be necessary).*

***Remark 2.8.*** *It is worth noting that the assumptions of Theorem 2.7 are satisfied whenever $\mu$ has compact support and $\mu(\{\inf\operatorname{supp}\mu\})>0$ (see Proposition 2.12).*

**Proof.** From similar arguments as in the proof of Lemma 2.2 it follows that condition (A2) implies $\int_{\mathbb{R}_+} q(y,y+ax)\mu(dx)<\infty$ for all $y\in I$ and $a>0$. Notice that the sup in (20) is attained, since $G_y$ is left-continuous. As in the proof of Lemma 2.5 one can show that $G_y:[0,\bar a(y)]\to[0,G_y(\bar a(y))]$ is bijective.

Now assume $q(m,l+)<\infty$. Equation (18) implies that $w=\mu(\{\inf\operatorname{supp}\mu\})$ is positive. Let $N\in\mathbb{N}$ and $\nu_N(y,B)=K(y,a_N(y),B)$ for $y\in I$ and $B\in\mathcal{B}(\mathbb{R})$. Similarly to the proof of Theorem 2.3 we construct a sequence of stopping times $(\tau(k))_{k\in\mathbb{N}_0}$ embedding the transition probabilities into $M$. Let $\tau(0)=0$. Suppose that $\tau(k)$ is defined. By Theorem 1.1 there exists an $(\mathcal{F}_{t+\tau(k)})$-stopping time $\rho(k+1)$ that embeds $\nu_N(M_{\tau(k)},\cdot)$ into the process $\tilde M_t=M_{t+\tau(k)}$, $t\geq 0$, and, with $Q(y)=G_y(a_N(y))$, satisfies

$$E\big[\rho(k+1)|\mathcal{F}_{\tau(k)}\big]=Q(M_{\tau(k)})=G_{M_{\tau(k)}}\big(a_N(M_{\tau(k)})\big)\leq\frac{1}{N}.$$

Define $\tau(k+1)=\tau(k)+\rho(k+1)$ if $M_{\tau(k)+\rho(k+1)}>l$, and $\tau(k+1)=\tau(k)+\rho(k+1)+\frac{1}{w}[\frac{1}{N}-Q(M_{\tau(k)})]$ if $M_{\tau(k)+\rho(k+1)}=l$. Then we have

$$\begin{aligned}E\big[\tau(k+1)|\mathcal{F}_{\tau(k)}\big]&=\tau(k)+Q(M_{\tau(k)})+\frac{1}{w}\bigg[\frac{1}{N}-Q(M_{\tau(k)})\bigg]P(M_{\tau(k)+\rho(k+1)}=l|\mathcal{F}_{\tau(k)})\\&=\tau(k)+\frac{1}{N}.\end{aligned}$$

Next assume that $q(m,l+)=\infty$. Due to (19) and Lemma 2.9 below, we have

$$\inf_{y\in I}G_y\big(\bar a(y)\big)>0.$$

Choosing $N_0\in\mathbb{N}$ such that $1/N_0<\inf_{y\in I}G_y(\bar a(y))$ yields that for every $N\geq N_0$ and $y\in I$ we have $G_y(a_N(y))=\frac{1}{N}$. The rest of the proof goes along the lines of the proof of Theorem 2.3. □

**Lemma 2.9.** *Suppose* (A2).

(i) *The function $y\mapsto G_y(\bar a(y))$ is a lower semicontinuous function $I\to(0,\infty]$.*
(ii) *For any compact subinterval $J\subset I$, we have*

$$\inf_{y\in J}G_y\big(\bar a(y)\big)>0.$$

**Proof.** To simplify notation we assume that $\inf\operatorname{supp}\mu=-1$. For $y>l$, we have $G_y(\bar a(y))>0$, $\bar a(y)=y-l$ and $g(y):=G_y(\bar a(y))=\int q(y,y+(y-l)x)\mu(dx)$. Since $q(y,\cdot)$ nonnegative, Fatou's lemma yields for $y_0\in I$

$$\liminf_{y\to y_0}g(y)\geq\int\liminf_{y\to y_0}q\big(y,y+(y-l)x\big)\mu(dx)=g(y_0).\tag{21}$$

This proves the first statement. The second statement immediately follows from the first one. □

***Remark 2.10.*** *In comparison with Case* 1 *the condition that determines the scale factor changes from* (17) *to a less pleasant one* (20). *Notice that in Case* 1, *conditions* (17) *and* (20) *are equivalent. As stated in Theorem 2.7, in Case* 2 *with $q(m,l+)=\infty$ again* (17) *holds true. Example 2.11 below shows that, in Case* 2 *with $q(m,l+)<\infty$, it* *can indeed happen that the scale factor does not satisfy* (17) *any longer. However, by Lemma* 2.9(ii), *the following statement holds*:

(Eq) *Suppose* (A2) *and, for* $N \in \mathbb{N}$, *consider the scale factor* $a_N$ *satisfying* (20). *Then, for any compact subinterval* $J \subset I$, *there exists* $N_1 \in \mathbb{N}$ *such that, for all* $N \geq N_1$, *the scale factor* $a_N$ *satisfies* (17) *on* $J$.

***Example 2.11.*** *Let* $M$ *be a Brownian motion starting at* $m > 0$ *and absorbed as it reaches zero, i.e. we have* $I = (0, \infty)$ *and* $\eta \equiv 1$. *Let* $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$. *A short computation shows that, for* $y > 0$,

$$G_y(a) = \begin{cases} a^2 & \text{if } a \in [0, y], \\ \infty & \text{if } a \in (y, \infty). \end{cases}$$

*Hence,* $a_N(y) = \frac{1}{\sqrt{N}} \wedge y$, *and* (17) *fails whenever* $y \in (0, \frac{1}{\sqrt{N}})$.

While (18) is a condition on the primitives of our problem, (19) is harder to verify. In the sequel we present sufficient conditions on $\mu$ (Proposition 2.12) and $\eta$ (Proposition 2.13) that imply (19).

**Proposition 2.12.** *Suppose* (A2). *If* $\mu(\{\inf \operatorname{supp} \mu\}) > 0$, *then* (19) *is satisfied. Moreover, Theorem* 2.7 *applies with* $N_0 = 1$.

**Proof.** From $q(m, l+) = \infty$ and $\mu(\{\inf \operatorname{supp} \mu\}) > 0$ it follows that $G_y(\bar{a}(y)) = \infty$ for all $y \in I$, which implies the claims. □

**Proposition 2.13.** *Under* (A2) *assume that*

$$\limsup_{x \searrow l} \frac{|\eta(x)|}{x - l} < \infty \quad \text{and} \quad \limsup_{x \nearrow \infty} \frac{|\eta(x)|}{x} < \infty.$$

*Then* (19) *is satisfied.*

**Proof.** To simplify notation we assume that $\inf \operatorname{supp} \mu = -1$. For $y > l$ we have $\bar{a}(y) = y - l$ and $g(y) := G_y(\bar{a}(y)) = \int h(x, y)\mu(dx)$ with $h(x, y) := q(y, y + (y - l)x)$. We need to show that $\liminf_{y \to y_0} g(y) > 0$ for $y_0 \in \{l, \infty\}$. Note that

$$h(x, y) = 2 \int_0^x \int_0^u \frac{(y - l)^2}{\eta^2((y - l)z + y)} \, dz \, du. \tag{22}$$

We have for every $z > -1$

$$\liminf_{y \searrow l} \left( \frac{y - l}{|\eta((y - l)z + y)|} \right) = \frac{1}{z + 1} \liminf_{y \searrow l} \left( \frac{(y - l)z + y - l}{|\eta((y - l)z + y)|} \right) = \frac{1}{z + 1} \liminf_{x \searrow l} \left( \frac{x - l}{|\eta(x)|} \right) > 0$$

and

$$\liminf_{y \nearrow \infty} \left( \frac{y - l}{|\eta((y - l)z + y)|} \right) = \frac{1}{z + 1} \liminf_{x \nearrow \infty} \left( \frac{x - l}{|\eta(x)|} \right) = \frac{1}{z + 1} \liminf_{x \nearrow \infty} \left( \frac{x}{|\eta(x)|} \right) > 0.$$

Thus, for $y_0 \in \{l, \infty\}$, applying Fatou's lemma in (22) (observe that the area of integration is positively oriented also for $x \leq 0$) yields $\liminf_{y \to y_0} h(x, y) > 0$ for every $x \in (-1, \infty) \setminus \{0\}$. Now the argument similar to (21) yields the claim. □

***Remark 2.14.*** *We can replace the conditions* $\limsup_{x \searrow l} \frac{|\eta(x)|}{x - l} < \infty$ *and* $\limsup_{x \nearrow \infty} \frac{|\eta(x)|}{x} < \infty$ *in the formulation of Proposition* 2.13 *by the weaker conditions* $\liminf_{y \searrow l} h(\cdot, y) > 0$ *and* $\liminf_{y \nearrow \infty} h(\cdot, y) > 0$ *on a set of positive mass with respect to* $\mu$, *where* $h$ *is defined as in the proof of Proposition* 2.13.

Let us illustrate in more detail how the assumptions in Theorem 2.7 work when $q(m,l+)=\infty$. Recall that, in the case $q(m,l+)=\infty$, the scale factor $a_N$ satisfies (17) (not only (20)). If, however, (A2) does not hold true, then a scale factor satisfying (17) does not necessarily exist. This can be shown by means of an example similar to Example 2.6. The role of condition (19) is as follows. Together with (A2) it guarantees that, in the case $q(m,l+)=\infty$, there is a scale factor satisfying (17). Examples 2.15 and 2.16 below show that (19) can fail and a scale factor satisfying (17) does not necessarily exist when we require (A2) alone.

***Example 2.15.*** *Let us consider the constant elasticity of variance* (*CEV*) *process with* $\alpha>1$, *i.e.* $l=0$, $r=\infty$ *and* $\eta(x)=x^\alpha$ *on* $I$. *For* $y>0$, *we have*

$$q(y,x)=\begin{cases}\frac{2}{2\alpha-1}\big[\frac{1}{2\alpha-2}\big(\frac{1}{x^{2\alpha-2}}-\frac{1}{y^{2\alpha-2}}\big)+\frac{x-y}{y^{2\alpha-1}}\big] & \text{if } x\ge 0,\\ \infty & \text{if } x<0,\end{cases}\tag{23}$$

*in particular*, $q(y,0+)=\infty$. *We see that any centered measure* $\mu\neq\delta_0$ *with* $\inf\operatorname{supp}\mu>-\infty$ *satisfies* (A2).

*Notice that* $q(y/a,x/a)=a^{2\alpha-2}q(y,x)$ *for* $a,x,y>0$. *With* $b=-\inf\operatorname{supp}\mu>0$ *we now calculate* $\bar a(y)=y/b$ *and*

$$G_y\big(\bar a(y)\big)=\frac{1}{y^{2\alpha-2}}\int_{\mathbb{R}}q\left(1,1+\frac{x}{b}\right)\mu(dx)=\frac{b^{2\alpha-2}}{y^{2\alpha-2}}\int_{\mathbb{R}}q(b,b+x)\mu(dx).$$

*Thus*, *Theorem* 2.7 *applies if and only if* $\mu\neq\delta_0$ *is centered*, $\inf\operatorname{supp}\mu>-\infty$ *and*, *for some* $\varepsilon>0$,

$$\int_{[\inf\operatorname{supp}\mu,\inf\operatorname{supp}\mu+\varepsilon]}q(-\inf\operatorname{supp}\mu,-\inf\operatorname{supp}\mu+x)\mu(dx)=\infty.\tag{24}$$

*Here we used that such an integral over* $\mathbb{R}$ *is infinite if and only if* (24) *is satisfied* (*for any* $y>0$, *the function* $x\mapsto q(y,x)$ *has linear growth as* $x\to\infty$). *Notice that a sufficient condition for* (24) *is* $\mu(\{\inf\operatorname{supp}\mu\})>0$.

The previous example shows that, choosing a centered measure $\mu\neq\delta_0$ with $\inf\operatorname{supp}\mu>-\infty$ in a way that (24) fails, we have (A2) but violate (19) in the way $G_y(\bar a(y))\to 0$ as $y\to\infty$. This raises the question of whether it is possible to violate (19), under (A2), in the way $G_y(\bar a(y))\to 0$ as $y\searrow l$. This must be more delicate because, on the one hand, the condition $|\eta(x)|\ge c(x-l)^\alpha$, for all $x\in(l,b)$, with some $c>0$, $b>l$ and $\alpha<1$, implies $q(m,l+)<\infty$, while, on the other hand, the condition $|\eta(x)|\le c(x-l)$, for all $x\in(l,b)$, with some $c>0$ and $b>l$, implies $\liminf_{y\searrow l}G_y(\bar a(y))>0$ by Proposition 2.13. Still this is possible as the following example shows.

***Example 2.16.*** *We consider again* $I=(0,\infty)$ *and define*

$$\eta(x)=\begin{cases}2\sqrt{2}x\frac{(-\log x)^{\frac{3}{4}}}{\sqrt{-1-2\log x}} & \text{if } x\in(0,1/2),\\ 1 & \text{if } x\in[1/2,\infty).\end{cases}$$

*Then one can verify that*

$$q(1/2,x)=\begin{cases}\sqrt{-\log x}+\frac{1}{\sqrt{\log 2}}(x-\frac{1}{2})-\sqrt{\log 2} & \text{if } x\in(0,1/2),\\ x^2-x+1/4 & \text{if } x\in[1/2,\infty),\end{cases}$$

*in particular*, $q(1/2,0+)=\infty$. *We see that any centered measure* $\mu\neq\delta_0$ *with* $\inf\operatorname{supp}\mu>-\infty$ *and* $\int_{\mathbb{R}_+}x^2\mu(dx)<\infty$ *satisfies* (A2).

*Let now* $\mu$ *be a centered measure with*

$$\inf\operatorname{supp}\mu=-1\quad\text{and}\quad\sup\operatorname{supp}\mu=1,$$

*in particular*, $\bar a(y)=y$ *for* $y>0$. *Moreover assume that*

$$\int_{\mathbb{R}}-\log(x+1)\mu(dx)<\infty.\tag{25}$$

*By formula* (7), *we have for* $x \in (-1, 1]$ *and* $y \in (0, 1/4]$

$$q(y, y + xy) = \sqrt{-\log[y(x+1)]} - \sqrt{-\log y} + \frac{x}{2\sqrt{-\log y}}.$$

*In particular, for any* $x \in (-1, 1]$, *the mapping* $y \mapsto q(y, y + xy)$ *is increasing with* $q(y, y + xy) \to 0$ *as* $y \searrow 0$. *Indeed, we have*

$$\sqrt{-\log[y(x+1)]} - \sqrt{-\log y} = \frac{-\log(x+1)}{\sqrt{-\log[y(x+1)]} + \sqrt{-\log y}}.$$

*Then dominated convergence* (*cf.* (25)) *ensures that* $G_y(\bar{a}(y)) \to 0$ *as* $y \searrow 0$.

Finally, we illustrate how Theorem 2.7 works when $q(m, l+) < \infty$.

***Example 2.17.*** *Let us now consider the CEV process with* $\alpha \in (-\infty, 1) \setminus \{0\}$, *i.e.* $I = (0, \infty)$ *and* $\eta(x) = x^\alpha$ *on* $I$. *In the case* $\alpha \neq \frac{1}{2}$, *for* $y > 0$, *the function* $q(y, \cdot)$ *is given by formula* (23). *In the case* $\alpha = \frac{1}{2}$, *for* $y > 0$, *we have*

$$q(y, x) = \begin{cases} 2x \log \frac{x}{y} - 2(x - y) & \text{if } x \geq 0, \\ \infty & \text{if } x < 0. \end{cases}$$

*In particular,* $q(y, 0+) < \infty$ *in both cases. Thus, Theorem* 2.7 *applies if and only if* $\mu \neq \delta_0$ *is centered,* $\inf \operatorname{supp} \mu > -\infty$, $\mu(\{\inf \operatorname{supp} \mu\}) > 0$ *and*

$$\text{if } \alpha = \frac{1}{2}, \quad \text{then } \int_{\mathbb{R}_+} x \log x \, \mu(dx) < \infty,$$

$$\text{if } \alpha < \frac{1}{2}, \quad \text{then } \int_{\mathbb{R}_+} x^{2-2\alpha} \mu(dx) < \infty.$$

### 2.3. *Case* 3: $l = -\infty$ *and* $r < \infty$

This case can be reduced to Case 2 by considering the diffusion $-M$.

### 2.4. *Case* 4: $l > -\infty$ *and* $r < \infty$

In this subsection we make the following assumption.

(A3) $\inf \operatorname{supp} \mu > -\infty$ and $\sup \operatorname{supp} \mu < \infty$.

For every $y \in I$ we set $\bar{a}(y) = \frac{l-y}{\inf \operatorname{supp} \mu} \wedge \frac{r-y}{\sup \operatorname{supp} \mu}$. Note that for all $a \leq \bar{a}(y)$ we have $a \operatorname{supp} \mu \subset [l - y, r - y]$.

A solution to Problem (P) in Case 4 is given in the next theorem.

**Theorem 2.18.** *Suppose that* (A3) *is satisfied and additionally that the following implications hold true*:

$$\text{if } q(m, l+) < \infty, \quad \text{then } \mu(\{\inf \operatorname{supp} \mu\}) > 0, \tag{26}$$

$$\text{if } q(m, l+) = \infty, \quad \text{then } \liminf_{y \searrow l} G_y(\bar{a}(y)) > 0, \tag{27}$$

$$\text{if } q(m, r-) < \infty, \quad \text{then } \mu(\{\sup \operatorname{supp} \mu\}) > 0, \tag{28}$$

$$\text{if } q(m, r-) = \infty, \quad \text{then } \liminf_{y \nearrow r} G_y(\bar{a}(y)) > 0. \tag{29}$$

*Then there exists* $N_0 \in \mathbb{N}$ *such that for all* $N \geq N_0$ *there exists a unique scale factor* $a_N$ *satisfying* (20) *and* $a_N(l) = a_N(r) = 0$. *Moreover, the random walk* $(Y_k^N)_{k \in \mathbb{N}_0}$, *scaled with* $a_N$ *and starting in* $m$, *is embeddable in* $M$ *with stopping times* $(\tau_k^N)_{k \in \mathbb{N}_0}$ *satisfying* (15).

**Proof.** Similar to the proof of Theorem 2.7. □

***Remark 2.19.*** *Notice that the assumptions of Theorem* 2.18 *are satisfied whenever* $\mu$ *has compact support*, $\mu(\{\inf \operatorname{supp} \mu\}) > 0$ *and* $\mu(\{\sup \operatorname{supp} \mu\}) > 0$ (*see Proposition* 2.20).

The next two propositions provide sufficient conditions for the properties (27) and (29) to hold true.

**Proposition 2.20.** *Suppose* (A3). *If* $\mu(\{\inf \operatorname{supp} \mu\}) > 0$, *then* (27) *is satisfied.*

**Proof.** Similar to the proof of Proposition 2.12. □

Similarly, the condition $\mu(\{\sup \operatorname{supp} \mu\}) > 0$ is sufficient for (29).

**Proposition 2.21.** *Suppose* (A3). *If* $\limsup_{x\searrow l} \frac{|\eta(x)|}{x-l} < \infty$, *then* (27) *is satisfied. If* $\limsup_{x\nearrow r} \frac{|\eta(x)|}{r-x} < \infty$, *then* (29) *is satisfied.*

**Proof.** Similar to the proof of Proposition 2.13. □

Finally, it is worth noting that the detailed discussions in Case 2 about the role of different assumptions, etc., have their analogues in Case 4. In particular, we have:

- The statements of Lemma 2.9 apply verbatim under (A3) instead of (A2);
- Statement (Eq) in Remark 2.10 applies verbatim under (A3) instead of (A2);
- The conclusion of Theorem 2.18 holds true with $N_0 = 1$ whenever (A3) is satisfied and we have

  $$\mu\big(\{\inf \operatorname{supp} \mu\}\big) > 0 \quad \text{and} \quad \mu\big(\{\sup \operatorname{supp} \mu\}\big) > 0$$

  (cf. with the statements in the end of Theorem 2.7 and Proposition 2.12);
- Under the assumptions of Theorem 2.18, for all $N \geq N_0$ and $y \in I$, the scale factors $a_N$ satisfy (17) (not only (20)) whenever $q(m, l+) = q(m, r-) = \infty$ (cf. with the statement in the end of Theorem 2.7).

## 3. Weak convergence

In this section we use the setting and notations of Section 2. In particular, we consider a weak solution $(M, W)$ of (13), denote by $(Y_k^N)_{k\in\mathbb{N}_0}$ the scaled random walk (14), and assume that $\mu \neq \delta_0$ is a centered probability measure on $\mathbb{R}$. Throughout this section we suppose that one of the sufficient conditions from Section 2 is satisfied that guarantees, for sufficiently large $N \in \mathbb{N}$, the existence of a scale factor $a_N$ satisfying (20) and solving Problem (P). Let us remark that under each of these sufficient conditions, we have

$$\int_{\mathbb{R}} q(m, m + ax)\mu(dx) < \infty \quad \text{for some } a > 0. \tag{30}$$

We extend $Y^N$ to a continuous-time process on $\mathbb{R}_+$ via linear interpolation, i.e. for all $t \geq 0$ we set $Y_t^N = Y_{\lfloor t\rfloor}^N + (t - \lfloor t\rfloor)(Y_{\lfloor t\rfloor+1}^N - Y_{\lfloor t\rfloor}^N)$.

In this section we present sufficient conditions ensuring that the sequence of continuous processes $(Y_{Nt}^N)_{t\in\mathbb{R}_+}$ converges in law to the process $(M_t)_{t\in\mathbb{R}_+}$, as $N \to \infty$ (see Theorem 3.1). For example this weak convergence holds true if the diffusion coefficient $\eta$ is locally bounded away from 0 and from $\pm\infty$ and $\mu$ has compact support. One can thus interpret $(Y_k^N)_{k\in\mathbb{N}_0}$ as a Markov chain approximating the diffusion $(M_t)_{t\in\mathbb{R}_+}$.

To simplify the analysis, we only show the weak convergence on the time interval [0, 1]. A straightforward generalization implies the weak convergence on $\mathbb{R}_+$.

**Theorem 3.1.** *Suppose that* $|\eta|$ *and* $\frac{1}{|\eta|}$ *are locally bounded on* $I$ *and the following implications hold true*:

$$\text{if } \sup \operatorname{supp}(\mu) = \infty, \quad \text{then } |\eta| \text{ and } \frac{1}{|\eta|} \text{ are bounded on } (m, r), \tag{31}$$

$$\text{if } \inf \operatorname{supp}(\mu) = -\infty, \quad \text{then } |\eta| \text{ and } \frac{1}{|\eta|} \text{ are bounded on } (l, m). \tag{32}$$

*Then the processes* $(Y^N_{Nt})_{t\in[0,1]}$ *converge to* $(M_t)_{t\in[0,1]}$ *in distribution, as* $N \to \infty$, *i.e. the associated measures on* $(C[0,1], \mathcal{B}(C[0,1]))$ *converge weakly.*

Let $(\tau^N(k))_{k\in\mathbb{N}_0}$ be a sequence of stopping times embedding $(Y^N_k)_{k\in\mathbb{N}_0}$ into $M$ and satisfying (15). The first aim in the proof of Theorem 3.1 is to show (up to a localization) that $\tau^N(\lfloor Nt\rfloor)$ converges to $t$ in probability as $N \to \infty$. To this end, denote the difference between two consecutive embedding times by $\rho^N(k) = \tau^N(k) - \tau^N(k-1)$, $k \in \mathbb{N}$. Notice that $\rho^N(k)$ is an $(\mathcal{F}_{\tau^N(k-1)+t})_{t\in\mathbb{R}_+}$-stopping time with $E(\rho^N(k)) = \frac{1}{N}$. Moreover, the sequence $(\rho^N(k))_{k\in\mathbb{N}}$ is pairwise uncorrelated. We aim at applying the following version of the weak law of large numbers to prove that the sum of the $\rho^N(k)$ converges.

**Theorem 3.2 (Weak LLN for uncorrelated arrays).** *Let* $(Z^n_k)_{n\in\mathbb{N},1\le k\le n}$ *be a triangular array of nonnegative and uniformly integrable random variables. Suppose that, for all* $n \in \mathbb{N}$, *the random variables* $Z^n_k$, $1 \le k \le n$, *are pairwise nonpositively correlated, i.e.* $E(Z^n_k Z^n_l) \le EZ^n_k EZ^n_l$ *for* $k \ne l$. *Then* $\frac{1}{n}\sum_{k=1}^n (Z^n_k - EZ^n_k)$ *converges to zero in* $L^1$ *and hence in probability.*

The corresponding statement of Theorem 3.2 for *sequences* of random variables is established in Theorem 2 of [11], which assumes that the random variables are nonnegative, Cesàro uniformly integrable and pairwise nonpositively correlated. It is straightforward to extend the proof of Theorem 2 in [11] to triangular arrays. Nevertheless, for the convenience of the reader we provide a short self-contained proof of Theorem 3.2. We remark that Example 4 in [11] shows that the statement does not hold true if one omits the nonnegativity assumption (see also Theorem 1 in [12]).

There are many different forms of weak laws of large numbers for arrays of random variables. Often it is assumed that the random variables in each row of the array are pairwise independent, which cannot yield Theorem 3.2. In several articles, including [6,8,17] and [18], pairwise independence is not assumed, and weak laws of large numbers for arrays are established under different Cesàro-type conditions and obtained in forms similar to the following one: $\frac{1}{n}\sum_{k=1}^n (Z^n_k - a^n_k)$ converges to zero in probability, where $a^n_k = E[Z^n_k | Z^n_i, 1 \le i \le k-1]$, $n \in \mathbb{N}$, $1 \le k \le n$. While weak laws of such type can be in principle applied in the situation of Theorem 3.2, they do not seem to imply our result because the important role of the nonnegativity assumption in Theorem 3.2 does not appear in the aforementioned line of research.

**Proof of Theorem 3.2.** Let us set $\overline{Z}^n_k = Z^n_k 1_{\{Z^n_k \le n\}}$ and define the sums $S_n = \sum_{k=1}^n Z^n_k$ and $\overline{S}_n = \sum_{k=1}^n \overline{Z}^n_k$. Notice that the aim is to prove $\frac{S_n - ES_n}{n} \xrightarrow{L^1} 0$. Since the family $(Z^n_k)_{n,k}$ is uniformly integrable, we have

$$D(y) := \sup_{n\in\mathbb{N},1\le k\le n} EZ^n_k 1_{\{Z^n_k > y\}} \to 0, \quad y \to \infty.$$

It follows from $S_n - \overline{S}_n \ge 0$ and

$$\frac{E(S_n - \overline{S}_n)}{n} = \frac{1}{n}\sum_{k=1}^n EZ^n_k 1_{\{Z^n_k > n\}} \le D(n) \to 0, \quad n \to \infty,$$

that $\frac{S_n-\overline{S}_n}{n} \xrightarrow{L^1} 0$. Subtracting the expectation yields $\frac{S_n-ES_n}{n} - \frac{\overline{S}_n-E\overline{S}_n}{n} \xrightarrow{L^1} 0$. Therefore, it is enough to prove that $\frac{\overline{S}_n-E\overline{S}_n}{n}$ converges to zero in $L^1$. We now show that the latter sequence converges to zero even in $L^2$. We have

$$E\left(\frac{\overline{S}_n - E\overline{S}_n}{n}\right)^2 = \frac{1}{n^2}\operatorname{Var}\overline{S}_n \le \frac{1}{n^2}\left(\sum_{k=1}^n E\left(\overline{Z}_k^n\right)^2 + 2\sum_{1\le k<l\le n}\operatorname{Cov}\left(\overline{Z}_k^n, \overline{Z}_l^n\right)\right). \tag{33}$$

Since $E(\overline{Z}_k^n)^2 = \int_0^\infty 2yP(\overline{Z}_k^n > y)\,dy \le \int_0^n 2yP(Z_k^n > y)\,dy \le \int_0^n 2D(y)\,dy$, we get

$$\frac{1}{n^2}\sum_{k=1}^n E\left(\overline{Z}_k^n\right)^2 \le \frac{1}{n}\int_0^n 2D(y)\,dy \to 0, \quad n\to\infty. \tag{34}$$

Using that the random variables $Z_k^n$, $1 \le k \le n$, are pairwise nonpositively correlated, we get, for $k \ne l$,

$$\begin{aligned}
\operatorname{Cov}\left(\overline{Z}_k^n, \overline{Z}_l^n\right) &= E\overline{Z}_k^n\overline{Z}_l^n - E\overline{Z}_k^n E\overline{Z}_l^n \le EZ_k^n Z_l^n - E\overline{Z}_k^n E\overline{Z}_l^n \\
&\le EZ_k^n EZ_l^n - \left(EZ_k^n - EZ_k^n 1_{\{Z_k^n > n\}}\right)\left(EZ_l^n - EZ_l^n 1_{\{Z_l^n > n\}}\right) \\
&\le C\left(EZ_k^n 1_{\{Z_k^n > n\}} + EZ_l^n 1_{\{Z_l^n > n\}}\right) \le 2CD(n),
\end{aligned}$$

where $C := \sup_{n\in\mathbb{N}, 1\le k\le n} EZ_k^n$ is finite due to the uniform integrability of $(Z_k^n)_{n,k}$. Hence,

$$\limsup_{n\to\infty}\sup_{1\le k<l\le n}\operatorname{Cov}\left(\overline{Z}_k^n, \overline{Z}_l^n\right) = 0.$$

Together with (34) and the fact that the right-hand side of (33) is nonnegative, this implies that the right-hand side of (33) converges to zero. The proof is completed. □

We apply Theorem 3.2 to a slight modification of $(N\rho^N(k))_{N\in\mathbb{N}, 1\le k\le N}$. This modified array will appear as a part of a localization argument, which we now prepare. In what follows, we fix an increasing sequence of intervals

$$I_n = (l_n, r_n) \subset I = (l, r), \quad n \in \mathbb{N}, \tag{35}$$

defined according to the following rules (cf. (31)–(32)):

(1) if $\sup\operatorname{supp}(\mu) = \infty$, then $r_n = r$ for all $n \in \mathbb{N}$ (note that $r = \infty$ in this case);
(2) if $\sup\operatorname{supp}(\mu) < \infty$, then $r_n \nearrow r$ with $r_n < r$ for all $n \in \mathbb{N}$;
(3) if $\inf\operatorname{supp}(\mu) = -\infty$, then $l_n = l$ for all $n \in \mathbb{N}$ (note that $l = -\infty$ in this case);
(4) if $\inf\operatorname{supp}(\mu) > -\infty$, then $l_n \searrow l$ with $l_n > l$ for all $n \in \mathbb{N}$.

By skipping finitely many elements if necessary, we can assume that $m \in I_n$ for all $n \in \mathbb{N}$.

**Lemma 3.3.** *Under the assumptions of Theorem 3.1, for every $I_n$, the following property is satisfied: there exists $A \in (0,\infty)$ and $N_0 \in \mathbb{N}$ such that for all $y \in I_n$, $x \in [\inf\operatorname{supp}(\mu), \sup\operatorname{supp}(\mu)]$ and $N \ge N_0$, we have*

$$\frac{1}{A} \le \left|\eta\left(y + a_N(y)x\right)\right| \le A \tag{36}$$

*and*

$$a_N(y) \le \frac{A}{\sqrt{N}}. \tag{37}$$

Observe that $A$ and $N_0$ in Lemma 3.3 depend on the localization parameter $n$.

**Proof.** Fix $n \in \mathbb{N}$. Suppose first that $\operatorname{supp}(\mu)$ is bounded. Choose $\varepsilon > 0$ such that $[l_n - \varepsilon, r_n + \varepsilon] \subset I$. Since $|\eta|$ and $\frac{1}{|\eta|}$ are locally bounded, there exists $C > 0$ such that $\frac{1}{C} \le |\eta(x)| \le C$ for all $x \in [l_n - \varepsilon, r_n + \varepsilon]$. This further implies

$$q(y, x) \ge \int_y^x \int_y^u \frac{2}{C^2} \, dz \, du = \frac{(x-y)^2}{C^2} \tag{38}$$

for all $y \in I_n$ and $x \in [l_n - \varepsilon, r_n + \varepsilon]$. Next define

$$b_N = \frac{1}{\sqrt{N}} \frac{C}{\sqrt{\int x^2 \mu(dx)}}$$

and choose $N_0 \in \mathbb{N}$ such that $b_N \le \frac{-\varepsilon}{\inf \operatorname{supp}(\mu)} \wedge \frac{\varepsilon}{\sup \operatorname{supp}(\mu)}$ for all $N \ge N_0$. Notice that for all $y \in I_n$ and $N \ge N_0$

$$\int q(y, y + b_N x) \mu(dx) \ge \frac{1}{C^2} \int b_N^2 x^2 \mu(dx) = \frac{1}{N}.$$

Therefore, $a_N(y) \le b_N$. This further implies $y + a_N(y)x \in [l_n - \varepsilon, r_n + \varepsilon]$, and consequently $\frac{1}{C} \le |\eta(y + a_N(y)x)| \le C$ for all $y \in I_n$, $x \in [\inf \operatorname{supp}(\mu), \sup \operatorname{supp}(\mu)]$ and $N \ge N_0$. Thus, by setting $A = \max(C, \frac{C}{\sqrt{\int x^2 \mu(dx)}})$ we have shown inequalities (36) and (37).

Now assume that $\sup \operatorname{supp}(\mu) = \infty$ and $\inf \operatorname{supp}(\mu) > -\infty$. Then the argumentation above works with $[l_n - \varepsilon, \infty)$ in place of $[l_n - \varepsilon, r_n + \varepsilon]$. Here we also need to check that $\int x^2 \mu(dx) < \infty$. The latter follows from (30) and (38), where (38) now holds for all $y \in I_n$ and $x \in [l_n - \varepsilon, \infty)$ (also recall that $m \in I_n$).

The remaining cases are considered in a similar way. □

We next modify the stopping times $(\rho^N(k))_{k \in \mathbb{N}}$ so as to make them satisfy the integrability condition of Theorem 3.2. Let $H(I_n)$ denote the first exit time of $M$ from $I_n$. For a fixed $n$ let $\widehat{\rho}^N(k+1) = \frac{1}{N}$ if $\tau^N(k) > H(I_n)$, and let $\widehat{\rho}^N(k+1) = \rho^N(k+1)$ otherwise. We set $\widehat{\tau}^N(0) = 0$ and $\widehat{\tau}^N(k) = \sum_{j=1}^k \widehat{\rho}^N(j)$ for $k \ge 1$. Notice that $(\widehat{\tau}^N(k))_{k \in \mathbb{N}_0}$ and $(\widehat{\rho}^N(k))_{k \in \mathbb{N}}$ depend on the localization parameter $n$.

**Lemma 3.4.** *Under the assumptions of Theorem 3.1, for any localization parameter $n$, the $(\mathcal{F}_{\tau^N(k-1)+t})_{t \ge 0}$-stopping times $\rho^N(k)$ can be chosen in such a way that the family $N\widehat{\rho}^N(k)$, $1 \le k \le N$, $N \in \mathbb{N}$, is uniformly integrable.*

**Proof.** Fix $n \in \mathbb{N}$. Choose $\rho^N(k)$ according to the construction method outlined in the Appendix. More precisely, suppose that $\rho^N(k) = \Delta(M_{\tau^N(k-1)})$ (see the last line of the Appendix for the definition), and let $\widehat{\rho}^N(k)$ be the associated modified version. We now show that the family $N\widehat{\rho}^N(k)$, $1 \le k \le N$, $N \in \mathbb{N}$, is uniformly integrable.

Below, for random variables $\xi$ and $\zeta$, we write $\xi \overset{d}{=} \zeta$ (resp. $\xi \overset{d}{\le} \zeta$) to indicate that $\xi$ and $\zeta$ have the same distribution (resp. $\zeta$ stochastically dominates $\xi$). Let $A > 0$ and $N_0 \in \mathbb{N}$ be as in Lemma 3.3. By Lemmas A.3 and 3.3, for $N \ge N_0$, we have

$$\rho^N(1) = \Delta(M_0) \overset{d}{=} \int_0^1 \frac{a_N^2(m) b_x^2(s, \tilde{W}_s)}{\eta^2(a_N(m)b(s, \tilde{W}_s) + m)} \, ds \ge \frac{a_N^2(m)}{A^2} \int_0^1 b_x^2(s, \tilde{W}_s) \, ds$$

(notice that $b(s, \tilde{W}_s)$ takes values only in $[\inf \operatorname{supp}(\mu), \sup \operatorname{supp}(\mu)]$; also recall that $m \in I_n$). Therefore, the random variable $Z := \int_0^1 b_x^2(s, W_s) \, ds$ is integrable. With $\tilde{M}_0$ having the distribution of $M_{\tau^N(k)}$, we get

$$\rho^N(k+1) = \Delta(M_{\tau^N(k)}) \overset{d}{=} \int_0^1 \frac{a_N^2(\tilde{M}_0) b_x^2(s, \tilde{W}_s)}{\eta^2(a_N(\tilde{M}_0) b(s, \tilde{W}_s) + \tilde{M}_0)} \, ds.$$

By Lemma 3.3, on the event $\{\tilde{M}_0 \in I_n\}$, for $N \geq N_0$, we have $a_N(\tilde{M}_0) \leq \frac{A}{\sqrt{N}}$ and $\eta(a_N(\tilde{M}_0)b(s, \tilde{W}_s) + \tilde{M}_0) \geq \frac{1}{A}$. Therefore, for $N \geq N_0$,

$$1_{\{M_{\tau^N(k)} \in I_n\}} \rho^N(k+1) \overset{d}{\leq} \frac{A^4}{N} Z.$$

It follows from the construction of the modified time $\widehat{\rho}^N(k+1)$ that, for $N \geq N_0$,

$$\widehat{\rho}^N(k+1) \leq \max\left(1_{\{M_{\tau^N(k)} \in I_n\}} \rho^N(k+1), \frac{1}{N}\right) \overset{d}{\leq} \frac{1}{N} \max\bigl(A^4 Z, 1\bigr).$$

In other words, the integrable random variable $\max(A^4 Z, 1)$ stochastically dominates every $N\widehat{\rho}^N(k)$, $1 \leq k \leq N$, $N \geq N_0$, which yields uniform integrability of the latter family. By adding finitely many integrable random variables (the ones with $N < N_0$) we obtain the result. □

**Lemma 3.5.** *Suppose the conditions of Theorem 3.1 are satisfied. We fix a localization parameter $n$, choose times $\rho^N(k)$ as in Lemma 3.4, denote by $\widehat{\rho}^N(k)$ the associated modified versions and consider the associated cumulative sums $\widehat{\tau}^N(k)$, $1 \leq k \leq N$, $N \in \mathbb{N}$. For all $s \in [0, 1]$, we have $\widehat{\tau}^N(\lfloor Ns \rfloor) \to s$ in probability as $N \to \infty$.*

**Proof.** Let $s \in [0, 1]$. Set $Z_k^N = N\widehat{\rho}^N(k)$ if $k \leq \lfloor Ns \rfloor$ and $Z_k^N = 0$ otherwise. The family $(Z_k^N)_{N \in \mathbb{N}, 1 \leq k \leq N}$ satisfies the assumptions of Theorem 3.2 (notice that the random variables in each row are pairwise uncorrelated). Hence, $\frac{1}{N}\sum_{k=1}^N (Z_k^N - E(Z_k^N))$ converges to zero in probability. Observe that

$$\frac{1}{N}\sum_{k=1}^{N} Z_k^N = \sum_{k=1}^{\lfloor Ns \rfloor} \widehat{\rho}^N(k) = \widehat{\tau}^N\bigl(\lfloor Ns \rfloor\bigr),$$

and $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N E(Z_k^N) = \lim_{N\to\infty} \frac{1}{N}\lfloor Ns \rfloor = s$. Consequently, $\widehat{\tau}(\lfloor Ns \rfloor)$ converges to $s$ in probability, as $N \to \infty$. □

By combining standard arguments (see Section 8.6 in [4]) with our localization we can now prove Theorem 3.1. We denote by $\|\cdot\|_{C[0,1]}$ the sup norm in $C[0, 1]$.

**Proof of Theorem 3.1.** We first show the result under the additional assumption that we can choose $I_n = I$ for all $n$. In this case $\widehat{\rho}^N(k) = \rho^N(k)$ and $\widehat{\tau}^N(k) = \tau^N(k)$. We can assume that $Y_k^N = M_{\tau^N(k)}$ and that the family $N\rho^N(k)$, $N \in \mathbb{N}$, $1 \leq k \leq N$, is uniformly integrable (see Lemma 3.4). Recall that $Y_{Nt}^N = Y_{\lfloor Nt \rfloor}^N + (Nt - \lfloor Nt \rfloor)(Y_{\lfloor Nt \rfloor + 1}^N - Y_{\lfloor Nt \rfloor}^N)$ for $t \in [0, 1]$.

First we show that $\|Y_{N\cdot}^N - M_\cdot\|_{C[0,1]} \to 0$ in probability. To this end let $\varepsilon > 0$. For $\delta > 0$ let

$$A(\delta) = \left\{|M_t - M_s| < \frac{\varepsilon}{2} \text{ for all } t, s \in [0, 1] \text{ such that } |t - s| \leq 2\delta\right\}.$$

We choose $\delta$ such that $\frac{1}{\delta} \in \mathbb{N}$ and $P(A(\delta)) > 1 - \frac{\varepsilon}{2}$. Next we define

$$C(N, \delta) = \left\{\bigl|\tau^N\bigl(\lfloor Nk\delta \rfloor\bigr) - k\delta\bigr| \leq \delta \text{ for } k = 1, \ldots, \frac{1}{\delta}\right\}.$$

By Lemma 3.5 there exists $N_1 \in \mathbb{N}$ such that for all $N \geq N_1$ we have $P(C(N, \delta)) > 1 - \frac{\varepsilon}{2}$.

Notice that on the event $C(N, \delta)$ we have $|\tau^N(\lfloor Ns \rfloor) - s| \leq 2\delta$ for all $s \in [0, 1]$. In the following suppose that $A(\delta) \cap C(N, \delta)$ occurs. Then for $s = \frac{m}{N}$ we have $|\tau^N(m) - \frac{m}{N}| \leq 2\delta$ and hence

$$\bigl|Y_{Ns}^N - M_s\bigr| = \bigl|Y_m^N - M_{\frac{m}{N}}\bigr| < \frac{\varepsilon}{2}.$$

Let now $s \in (\frac{m}{N}, \frac{m+1}{N})$. Set $\theta = s - \frac{m}{N}$ and notice that for all $N \geq \frac{1}{2\delta}$

$$\begin{aligned}\left|Y^N_{Ns} - M_s\right| &\leq \theta\left|Y^N_m - M_{\frac{m}{N}}\right| + (1-\theta)\left|Y^N_{m+1} - M_{\frac{m+1}{N}}\right| \\ &\quad + \theta|M_{\frac{m}{N}} - M_s| + (1-\theta)|M_{\frac{m+1}{N}} - M_s| < \varepsilon.\end{aligned}$$

Consequently, for all $N \geq N_1 \vee \frac{1}{2\delta}$ we have $P(\|Y^N_{N\cdot} - M_\cdot\|_{C[0,1]} > \varepsilon) < \varepsilon$. Since $\varepsilon$ is arbitrary, we obtain that $\|Y^N_{N\cdot} - M_\cdot\|_{C[0,1]} \to 0$ in probability.

Finally, let $\psi: C[0,1] \to \mathbb{R}$ be a bounded function that is continuous with respect to the sup norm. It is straightforward to show that $\lim_{N\to\infty} E\psi(Y^N_{N\cdot}) = E\psi(M_\cdot)$, and hence the theorem is proved in the case $I_n = I$ for sufficiently large $n$.

It remains to prove the result in the case $I_n \neq I$ for all $n$. Let $Y^N_k = M_{\tau^N(k)}$ and, for any fixed $n$, $\widehat{Y}^N_k = M_{\widehat{\tau}^N(k)}$. In the same way as above one can show that, for any fixed $n$, $\|\widehat{Y}^N_{N\cdot} - M_\cdot\|_{C[0,1]} \to 0$.

In order to show that convergence holds true also for $Y^N_{N\cdot}$, we consider several subcases. First let $q(m, l+) = q(m, r-) = \infty$, that is, both endpoints $l$ and $r$ are inaccessible. Recall that $H(I_n)$ denotes the first exit time of $M$ from $I_n$. Fix $\varepsilon > 0$ and choose the localization parameter $n$ such that $P(H(I_n) \geq 2) > 1 - \frac{\varepsilon}{2}$. Then we choose $N_1$ such that, for any $N \geq N_1$, we have $P(\widehat{\tau}^N(N) \leq 2) > 1 - \frac{\varepsilon}{2}$. On the event $\{H(I_n) \geq 2, \widehat{\tau}^N(N) \leq 2\}$ of probability of at least $1 - \varepsilon$ we have $Y^N_k = \widehat{Y}^N_k$ for all $0 \leq k \leq N$. Thus, $\|Y^N_{N\cdot} - M_\cdot\|_{C[0,1]} \to 0$ in probability.

Let now $q(m, l+) < \infty$ and $q(m, r-) = \infty$, i.e. $l$ is accessible, $r$ is inaccessible. In this case, $l > -\infty$ and $H(I_n) \to H(l)$ a.s. as $n \to \infty$, where $H(l)$ denotes the hitting time of $l$ by the process $M$. Fix $\varepsilon > 0$ and choose the localization parameter $n$ such that $P(A_n) < \frac{\varepsilon}{3}$ and $P(B_n) < \frac{\varepsilon}{3}$ with

$$\begin{aligned}A_n &= \left\{H(l) \leq 3 \text{ and } \exists s \in \left[H(I_n), H(l)\right] \text{ such that } M_s > l + \varepsilon\right\}, \\ B_n &= \left\{H(l) > 3 \text{ and } H(I_n) < 2\right\}.\end{aligned}$$

After $n$ is fixed, choose $N_1$ such that, for any $N \geq N_1$, we have $P(\widehat{\tau}^N(N) \leq 2) > 1 - \frac{\varepsilon}{3}$. Given an event $A$, we denote by $A^c$ the complement of $A$. For $N \geq N_1$, on the event

$$\left\{\widehat{\tau}^N(N) \leq 2\right\} \cap (A_n \cup B_n)^c$$

of probability of at least $1 - \varepsilon$ we have either

$$H(l) > 3, \qquad H(I_n) \geq 2, \quad \text{hence } Y^N_k = \widehat{Y}^N_k \text{ for all } 0 \leq k \leq N,$$

or

$$\begin{aligned}&H(l) \leq 3, \qquad M_s \in [l, l+\varepsilon] \quad \text{for } s \in \left[H(I_n), H(l)\right], \\ &\quad \text{hence } \left|Y^N_{Ns} - M_s\right| \leq \varepsilon \text{ whenever } Y^N_{Ns} \neq \widehat{Y}^N_{Ns}, s \in [0,1].\end{aligned}$$

Thus, $\|Y^N_{N\cdot} - M_\cdot\|_{C[0,1]} \to 0$ in probability.

The remaining cases are considered in a similar way. $\square$

*Examples*

We close the section by illustrating our results with several examples.

***Example 3.6 (Brownian motion).*** *Let $M$ be a Brownian motion starting from some $m \in \mathbb{R}$, i.e. we have $l = -\infty$, $r = \infty$ and $\eta \equiv 1$. Then $q(y, x) = (x - y)^2$, for $y, x \in \mathbb{R}$, and*

$$G_y(a) = a^2 \int x^2 \mu(dx), \quad y \in \mathbb{R}, a \geq 0.$$

*Therefore, condition* (A1) *of Section* 2.1 *is satisfied if and only if* $\sigma^2 := \int x^2\mu(dx) < \infty$. *In this case, the scaled random walk* $(Y_k^N)_{k\in\mathbb{N}_0}$ *is determined by the scale factor*

$$a_N(y) = \frac{1}{\sqrt{N\sigma^2}},$$

*which does not depend on the state* $y$. *Theorem* 3.1 *yields weak convergence of* $(Y_{Nt}^N)_{t\in\mathbb{R}_+}$ *to* $(M_t)_{t\in\mathbb{R}_+}$ *under the assumptions that* $\mu \neq \delta_0$ *is centered and* $\int x^2\mu(dx) < \infty$. *This is exactly the Donsker–Prokhorov invariance principle.*

***Example 3.7 (Diffusion between two media).*** *Let* $l = -\infty$, $r = \infty$ *and, with some* $A \in \mathbb{R}\setminus\{0\}$,

$$\eta(x) = 1_{(0,\infty)}(x) + A1_{(-\infty,0]}(x), \quad x \in \mathbb{R}.$$

*Notice that we have*

$$\text{for } y \geq 0: \quad q(y,x) = \begin{cases} (x-y)^2, & x \geq 0, \\ y^2 - 2xy + \frac{1}{A^2}x^2, & x < 0, \end{cases}$$

$$\text{for } y \leq 0: \quad q(y,x) = \begin{cases} \frac{1}{A^2}(x-y)^2, & x < 0, \\ \frac{1}{A^2}y^2 - \frac{2}{A^2}xy + x^2, & x \geq 0. \end{cases}$$

*Since, for appropriate* $0 < c_1 < c_2 < \infty$, *we have* $c_1(x-y)^2 \leq q(y,x) \leq c_2(x-y)^2$, *condition* (A1) *is satisfied if and only if* $\mu$ *has a finite second moment. By Theorem* 3.1 *the processes* $(Y_{Nt}^N)_{t\in\mathbb{R}_+}$ *converge in distribution to* $(M_t)_{t\in\mathbb{R}_+}$ *for any such* $\mu$.

***Example 3.8 (Geometric Brownian motion).*** *Let* $l = 0$, $r = \infty$ *and* $\eta(x) = x$ *on* $I$. *For* $y > 0$, *we have*

$$q(y,x) = \begin{cases} 2\frac{x-y}{y} - 2\log\frac{x}{y}, & \text{if } x > 0, \\ \infty, & \text{if } x \leq 0. \end{cases}$$

*Since, for fixed* $y > 0$, $q(y,x)$ *has linear growth as* $x \to \infty$, *condition* (A2) *of Section* 2.2 *is satisfied if and only if* $\inf \operatorname{supp} \mu > -\infty$. *For all such measures* $\mu$, (19) *is satisfied due to Proposition* 2.13, *and hence Theorem* 2.7 *applies*; *that is, for sufficiently large* $N \in \mathbb{N}$, *Problem* (P) *has a solution with scale factor* $a_N$ *satisfying* (17). *By Theorem* 3.1, *the processes* $(Y_{Nt}^N)_{t\in\mathbb{R}_+}$ *converge in distribution to* $(M_t)_{t\in\mathbb{R}_+}$ *for any* $\mu$ *with a compact support.*

## Appendix

We use the setting and notations of Section 1. In particular, we consider a weak solution $(M, W)$ of (6), where the initial condition $M_0$ has distribution $\gamma$, and we treat the embedding problem (9), where $a\colon I \to (0,\infty)$ is a given Borel function. Let us now briefly explain, following [1], a solution method of (9), which gives an embedding stopping time satisfying (12) provided (11) holds true.

Let $\tilde{W}$ be an $(\tilde{\mathcal{F}}_t)$-Brownian motion on some $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{P})$ and $\tilde{M}_0$ an $\tilde{\mathcal{F}}_0$-measurable random variable with distribution $\gamma$. For $y \in I$, let $F_y$ and $F_\mu$ be the distribution functions of $K(y, a(y), \cdot)$ and of $\mu$, as well as $F_y^{-1}$ and $F_\mu^{-1}$ their generalized inverse functions (that is, $F_y^{-1}(r) = \inf\{x \in \mathbb{R} : F_y(x) > r\}$, $r \in (0,1)$, and the same formula holds for $F_\mu^{-1}$). For $y \in I$, $t \in [0,1]$ and $x \in \mathbb{R}$, we define

$$g(y,t,x) = \tilde{E}\big[F_y^{-1} \circ \Phi(\tilde{W}_1) | \tilde{W}_t = x\big],$$

$$b(t,x) = \tilde{E}\big[F_\mu^{-1} \circ \Phi(\tilde{W}_1) | \tilde{W}_t = x\big],$$

where $\Phi$ denotes the standard normal distribution function, and notice that

$$g(y,t,x) = y + a(y)b(t,x). \tag{39}$$

Let us define the $(\tilde{\mathcal{F}}_t)$-martingale $N_t = b(t, \tilde{W}_t)$, $t \in [0,1]$, and the process $L_t = g(\tilde{M}_0, t, \tilde{W}_t) = \tilde{M}_0 + a(\tilde{M}_0)N_t$, $t \in [0,1]$ (the latter process can fail to be a martingale because it can fail to be integrable). Observe that $N_1$ has the distribution $\mu$, hence

$$\mathrm{Law}(L_1|\tilde{\mathcal{F}}_0) = K\big(\tilde{M}_0, a(\tilde{M}_0), \cdot\big). \tag{40}$$

Moreover, we have

$$\tilde{P}\big((L_t)_{t\in[0,1]} \in A|\tilde{\mathcal{F}}_0\big) = G(\tilde{M}_0, A), \quad A \in \mathcal{B}\big(C[0,1]\big), \tag{41}$$

where the kernel $G$ is given by the formula

$$G(y, A) = \tilde{P}\big(\big(g(y, t, \tilde{W}_t)\big)_{t\in[0,1]} \in A\big), \quad y \in I, A \in \mathcal{B}\big(C[0,1]\big). \tag{42}$$

One can also check that the function $b$ is smooth on $[0,1) \times \mathbb{R}$ and, for any $t \in [0,1)$, the function $b(t,\cdot)$ is a strictly increasing bijective mapping $\mathbb{R} \to (\inf\operatorname{supp}\mu, \sup\operatorname{supp}\mu)$. Let $g^{-1}$ denote the inverse of $g$ in the last argument, which is well defined when the second argument $t \in [0,1)$.

A straightforward generalization of Theorems 1 and 3 and Lemma 2 in [1] now yields the following statement.

**Proposition A.1.** *Assume that* (11) *holds true. Then the ODE*

$$\delta'(t) = \frac{a^2(M_0)b_x^2(t, g^{-1}(M_0, t, M_{\delta(t)}))}{\eta^2(M_{\delta(t)})}, \quad t \in [0,1), \delta(0) = 0, \tag{43}$$

*has a solution on* $[0,1)$ *for* $P$*-almost all paths. Here,* $b_x$ *denotes the partial derivative of b with respect to the second argument. We set*

$$\delta(1) = \lim_{t\uparrow 1}\delta(t), \tag{44}$$

*which is well defined* $P$*-a.s. because* $\delta$ *is nondecreasing. Moreover,* $(\delta(t))_{t\in[0,1]}$ *is an* $(\mathcal{F}_t)$*-time change, the* $(\mathcal{F}_t)$*-stopping time* $\delta(1)$ *satisfies*

$$E\big[\delta(1)|\mathcal{F}_0\big] = Q(M_0) \quad P\text{-a.s.}, \tag{45}$$

*the process*

$$Z_t = \frac{1}{a(M_0)}(M_{\delta(t)} - M_0), \quad t \in [0,1], \tag{46}$$

*is an* $(\mathcal{F}_{\delta(t)})$*-martingale, and*

$$\mathrm{Law}\big(Z_t; t \in [0,1]|\mathcal{F}_0\big) = \mathrm{Law}\big(N_t; t \in [0,1]\big) \quad P\text{-a.s.}, \tag{47}$$

*where the left-hand side is the notation for the regular conditional distribution of the process* $(Z_t)_{t\in[0,1]}$ *with respect to* $\mathcal{F}_0$*, while the right-hand side is the notation for the unconditional distribution of the process* $(N_t)_{t\in[0,1]}$ *(that is, the former, which is in general a kernel depending on* $\omega$*, equals the latter for almost all paths).*

**Corollary A.2.** *Assume that* (11) *holds true. Then*

$$\mathrm{Law}\big(M_{\delta(t)}; t \in [0,1]|\mathcal{F}_0\big) = G(M_0, \cdot) \quad P\text{-a.s.}, \tag{48}$$

*where the kernel G is given by* (42) *(recall the relation between the processes* $(N_t)$ *and* $(L_t)$ *right after* (39)*). In particular,* $\delta(1)$ *is a solution of the embedding problem* (9) *satisfying* (45) *(see* (40) *and* (41)*).*

The next lemma summarizes the properties we need in this paper.

**Lemma A.3.** *Assume* (11). *Then the following holds true.*

(i) *The process*

$$X_t = \frac{1}{a(M_0)}(M_{\delta(1)\wedge t} - M_0), \quad t \geq 0,$$

*is a uniformly integrable* $(\mathcal{F}_t)$*-martingale.*

(ii) *The* $(\mathcal{F}_t)$*-stopping time* $\delta(1)$ *has the same distribution as the random variable*

$$\xi = \int_0^1 \frac{a^2(\tilde{M}_0) b_x^2(s, \tilde{W}_s)}{\eta^2(\tilde{M}_0 + a(\tilde{M}_0) b(s, \tilde{W}_s))} \, ds. \tag{49}$$

(*Of course one can drop the tildes in the latter formula.*)

**Proof.** (i) First observe that the $(\mathcal{F}_t)$-time change $(\delta(t))_{t\in[0,1]}$ is $P$-a.s. strictly increasing on [0, 1]. Indeed, if it had an interval of constancy, then, by (46) and (47), the process $(N_t)_{t\in[0,1]}$ would have an interval of constancy, which is impossible because $N_t = b(t, \tilde{W}_t)$ and, for $t \in [0, 1)$, $b$ is smooth in both arguments and $b(t, \cdot)$ is strictly increasing. Thus, the inverse $\delta^{-1}$ is well defined.

Now, for a fixed $t \geq 0$, define $\eta = \delta^{-1}(\delta(1) \wedge t)$. Since $\delta(1) \wedge t$ is an $(\mathcal{F}_t)$-stopping time, $\eta$ is an $(\mathcal{F}_{\delta(t)})$-stopping time. Clearly, $\eta \leq 1$. Doob's optional sampling theorem applied to the $(\mathcal{F}_{\delta(t)})$-martingale $(Z_t)_{t\in[0,1]}$ (see (46)) and to the bounded $(\mathcal{F}_{\delta(t)})$-stopping times $\eta$ and 1 yields $E(Z_1|\mathcal{F}_{\delta(\eta)}) = Z_\eta$, $P$-a.s., which is equivalent to

$$E(X_\infty|\mathcal{F}_{\delta(1)\wedge t}) = X_t \quad P\text{-a.s.}$$

A short calculation reveals that, since the process $(X_t)_{t\in\mathbb{R}_+}$ is stopped at $\delta(1)$, we also have

$$E(X_\infty|\mathcal{F}_t) = X_t \quad P\text{-a.s.}$$

This concludes the proof of (i).

(ii) Formulas (43), (48), (42), (41) as well as

$$g^{-1}(\tilde{M}_0, t, L_t) = \tilde{W}_t \quad \text{and} \quad L_t = \tilde{M}_0 + a(\tilde{M}_0) b(t, \tilde{W}_t)$$

immediately imply

$$\operatorname{Law}\bigl(\delta(1)|\mathcal{F}_0\bigr) = H(M_0, \cdot) \quad P\text{-a.s.}, \tag{50}$$

where the kernel $H$ is given by the formula

$$H(y, \cdot) = \operatorname{Law}\biggl(\int_0^1 \frac{a^2(y) b_x^2(s, \tilde{W}_s)}{\eta^2(y + a(y) b(s, \tilde{W}_s))} \, ds\biggr), \quad y \in I.$$

Since $\tilde{M}_0$ is $\tilde{\mathcal{F}}_0$-measurable and the process $(\tilde{W}_s)$ is independent of $\tilde{\mathcal{F}}_0$, then for the random variable $\xi$ of (49) we get

$$\operatorname{Law}(\xi|\tilde{\mathcal{F}}_0) = H(\tilde{M}_0, \cdot) \quad \tilde{P}\text{-a.s.} \tag{51}$$

The statement now follows from (50), (51) and the fact that $M_0$ and $\tilde{M}_0$ have the same distribution. □

Sometimes we use the notation $\Delta = \delta(1)$ and also write $\Delta(M_0)$ instead of $\Delta$ whenever we want to stress the dependence on $M_0$.

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## References


[1] S. Ankirchner, D. Hobson and P. Strack. Finite, integrable and bounded time embeddings for diffusions. *Bernoulli* **21** (2) (2015) 1067–1088. MR3338657

[2] S. Athreya, W. Löhr and A. Winter. Invariance principle for variable speed random walks on trees. *Ann. Probab.* **45** (2017) 625–667.

[3] M. D. Donsker. An invariance principle for certain probability limit theorems. *Mem. Amer. Math. Soc.* **6** (1951) 1–12. MR0040613

[4] R. Durrett. *Probability: Theory and Examples*, 4th edition. *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2010. MR2722836

[5] H. J. Engelbert and W. Schmidt. On solutions of one-dimensional stochastic differential equations without drift. *Z. Wahrsch. Verw. Gebiete* **68** (3) (1985) 287–314. MR0771468

[6] A. Gut. The weak law of large numbers for arrays. *Statist. Probab. Lett.* **14** (1) (1992) 49–52. MR1172289

[7] D. G. Hobson. The Skorokhod embedding problem and model independent bounds for option prices. In *Paris-Princeton Lectures on Mathematical Finance 2010* 267–318. *Lecture Notes in Math.* **2003**. Springer, Berlin, 2011. MR2762363

[8] D. H. Hong and K. S. Oh. On the weak law of large numbers for arrays. *Statist. Probab. Lett.* **22** (1) (1995) 55–57. MR1327729

[9] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*, 2nd edition. *Graduate Texts in Mathematics* **113**. Springer-Verlag, New York, 1991. MR1121940

[10] P. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*, **23**. Springer, Berlin, 1992. MR1214374

[11] D. Landers and L. Rogge. Laws of large numbers for uncorrelated Cesàro uniformly integrable random variables. *Sankhya, Ser. A* **59** (3) (1997) 301–310. MR1665769

[12] D. Landers and L. Rogge. Identically distributed uncorrelated random variables not fulfilling the WLLN. *Bull. Korean Math. Soc.* **38** (3) (2001) 605–610. MR1851325

[13] J. Obloj. The Skorokhod embedding problem and its offspring. *Probab. Surv.* **1** (September) (2004) 321–392. MR2068476

[14] Y. V. Prohorov. Convergence of random processes and limit theorems in probability theory. *Teor. Veroyatn. Primen.* **1** (1956) 177–238. MR0084896

[15] A. V. Skorokhod. *Studies in the Theory of Random Processes*. Translated from the Russian by Scripta Technica, Inc. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965. MR0185620

[16] C. Stone. Limit theorems for random walks, birth and death processes, and diffusion processes. *Illinois J. Math.* **7** (1963) 638–660. MR0158440

[17] S. H. Sung. Weak law of large numbers for arrays. *Statist. Probab. Lett.* **38** (2) (1998) 101–105. MR1627902

[18] S. H. Sung, T. C. Hu and A. Volodin. On the weak laws for arrays of random variables. *Statist. Probab. Lett.* **72** (4) (2005) 291–298. MR2153126