Rescaled bipartite planar maps converge to the Brownian map

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Abstract. For every integer $n \geq 1$, we consider a random planar map $\mathcal{M}_n$ which is uniformly distributed over the class of all rooted bipartite planar maps with n edges. We prove that the vertex set of $\mathcal{M}_n$ equipped with the graph distance rescaled by the factor $(2n)^{-1/4}$ converges in distribution, in the Gromov–Hausdorff sense, to the Brownian map. This complements several recent results giving the convergence of various classes of random planar maps to the Brownian map.

Résumé. Pour tout entier n strictement positif, on considère une carte planaire aléatoire $\mathcal{M}_n$ de loi uniforme sur l'ensemble des cartes biparties enracinées à n arêtes. On montre que l'ensemble des sommets de $\mathcal{M}_n$ muni de la distance de graphe renormalisée par $(2n)^{-1/4}$ converge en loi au sens de Gromov–Hausdorff vers la carte brownienne. Ce travail complète une série de résultats de convergence de différents modèles de cartes aléatoires vers la carte brownienne.

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1. Introduction

Much attention has been given recently to the convergence of large random planar maps viewed as metric spaces to the continuous random metric space known as the Brownian map. See in particular [1,2,11,15]. The main goal of the present work is to provide another interesting example of these limit theorems, in the case of bipartite planar maps with a fixed number of edges.

Recall that a planar map is a proper embedding of a finite connected graph in the two-dimensional sphere, viewed up to orientation-preserving homeomorphisms of the sphere. The faces of the map are the connected components of the complement of edges. The degree of a face is the number of edges incident to it, with the convention that, if both sides of an edge are incident to the same face, then this edge is counted twice in the degree of the face. A planar map is rooted if there is a distinguished oriented edge, which is called the root edge.

We consider only bipartite planar maps in the present work. A planar map is bipartite if its vertices can be colored with two colors, in such a way that two vertices that have the same color are not connected by an edge (in particular, there are no loops). This is equivalent to the property that all faces of the map have an even degree.

If M is a planar map, the vertex set of M is denoted by $V(M)$, and the usual graph distance on $V(M)$ is denoted by d_{gr}^M. Let $\mathbf{M}_n^b$ stand for the set of all rooted bipartite maps with n edges.

Theorem 1. *For every $n \geq 1$, let $\mathcal{M}_n$ be uniformly distributed over $\mathbf{M}_n^b$. Then,*

$$\left(V(\mathcal{M}_n), 2^{-1/4}n^{-1/4}d_{\mathrm{gr}}^{\mathcal{M}_n}\right) \xrightarrow[n \to \infty]{(d)} \left(\mathbf{m}_\infty, D^*\right),$$

where $(\mathbf{m}_\infty, D^*)$ is the Brownian map. The convergence holds in distribution in the space $(\mathbb{K}, d_{\mathrm{GH}})$, where $\mathbb{K}$ is the set of all isometry classes of compact metric spaces and d_{GH} is the Gromov–Hausdorff distance.

A brief presentation of the Brownian map will be given in Section 5 below. See [11] and the references therein for more information about this random compact metric space.

As mentioned above, several limit theorems analogous to Theorem 1 have been proved for other classes of random planar maps. The case of p-angulations, which are planar maps where all faces have the same degree p, has received particular attention. Le Gall [11] proved the convergence in distribution of rescaled p-angulations with a fixed number of faces to the Brownian map, both when $p = 3$ (triangulations) and when $p \geq 4$ is even. The case of quadrangulations ($p = 4$) has been treated independently by Miermont [15]. More recently, similar results have been obtained for random planar maps with local constraints: Beltran and Le Gall [2] proved the convergence to the Brownian map for quadrangulations with no pendant vertices, and Addario-Berry and Albenque [1] discussed similar results for simple triangulations or quadrangulations, where there are no loops or multiple edges.

All these papers however deal with random planar maps conditioned to have a fixed number of faces. In our setting, it would make no sense to consider the uniform distribution over all bipartite planar maps with a given number of faces, since there are infinitely many such planar maps. Similarly it would make no sense to condition on the number of vertices, and for this reason we consider conditioning on the number of edges, which results in certain additional technical difficulties.

In order to prove Theorem 1, we first establish a similar result for planar maps that are both rooted and pointed (this means that, in addition to the root edge there is a distinguished vertex, which we call the origin of the map).

As in several of the previously mentioned papers, the proof of this result relies on the combinatorial bijections of Bouttier, di Francesco and Guitter [4] between (rooted and pointed) bipartite planar maps and certain labeled two-type plane trees. Let $\mathbf{M}_n^{b\bullet}$ denote the set of all rooted and pointed planar bipartite maps with n edges, and let $\mathcal{M}_n^\bullet$ be uniformly distributed over $\mathbf{M}_n^{b\bullet}$. The random tree associated with $\mathcal{M}_n^\bullet$ via the Bouttier, di Francesco, Guitter bijection is identified as a (labeled) two-type Galton–Watson tree with explicit offspring distributions, conditioned to have a fixed total progeny (see Proposition 2 below). In order to prove the convergence to the Brownian map, an important technical step is then to derive asymptotics for the contour and label functions associated with this conditioned tree (Theorem 7). Such asymptotics for conditioned two-type Galton–Watson trees have been discussed in [12] and [14]. However both these papers consider conditioning on the number of vertices of one type, which makes it easier to derive the desired asymptotics from the case of usual (one-type) Galton–Watson trees. The fact that we are here conditioning on the total number of vertices creates a significant additional difficulty, which we handle through an absolute continuity argument similar to the ones used in Section 6 of [10]. A useful technical ingredient is a seemingly new definition of a "modified" Lukasiewicz path associated with a two-type tree, which might be of independent interest. This new definition is somehow related to a bijection of Janson and Stefánsson [6] between one-type and two-type trees.

As we were finishing the first version of the present article, we learnt of the very recent paper [3], which obtains a result similar to ours for *general* planar maps. The arguments of [3] might also be applicable to the bipartite case, but the methods seem quite different from the ones that are presented here.

We finally note the simple scaling constant $2^{-1/4}$ in Theorem 1. As far as we know, this value is different from the ones already computed for other classes of maps. The analogous constant for uniform general maps with n edges [3] is $(9/8)^{1/4}$ and the one for uniform quadrangulations with n edges (n even) [11,15] is $(9/4)^{1/4}$.

The paper is organized as follows. Section 2 introduces our main notation and definitions, and recalls the key bijection of [4] between rooted and pointed bipartite maps and labeled two-type trees. In Section 3, we identify the distribution of the random two-type tree associated to a map uniformly distributed over $\mathbf{M}_n^{b\bullet}$, and we introduce its "modified" Lukasiewicz path. Section 4 is devoted to the asymptotics of the contour and label functions coding the two-type tree. Section 5 gives the proof of the statement analogous to Theorem 1 for rooted and pointed maps. Finally, Section 6 explains how to derive Theorem 1 from the latter statement.

2. Bipartite planar maps and trees

2.1. *Trees*

We set $\mathbb{N} = \{1, 2, \ldots\}$ and by convention $\mathbb{N}^0 = \{\varnothing\}$. We introduce the set

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n.$$

An element of $\mathcal{U}$ is a sequence $u = (u^1, \ldots, u^n)$ of elements of $\mathbb{N}$, and we set $|u| = n$ so that $|u|$ represents the "generation" of u. If $u = (u^1, \ldots, u^n)$ and $v = (v^1, \ldots, v^m)$ are two elements of $\mathcal{U}$, then $uv = (u^1, \ldots, u^n, v^1, \ldots, v^m)$ is the concatenation of u and v. The mapping $\pi : \mathcal{U} \setminus \{\varnothing\} \to \mathcal{U}$ is defined by $\pi((u^1, \ldots, u^n)) = (u^1, \ldots u^{n-1})$. One says that $\pi(u)$ is the parent of u, or that u is a child of $\pi(u)$. A plane tree T is a finite subset of $\mathcal{U}$ such that

  (i) $\varnothing \in T$;
 (ii) if $u \in T \setminus \{\varnothing\}$, then $\pi(u) \in T$;
(iii) for every $u \in T$, there exists an integer $k_u(T) \geq 0$ such that, for every $j \in \mathbb{N}$, $uj \in T$ if and only if $1 \leq j \leq k_u(T)$.

In (iii), the number $k_u(T)$ is interpreted as the number of children of u in T. The size of a plane tree T is $|T| = \#T - 1$, which is the number of edges of T. We denote the set of all plane trees by $\mathbf{A}$.

Consider now a plane tree T and $n = |T|$. We introduce the contour sequence $(u_0, u_1, \ldots, u_{2n})$ of T, which is defined by induction as follows: $u_0 = \varnothing$ and for $i \in \{0, \ldots, 2n - 1\}$, u_{i+1} is either the first child of u_i that has not appeared yet in the sequence $(u_0, \ldots, u_i)$, or the parent of u_i if all the children of u_i already appeared in the sequence $(u_0, \ldots, u_i)$. Note that $u_{2n} = \varnothing$ and that all vertices of T appear in the sequence $(u_0, \ldots, u_{2n})$ (some appear more than once).

The white vertices of a tree T are all vertices u such that $|u|$ is even and similarly the black vertices are all vertices such that $|u|$ is odd. We denote the sets of white and black vertices of T by T^0 and T^1 respectively.

We will be interested in certain two-type Galton–Watson trees, which we briefly describe here. Let (μ_0, μ_1) be a pair of probability distributions on $\mathbb{Z}_+$ with respective (finite) means m_0 and m_1. We only consider pairs such that $\mu_0(1) + \mu_1(1) < 2$ and $m_0 m_1 \neq 0$. We say that (μ_0, μ_1) is subcritical if $m_0 m_1 < 1$ and critical if $m_0 m_1 = 1$. Assume that the pair (μ_0, μ_1) is critical or subcritical. A random tree ξ whose distribution is specified by

$$P(\xi = T) = \prod_{u \in T^0} \mu_0\big(k_T(u)\big) \prod_{u \in T^1} \mu_1\big(k_T(u)\big), \quad \forall T \in \mathbf{A}$$

is called a two-type Galton–Watson tree with offspring distributions (μ_0, μ_1). Informally, white vertices have children according to the offspring distribution μ_0 and black vertices have children according to μ_1.

We now introduce labeled trees. A labeled tree is a pair $(T, (\ell(u))_{u \in T^0})$ where T is a plane tree and $(\ell(u))_{u \in T^0}$ is a collection of labels assigned to the white vertices of T, which must satisfy the following properties.

 (i) For every $u \in T$, $\ell(u) \in \mathbb{Z}$.
(ii) Let $v \in T^1$ and $k = k_v(T)$. Let $v_1 = v1, \ldots, v_k = vk$ be the children of v in T, and set also $v_0 = v_{k+1} = \pi(v)$. Then, for every $i \in \{0, 1, 2, \ldots, k\}$, $\ell(v_{i+1}) \geq \ell(v_i) - 1$.

The number $\ell(u)$ is called the label of u. Property (ii) means that, if v is a black vertex, the labels i and j of two white vertices adjacent to v and consecutive in clockwise order around v satisfy $j \geq i - 1$.

We denote the set of all labeled trees with n edges by $\mathbf{T}_n$.

A labeled tree $(T, (\ell(u))_{u \in T^0})$ can be coded by a pair of functions. Recall that if $|T| = n$, $(u_0, \ldots, u_{2n})$ is the contour sequence of T. Note that u_i is white if i is even and black if i is odd. We define for $0 \leq i \leq 2n$,

$$C_i^T = |u_i|.$$

We extend C^T to the real interval $[0, 2n]$ by linear interpolation. The function C^T is the contour function of the tree T. For $0 \leq i \leq n$, set $v_i = u_{2i}$. The sequence $(v_0, \ldots, v_n)$ is called the white contour sequence. We then set, for $0 \leq i \leq n$,
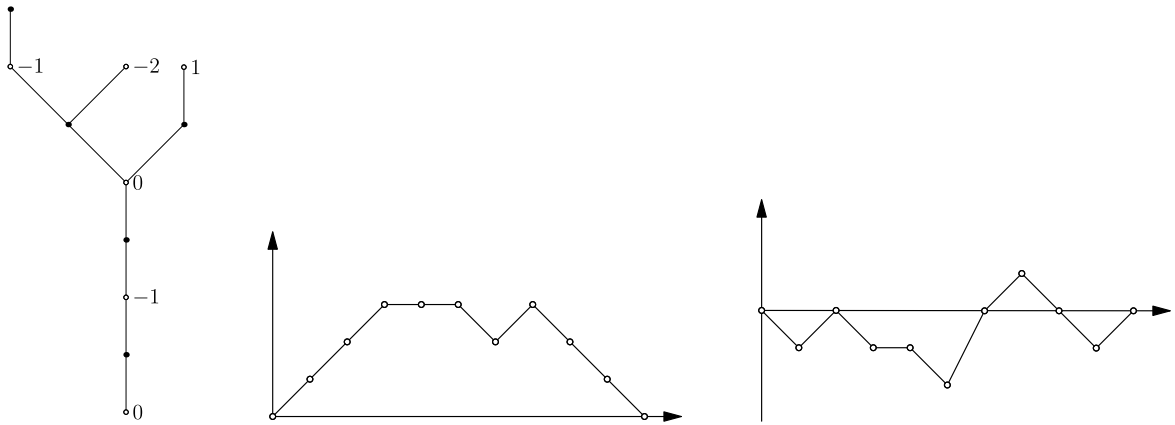
$$C_i^{T^0} = \frac{1}{2}|v_i|$$

Fig. 1. A labeled tree T with $n = 10$ edges, the contour function C^{T^0} and the label function L^{T^0}.

and

$$L_i^{T^0} = \ell(v_i).$$

We notice that for $0 \le i \le n$, we have $C_i^{T^0} = \frac{1}{2}C_{2i}^T$. We also extend both C^{T^0} and L^{T^0} to the real interval $[0, n]$ by linear interpolation. The function C^{T^0} is called the contour function of T^0 (or the white contour function) and L^{T^0} is called the label function of T^0. See Figure 1 for an example. It is easy to verify that the labeled tree $(T, (\ell(u))_{u \in T^0})$ is uniquely determined by the pair (C^T, L^{T^0}) (on the other hand, the pair (C^{T^0}, L^{T^0}) does not give enough information to recover the tree).

2.2. *The Bouttier–Di Francesco–Guitter bijection*

In this section we describe the Bouttier–Di Francesco–Guitter bijection (BDG bijection) between $\mathbf{T}_n \times \{0, 1\}$ and $\mathbf{M}_n^{b\bullet}$. This construction can be found in [4] and in [11] in the particular case of $2p$-angulations.

We start with a labeled tree $(T, (\ell(u))_{u \in T^0}) \in \mathbf{T}_n$ and $\varepsilon \in \{0, 1\}$. As above, $(v_0, \dots, v_n)$ stands for the white contour sequence of T. We suppose that the tree T is represented in the plane in the (obvious) way as suggested by Figure 1. A corner of T is a sector around a vertex of T delimited by two consecutive edges in clockwise order. Each corner is given the label of its associated vertex. We note that every $i \in \{0, 1, \dots, n-1\}$ corresponds to exactly one corner of the vertex v_i (if we move around the tree in clockwise order, the successive white vertices that are visited are $v_0, v_1, \dots, v_{n-1}, v_n = v_0$ and each visit but the last one corresponds to a new corner), and we will abuse terminology by calling this corner the corner v_i.

We then add an extra vertex ∂ outside the tree T, and we construct a planar map $M^\bullet$, whose vertex set is the union of T^0 and of the extra vertex ∂, as follows: For every $i \in \{0, \dots, n-1\}$,

- if $\ell(v_i) = \min\{\ell(v), v \in T^0\}$, then we draw an edge of $M^\bullet$ between the corner v_i and ∂;
- if $\ell(v_i) > \min\{\ell(v), v \in T^0\}$, then we draw an edge of $M^\bullet$ between the corner v_i and the corner v_j, where $j = \min\{k > i: \ell(v_k) = \ell(v_i) - 1\}$ if $\{k > i: \ell(v_k) = \ell(v_i) - 1\}$ is nonempty, $j = \min\{k \ge 0: \ell(v_k) = \ell(v_i) - 1\}$ otherwise.

Thanks to property (ii) of the labels, it is possible to achieve this construction in such a way that edges do not intersect (except at their ends) and do not cross the edges of the tree. The collection of all edges drawn in the preceding construction gives a bipartite planar map $M^\bullet$ with n edges. We then declare that the vertex ∂ is the distinguished vertex of this map and that its root edge is the edge obtained at step $i = 0$ of the preceding construction. The parameter ε gives the orientation of this root edge: the root vertex is $\varnothing$ if and only if $\varepsilon = 0$. In this way we get a pointed and rooted bipartite planar map $M^\bullet$. See Figure 2 for an example with $\varepsilon = 0$.
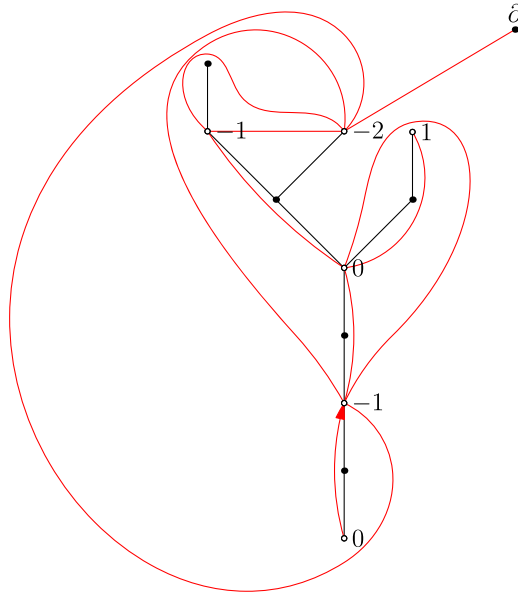
Fig. 2. The labeled tree T of Figure 1 and the associated rooted and pointed bipartite map $M^\bullet$.

The preceding construction yields a bijection from $\mathbf{T}_n \times \{0, 1\}$ onto $\mathbf{M}_n^{b\bullet}$, which is called the Bouttier–Di Francesco–Guitter (BDG) bijection. In this bijection, white vertices of the tree T are identified with vertices of the map $M^\bullet$ other than ∂, and moreover graph distances (in $M^\bullet$) from ∂ are related to labels on T by the formula

$$d_{\mathrm{gr}}^{M^\bullet}(\partial, u) = \ell(u) - \min\{\ell(v), v \in T^0\} + 1, \tag{1}$$

for every $u \in T^0$. There is no such expression for $d_{\mathrm{gr}}^{M^\bullet}(u, v)$ when u and v are arbitrary vertices of $M^\bullet$, but the following bound will be very useful. Let $i, j \in \{0, \dots, n\}$ such that $i < j$. Then,

$$d_{\mathrm{gr}}^{M^\bullet}(v_i, v_j) \le \ell(v_i) + \ell(v_j) - 2\max\{\min\{\ell(v_k), i \le k \le j\}, \min\{\ell(v_k), j \le k \le i + n\}\} + 2, \tag{2}$$

where we made the convention that $v_{n+k} = v_k$ for $0 \le k \le n$. The proof of this bound is easily adapted from [9], Lemma 3.1.

3. Random trees and their contour functions

3.1. *The tree associated with a map chosen uniformly in* $\mathbf{M}_n^{b\bullet}$

Let $\mathcal{M}_n^\bullet$ be uniformly distributed over the set $\mathbf{M}_n^{b\bullet}$, as in Section 1. We let $(\mathcal{T}_n, (\ell_n(u))_{u \in \mathcal{T}_n^0})$ be the random labeled tree associated with $\mathcal{M}_n^\bullet$ by the previously described BDG bijection. The next proposition determines the distribution of this random tree.

Proposition 2. *Let (μ_0, μ_1) be the pair of probability measures on $\mathbb{Z}_+$ defined by*

$$\begin{cases} \mu_0(k) = \frac{2}{3}(\frac{1}{3})^k, \\ \mu_1(k) = \frac{3}{8}\binom{2k+1}{k}(\frac{3}{16})^k \end{cases}$$

for every integer $k \ge 0$. The mean of μ_0 is $1/2$ and the mean of μ_1 is 2, so that the pair (μ_0, μ_1) is critical.

Then the random tree $\mathcal{T}_n$ is a two-type Galton–Watson tree with offspring distributions (μ_0, μ_1) conditioned to have n edges. Furthermore, conditionally given $\mathcal{T}_n$, the labels $(\ell_n(u))_{u \in \mathcal{T}_n^0}$ are uniformly distributed over all admissible labelings.

Proof. Clearly it is enough to determine the law of $\mathcal{T}_n$. We observe that, if T is a plane tree and if u is a black vertex of T with k children, there are $\binom{2k+1}{k}$ possible choices for the increments of labels of white vertices around u. Fix $a \in (0, 1)$ and $b \in (0, 1/4)$, and set for every $k \geq 0$,

$$\begin{cases} \nu_0(k) = (1-a)a^k, \\ \nu_1(k) = B\binom{2k+1}{k}b^k, \end{cases}$$

where B is determined by the requirement that ν_1 is a probability measure on $\mathbb{Z}_+$:

$$B = \frac{2b\sqrt{1-4b}}{1-\sqrt{1-4b}}.$$

Assume that (ν_0, ν_1) is subcritical or critical. If θ is a two-type Galton–Watson tree with offspring distributions (ν_0, ν_1), then, for every plane tree T with n edges,

$$P(\theta = T) = \prod_{u \in T^0} \nu_0\big(k_T(u)\big) \prod_{u \in T^1} \nu_1\big(k_T(u)\big).$$

Writing N_0, respectively N_1, for the number of white, respectively black, vertices of T, we get

$$P(\theta = T) = (1-a)^{N_0} a^{N_1} B^{N_1} b^{N_0-1} \prod_{u \in T^1} \binom{2k_T(u)+1}{k_T(u)}$$

$$= \frac{1}{b}\big((1-a)b\big)^{N_0}(aB)^{N_1} \prod_{u \in T^1} \binom{2k_T(u)+1}{k_T(u)}.$$

On the other hand, the quantity $P(\mathcal{T}_n = T)$ is proportional to the number of possible labelings of T, so that

$$P(\mathcal{T}_n = T) = c_n \prod_{u \in T^1} \binom{2k_T(u)+1}{k_T(u)},$$

where c_n is the appropriate normalizing constant. If a and b are such that

$$(1-a)b = aB, \tag{3}$$

noting that $N_0 + N_1 = n + 1$, we see that $P(\mathcal{T}_n = T)$ coincides with $P(\theta = T)$ up to a multiplicative constant that depends only on n, and it follows that

$$P(\mathcal{T}_n = T) = P\big(\theta = T \mid |\theta| = n\big). \tag{4}$$

The condition (3) holds if

$$a = \frac{1}{3}, \qquad b = \frac{3}{16}.$$

Furthermore, for these values of a and b, we can verify that the mean of ν_0 is $1/2$ and the mean of ν_1 is 2, so that the pair (ν_0, ν_1) is critical. It then follows from the preceding considerations and in particular from (4) that the law of $\mathcal{T}_n$ is as stated in the proposition. $\square$

Remark 3. *One can easily compute the respective variances σ_0^2 and σ_1^2 of the probability measures μ_0 and μ_1. For future reference, we record that*

$$\sigma_0^2 = \frac{3}{4}, \qquad \sigma_1^2 = \frac{15}{2}.$$

3.2. *The white contour function and an associated random walk*

Consider a random labeled tree $(\mathcal{T}, (\ell(u))_{u \in \mathcal{T}})$, such that $\mathcal{T}$ is a two-type Galton–Watson tree with offspring distributions (μ_0, μ_1) given by Proposition 2, and conditionally on $\mathcal{T}$ the labels $(\ell(u))_{u \in \mathcal{T}}$ are uniformly distributed among admissible labelings. Let N denote the (random) number of edges of $\mathcal{T}$, and write $(u_0, \ldots, u_{2N})$ for the contour sequence of $\mathcal{T}$.

For every integer $k \geq 0$, we let the σ-field $\mathcal{F}_k$ be generated by the following random variables:

- the quantity $k \wedge N$ and the vertices $u_0, u_1, \ldots, u_{2(k \wedge N)}$ of $\mathcal{T}$;
- the labels $\ell(u_0), \ell(u_2), \ldots, \ell(u_{2(k \wedge N)})$ of the white vertices $u_0, u_2, \ldots, u_{2(k \wedge N)}$;
- for every odd integer i such that $0 < i < 2(k \wedge N)$, the quantity $k_{\mathcal{T}}(u_i)$ and the labels $\ell(u_i j)$, $1 \leq j \leq k_{\mathcal{T}}(u_i)$, of the (white) children of the black vertex u_i.

Figure 3 below gives a realization of the tree $\mathcal{T}$ and Figure 4 shows the information discovered by the σ-field $\mathcal{F}_k$ for $k = 0, 1, \ldots, 5$. This information should also include the labels of the white vertices that are successively revealed, but these labels are not shown here.

We also introduce a random sequence $(Y_0, Y_1, \ldots, Y_{N+1})$, which is defined by induction by setting $Y_0 = 1$ and, for every $0 \leq k \leq N$:

- if u_{2k} has at least one child that does not appear among $u_0, u_1, \ldots, u_{2k-1}$, then $Y_{k+1} = Y_k + k_{\mathcal{T}}(u_{2k+1})$,
- otherwise $Y_{k+1} = Y_k - 1$.

Informally, for $0 \leq k \leq N$, Y_k counts the number of white vertices that have been visited before time $2k$ by the contour sequence, or are children of black vertices visited before time $2k$, and are still "active" at time $2k$. Saying that a white vertex is still active means that it may have children that have not yet been visited at time $2k$. It is easy to verify that the random variable Y_k (which is only defined on the $\mathcal{F}_k$-measurable set $\{k \leq N + 1\}$) is $\mathcal{F}_k$-measurable and $Y_k \geq 1$ if $0 \leq k \leq N$, whereas $Y_{N+1} = 0$.
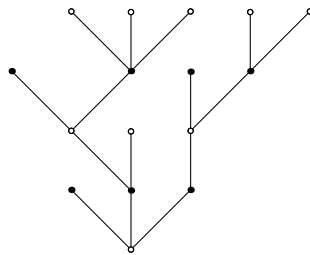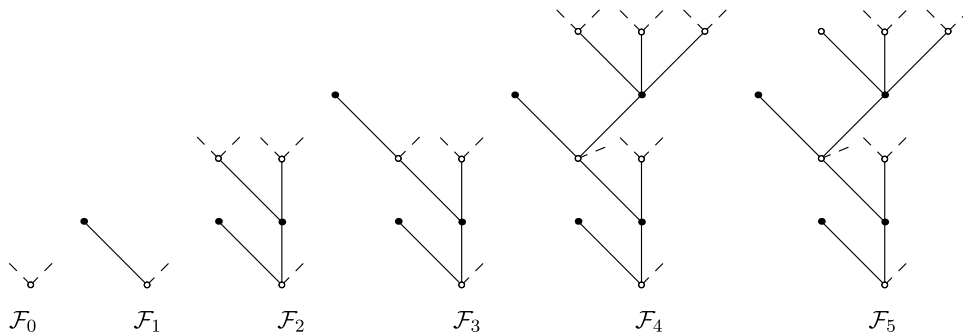

Fig. 3. A realization of the tree $\mathcal{T}$.

Fig. 4. The information about the tree $\mathcal{T}$ of Figure 3 given by the σ-field $\mathcal{F}_k$ for $k = 0, 1, \ldots, 5$. The dashed lines correspond to the "active" white vertices. In this example, $Y_0 = Y_1 = 1$, $Y_2 = Y_3 = 3$, $Y_4 = 6$, $Y_5 = 5$, etc.

The white contour function of $\mathcal{T}$ can be expressed in terms of the sequence $(Y_k)_{0 \le k \le N+1}$ via the formula: for $0 \le k \le N$,

$$C_k^{\mathcal{T}^0} = \mathrm{Card}\{j \in \{0, \ldots, k-1\}: Y_j < \inf\{Y_l: j+1 \le l \le k\}\}. \tag{5}$$

We leave the easy verification of (5) to the reader. Note that the sequence $(Y_0, Y_1, \ldots, Y_{N+1})$ is a kind of "Lukasiewicz path" for our two-type tree, and that the preceding display is analogous to the formula relating the Lukasiewicz path of a (one-type) tree to its height function, see e.g. [8], Proposition 1.2. We also notice that the indices j counted in $C_k^{\mathcal{T}^0}$ correspond to white vertices on the lineage path of v_k in the tree $\mathcal{T}$.

For every $k \ge 0$, we denote the indicator function of the event

$$\{k \le N \text{ and the vertex } u_{2k} \text{ still has a non-visited black child at instant } 2k\}$$

by η_k. Then, conditionally on $\mathcal{F}_k$ and on the event $\{k \le N\}$, η_k is distributed as a Bernoulli random variable with parameter $\frac{1}{3}$. Furthermore, conditionally on $\mathcal{F}_k$ and on the event $\{\eta_k = 1\}$, $k_{\mathcal{T}}(u_{2k+1}) = Y_{k+1} - Y_k$ is distributed according to μ_1. On the other hand, if $k \le N$ and $\eta_k = 0$, we have $Y_{k+1} = Y_k - 1$.

Let ν be the probability measure on $\{-1, 0, 1, \ldots\}$ defined by

$$\begin{cases} \nu(-1) = \frac{2}{3}, \\ \nu(k) = \frac{1}{3}\mu_1(k) & \text{for } k \ge 0 \end{cases}$$

and let $(S_k)_{k \ge 0}$ be a random walk with jump distribution ν starting from $S_0 = 1$. It follows from the preceding discussion that $(Y_0, Y_1, \ldots, Y_{N+1})$ has the same distribution as $(S_0, S_1, \ldots, S_\tau)$ where $\tau = \inf\{n \ge 0, S_n = 0\}$. The distribution ν is centered and has a finite variance $\sigma^2 = 9/2$.

Remark 4. *It follows that $N+1$ (which is the total progeny of the two-type tree $\mathcal{T}$) has the same distribution as τ, and it is well known that this distribution is the same as the total progeny of a (one-type) Galton–Watson tree with offspring distribution $\mu(k) = \nu(k-1)$ for every $k \ge 0$. A similar fact would hold for any (critical or subcritical) two-type Galton–Watson tree such that the offspring distribution of white vertices is geometric. This was already observed in the recent article of Janson and Stefánsson [6], with a different approach involving a bijection between one-type and two-type trees: See [5], Proposition 3.6, for a statement derived from [6], which corresponds exactly to the previous discussion.*

In the remaining part of this section, we state a couple of useful facts about the random walk S, which are variants of results than can be found in [8], Lemmas 1.9 to 1.12. For $m \in \mathbb{Z}_+$, we introduce the "time-reversed" random walk $\hat{S}^m$ defined by

$$\hat{S}_k^m = S_m - S_{m-k} + 1$$

for $0 \le k \le m$. The random walk $(\hat{S}_k^m, 0 \le k \le m)$ has the same distribution as $(S_k, 0 \le k \le m)$. We set

$$M_m = \sup\{S_k, 0 \le k \le m\}$$

and

$$I_m = \inf\{S_k, 0 \le k \le m\}.$$

For every sequence $\omega = (\omega(0), \omega(1), \ldots)$ of integers of length at least m, we set

$$F_m(\omega) = \mathrm{Card}\{k \in \{1, \ldots, m\}: \omega(k) > \sup\{\omega(j), 0 \le j \le k-1\}\}.$$

We then define $(R_m)_{m \ge 0}$ and $(K_m)_{m \ge 0}$ by

$$R_m = F_m(\hat{S}^m), \qquad K_m = F_m(S).$$

Note that we have

$$R_m = \mathrm{Card}\{j \in \{0, \ldots, m-1\}: \ S_j < \inf\{S_l: \ j+1 \le l \le m\}\} \tag{6}$$

(compare with (5)).

Lemma 5. *We define by induction $T_0 = 0$, and for every integer $j \ge 1$,*

$$T_{j+1} = \inf\{k > T_j: \ S_k > S_{T_j}\}.$$

Then the random variables $(S_{T_j} - S_{T_{j-1}})_{j \ge 1}$ are independent and identically distributed, and the distribution of $S_{T_1} - S_{T_0} = S_{T_1} - 1$ is given by

$$P(S_{T_1} - 1 = k) = \frac{3}{2} \nu\big([k, \infty)\big)$$

for $k \ge 1$.

Proof. The fact that the random variables $(S_{T_j} - S_{T_{j-1}})_{j \ge 1}$ are i.i.d. is immediate from the strong Markov property. Let S' be a random walk with jump distribution ν, starting from $S'_0 = 0$, and $T'_1 = \inf\{k > 0, S'_k \ge 0\}$. By [8], Lemma 1.9, we have $P(S'_{T'_1} = k) = \nu([k, \infty))$ for every $k \ge 0$. Next it is clear that the law of $S_{T_1} - 1$ coincides with the conditional law of $S'_{T'_1}$ knowing that $\{S'_{T'_1} > 0\}$. The desired result easily follows. $\square$

It follows that the distribution of $S_{T_1} - 1$ has a finite first moment, given by $E(S_{T_1} - 1) = 3\sigma^2/4$. A simple argument using the law of large numbers then shows that

$$\frac{M_m}{K_m} \xrightarrow[m \to \infty]{} \frac{3\sigma^2}{4}$$

almost surely. The next lemma provides estimates for "moderate deviations" in this convergence.

Lemma 6. *Let $\varepsilon \in (0, \frac{1}{4})$. We can find $\varepsilon' > 0$ and an integer $n_0 \ge 1$ such that for $m \ge n_0$ et $l \in \{0, \ldots, m\}$, we have the bound*

$$P\left(\left| M_l - \frac{3\sigma^2}{4} K_l \right| > m^{1/4+\varepsilon} \right) < \exp\big(-m^{\varepsilon'}\big).$$

Proof. The arguments are easily adapted from the proof of Lemma 1.11 in [8]. $\square$

4. Convergence of the contour and the label functions

We keep the notation $(\mathcal{T}, (\ell(u))_{u \in \mathcal{T}})$ for a random labeled tree such that $\mathcal{T}$ is a two-type Galton–Watson tree with offspring distributions (μ_0, μ_1) given by Proposition 2, and conditionally on $\mathcal{T}$ the labels $(\ell(u))_{u \in \mathcal{T}}$ are uniformly distributed among admissible labelings. As previously, $N = |\mathcal{T}|$. In this section, we discuss the convergence as $n \to \infty$ of the conditional distribution of the pair $(n^{-1/2} C_{nt}^{\mathcal{T}^0}, n^{-1/4} L_{nt}^{\mathcal{T}^0})_{0 \le t \le 1}$ knowing that $N = n$ (recall the notation $C^{\mathcal{T}^0}$ and $L^{\mathcal{T}^0}$ for the contour function and the label function of $\mathcal{T}^0$, see the end of Section 2.1). The whole section is devoted to the proof of the next theorem.

Theorem 7. *The conditional distribution of*

$$\left(\frac{1}{\sqrt{n}} C_{nt}^{\mathcal{T}^0}, \frac{1}{n^{1/4}} L_{nt}^{\mathcal{T}^0} \right)_{0 \le t \le 1}$$

knowing that $N = n$ converges as $n \to \infty$ to the law of

$$\left(\frac{4\sqrt{2}}{9} \mathbf{e}_t, 2^{1/4} Z_t \right)_{0 \le t \le 1},$$

where $\mathbf{e}$ is a normalized Brownian excursion and Z is the Brownian snake driven by this excursion.

Remark 8. *We note that, for every $i \in \{0, \dots, N+1\}$, $C_{2i}^{\mathcal{T}} = 2C_i^{\mathcal{T}^0}$ and $|C_{2i+1}^{\mathcal{T}} - C_{2i}^{\mathcal{T}}| = 1$. From this trivial observation, the convergence in distribution of Theorem 7 also implies that $(\frac{1}{2\sqrt{n}} C_{2nt}^{\mathcal{T}})_{0 \le t \le 1}$ converges to $(\frac{4\sqrt{2}}{9} \mathbf{e}_t)_{0 \le t \le 1}$, and the latter convergence holds jointly with that of Theorem 7. This simple remark will be useful later.*

We recall that a normalized Brownian excursion $\mathbf{e}$ is just a Brownian excursion conditioned to have duration 1, and that the distribution of Z can be described by saying that, conditionally on $\mathbf{e}$, $(Z_t)_{0 \le t \le 1}$ is a centered Gaussian process with continuous sample paths, with covariance

$$E[Z_s Z_t \mid \mathbf{e}] = \min_{s \wedge t \le r \le s \vee t} \mathbf{e}_r.$$

It will sometimes be convenient to make the convention that $\mathbf{e}_t = Z_t = 0$ for $t > 1$. Later we will consider the Brownian snake driven by other types of Brownian excursion, or by reflected linear Brownian motion. Obviously this is defined by the same conditional distribution as above.

As we already mentioned in the introduction, Theorem 7 is closely related to analogous statements proved in [12, 14] for multitype Galton–Watson trees. A major difference however is the fact that [12,14] condition on the number of vertices of one particular type, and not on the total number of vertices in the tree. Apparently the latter conditioning (on the total size of the tree) cannot be handled easily by the methods of [12,14]. See in particular the remarks in [12], p. 1682.

Let us turn to the proof. We will rely on formula (5) for $C^{\mathcal{T}^0}$. In connection with this formula, we recall that $(Y_0, Y_1, \dots, Y_{N+1})$ has the same distribution as $(S_0, S_1, \dots, S_\tau)$, where $(S_k)_{k \ge 0}$ is a random walk with jump distribution ν starting from 1, and $\tau = \inf\{n \ge 0 : S_n = 0\}$. It will be convenient to use the notation P_j for a probability measure under which the random walk S starts from j. By standard local limit theorems (see e.g. Theorems 2.3.9 and 2.3.10 in [7]), we have

$$\lim_{m \to \infty} \sup_{j \in \mathbb{Z}} \left(1 \vee \frac{|j|^2}{m} \right) \left| \sqrt{m} P_j(S_m = 0) - \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{j^2}{2\sigma^2 m} \right) \right| = 0. \tag{7}$$

Here $\sigma^2 = 9/2$ is the variance of the distribution ν. We also recall Kemperman's formula (see e.g. [16], p. 122). Let $m \ge j \ge 1$ be two integers. Then,

$$P_j(\tau = m) = \frac{j}{m} P_j(S_m = 0). \tag{8}$$

Since $N + 1$ has the same distribution as τ under P_1, by combining Kemperman's formula with (7), we immediately get

$$n^{3/2} P(N = n) \xrightarrow[n \to \infty]{} \frac{1}{\sigma\sqrt{2\pi}} \quad \text{and} \quad n^{1/2} P(N \ge n) \xrightarrow[n \to \infty]{} \frac{2}{\sigma\sqrt{2\pi}}. \tag{9}$$

First step

Let $\delta \in (0, 1)$ and let Ψ be a bounded continuous function on the space $\mathcal{C}([0, 1], \mathbb{R}^2)$ of all continuous functions from $[0, 1]$ into $\mathbb{R}^2$. Recall the definition of the σ-fields $\mathcal{F}_k$. We have

$$E\left[\Psi\left(\left(\frac{1}{\sqrt{n}}C_{nt}^{\mathcal{T}^0}, \frac{1}{n^{1/4}}L_{nt}^{\mathcal{T}^0}\right), 0 \leq t \leq 1 - \delta\right)\mathbf{1}_{\{N=n\}}\right]$$

$$= E\left[\Psi\left(\left(\frac{1}{\sqrt{n}}C_{nt}^{\mathcal{T}^0}, \frac{1}{n^{1/4}}L_{nt}^{\mathcal{T}^0}\right), 0 \leq t \leq 1 - \delta\right)\mathbf{1}_{\{N \geq \lceil(1-\delta)n\rceil\}}P(N = n \mid \mathcal{F}_{\lceil(1-\delta)n\rceil})\right]. \tag{10}$$

We then need to study the term $P(N = n \mid \mathcal{F}_{\lceil(1-\delta)n\rceil})$.

We notice that, conditionally on $\{N \geq \lceil(1 - \delta)n\rceil\}$ and on the σ-field $\mathcal{F}_{\lceil(1-\delta)n\rceil}$, the sequence $(Y_{\lceil(1-\delta)n\rceil}, Y_{\lceil(1-\delta)n\rceil+1}, \ldots, Y_{N+1})$ has the same distribution as a random walk with jump distribution ν starting from $Y_{\lceil(1-\delta)n\rceil}$ and stopped when it hits 0. Thus, we apply Kemperman's formula (8), and we obtain, still on the event $\{N \geq \lceil(1 - \delta)n\rceil\}$,

$$P(N = n \mid \mathcal{F}_{\lceil(1-\delta)n\rceil}) = P_{Y_{\lceil(1-\delta)n\rceil}}\left(\tau = n + 1 - \lceil(1 - \delta)n\rceil\right) = \Phi_n(Y_{\lceil(1-\delta)n\rceil}), \tag{11}$$

where $\Phi_n(j) = \frac{j}{m_n}P_j(S_{m_n} = 0)$, for $0 \leq j \leq m_n$, and $m_n = n + 1 - \lceil(1 - \delta)n\rceil = \lfloor\delta n\rfloor + 1$.

Lemma 9. *We have*

$$\lim_{n \to \infty} \sqrt{n}E\left[\mathbf{1}_{\{N \geq \lceil(1-\delta)n\rceil\}}\left|n\Phi_n(Y_{\lceil(1-\delta)n\rceil}) - f_\delta\left(\frac{Y_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)\right|\right] = 0,$$

where for every $x \geq 0$,

$$f_\delta(x) = \frac{x}{\delta\sigma\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Proof. We use the local limit theorem (7) to evaluate $n\Phi_n(j)$. Remark that

$$n\Phi_n(j) = \frac{n}{m_n}jP_j(S_{m_n} = 0)$$

and $n/m_n \longrightarrow 1/\delta$ as $n \to \infty$. It easily follows from (7) that

$$\lim_{n \to \infty}\sup_{0 \leq j \leq m_n}\left|jP_j(S_{m_n} = 0) - \frac{1}{\sigma\sqrt{2\pi}}\frac{j}{\sqrt{m_n}}\exp\left(-\frac{j^2}{2\sigma^2 m_n}\right)\right| = 0.$$

Thus we have

$$\lim_{n \to \infty}\sup_{0 \leq j \leq m_n}\left|n\Phi_n(j) - \frac{1}{\sigma\delta\sqrt{2\pi}}\frac{j}{\sqrt{m_n}}\exp\left(-\frac{j^2}{2\sigma^2 m_n}\right)\right| = 0.$$

Recalling the definition of f_δ, we have thus obtained

$$\lim_{n \to \infty}\sup_{0 \leq j \leq m_n}\left|n\Phi_n(j) - f_\delta\left(\frac{j}{\sqrt{m_n}}\right)\right| = 0, \tag{12}$$

and the result of the lemma follows using also (9). $\qquad\square$

The next step is given by the following lemma.

Lemma 10. *We have*

$$\sqrt{n}E\left[\mathbf{1}_{\{N\geq\lceil(1-\delta)n\rceil\}}\left|f_\delta\left(\frac{Y_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)-f_\delta\left(\frac{3\sigma^2}{4}\frac{C_{\lceil(1-\delta)n\rceil}^{\mathcal{T}^0}}{\sqrt{m_n}}\right)\right|\right]\underset{n\to\infty}{\longrightarrow}0.$$

Proof. From the fact that $(Y_0,\ldots,Y_{N+1})$ has the same distribution as $(S_0,\ldots,S_\tau)$ under P_1, and formula (5), we get that the distribution of $(Y_{\lceil(1-\delta)n\rceil},C_{\lceil(1-\delta)n\rceil}^{\mathcal{T}^0},N)$ conditionally on $\{N\geq\lceil(1-\delta)n\rceil\}$ is the same as the distribution of $(S_{\lceil(1-\delta)n\rceil},R_{\lceil(1-\delta)n\rceil},\tau-1)$ under P_1 conditionally on $\{\tau>\lceil(1-\delta)n\rceil\}$. Thus the left-hand side of (10) can be written as

$$\sqrt{n}E_1\left[\mathbf{1}_{\{\tau>\lceil(1-\delta)n\rceil\}}\left|f_\delta\left(\frac{S_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)-f_\delta\left(\frac{3\sigma^2}{4}\frac{R_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)\right|\right].$$

By time reversal, the following identity in distribution holds under P_1, for $0\leq l\leq m$:

$$(S_l-I_l,R_l)\overset{(d)}{=}(M_l-1,K_l).$$

So Lemma 6 can be rephrased as follows. Let $\varepsilon\in(0,1/4)$. We can find $\varepsilon'>0$ and $n_0\geq1$ such that for $m\geq n_0$ and $l\in\{0,\ldots,m\}$, we have

$$P_1\left(\left|\frac{S_l-I_l+1}{\sqrt{m}}-\frac{3\sigma^2}{4}\frac{R_l}{\sqrt{m}}\right|>m^{-1/4+\varepsilon}\right)<\exp(-m^{\varepsilon'}). \tag{13}$$

Then, since the function f_δ is bounded and Lipschitz, we have

$$\sqrt{n}E_1\left[\mathbf{1}_{\{\tau>\lceil(1-\delta)n\rceil\}}\left|f_\delta\left(\frac{S_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)-f_\delta\left(\frac{3\sigma^2}{4}\frac{R_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)\right|\right]$$

$$\leq\sqrt{n}K_\delta E_1\left[\mathbf{1}_{\{\tau>\lceil(1-\delta)n\rceil\}}\left(\left|\frac{S_{\lceil(1-\delta)n\rceil}-(3\sigma^2/4)R_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right|\wedge1\right)\right],$$

where the constant K_δ only depends on δ. It follows that

$$\sqrt{n}E_1\left[\mathbf{1}_{\{\tau>\lceil(1-\delta)n\rceil\}}\left|f_\delta\left(\frac{S_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)-f_\delta\left(\frac{3\sigma^2}{4}\frac{R_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)\right|\right]$$

$$\leq\sqrt{n}K_\delta\frac{1}{\sqrt{m_n}}n^{1/4+\varepsilon}E_1[\mathbf{1}_{\{\tau>\lceil(1-\delta)n\rceil\}}]$$

$$+\sqrt{n}K_\delta P_1\left(\left|S_{\lceil(1-\delta)n\rceil}-\frac{3\sigma^2}{4}R_{\lceil(1-\delta)n\rceil}\right|>n^{1/4+\varepsilon},\tau>\lceil(1-\delta)n\rceil\right).$$

The first term in the sum tends to 0 as $n\to\infty$ thanks to (9). We then use the fact that $I_{\lceil(1-\delta)n\rceil}=1$ on the event $\{\tau>\lceil(1-\delta)n\rceil\}$ and the bound (13) to see that the second term also tends to 0. We thus get

$$\sqrt{n}E_1\left[\mathbf{1}_{\{\tau>\lceil(1-\delta)n\rceil\}}\left|f_\delta\left(\frac{S_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)-f_\delta\left(\frac{3\sigma^2}{4}\frac{R_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)\right|\right]\underset{n\to\infty}{\longrightarrow}0$$

and our claim follows. $\square$

It follows from Lemmas 9 and 10 that

$$\lim_{n\to\infty}\sqrt{n}E\left[\left|n\Phi_n(Y_{\lceil(1-\delta)n\rceil})-f_\delta\left(\frac{3\sigma^2}{4}\frac{C_{\lceil(1-\delta)n\rceil}^{\mathcal{T}^0}}{\sqrt{m_n}}\right)\right|\mathbf{1}_{\{N\geq\lceil(1-\delta)n\rceil\}}\right]=0. \tag{14}$$

From (10) and (11), we now obtain

$$
\lim_{n\to\infty}\left| n^{3/2}E\left[\Psi\left(\left(\frac{1}{\sqrt{n}}C^{\mathcal{T}^0}_{nt}, \frac{1}{n^{1/4}}L^{\mathcal{T}^0}_{nt}\right), 0\le t\le 1-\delta\right)\mathbf{1}_{\{N=n\}}\right]\right.
$$
$$
\left. -\sqrt{n}E\left[\Psi\left(\left(\frac{1}{\sqrt{n}}C^{\mathcal{T}^0}_{nt}, \frac{1}{n^{1/4}}L^{\mathcal{T}^0}_{nt}\right), 0\le t\le 1-\delta\right)f_\delta\left(\frac{3\sigma^2}{4}\frac{C^{\mathcal{T}^0}_{\lceil(1-\delta)n\rceil}}{\sqrt{m_n}}\right)\mathbf{1}_{\{N\ge\lceil(1-\delta)n\rceil\}}\right]\right| = 0. \tag{15}
$$

Second step

In view of (15), we now need to get a limit in distribution for the (rescaled) pair $(C^{\mathcal{T}^0}_{nt}, L^{\mathcal{T}^0}_{nt})_{0\le t\le 1-\delta}$ conditioned on the event $\{N\ge\lceil(1-\delta)n\rceil\}$. This is the goal of the next lemma, which is essentially a consequence of results found in [14].

Lemma 11. *Let $a > 0$. The law under $P(\cdot\mid N\ge an)$ of the process*

$$
\left(\left(\frac{1}{\sqrt{n}}C^{\mathcal{T}^0}_{(nt)\wedge N}, \frac{1}{n^{1/4}}L^{\mathcal{T}^0}_{(nt)\wedge N}\right), t\ge 0\right)
$$

converges when $n\to\infty$ to the law of

$$
\left(\left(\frac{1}{\tilde\sigma}\mathbf{e}^{(a)}_t, \Sigma\sqrt{\frac{2}{\tilde\sigma}}Z^{(a)}_t\right), t\ge 0\right),
$$

where $\mathbf{e}^{(a)}$ is a Brownian excursion conditioned to have duration greater than a, $Z^{(a)}$ is the Brownian snake driven by this excursion, and the constants are given by

$$
\tilde\sigma = \frac{9}{4\sqrt{2}}, \qquad \Sigma = \sqrt{\frac{9}{8}}.
$$

Proof. To relate the convergence of the lemma to the results of [14], we first recall the contour function $C^{\mathcal{T}}$ and introduce a label function $L^{\mathcal{T}}$ defined as follows. If $(u_0, u_1, \ldots, u_{2N})$ is the contour sequence of $\mathcal{T}$, we already saw that $C^{\mathcal{T}}_i = |u_i|$ and we put $L^{\mathcal{T}}_i = \ell(u_i)$, for every $i\in\{0, 1, \ldots, 2N\}$, where by convention we have assigned to each black vertex the label of its parent. We then interpolate linearly to define $C^{\mathcal{T}}_t$ and $L^{\mathcal{T}}_t$ for every real $t\in[0, 2N]$. It is then enough to verify that the convergence of the lemma holds when $(n^{-1/2}C^{\mathcal{T}^0}_{(nt)\wedge N}, n^{-1/4}L^{\mathcal{T}^0}_{(nt)\wedge N})_{t\ge 0}$ is replaced by $(2^{-1}n^{-1/2}C^{\mathcal{T}}_{(2nt)\wedge(2N)}, n^{-1/4}L^{\mathcal{T}}_{(2nt)\wedge(2N)})_{t\ge 0}$ (see Remark 8).

We also introduce the variant of the contour function called the height function, and the corresponding variant of the label function. The height function of $\mathcal{T}$ is defined by setting $H^{\mathcal{T}}_i = |w_i|$ for $0\le i\le N$, where $w_0, w_1, \ldots, w_N$ are the vertices of $\mathcal{T}$ listed in lexicographical order, and the modified label function is defined by $\tilde{L}^{\mathcal{T}}_i = \ell(w_i)$ (again we assign to each black vertex the label of its parent). By convention we set $H^{\mathcal{T}}_{N+1} = 0$ and $\tilde{L}^{\mathcal{T}}_{N+1} = 0$. Both $H^{\mathcal{T}}$ and $\tilde{L}^{\mathcal{T}}$ are interpolated linearly to give processes indexed by $[0, N+1]$. Then we may replace $(2^{-1}n^{-1/2}C^{\mathcal{T}}_{(2nt)\wedge(2N)}, n^{-1/4}L^{\mathcal{T}}_{(2nt)\wedge(2N)})_{t\ge 0}$ by $(2^{-1}n^{-1/2}H^{\mathcal{T}}_{(nt)\wedge(N+1)}, n^{-1/4}\tilde{L}^{\mathcal{T}}_{(nt)\wedge(N+1)})_{t\ge 0}$. Indeed it is well known that asymptotics for the height functions, of the type of the convergence (16), imply similar asymptotics for the contour functions (and similarly for the label functions) modulo an extra multiplicative factor 2 in the time scaling. See e.g. Section 1.6 in [8] for a precise justification in a slightly different setting. In the case of Galton–Watson trees with a fixed size, the fact that the height process and the contour function converge jointly to the same Brownian excursion is due to Marckert and Mokkadem in [13].

Consider then a sequence $(\mathcal{T}_{(k)}, (\ell_{(k)}(u))_{u\in\mathcal{T}^0_{(k)}})_{k\ge 1}$ of independent labeled trees distributed as $(\mathcal{T}, (\ell(u))_{u\in\mathcal{T}^0})$. Set $N_{(k)} = |\mathcal{T}_{(k)}|$ for every $k\ge 1$. Define the height function H^∞, respectively the label function $\tilde{L}^\infty$, by concatenating the

height functions $(H_t^{\mathcal{T}_{(k)}})_{0 \le t \le N_{(k)}+1}$, resp. the label functions $(\tilde{L}_t^{\mathcal{T}_{(k)}})_{0 \le t \le N_{(k)}+1}$. Then a very special case of Theorems 1 and 3 in [14] gives the convergence in distribution

$$\left(\left(\frac{1}{\sqrt{n}} H_{nt}^{\infty}, \frac{1}{n^{1/4}} \tilde{L}_{nt}^{\infty} \right), t \ge 0 \right) \xrightarrow[n \to \infty]{(d)} \left(\left(\frac{2}{\tilde{\sigma}} \beta_t, \Sigma \sqrt{\frac{2}{\tilde{\sigma}}} W_t \right), t \ge 0 \right), \tag{16}$$

where β is a standard reflected linear Brownian motion, and W is the Brownian snake driven by β. Furthermore, the constants $\tilde{\sigma}$ and Σ are as in the statement of the lemma.

Let us comment on the numerical values of the constants $\tilde{\sigma}$ and Σ. Both these constants can be calculated using the formulas found in [14]. More precisely, $\tilde{\sigma}$ is evaluated from formula (2) in [14], using also the numerical values $\sigma_0^2 = 3/4$ and $\sigma_1^2 = 15/2$ for the respective variances of μ_0 and μ_1. Similarly, Σ is computed from the formula in [14], Theorem 3. When applying this formula, we need to calculate the variance of the difference between the label of the ith child of a black vertex and the label of the parent of this black vertex, conditionally on the event that the black vertex in consideration has p children (with of course $p \ge i$). This variance is equal to $2i(p - i + 1)/(p + 2)$, by a calculation found on p. 1664 of [12]. The remaining part of the calculation is straightforward, and we leave the details to the reader.

Finally we observe that if $K = \min\{k \ge 1 : N_{(k)} \ge an\}$, the law of the labeled tree $(\mathcal{T}_{(K)}, (\ell_{(K)}(u))_{u \in \mathcal{T}_{(K)}^0})$ is the same as the conditional law $(\mathcal{T}, (\ell(u))_{u \in \mathcal{T}^0})$ knowing that $N \ge an$. On the other hand, the process $(n^{-1/2} H_{nt}^{\mathcal{T}_{(K)}})_{0 \le t \le n^{-1}(N_{(K)}+1)}$ corresponds to the first excursion of $(n^{-1/2} H_{nt}^{\infty})_{t \ge 0}$ away from 0 with length greater than or equal to $a + n^{-1}$. By arguments very similar to [8], Proof of Corollary 1.13, we deduce from (16) that $(n^{-1/2} H_{(nt) \wedge (N_{(K)}+1)}^{\mathcal{T}_{(K)}})_{t \ge 0}$ converges in distribution to the first excursion of $(\frac{2}{\tilde{\sigma}} \beta_t)_{t \ge 0}$ away from 0 with duration greater than a. This gives the convergence of the first component in Lemma 11. The convergence of the second component (and the fact that it holds jointly with the first one) is obtained by the same argument. $\square$

By (9), we have

$$n^{3/2} P(N = n) \xrightarrow[n \to \infty]{} \frac{1}{\sigma \sqrt{2\pi}}, \qquad \sqrt{n} P\left(N \ge \lceil (1-\delta)n \rceil\right) \xrightarrow[n \to \infty]{} \frac{2}{\sigma \sqrt{2\pi}} (1-\delta)^{-1/2}.$$

From (15) and Lemma 11, we now get

$$\lim_{n \to \infty} E\left[\Psi\left(\left(\frac{1}{\sqrt{n}} C_{nt}^{\mathcal{T}^0}, \frac{1}{n^{1/4}} L_{nt}^{\mathcal{T}^0} \right), 0 \le t \le 1 - \delta \right) \big| N = n \right]$$

$$= 2(1-\delta)^{-1/2} E\left[\Psi\left(\left(\frac{1}{\tilde{\sigma}} \mathbf{e}_t^{(1-\delta)}, \Sigma \sqrt{\frac{2}{\tilde{\sigma}}} Z_t^{(1-\delta)} \right), 0 \le t \le 1 - \delta \right) f_\delta\left(\frac{3}{\sqrt{2\delta}} \mathbf{e}_{1-\delta}^{(1-\delta)} \right) \right]$$

$$= E\left[\Psi\left(\left(\frac{1}{\tilde{\sigma}} \mathbf{e}_t^{(1-\delta)}, \Sigma \sqrt{\frac{2}{\tilde{\sigma}}} Z_t^{(1-\delta)} \right), 0 \le t \le 1 - \delta \right) g_\delta\left(\mathbf{e}_{1-\delta}^{(1-\delta)} \right) \right],$$

where, for every $x \ge 0$,

$$g_\delta(x) = 2(1-\delta)^{-1/2} f_\delta\left(\frac{3}{\sqrt{2\delta}} x \right).$$

Recalling the definition of f_δ, and the fact that $\sigma^2 = 9/2$, we obtain

$$g_\delta(x) = \frac{2x}{\sqrt{2\pi \delta^3 (1-\delta)}} \exp\left(-\frac{x^2}{2\delta} \right).$$

It is well known (see formula (1) in [10]) that the function $\omega \longrightarrow g_\delta(\omega(1-\delta))$ is the density (on the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$) of the law of the normalized Brownian excursion with respect to the law of the Brownian excursion conditioned to

have length greater than $1 - \delta$, on the σ-field generated by the coordinates up to time $1 - \delta$. Hence we conclude that we have also

$$\lim_{n\to\infty} E\left[\Psi\left(\left(\frac{1}{\sqrt{n}}C_{nt}^{\mathcal{T}^0}, \frac{1}{n^{1/4}}L_{nt}^{\mathcal{T}^0}\right), 0 \le t \le 1 - \delta\right)|N = n\right]$$

$$= E\left[\Psi\left(\left(\frac{1}{\tilde{\sigma}}\mathbf{e}_t, \Sigma\sqrt{\frac{2}{\tilde{\sigma}}}Z_t\right), 0 \le t \le 1 - \delta\right)\right], \tag{17}$$

where $\mathbf{e}$ and Z are as in the statement of Theorem 7. Since this holds for every $\delta \in (0, 1)$ and since we have $C_n^{\mathcal{T}^0} = L_n^{\mathcal{T}^0} = 0$ on the event $\{N = n\}$, we have obtained the convergence of finite-marginal distributions in the convergence of Theorem 7 (note that $\frac{1}{\tilde{\sigma}} = \frac{4\sqrt{2}}{9}$ and $\Sigma\sqrt{\frac{2}{\tilde{\sigma}}} = 2^{1/4}$).

To complete the proof, we still need a tightness argument. But tightness holds if we restrict our processes to $[0, 1 - \delta]$ by (17), and we can then use a time-reversal argument. Indeed $(C_0^{\mathcal{T}^0}, C_1^{\mathcal{T}^0}, \ldots, C_n^{\mathcal{T}^0})$ and $(C_n^{\mathcal{T}^0}, C_{n-1}^{\mathcal{T}^0}, \ldots, C_0^{\mathcal{T}^0})$ have the same distribution under $P(\cdot|N = n)$. The similar property does not hold for the label process, but $(L_n^{\mathcal{T}^0}, L_{n-1}^{\mathcal{T}^0}, \ldots, L_0^{\mathcal{T}^0})$ corresponds to the label process for a (conditioned) tree where labels would be generated by using the counterclockwise order instead of the clockwise order, in the constraints of the definition of a labeled tree in Section 2.1. Clearly, our arguments would go through with this different convention, and so we get the desired tightness also for the label process. This completes the proof of Theorem 7.

Remark 12. *The difficulty in proving Theorem 7 comes from the convergence of labels. If we had been interested only in the convergence of the rescaled contour functions $(\frac{1}{\sqrt{n}}C_{nt}^{\mathcal{T}^0})$, we could have used formula (5) more directly, following the ideas of Marckert and Mokkadem [13]. See also [8], Chapter 1.*

5. Convergence towards the Brownian map for rooted and pointed maps

Recall that $\mathcal{M}_n^\bullet$ is a random bipartite planar map uniformly distributed over the set $\mathbf{M}_n^{b\bullet}$ of all bipartite planar rooted and pointed maps with n edges. In this section, we prove the analog of Theorem 1 when $\mathcal{M}_n$ is replaced by $\mathcal{M}_n^\bullet$, namely

$$\left(V\left(\mathcal{M}_n^\bullet\right), 2^{-1/4}n^{-1/4}d_{\mathrm{gr}}^{\mathcal{M}_n^\bullet}\right) \xrightarrow[n\to\infty]{(d)} \left(\mathbf{m}_\infty, D^*\right), \tag{18}$$

where $(\mathbf{m}_\infty, D^*)$ is the Brownian map.

5.1. *Definition of the Brownian map*

We define the Brownian map following [11], Section 2.4. We first need to introduce the CRT (Continuous Real Tree). Let $(\mathbf{e}_s)_{0 \le s \le 1}$ be a normalized Brownian excursion. For $s, t \in [0, 1]$, we set

$$d_{\mathbf{e}}(s, t) = \mathbf{e}_s + \mathbf{e}_t - 2\min\{\mathbf{e}_r : s \wedge t \le r \le s \vee t\}.$$

We notice that $d_{\mathbf{e}}$ is a random pseudo-metric on $[0, 1]$. Consider the equivalence relation defined for $s, t \in [0, 1]$ by

$$s \sim_{\mathbf{e}} t \quad \text{iff} \quad d_{\mathbf{e}}(s, t) = 0.$$

The CRT is then the quotient space $\mathcal{T}_{\mathbf{e}} = [0, 1]/ \sim_{\mathbf{e}}$, which is equipped with the distance induced by $d_{\mathbf{e}}$. We denote the canonical projection $[0, 1] \to \mathcal{T}_{\mathbf{e}}$ by $p_{\mathbf{e}}$.

We then let $Z = (Z_s)_{0 \le s \le 1}$ be the Brownian snake driven by $\mathbf{e}$, as in Theorem 7. We note that $Z_0 = 0$ and $E((Z_s - Z_t)^2|\mathbf{e}) = d_{\mathbf{e}}(s, t)$. From the last relation, one obtains that $Z_s = Z_t$ for every $s, t \in [0, 1]$ such that $d_{\mathbf{e}}(s, t) = 0$, a.s. Thus the process Z can be viewed as indexed by the CRT $\mathcal{T}_{\mathbf{e}}$, in such a way that $Z_s = Z_{p_{\mathbf{e}(s)}}$ for $s \in [0, 1]$. In the sequel, we will use the notation $Z_s = Z_a$ if $s \in [0, 1]$ and $a = p_{\mathbf{e}}(s)$. Using similar techniques as in the proof of the

Kolmogorov regularity theorem, one can show that the mapping $a \mapsto Z_a$ is Hölder continuous with exponent $\frac{1}{2} - \varepsilon$ with respect to $d_\mathbf{e}$, for every $\varepsilon \in]0, \frac{1}{2}[$. The pair $(\mathcal{T}_\mathbf{e}, (Z_a)_{a \in \mathcal{T}_\mathbf{e}})$ is then a continuous analog of discrete labeled trees.

We can now define the Brownian map, as a quotient space of the CRT. For $s, t \in [0, 1]$ such that $s \le t$, we set

$$D^0(s, t) = D^0(t, s) = Z_s + Z_t - 2 \max\big(\min\{Z_r, r \in [s, t]\}, \min\{Z_r, r \in [0, s] \cup [t, 1]\}\big)$$

and for $a, b \in \mathcal{T}_\mathbf{e}$,

$$D^0(a, b) = \min\big\{D^0(s, t): (s, t) \in [0, 1]^2, \, p_\mathbf{e}(s) = a, \, p_\mathbf{e}(t) = b\big\}.$$

Finally, for $a, b \in \mathcal{T}_\mathbf{e}$, let

$$D^*(a, b) = \inf\left\{\sum_{i=1}^{k} D^0(a_{i-1}, a_i)\right\},$$

where the infimum is over all choices of the integer $k \ge 1$ and of the finite sequence $(a_0, \ldots, a_k)$ of elements of $\mathcal{T}_\mathbf{e}$ such that $a_0 = a$ and $a_k = b$. Then, D^* is a pseudo-metric on the CRT $\mathcal{T}_\mathbf{e}$, which satisfies $D^* \le D^0$. One can also interpret D^* as a function on $[0, 1]^2$ by setting $D^*(s, t) = D^*(p_\mathbf{e}(s), p_\mathbf{e}(t))$ for $(s, t) \in [0, 1]^2$. Let $\simeq$ be the equivalence relation on $\mathcal{T}_\mathbf{e}$ given by

$$a \simeq b \quad \text{iff} \quad D^*(a, b) = 0.$$

We set

$$\mathbf{m}_\infty = \mathcal{T}_\mathbf{e}/\simeq$$

and let $\Pi : \mathcal{T}_\mathbf{e} \to \mathbf{m}_\infty$ be the canonical projection. The Brownian map is the space $\mathbf{m}_\infty$ equipped with the distance induced by D^*.

5.2. *Proof of the convergence towards the Brownian map*

As previously, we let $(\mathcal{T}_n, (\ell_n(v))_{v \in \mathcal{T}_n^0})$ be the random labeled tree associated with $\mathcal{M}_n^\bullet$ via the BDG bijection. Recall that $\mathcal{T}_n$ is a two-type Galton–Watson tree with offspring distributions μ_0 and μ_1, conditioned to have n edges. We use the notation $(v_0^n, \ldots, v_n^n)$ for the white contour sequence of $\mathcal{T}_n$. Recall that the white vertices in $\mathcal{T}_n$ are identified to vertices of the map $\mathcal{M}_n^\bullet$. For $(i, j) \in \{0, \ldots, n\}^2$, we set

$$d_n(i, j) = d_{\mathrm{gr}}^{\mathcal{M}_n^\bullet}(v_i^n, v_j^n).$$

We then extend this definition to noninteger values of i and j by putting for $s, t \in [0, n]^2$

$$d_n(s, t) = (s - \lfloor s \rfloor)(t - \lfloor t \rfloor)d_n(\lceil s \rceil, \lceil t \rceil) + (s - \lfloor s \rfloor)(\lceil t \rceil - t)d_n(\lceil s \rceil, \lfloor t \rfloor)$$
$$+ (\lceil s \rceil - s)(t - \lfloor t \rfloor)d_n(\lfloor s \rfloor, \lceil t \rceil) + (\lceil s \rceil - s)(\lceil t \rceil - t)d_n(\lfloor s \rfloor, \lfloor t \rfloor).$$

Recall our convention $v_{n+i}^n = v_i^n$ for $0 \le i \le n$. From the bound (2), we have for $0 \le i < j \le n$,

$$d_n(i, j) \le \ell_n(v_i^n) + \ell_n(v_j^n) - 2 \max\big\{\min\{\ell_n(v_k^n), i \le k \le j\}, \min\{\ell_n(v_k^n), j \le k \le i + n\}\big\} + 2$$

$$= L_i^{\mathcal{T}_n^0} + L_j^{\mathcal{T}_n^0} - 2 \max\big\{\min\{L_k^{\mathcal{T}_n^0}, k \in [i, j]\}, \min\{L_k^{\mathcal{T}_n^0}, k \in [j, n] \cup [0, i]\}\big\}. \tag{19}$$

From the last bound and the convergence in distribution of the sequence of processes $(n^{-1/4} L_{nt}^{\mathcal{T}_n^0})_{0 \le t \le 1}$ (Theorem 7), one gets that the sequence of the distributions of the processes

$$\big(n^{-1/4} d_n(ns, nt), 0 \le s, t \le 1\big)$$

is tight. Using Theorem 7 and Remark 8, we see that we can find a sequence $(n_k)_{k\geq 1}$ tending to infinity and a continuous random process $(D(s,t))_{0\leq s,t\leq 1}$ such that, along $(n_k)_{k\geq 1}$, the following joint convergence in distribution in $\mathcal{C}([0,1]^2, \mathbb{R}^3)$ holds:

$$\left(\frac{9}{8\sqrt{2}} \frac{C_{2nt}^{\mathcal{T}_n}}{n^{1/2}}, 2^{-1/4} \frac{L_{nt}^{\mathcal{T}_n^0}}{n^{1/4}}, 2^{-1/4} \frac{d_n(s,t)}{n^{1/4}} \right)_{0\leq s,t\leq 1} \xrightarrow[n\to\infty]{} \left(\mathbf{e}_t, Z_t, D(s,t) \right)_{0\leq s,t\leq 1}. \tag{20}$$

Using the Skorokhod representation theorem (and recalling that $(\mathcal{T}_n, (\ell_n(v))_{v\in\mathcal{T}_n^0})$ is determined by the pair $(C^{\mathcal{T}_n}, L^{\mathcal{T}_n^0})$), we may and will assume that the convergence (20) holds a.s. along the sequence $(n_k)_{k\geq 1}$. From the definition of $D^0(s,t)$ and the bound (19), we obtain that for every $(s,t) \in [0,1]^2$,

$$D(s,t) \leq D^0(s,t). \tag{21}$$

Similarly, a passage to the limit from the identity (1) gives

$$D(0,t) = Z_t - \min\{Z_s \colon 0 \leq s \leq 1\}, \tag{22}$$

for every $t \in [0,1]$, a.s.

The function $(s,t) \mapsto D(s,t)$ is clearly symmetric and satisfies the triangle inequality since the functions d_n do. Moreover, the fact that $d_n(i,j) = 0$ if $v_i^n = v_j^n$ easily implies that $D(s,t) = 0$ for s,t such that $s \sim_{\mathbf{e}} t$ a.s. (see the proof of Proposition 3.3 in [9] for a similar argument). Hence $D(s,t)$ only depends on $p_{\mathbf{e}}(s)$ and $p_{\mathbf{e}}(t)$, and D can be viewed as a pseudo-metric on the CRT $\mathcal{T}_{\mathbf{e}}$, which satisfies $D(a,b) \leq D^0(a,b)$ for every $a,b \in \mathcal{T}_{\mathbf{e}}$, by (21). Since D verifies the triangle inequality, the latter bound also implies

$$D(a,b) \leq D^*(a,b)$$

for every $a,b \in \mathcal{T}_{\mathbf{e}}$ a.s. To complete the proof, we need the next lemma.

Lemma 13. *We have*

$$D(a,b) = D^*(a,b)$$

for every $a,b \in \mathcal{T}_{\mathbf{e}}$ a.s.

The statement of the theorem easily follows from the lemma. Indeed, we introduce a correspondence between the metric spaces $(V(\mathcal{M}_n^\bullet) \setminus \{\partial\}, 2^{-1/4} n^{-1/4} d_{gr}^{\mathcal{M}_n^\bullet})$ and $(\mathbf{m}_\infty, D^*)$ by setting

$$\mathcal{R}_n = \left\{ \left(v_{\lfloor nt \rfloor}^n, \Pi(p_{\mathbf{e}}(t)) \right) \colon t \in [0,1] \right\}.$$

From the (almost sure) convergence (20), and the equality $D = D^*$, we easily get that the distortion of $\mathcal{R}_n$ tends to 0 as $n \to \infty$ along the sequence $(n_k)_{k\geq 1}$. It follows that the random metric space $(V(\mathcal{M}_n^\bullet) \setminus \{\partial\}, 2^{-1/4} n^{-1/4} d_{gr}^{\mathcal{M}_n^\bullet})$ converges a.s. to $(\mathbf{m}_\infty, D^*)$ as $n \to \infty$ along the sequence $(n_k)_{k\geq 1}$, in the Gromov–Hausdorff sense. Clearly, this convergence still holds if we replace $V(\mathcal{M}_n^\bullet) \setminus \{\partial\}$ by $V(\mathcal{M}_n^\bullet)$. The previous discussion shows that from every sequence of integers going to infinity, we can extract a subsequence along which the convergence stated in (18) holds. This suffices to complete the proof of (18).

It only remains to prove Lemma 13.

5.3. *Proof of Lemma* 13

Here we follow closely [11], Section 8.3. By a continuity argument, it is enough to show that if X and Y are two independent random variables uniformly distributed over $[0,1]$, which are also independent of the sequence $(\mathcal{M}_n^\bullet)_{n\geq 1}$ and of the triplet $(\mathbf{e}, Z, D)$, we have

$$D\left(p_{\mathbf{e}}(X), p_{\mathbf{e}}(Y) \right) = D^*\left(p_{\mathbf{e}}(X), p_{\mathbf{e}}(Y) \right) \quad \text{a.s.}$$

Since one already knows that

$$D\big(p_{\mathbf{e}}(X),\, p_{\mathbf{e}}(Y)\big) \le D^*\big(p_{\mathbf{e}}(X),\, p_{\mathbf{e}}(Y)\big),$$

it is enough to prove that these two random variables have the same distribution.

First, the distribution of $D^*(p_{\mathbf{e}}(X), p_{\mathbf{e}}(Y))$ can be found in [11], Corollary 7.3:

$$D^*\big(p_{\mathbf{e}}(X),\, p_{\mathbf{e}}(Y)\big) \overset{(d)}{=} Z_X - \min\{Z_s \colon 0 \le s \le 1\}. \tag{23}$$

We then want to determine the distribution of $D(p_{\mathbf{e}}(X), p_{\mathbf{e}}(Y)) = D(X, Y)$. We set for $n \ge 1$,

$$i_n = \lfloor nX \rfloor, \qquad j_n = \lfloor nY \rfloor.$$

The random variables i_n and j_n are independent, independent of $\mathcal{M}_n^\bullet$ and uniformly distributed over $\{0, \ldots, n-1\}$. As we already explained in Section 2.2, every integer between 0 and $n-1$ corresponds to a corner of a white vertex in the tree $\mathcal{T}_n$, and thus by the BDG bijection to an edge of $\mathcal{M}_n^\bullet$. We introduce a new planar map $\mathcal{M}_n^{\bullet\prime}$ in $\mathbf{M}_n^{b\bullet}$ defined by saying that $\mathcal{M}_n^{\bullet\prime}$ has the same vertices, edges, faces and origin vertex as $\mathcal{M}_n^\bullet$, but a different root edge, which is the edge associated with the corner corresponding to i_n in the BDG bijection between $\mathcal{T}_n$ and $\mathcal{M}_n^\bullet$. The orientation of this root edge is chosen with probability $\frac{1}{2}$ among the two possible ones. Since what we have done is just replacing the root edge by another oriented edge chosen uniformly at random over the $2n$ possible choices, it is easy to see that the map $\mathcal{M}_n^{\bullet\prime}$ is also uniformly distributed over $\mathbf{M}_n^{b\bullet}$.

The tree associated with $\mathcal{M}_n^{\bullet\prime}$ via the BDG bijection is denoted by $\mathcal{T}_n'$. We let $v_0'^n, \ldots, v_n'^n$ be the white contour sequence of $\mathcal{T}_n'$ and we also let d_n' be the analog of d_n when $\mathcal{M}_n^\bullet$ is replaced by $\mathcal{M}_n^{\bullet\prime}$.

Let $k_n \in \{0, \ldots, n-1\}$ be the index of the white corner of $\mathcal{T}_n'$ corresponding via the BDG bijection to the edge of $\mathcal{M}_n^\bullet$ starting from the corner j_n in $\mathcal{T}_n$. Conditionally on the pair $(\mathcal{M}_n^\bullet, \mathcal{M}_n^{\bullet\prime})$, the latter edge is uniformly distributed over the set of all edges of $\mathcal{M}_n^\bullet$ (thus also over the set of all edges of $\mathcal{M}_n^{\bullet\prime}$). It follows that, conditionally to $(\mathcal{M}_n^\bullet, \mathcal{M}_n^{\bullet\prime})$, the index k_n is uniformly distributed over $\{0, \ldots, n-1\}$, so it is independent of $\mathcal{M}_n^{\bullet\prime}$. From the definition of $\mathcal{M}_n^{\bullet\prime}$, the vertex $v_{i_n}'^n$ is either equal or adjacent to $v_0'^n$ and in a similar way the vertex $v_{j_n}'^n$ is either equal or adjacent to $v_{k_n}'^n$. This leads to the bound.

$$\big| d_n(i_n, j_n) - d_n'(0, k_n) \big| \le 2. \tag{24}$$

Moreover we observe that

$$d_n'(0, k_n) \overset{(d)}{=} d_n(0, i_n) \tag{25}$$

because k_n is independent of $\mathcal{M}_n^{\bullet\prime}$ and uniformly distributed over $\{0, \ldots, n-1\}$, and i_n satisfies the same properties with respect to $\mathcal{M}_n^\bullet$. We now use the a.s. convergence (20) to get

$$2^{-1/4} n^{-1/4} d_n(0, i_n) \underset{n \to \infty}{\longrightarrow} D(0, X) = Z_X - \min\{Z_s \colon 0 \le s \le 1\}, \tag{26}$$

where the last equality holds by (22), and

$$2^{-1/4} n^{-1/4} d_n(i_n, j_n) \underset{n \to \infty}{\longrightarrow} D(X, Y). \tag{27}$$

Both (26) and (27) hold a.s along the subsequence $(n_k)_{k \ge 1}$. On the other hand, (24) and (25) show that the limit in (26) must have the same distribution as the limit in (27), and we get

$$D(X, Y) \overset{(d)}{=} Z_X - \min\{Z_s \colon 0 \le s \le 1\}.$$

Recalling (23), we see that $D(p_{\mathbf{e}}(X), p_{\mathbf{e}}(Y))$ and $D^*(p_{\mathbf{e}}(X), p_{\mathbf{e}}(Y))$ have the same distribution, which completes the proof of Lemma 13.

6. Convergence of rooted maps

In this section, we derive Theorem 1 from the convergence (18) for rooted and pointed maps. Notice that similar arguments appear in [3], Proposition 4. As previously, $\mathcal{M}_n^\bullet$ is uniformly distributed over $\mathbf{M}_n^{b\bullet}$, but it will be sometimes be convenient to view $\mathcal{M}_n^\bullet$ as a random element of $\mathbf{M}_n^b$, just by "forgetting" the distinguished vertex. In particular, if F is a function on $\mathbf{M}_n^b$, the notation $F(\mathcal{M}_n^\bullet)$ means that we apply F to the rooted map obtained by forgetting the distinguished vertex of $\mathcal{M}_n^\bullet$. Similarly, we will write $\mu_n^\bullet$ for the law of $\mathcal{M}_n^\bullet$ viewed as a random element of $\mathbf{M}_n^b$. The notation μ_n will then stand for the law of $\mathcal{M}_n$, that is, the uniform probability measure on $\mathbf{M}_n^b$. Let $\|\cdot\|$ stand for the total variation norm. In order to get Theorem 1 from (18), it is sufficient to prove the following result.

Proposition 14. *The following convergence holds.*

$$\left\|\mu_n - \mu_n^\bullet\right\| \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. We have

$$\left\|\mu_n - \mu_n^\bullet\right\| = \frac{1}{2} \sup_{-1\le F\le 1} \left|E\big(F(\mathcal{M}_n)\big) - E\big(F\big(\mathcal{M}_n^\bullet\big)\big)\right|,$$

where the supremum is over all functions $F : \mathbf{M}_n^{b\bullet} \longrightarrow [-1,1]$. The quantity $E(F(\mathcal{M}_n^\bullet))$ can be expressed in terms of $E(F(\mathcal{M}_n))$ as

$$E\big(F\big(\mathcal{M}_n^\bullet\big)\big) = \frac{E(F(\mathcal{M}_n)\,\mathrm{Card}\,V(\mathcal{M}_n))}{E(\mathrm{Card}\,V(\mathcal{M}_n))},$$

which implies

$$E\big(F(\mathcal{M}_n)\big) = E\left(\frac{F(\mathcal{M}_n^\bullet)}{\mathrm{Card}\,V(\mathcal{M}_n^\bullet)}\right)\frac{1}{E(1/\,\mathrm{Card}\,V(\mathcal{M}_n^\bullet))}. \tag{28}$$

We then need an estimate of $\mathrm{Card}\,V(\mathcal{M}_n^\bullet)$, which is given by the next lemma.

Lemma 15. *Let $\delta > 0$. There exists a positive constant C_δ such that*

$$P\left(\left|\mathrm{Card}\,V\big(\mathcal{M}_n^\bullet\big) - \frac{2n}{3}\right| > \delta n\right) \le \exp(-C_\delta n)$$

for all n sufficiently large.

Proof. We start by observing that the number $\mathrm{Card}\,V(\mathcal{M}_n^\bullet)$ corresponds via the BDG bijection to (1 plus) the number of white vertices of a two-type Galton–Watson tree with offspring distributions (μ_0, μ_1) given by Proposition 2, conditioned to have n edges.

Let us consider a sequence of independent two-type Galton–Watson trees with offspring distributions (μ_0, μ_1). Suppose that the white vertices of these trees are listed in lexicographical order for each tree, one tree after another, and write $A_1, A_2, \ldots$ for the respective numbers of black children of the white vertices in this enumeration. Then $A_1, A_2, \ldots$ are i.i.d. random variables with distribution μ_0, and we recall that μ_0 is a geometric distribution with mean $\frac{1}{2}$. We can apply Cramer's theorem to get the exponential bound, for every $n \ge 1$,

$$P\left(\left|\frac{A_1 + \cdots + A_n}{n} - \frac{1}{2}\right| \ge \delta\right) \le \exp(-K_\delta n), \tag{29}$$

where K_δ is a positive constant.

Let N_0 and N_1 be respectively the numbers of white and black vertices in the first tree in our sequence, and let $N = N_0 + N_1 - 1$, which is the number of edges of this tree. The point now is the fact that if we condition on the event

$\{N = n\}$, the planar map associated with the first tree becomes uniform on $\mathbf{M}_n^{b\bullet}$. Since this planar map has $N_0 + 1$ vertices, the result of the lemma will follow if we can prove that, for n sufficiently large,

$$P\left[\left|N_0 - \frac{2}{3}(n+1)\right| > \delta(n+1)|N = n\right] \leq \exp(-C_\delta n)$$

for some positive constant C_δ.

Recall from (9) that $n^{3/2} P(N = n) \longrightarrow (\sigma\sqrt{2\pi})^{-1}$ as $n \to \infty$. Therefore the preceding exponential bound will follow if we can verify that for all n large enough,

$$P\left[\left\{\left|N_0 - \frac{2}{3}(n+1)\right| > \delta(n+1)|\right\} \cap \{N = n\}\right] \leq \exp(-c_\delta n)$$

with some positive constant c_δ.

We first observe that the event $\mathcal{E}_1 := \{N_0 - \frac{2}{3}(n+1) > \delta(n+1)\} \cap \{N = n\}$ is contained in

$$\left\{n+1 \geq N_0 > \left(\frac{2}{3} + \delta\right)(n+1)\right\} \cap \left\{\frac{N_1}{N_0} < \frac{1/3 - \delta}{2/3 + \delta}\right\}.$$

Therefore if we set $a_\delta = (\frac{1}{3} - \delta)/(\frac{2}{3} + \delta) < \frac{1}{2}$, the event $\mathcal{E}_1$ may only hold if, for some k such that $(\frac{2}{3} + \delta)(n+1) < k \leq n+1$, the first k white vertices of our sequence of trees have less than $a_\delta k$ black children. Using (29), we obtain that

$$P(\mathcal{E}_1) \leq \sum_{(2/3+\delta)(n+1)<k\leq n+1} \exp(-K'_\delta k) \leq \exp(-c'_\delta n)$$

for some positive constants K'_δ and c'_δ. Similar arguments give an analogous exponential bound for the probability of the event $\mathcal{E}_2 := \{N_0 - \frac{2}{3}(n+1) < -\delta(n+1)\} \cap \{N = n\}$. This completes the proof of the lemma. □

Set $X_n = (2n/3)^{-1} \operatorname{Card} V(\mathcal{M}_n^\bullet)$ for every $n \geq 1$.

Lemma 16. *The random variables X_n^{-1} converge to* 1 *in* L^1 *when n tends to infinity.*

Proof. First, as $\operatorname{Card} V(\mathcal{M}_n^\bullet) \geq 1$, we have $X_n^{-1} \leq \frac{2n}{3}$. Let $\delta > 0$. The event $\{|X_n^{-1} - 1| > \delta\}$ is contained in $\{X_n < \frac{1}{2}\} \cup \{|X_n - 1| > \frac{\delta}{2}\}$. This leads to

$$E(|X_n^{-1} - 1|) \leq \delta + E(|X_n^{-1} - 1|\mathbf{1}_{\{|X_n^{-1}-1|>\delta\}}) \leq \delta + \frac{2n}{3}P\left(|X_n - 1| > \frac{\delta}{2} \wedge \frac{1}{2}\right).$$

Hence, by Lemma 15,

$$\limsup_{n\to\infty} E(|X_n^{-1} - 1|) \leq \delta$$

and the desired result follows since δ was arbitrary. □

Finally we use (28) and Lemma 16 to get

$$
\begin{aligned}
\|\mu_n - \mu_n^\bullet\| &= \frac{1}{2} \sup_{-1 \le F \le 1} \left| E\left[F\big(\mathcal{M}_n^\bullet\big)\left(1 - \frac{1}{\operatorname{Card} V(\mathcal{M}_n^\bullet)} \frac{1}{E(1/\operatorname{Card} V(\mathcal{M}_n^\bullet))} \right) \right] \right| \\
&\le E\left[\left| 1 - \frac{1}{\operatorname{Card} V(\mathcal{M}_n^\bullet)} \frac{1}{E(1/\operatorname{Card} V(\mathcal{M}_n^\bullet))} \right| \right] \\
&= E\left[\left| 1 - \frac{1/X_n}{E(1/X_n)} \right| \right] \\
&\xrightarrow[n \to \infty]{} 0.
\end{aligned}
$$

This completes the proof of Proposition 14. □

Acknowledgements

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