Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2014, Vol. 50, No. 4, 1213–1230
DOI: 10.1214/13-AIHP560

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Spectral condition, hitting times and Nash inequality

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Received 6 February 2012; revised 21 November 2012; accepted 21 March 2013

Abstract. Let X be a μ-symmetric Hunt process on a LCCB space $\mathtt{E}$. For an open set $\mathtt{G} \subseteq \mathtt{E}$, let $\tau_{\mathtt{G}}$ be the exit time of X from $\mathtt{G}$ and $A^{\mathtt{G}}$ be the generator of the process killed when it leaves $\mathtt{G}$. Let $r : [0,\infty[\to [0,\infty[$ and $R(t) = \int_0^t r(s)\,\mathrm{d}s$.

We give necessary and sufficient conditions for $\mathbb{E}_\mu R(\tau_{\mathtt{G}}) < \infty$ in terms of the behavior near the origin of the spectral measure of $-A^{\mathtt{G}}$. When $r(t) = t^l$, $l \geq 0$, by means of this condition we derive the Nash inequality for the killed process.

In the diffusion case this permits to show that the existence of moments of order $l+1$ for $\tau_{\mathtt{G}}$ implies the Nash inequality of order $p = \frac{l+2}{l+1}$ for the whole process. The associated rate of convergence of the semi-group in $\mathbb{L}^2(\mu)$ is bounded by $t^{-(l+1)}$.

Finally, we show for general Hunt processes that the Nash inequality giving rise to a convergence rate of order $t^{-(l+1)}$ of the semi-group implies the existence of moments of order $l+1-\varepsilon$ for $\tau_{\mathtt{G}}$, for all $\varepsilon > 0$.

Résumé. Soit X un processus de Hunt μ-symétrique à valeurs dans un espace LCCB $\mathtt{E}$. Pour un ouvert $\mathtt{G} \subseteq \mathtt{E}$, soit $\tau_{\mathtt{G}}$ le temps de sortie de $\mathtt{G}$ par X et $A^{\mathtt{G}}$ le générateur du processus tué lorsqu'il quitte $\mathtt{G}$. Soit $r : [0,\infty[\to [0,\infty[$ et $R(t) = \int_0^t r(s)\,\mathrm{d}s$.

Nous établissons des conditions nécéssaires et suffisantes pour que $\mathbb{E}_\mu R(\tau_{\mathtt{G}}) < \infty$. Ces conditions sont données en termes du comportement au voisinage de zéro de la mesure spectrale de $-A^{\mathtt{G}}$ Dans le cas ou $r(t) = t^l$, $l \geq 0$, en utilisant ces conditions, à partir de $\mathbb{E}_\mu R(\tau_{\mathtt{G}}) < \infty$ nous déduisons l'inégalité de Nash pour le processes tué.

Dans le cas d'un processus de diffusion cela permet de montrer que l'existence des moments d'ordre $l+1$ pour $\tau_{\mathtt{G}}$ implique l'inégalité de Nash d'ordre $p = \frac{l+2}{l+1}$ pour le processus X. La vitesse de convergence du semi-groupe dans $\mathbb{L}^2(\mu)$ est donnée par $t^{-(l+1)}$.

Finalement pour un processus de Hunt μ-symétrique à valeurs dans un espace LCCB nous montrons que l'inégalité de Nash donnant lieu à la convergence du semi-groupe avec la vitesse $t^{-(l+1)}$ implique l'existence des moments d'ordre $l+1-\varepsilon$ pour $\tau_{\mathtt{G}}$, pour tout $\varepsilon > 0$.

MSC: 60J25; 60J35; 60J60

Keywords: Recurrence; Hitting times; Dirichlet form; Nash inequality; Weak Poincaré inequality; α-mixing; Continuous time Markov processes

1. Introduction

In the recent literature on convergence rates for continuous time Markov processes, the link between functional inequalities and the integrability of hitting times has regained a new interest.

The most studied case is undoubtedly the exponential one. It is known since Carmona–Klein [5] that for a very general Markov process with invariant probability μ and Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $\mathbb{L}^2(\mu)$, the Poincaré inequality

$$\mu(f^2) \leq C_P \mathcal{E}(f,f), \quad f \in \mathcal{D}(\mathcal{E}), \mu(f) = 0,$$

implies the exponential μ-integrability of hitting times of open sets. The converse implication for reversible diffusions can be deduced from the work of Down–Meyn–Tweedie [6]. In the particular case of linear diffusions, a simple proof of the equivalence between Poincaré inequality and exponential integrability of hitting times, with explicit estimations, was given in Loukianov, Loukianova and Song [10]. In a recent preprint by Cattiaux, Guillin and Zitt [4] the authors show that for symmetric hypo-elliptic diffusions in $\mathbb{R}^n$, both Poincaré inequality and exponential integrability of hitting times are equivalent to the existence of Lyapunov functions.

Although the exponential case, at least for diffusion processes, is now fairly well understood, the sub-exponential, and in particular the polynomial one, is less studied. To the best of our knowledge, the first work in this direction was done by Mathieu [11]. For a diffusion driven by a logarithmically decreasing potential, he gives a bound for the first moment of hitting times and relates this bound to some functional inequality.

More recently, the last chapter of [4] is devoted to the study of the relation between polynomial moments of hitting times and the weak Poincaré inequality

$$\mu\big(f^2\big) \le \beta(s)\mathcal{E}(f,f) + s\Phi(f), \quad s>0,\ f\in\mathcal{D}(\mathcal{E}),\ \mu(f)=0, \tag{1.1}$$

where

$$\Phi : \mathbb{L}^2(\mu) \to [0,\infty], \quad \Phi(cf) = c^2\Phi(f), \forall c\in\mathbb{R},\ f\in\mathbb{L}^2(\mu). \tag{1.2}$$

For uniformly strongly hypo-elliptic symmetric diffusions on $\mathbb{R}^n$, using Lyapunov functions, the authors show that the finiteness of polynomial moments $v_m(x) = \mathbb{E}_x(T_U^m)$, $m\in\mathbb{N}$, $T_U = \inf\{t>0\colon X_t\in U\}$ for some bounded open set U together with a local Poincaré inequality implies the weak Poincaré inequality with $\Phi(f) = (\operatorname{Osc} f)^2$ and with rate-generating function β given by

$$\beta(s) = C\left(\inf\left\{u : \mu\left(\frac{v_{m-1}}{1+v_m} < u\right) > s\right\}\right)^{-1}. \tag{1.3}$$

It is known since the work of Liggett [9] and its generalization by Röckner and Wang [14], and Wang [16], that the weak Poincaré inequality (1.1) gives rise to the $\mathbb{L}^2$-convergence of the semigroup

$$\mu\big((P_t f)^2\big) \le \xi(t)\big[\Phi(f) + \mu\big(f^2\big)\big], \quad t>0,\ \mu(f)=0,\ f\in\mathbb{L}^2(\mu),$$

with speed at least

$$\xi(t) := \inf\big\{s>0; -(1/2)\beta(s)\log s \le t\big\}.$$

When the weak Poincaré inequality is deduced as a consequence of the finiteness of the mth moment of hitting time, one interesting question is the explicit dependence of $\xi(t)$ on m. Unfortunately, the implicit form of $\beta(s)$ in (1.3) makes it difficult to obtain this dependence.

The aim of the present work is to describe more explicitly an inequality which corresponds to the finiteness of polynomial moments of hitting times. It is well known (see [14]) that in the case $\beta(s) = cs^{1-p}$ with $p>1$ and some $c>0$, the weak Poincaré inequality (1.1) is equivalent to the following Nash inequality of order p:

$$\mu\big(f^2\big) \le C\mathcal{E}^{1/p}(f,f)\Phi^{1/q}(f), \quad f\in\mathcal{D}(\mathcal{E}),\ \mu(f)=0,\ \frac{1}{p}+\frac{1}{q}=1. \tag{1.4}$$

In this article we show that the finiteness of polynomial (not necessarily integer) moments of hitting times is related to the Nash inequality with explicit relation between the order of the moment, the order of the inequality and the speed of convergence of the semigroup. Our result can be summarized in the following scheme: For any open set G, let $\tau_{\mathsf{G}} = \inf\{t\ge 0\colon X_t\notin \mathsf{G}\}$, $l\ge 0$. Then the following implication holds.

$$\mathbb{E}_\mu \tau_{\mathsf{G}}^{l+1} < \infty \text{ for suitable } \mathsf{G} \quad \Longrightarrow \quad \text{Nash inequality of order } \frac{l+2}{l+1}, \tag{1.5}$$

where the functional $\Phi(f)$ depends both on $\mathbb{E}_\mu \tau_{\mathrm{G}}^{l+1}$ and on $\mathrm{Osc}(f)$. On the other hand, for all $l > 0$ and any Φ satisfying (1.2),

$$\text{Nahs inequality of order } \frac{l+2}{l+1} \quad \Longrightarrow \quad \mathbb{E}_\mu \tau_{\mathrm{G}}^{l+1-\varepsilon} < \infty \tag{1.6}$$

for all $\varepsilon > 0$ and for all open G such that $\mu(\mathrm{G}^c) > 0$. Moreover it is well known since [9] that for symmetric semigroups, the Nash inequality of order $\frac{l+2}{l+1}$ is equivalent to the following speed of convergence for the semigroup

$$\mu\big((P_t f)^2\big) \le C\Phi(f)t^{-(l+1)}, \quad \mu(f) = 0,\ f \in \mathbb{L}^2(\mu).$$

The implication (1.5) is proved only in the diffusion case, but (1.6) is valid for a very general Markov process. The method to prove the first implication relies on the use of killed processes. We establish a condition for the existence of general hitting time moments in terms of spectral properties of the killed process. This spectral condition generalizes the well known equivalence "exponential moments $\Longleftrightarrow$ spectral gap."

Let us now give the precise statement of our results. X will be a μ-symmetric Hunt process on a locally compact separable Hausdorff space E where μ is a bounded Radon measure (wlog we suppose that μ is a probability measure). For an open set $\mathrm{G} \subseteq \mathrm{E}$, we put $P_t^{\mathrm{G}}[\mathrm{A}](x) = \mathbb{P}_x[X_t \in \mathrm{A}; t < \tau_{\mathrm{G}}]$ for a measurable subset A of E. Denote A^{G} the infinitesimal generator of (P_t^{G}) in $\mathbb{L}^2_{\mathrm{G}}(\mu) = \{f \in \mathbb{L}^2(\mu):\ f = 0\ \mu\text{-a.s. on } \mathrm{G}^c\}$ and let $(E_\xi^{\mathrm{G}}, \xi \ge 0)$ be its spectral family.

It is known, see e.g. Friedman [7], or Loukianova, Loukianov and Song [10], that $\mathbb{E}_\mu \exp(\lambda\tau_{\mathrm{G}}) < \infty$; $\lambda < \lambda_0$, is equivalent to the fact that $-A^{\mathrm{G}}$ has a spectral gap of width at least equal to λ_0. It turns out that hitting time moments generated by other functions than the exponential ones are still related with the spectral properties of $-A^{\mathrm{G}}$ in the following sense: Let $r : [0,\infty[\to [0,\infty[$, $R(t) = \int_0^t r(s)\,\mathrm{d}s$ and denote by $\Lambda_r : [0,\infty[\to [0,\infty]$ the Laplace transform of r:

$$\forall \xi \ge 0, \quad \Lambda_r(\xi) = \int_0^\infty r(t)\mathrm{e}^{-\xi t}\,\mathrm{d}t. \tag{1.7}$$

We show in Theorem 2.2 that $\mathbb{E}_\mu R(\tau_{\mathrm{G}}) < \infty$ if and only if the spectral measure of $-A^{\mathrm{G}}$ integrates Λ_r:

$$\forall f : \mathrm{G} \to \mathbb{R},\ \|f\|_\infty < \infty, \quad \int_{[0,\infty[} \Lambda_r(\xi)\,\mathrm{d}\big(E_\xi^{\mathrm{G}} f, f\big) < \infty.$$

This condition on the spectral measure will be called in the sequel the *r-spectral condition*. Then we show how we can derive in a very elementary way the Nash inequality for the killed process X^{G} with the help of the spectral condition specified by $r(t) = t^l$ (Proposition 2.6). In this case the corresponding rate of transience of the killed process, i.e. the rate of convergence of P_t^{G} to zero, is given by $t^{-(l+1)}$. All this is the content of Section 2, which is entirely devoted to the study of the killed process.

In Section 3 we address the question how the polynomial spectral condition for the killed process (equivalently the existence of polynomial moments of hitting times) can be used to derive the Nash inequality for the non-killed process. In this section, our method applies only in the case when the Dirichlet form is local, i.e. in the diffusion case, in the sense that X has a.s. continuous trajectories. But we do not need to suppose that the process is driven by a stochastic differential equation.

In the one-dimensional diffusion case, from the existence of polynomial moments of order $l+1$, $l \ge 0$, we derive the Nash inequality specified by $p = \frac{l+2}{l+1}$ without any further assumptions.

The multidimensional diffusion case is treated as well. Here we need an additional non-degeneracy condition on the diffusion. Like in [4], we have to suppose that a local Poincaré inequality on some small domain holds. At the end of this section we provide the example of a multidimensional diffusion for which our result holds.

Finally, in Section 4 we study the implication "Nash inequality $\Longrightarrow$ polynomial moments." The Nash inequality gives an explicit α-mixing rate of the process, and then the main idea is to use this mixing rate in order to obtain a deviation inequality to estimate $\mathbb{P}_\mu(\tau_{\mathrm{G}} > t)$. This nice idea is borrowed from Cattiaux and Guillin [3]. As a consequence, for all $l > 0$, the Nash inequality of order $p = \frac{l+2}{l+1}$ implies the existence of the polynomial moments of hitting times of order $l + 1 - \varepsilon$, for any $\varepsilon > 0$. Note that the main tool of this section is the bound on the variance of the sum of strictly mixing variables. This bound is only valid for $l > 0$, hence this is a technical hypothesis for the last section. Note also that this last section is valid for general Hunt processes.

2. Killed process

2.1. *Modulated moments and spectral condition for the killed process*

Consider a Hunt process X on a locally compact separable Hausdorff space $\mathtt{E}$ in the sense of Fukushima, Oshima, Takeda [8]. Let μ be a Radon measure on $\mathtt{E}$. Suppose that μ is bounded (wlog μ is supposed to be a probability measure), everywhere dense in $\mathtt{E}$, and that X is a μ-symmetric process. Let $(P_t)_{t\geq 0}$ be the transition semigroup of X with associated Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ on $\mathbb{L}^2(\mu)$. We suppose that $\mathcal{E}$ is regular. Denote by $\mathbb{P}_x$ the law of the process X starting from $x\in\mathtt{E}$. For an open set $\mathtt{G}\subseteq\mathtt{E}$, set $\tau_{\mathtt{G}}=\inf\{t\geq 0\colon\ X_t\notin\mathtt{G}\}$ the exit time of X from $\mathtt{G}$. Throughout this section we suppose $\tau_{\mathtt{G}}<\infty$ $\mathbb{P}_\mu$-almost surely. Introduce

$$P_t^{\mathtt{G}}[\mathtt{A}](x)=\mathbb{P}_x[X_t\in\mathtt{A};t<\tau_{\mathtt{G}}],\quad x\in\mathtt{G},$$

for a measurable subset $\mathtt{A}$ of $\mathtt{E}$, and set

$$X_t^{\mathtt{G}}=\begin{cases}X_t, & 0\leq t<\tau_{\mathtt{G}},\\ \Delta, & t\geq\tau_{\mathtt{G}},\end{cases}$$

where we adjoin an extra point Δ to the space $\mathtt{E}$. Δ plays the role of a cemetery, and we put $P_t^{\mathtt{G}}(\Delta,\cdot)=\delta_\Delta$. Any measurable function defined on $\mathtt{E}$ is extended in a natural way to $\mathtt{E}\cup\{\Delta\}$ by defining $f(\Delta)=0$. Then, according to [8], $X^{\mathtt{G}}$ is a Hunt process on the state space $\mathtt{G}\cup\Delta$, symmetric with respect to the measure $\mathbb{I}_{\mathtt{G}}\cdot\mu(\mathrm{d}x)$, with transition semi-group $(P_t^{\mathtt{G}})_{t\geq 0}$ on $\mathbb{L}^2_{\mathtt{G}}(\mu)=\{f\in\mathbb{L}^2(\mu)\colon\ f=0\ \mu\text{-a.s. on }\mathtt{G}^c\}$. We write $A^{\mathtt{G}}$ for the infinitesimal generator of $(P_t^{\mathtt{G}})$ in $\mathbb{L}^2_{\mathtt{G}}(\mu)$: With

$$\mathcal{D}\bigl(A^{\mathtt{G}}\bigr)\colon\ \left\{u\in\mathbb{L}^2_{\mathtt{G}}(\mu)\colon\ \lim_{t\to 0}\frac{P_t^{\mathtt{G}}u-u}{t}\text{ exists in }\mathbb{L}^2_{\mathtt{G}}(\mu)\right\},$$

we define $A^{\mathtt{G}}f$ for any $f\in\mathcal{D}(A^{\mathtt{G}})$ as follows

$$A^{\mathtt{G}}f=\lim_{t\to 0}\frac{P_t^{\mathtt{G}}f-f}{t}\quad\text{in }\mathbb{L}^2_{\mathtt{G}}(\mu).$$

$A^{\mathtt{G}}$ is a self-adjoint negative operator. Let us denote by $(\cdot,\cdot)$ the scalar product in $\mathbb{L}^2_{\mathtt{G}}(\mu)$ and by $(E_\xi^{\mathtt{G}},\xi\geq 0)$ the spectral family of $-A^{\mathtt{G}}$.

$(E_\xi^{\mathtt{G}},\xi\geq 0)$ is a right-continuous and increasing family of projection operators such that for any bounded Borel measurable function f defined on $[0,\infty[$, the operator $f(-A^{\mathtt{G}})$ is given by

$$f\bigl(-A^{\mathtt{G}}\bigr)u=\int_{[0,\infty[}f(\xi)\,\mathrm{d}E_\xi^{\mathtt{G}}u,\quad u\in\mathbb{L}^2_{\mathtt{G}}(\mu).$$

Moreover, for all $u\in\mathbb{L}^2_{\mathtt{G}}(\mu)$,

$$Idu=\int_{[0,\infty[}\mathrm{d}E_\xi^{\mathtt{G}}u,\qquad P_t^{\mathtt{G}}u=\exp\bigl(tA^{\mathtt{G}}\bigr)u=\int_{[0,\infty[}\mathrm{e}^{-\xi t}\,\mathrm{d}E_\xi^{\mathtt{G}}u$$

and

$$-A^{\mathtt{G}}u=\int_{[0,\infty[}\xi\,\mathrm{d}E_\xi^{\mathtt{G}}u,\quad u\in\mathcal{D}\bigl(A^{\mathtt{G}}\bigr).$$

The above integrals can be understood in a weak sense, that is, for all $u,v\in\mathbb{L}^2_{\mathtt{G}}(\mu)$,

$$\bigl(f\bigl(-A^{\mathtt{G}}\bigr)u,g\bigl(-A^{\mathtt{G}}\bigr)v\bigr)=\int_{[0,\infty[}f(\xi)g(\xi)\,\mathrm{d}\bigl(E_\xi^{\mathtt{G}}u,v\bigr).\tag{2.1}$$

Actually, the bounded variation function $\xi \to (E^{\mathrm{G}}_\xi u, u)$ is only increasing on the spectrum of $-A^{\mathrm{G}}$ and its discontinuity points are eigenvalues of $-A^{\mathrm{G}}$. Denote by $\mathcal{E}_{\mathrm{G}}$ the Dirichlet form associated with $-A^{\mathrm{G}}$ on $\mathbb{L}^2_{\mathrm{G}}(\mu)$. We have

$$\mathcal{E}_{\mathrm{G}}(u,v) = \int_{[0,\infty[} \xi \,\mathrm{d}\big(E^{\mathrm{G}}_\xi u, v\big)$$

for all $u, v \in \mathcal{D}(\mathcal{E}_{\mathrm{G}}) = \{f \in \mathbb{L}^2_{\mathrm{G}}(\mu)\colon \int_{[0,\infty[} \xi \,\mathrm{d}(E^{\mathrm{G}}_\xi f, f) < \infty\}$. Let H^{G}_ξ be the image space of E^{G}_ξ.

Proposition 2.1. *Under the condition $\tau_{\mathrm{G}} < \infty$ almost surely, we have $H^{\mathrm{G}}_0 = \{0\}$.*

Proof. H^{G}_0 is invariant under P^{G}_t for all $t > 0$. Indeed, using $E^{\mathrm{G}}_\lambda E^{\mathrm{G}}_0 = E^{\mathrm{G}}_0\ \forall \lambda \geq 0$ we see that $\forall u \in H^{\mathrm{G}}_0$, $\forall t \geq 0$

$$P^{\mathrm{G}}_t u = \int_{[0,\infty[} \mathrm{e}^{-\xi t}\,\mathrm{d}E^{\mathrm{G}}_\xi u = \mathrm{e}^0 E^{\mathrm{G}}_0 u = u.$$

For all $v \geq 0$, bounded, $\lim_{t\to\infty} P^{\mathrm{G}}_t v = \lim_{t\to\infty} \mathbb{E}[v(X_t)\mathbf{1}_{t<\tau_{\mathrm{G}}}] = 0$ and hence $(u,v) = (P^{\mathrm{G}}_t u, v) = (u, P^{\mathrm{G}}_t v) \to 0$, $t \to \infty$. Positive bounded functions being dense in $\mathbb{L}^2$, we conclude that $u = 0$. $\square$

Since $E^{\mathrm{G}}_0 = 0$, one has $\int_{[0,\infty[} f(\xi)\,\mathrm{d}E^{\mathrm{G}}_\xi u = \int_{]0,\infty[} f(\xi)\,\mathrm{d}E^{\mathrm{G}}_\xi u$. In what follows we give necessary and sufficient conditions for the existence of arbitrary moments of τ_{G} in terms of the behavior near the origin of the spectral measure $\mathrm{d}E^{\mathrm{G}}_\xi$. Let $r : [0,+\infty[\to [0,+\infty[$ be some measurable function, and denote $\Lambda_r : [0,\infty[\to [0,\infty]$ its Laplace transform:

$$\forall \xi \geq 0, \quad \Lambda_r(\xi) = \int_0^\infty r(t)\mathrm{e}^{-\xi t}\,\mathrm{d}t. \tag{2.2}$$

Instead of hitting time moments, we consider more generally modulated moments defined by $\int_0^{\tau_{\mathrm{G}}} r(t) f(X_t)\,\mathrm{d}t$. Denote by $\mathcal{B}_b(\mathrm{G}) \subset \mathbb{L}^2_{\mathrm{G}}(\mu)$ the space of Borel-measurable and bounded real functions which vanish μ-almost surely on G^c. Let $R(t) = \int_0^t r(s)\,\mathrm{d}s$ and $\|\cdot\|_1 := \|\cdot\|_{L^1(\mu)}$.

Theorem 2.2. *The following four conditions are equivalent*:

1. $\mathbb{E}_\mu R(\tau_{\mathrm{G}}) < \infty$;
2. *For all* $f \in \mathcal{B}_b(\mathrm{G})$, $x \to f(x) \times \mathbb{E}_x \int_0^{\tau_{\mathrm{G}}} r(t) f(X_t)\,\mathrm{d}t \in L^1(\mathbb{I}_{\mathrm{G}} \cdot \mu(\mathrm{d}x))$;
3. *For all* $f \in \mathcal{B}_b(\mathrm{G})$, $\int_{[0,\infty[} \Lambda_r(\xi)\,\mathrm{d}(E^{\mathrm{G}}_\xi f, f) < \infty$;
4. *For all* $f \in \mathcal{B}_b(\mathrm{G})$, $\int_0^\infty r(t)\|P^{\mathrm{G}}_{t/2} f\|^2\,\mathrm{d}t < \infty$.

Moreover, for any $f \in \mathcal{B}_b(\mathrm{G})$,

$$\left\| f \times \mathbb{E}_\cdot \int_0^{\tau_{\mathrm{G}}} r(t) f(X_t)\,\mathrm{d}t \right\|_1 = \int_{[0,\infty[} \Lambda_r(\xi)\,\mathrm{d}\big(E^{\mathrm{G}}_\xi f, f\big) = \int_0^\infty r(t) \big\| P^{\mathrm{G}}_{t/2} f \big\|^2\,\mathrm{d}t. \tag{2.3}$$

Definition 2.3. *In the sequel the condition* 3 *of Theorem* 2.2 *will be called the r-spectral condition for the killed process.*

Proof of Theorem 2.2. The equivalence $1 \iff 2$ is obvious. The following calculus yields $2 \iff 3 \iff 4$ and the equality (2.3) for positive bounded functions.

$$\begin{aligned}\left(f, \mathbb{E}_\cdot \int_0^{\tau_{\mathrm{G}}} r(t) f(X_t)\,\mathrm{d}t\right) &= \left(f, \int_0^\infty r(t) P^{\mathrm{G}}_t f(\cdot)\,\mathrm{d}t\right) = \int_0^\infty r(t)\big(f, P^{\mathrm{G}}_t f\big)\,\mathrm{d}t \\ &= \int_0^\infty r(t)\big\|P^{\mathrm{G}}_{t/2} f\big\|^2\,\mathrm{d}t = \int_0^\infty r(t)\left(f, \int_{[0,\infty[} \mathrm{e}^{-\xi t}\,\mathrm{d}E^{\mathrm{G}}_\xi f\right)\mathrm{d}t\end{aligned}$$

$$= \int_0^\infty r(t) \int_{[0,\infty[} \mathrm{e}^{-\xi t}\, \mathrm{d}\big(E_\xi^{\mathrm{G}} f, f\big)\, \mathrm{d}t = \int_{[0,\infty[} \int_0^\infty r(t)\mathrm{e}^{-\xi t}\, \mathrm{d}t\, \mathrm{d}\big(E_\xi^{\mathrm{G}} f, f\big)$$
$$= \int_{[0,\infty[} \Lambda_r(\xi)\, \mathrm{d}\big(E_\xi^{\mathrm{G}} f, f\big).$$

If $\mathbb{E}_\mu R(\tau_{\mathrm{G}}) < \infty$ and f is only bounded, we have

$$\mathbb{E}_x \int_0^{\tau_{\mathrm{G}}} r(t) f(X_t)\, \mathrm{d}t = \int_0^\infty r(t) P_t^{\mathrm{G}} f(x)\, \mathrm{d}t, \quad \mu(\mathrm{d}x)\text{-almost surely},$$

which follows from Fubini's theorem and the fact that $\mu(\mathrm{d}x)$-almost surely, $(t,\omega) \mapsto r(t) f(X_t(\omega)) 1_{t<\tau_{\mathrm{G}}}(\omega) \in \mathbb{L}^1(P_x(\mathrm{d}\omega) \otimes \mathrm{d}t)$, since

$$E_x \int_0^\infty 1_{t<\tau_{\mathrm{G}}} r(t) \big| f(X_t)\big|\, \mathrm{d}t \le \|f\|_\infty \mathbb{E}_x R(\tau_{\mathrm{G}}) < \infty, \quad \mu(\mathrm{d}x)\text{-almost surely.}$$

A second application of Fubini's theorem implies that

$$\left(f, \int_0^\infty r(t) P_t^{\mathrm{G}} f(\cdot)\, \mathrm{d}t\right) = \int_0^\infty r(t) \big(f, P_t^{\mathrm{G}} f\big)\, \mathrm{d}t,$$

which follows from the product integrability of $(t,x) \mapsto f(x) P_t^{\mathrm{G}} f(x) r(t)$ with respect to $\mathrm{d}t\mu(\mathrm{d}x)$ which in turn is granted by the following upper bound.

$$\left(|f|, \int_0^\infty r(t) \big|P_t^{\mathrm{G}} f(\cdot)\big|\, \mathrm{d}t\right) \le \left(|f|, \int_0^\infty r(t) P_t^{\mathrm{G}} |f|(\cdot)\, \mathrm{d}t\right) \le \|f\|_\infty^2 \mathbb{E}_\mu R(\tau_{\mathrm{G}}).$$

This shows that 2 implies 3 and 4 as well as the equalities of (2.3). □

Remark 1. *Note that the equivalence between points* 3 *and* 4 *of Theorem* 2.2 *remains true for the non-killed process, after removing the spectral projection on the* 0*-eigenspace. More precisely, we have for all bounded measurable functions f such that $\mu(f) = 0$,*

$$\int_{]0,\infty[} \Lambda_r(\xi)\, \mathrm{d}(E_\xi f, f) = \int_0^\infty r(t) \|P_{t/2} f\|^2\, \mathrm{d}t.$$

Example 2.4. *Consider the case $r(t) = t^l$, $l \ge 0$. We have for $\xi \ge 0$ $\Lambda_r(\xi) = \Gamma(l+1)\xi^{-(l+1)}$. Hence*

$$\mathbb{E}_\mu \tau_{\mathrm{G}}^{l+1} < \infty \quad \Longleftrightarrow \quad \int_{[0,\infty[} \xi^{-(l+1)}\, \mathrm{d}\big(E_\xi^{\mathrm{G}} f, f\big) < \infty, \tag{2.4}$$

for all f non-negative and bounded. In the next section we will explain how to use the spectral condition to obtain functional inequalities for X^{G} and then for X.

Example 2.5. *Consider the case $r(t) = \mathrm{e}^{\lambda t}$, $\lambda > 0$. We have*

$$\Lambda_r(\xi) = \frac{1}{\xi - \lambda}, \quad \textit{if } \xi > \lambda, \qquad \Lambda_r(\xi) = +\infty \quad \textit{otherwise.}$$

Put

$$\lambda_0 = \sup\big\{\lambda > 0, \mathbb{E}_\mu \mathrm{e}^{\lambda \tau_{\mathrm{G}}} < \infty\big\}.$$

We obtain that $\int_{]0,\lambda_0[} \mathrm{d}E_\xi^{\mathrm{G}} = 0$, i.e. spectral measure does not charge $]0, \lambda_0[$.

2.2. *Polynomial spectral condition and Nash inequality for the killed process*

In [9], Liggett introduced the following Nash inequality for a Dirichlet form $\mathcal{E}(f,f)$ associated to a linear operator generating a strongly continuous Markovian semigroup with invariant probability measure μ.

$$\mu\big(\big(f-\mu(f)\big)^2\big) \le C\mathcal{E}^{1/p}(f,f)\Phi^{1/q}(f), \quad f \in \mathcal{D}(\mathcal{E}). \tag{2.5}$$

Here $1 < p, q < \infty$ with $1/p + 1/q = 1$, C is a positive constant, and $\Phi : \mathbb{L}^2(\mu) \to [0,\infty]$ satisfies $\Phi(cf) = c^2\Phi(f)$, for any $c \in \mathbb{R}$ and $f \in \mathbb{L}^2(\mu)$. It is shown in [9] that if in addition $\Phi(P_t f) \le \Phi(f)\ \forall f \in \mathbb{L}^2(\mu), \forall t > 0$, then the inequality (2.5) is equivalent to

$$\exists C > 0, \quad \big\|P_t(f) - \mu(f)\big\|_2^2 \le C\frac{\Phi(f)}{t^{q-1}} \tag{2.6}$$

for all $f \in \mathbb{L}^2(\mu)$ and $t > 0$. If the semi-group of X is conservative, symmetric and ergodic, $\mu(f) = E_0 f$. Hence we will consider the following form of the Nash inequality:

$$\big\|f - E_0(f)\big\|^2 \le C\mathcal{E}^{1/p}(f,f)\Phi^{1/q}(f), \quad f \in \mathcal{D}(\mathcal{E}). \tag{2.7}$$

Let us point out again that for the killed process the semi-group is not conservative, transient, and $E_0 = 0$. The following proposition shows that the condition $\mathbb{E}_{\mathbb{I}_{\mathrm{G}}\mu}\tau_{\mathrm{G}}^{l+1} < \infty$ implies the Nash inequality in the form (2.7) for the killed process.

Proposition 2.6. *Let $l \ge 0$ and suppose that $\mathbb{E}_\mu \tau_{\mathrm{G}}^{l+1} < \infty$. Let*

$$\Phi(f) = \int_{[0,\infty[} \xi^{-(l+1)}\,\mathrm{d}\big(E_\xi^{\mathrm{G}} f, f\big), \quad f \in \mathbb{L}^2_{\mathrm{G}}(\mu). \tag{2.8}$$

Then the Nash inequality (2.7) holds for the killed process with $p = \frac{l+2}{l+1}$ and $q = l+2$. Furthermore, Φ satisfies $\Phi(f) \le \frac{1}{\Gamma(l+2)}\|f\|_\infty^2 \mathbb{E}_\mu \tau_{\mathrm{G}}^{l+1}$.

Proof. In virtue of Theorem 2.2, the condition $\mathbb{E}_\mu \tau_{\mathrm{G}}^{l+1} < \infty$ is equivalent to

$$\int_{[0,\infty[} \xi^{-(l+1)}\,\mathrm{d}\big(E_\xi^{\mathrm{G}} f, f\big) < \infty, \tag{2.9}$$

for all bounded $f \in \mathcal{B}_b(\mathrm{G})$. Let $f \in \mathcal{D}(\mathcal{E}_{\mathrm{G}})$. Suppose that $p^{-1} + q^{-1} = 1$ and write, using Hölder's inequality:

$$\begin{aligned}\|f - E_0 f\|^2 = \|f\|^2 &= \int_{[0,\infty[} \mathrm{d}\big(E_\xi^{\mathrm{G}} f, f\big) = \int_{[0,\infty[} \xi^{1/p}\xi^{-1/p}\,\mathrm{d}\big(E_\xi^{\mathrm{G}} f, f\big) \\ &\le \left(\int_{[0,\infty[} \xi\,\mathrm{d}\big(E_\xi^{\mathrm{G}} f, f\big)\right)^{1/p} \times \left(\int_{[0,\infty[} \xi^{-q/p}\,\mathrm{d}\big(E_\xi^{\mathrm{G}} f, f\big)\right)^{1/q} = \mathcal{E}_{\mathrm{G}}^{1/p}(f)\Phi^{1/q}(f),\end{aligned}$$

where

$$\Phi(f) = \int_{[0,\infty[} \xi^{-q/p}\,\mathrm{d}\big(E_\xi^{\mathrm{G}} f, f\big).$$

Now we choose p and q in such a way that

$$\Phi(f) = \int_{[0,\infty[} \xi^{-(l+1)}\,\mathrm{d}\big(E_\xi^{\mathrm{G}} f, f\big).$$

This choice is given by $p = \frac{l+2}{l+1}$ and $q = l+2$. Finally we obtain for all $f \in D(\mathcal{E}_{\mathsf{G}})$

$$\|f - E_0 f\|^2 \le \mathcal{E}_{\mathsf{G}}(f)^{(l+1)/(l+2)} \times \Phi^{1/(l+2)}(f),$$

where Φ satisfies $\Phi(cf) = c^2\Phi(f)$ for any $c \in \mathbb{R}$ and $\Phi(f) < \infty$ for all bounded f. Also

$$\Phi\big(P_t^{\mathsf{G}} f\big) = \int_{[0,\infty[} \xi^{-(l+1)} \mathrm{e}^{-2\xi t}\, \mathrm{d}\big(E_\xi^{\mathsf{G}} f, f\big) \le \Phi(f). \qquad \square$$

Remark 2. *Since the proof of the above Proposition 2.6 uses only the condition (2.9), we deduce from this the following: If for the non-killed process the spectral condition*

$$\int_{]0,\infty[} \xi^{-(l+1)}\, \mathrm{d}(E_\xi f, f) < \infty \tag{2.10}$$

holds for all $f \in \mathcal{B}_b$ such that $\mu(f) = 0$, then the Nash inequality (2.7) holds, with $\Phi(f) = \int_{]0,\infty[} \xi^{-(l+1)}\, \mathrm{d}(E_\xi f, f)$ and $q = l+2$.

However, in general, it is difficult to find sufficient conditions, expressed in terms of the generator of the process, ensuring condition (2.10) for the non-killed process, whereas conditions ensuring $\mathbb{E}_\mu {\tau_{\mathsf{G}}}^{l+1} < \infty$ are by now classical, for suitable choices of the set G, see Section 3.3 below.

3. Polynomial moments and Nash inequality for the non-killed process

In this section we show how polynomial modulated moments are related to the Nash inequality for the non-killed process. The result can be summarized as follows. For all $l > 0$, for all $\varepsilon > 0$ we have: "integrability of moments of order $l+1 \Longrightarrow$ Nash inequality giving rise to $\mathbb{L}^2$ convergence of the semigroup with speed $t^{-(l+1)} \Longrightarrow$ existence of moments of order $l+1-\varepsilon$."

For the second implication we work under the general conditions of Section 2. For the first implication "moments imply Nash" we work in the diffusion case only. In dimension 1, no hypothesis on the diffusion is imposed. In higher dimension, however, we need a non-degeneracy condition which is a local Poincaré inequality (see the comments in Remark 5).

3.1. *Polynomial moments $\Longrightarrow$ Nash inequality. One-dimensional diffusion case*

In this subsection we show that the Nash inequality for a killed diffusion process on $\mathbb{R}$ implies the Nash inequality for the non-killed process. Fix some $a \in \mathbb{R}$ and let $\mathsf{G}^- =]-\infty, a[$, $\mathsf{G}^+ =]a, \infty[$. We use some well-known techniques which are specific to the one-dimensional case.

Since X is a diffusion, it possesses a scale function S and a corresponding speed measure m, see e.g. Revuz and Yor [12], Chapter VII. We suppose that X is recurrent. This implies that $\lim_{x\to\pm\infty} S(x) = \pm\infty$. Note moreover that m charges any non-empty open set. Denote by $\mathrm{d}S$ the measure induced by $S(x)$. Let $F(x)$ be a real function on $\mathbb{R}$. We shall write $\mathrm{d}F \ll \mathrm{d}S$, if there exists a function $f(x)$ in $\mathbb{L}^1_{\mathrm{loc}}(\mathrm{d}S)$ such that

$$\int_a^b f(x)\, \mathrm{d}S(x) = F(b) - F(a), \quad \forall a < b.$$

The function $f(x)$ will be denoted $\frac{\mathrm{d}F}{\mathrm{d}S}(x)$. Introduce then the function spaces

$$\begin{aligned}
&\mathcal{F} = \left\{F \in \mathbb{L}^2(m)\colon\ \mathrm{d}F \ll \mathrm{d}S, \frac{\mathrm{d}F}{\mathrm{d}S} \in \mathbb{L}^2(\mathrm{d}S)\right\},\\
&\mathcal{F}_{]a,\infty[} = \big\{F \in \mathcal{F}\colon\ F(x) = 0, x \le a\big\},\\
&\mathcal{F}_{]-\infty,a[} = \big\{F \in \mathcal{F}\colon\ F(x) = 0, x \ge a\big\}.
\end{aligned} \tag{3.1}$$

We do not assume that $\mathrm{d}S$ and m are absolutely continuous with respect to the Lebesgue measure. We cite the following theorem from [10].

Theorem 3.1 ([10]). *The diffusion X is m-symmetric. The Dirichlet space associated with X is the function space $\mathcal{F}$ given by* (3.1), *and the Dirichlet form has the expression*

$$\mathcal{E}(F,F)=\int_{-\infty}^{\infty}\left(\frac{\mathrm{d}F}{\mathrm{d}S}\right)^2(x)\,\mathrm{d}S(x),\qquad F\in\mathcal{F}.$$

The restriction of the Dirichlet form $\mathcal{E}$ on $\mathcal{F}_{]a,\infty[}$ is the Dirichlet form $\mathcal{E}_{]a,\infty[}$ associated with the semigroup $(P_t^{]a,\infty[})_{t\ge 0}$ of the process X killed when it exits $]a,\infty[$. The killed process $X^{]a,\infty[}$ is symmetric with respect to the measure $\mathbb{I}_{]a,\infty[}\cdot m(\mathrm{d}x)$.

The same is true (*with obvious modifications*) *for $\mathcal{E}_{]-\infty,a[}$.*

The proof of this theorem is given in [10].

We can now state the Nash inequality for the non-killed process X. For $a\in\mathbb{R}$ introduce the hitting time $T_a=\inf\{t\ge 0, X_t=a\}$. We write $\mu(\cdot)=\frac{1}{m(\mathbb{R})}m(\cdot)$ for the renormalized speed measure.

Theorem 3.2. *Let $l>0$. Suppose that for some $a\in\mathbb{R}$*

$$\int_{-\infty}^{+\infty}\mathbb{E}_x T_a^{l+1}m(\mathrm{d}x)<\infty. \tag{3.2}$$

Then there exists a functional $\Phi:\mathbb{L}^2(\mu)\to[0,+\infty]$ such that the Nash inequality

$$\mu\big(\big(F-\mu(F)\big)^2\big)\le\mathcal{E}^{1/p}(F,F)\Phi^{1/q}(F),\qquad F\in\mathcal{F}, \tag{3.3}$$

holds with $p=\frac{l+2}{l+1}$ and $q=l+2$. The functional Φ satisfies $\Phi(cF)=c^2\Phi(F)$ for all $c\in\mathbb{R}$, $F\in\mathbb{L}^2(\mu)$, and $\Phi(P_tF)\le\Phi(F)$ for all $t>0$. Moreover, there exists a finite constant $C>0$ such that

$$\Phi(F)\le C\|F-\mu(F)\|_\infty^2\quad\forall F\in\mathbb{L}^2. \tag{3.4}$$

Remark 3. *Note also that*

$$\Phi(F)\le C\Big(\sup_{\mathbb{R}}F-\inf_{\mathbb{R}}F\Big)^2=C\operatorname{Osc}(F)^2. \tag{3.5}$$

Proof. Fix a point $a\in\mathbb{R}$. Then the variational formula for the variance gives for all $F\in\mathcal{F}$,

$$\begin{aligned}\int_{-\infty}^{+\infty}\big(F(x)-\mu(F)\big)^2\mu(\mathrm{d}x)&\le\int_{-\infty}^{+\infty}\big(F(x)-F(a)\big)^2\mu(x)\\&=\int_{-\infty}^{a}\big(F(x)-F(a)\big)^2\mu(\mathrm{d}x)+\int_{a}^{+\infty}\big(F(x)-F(a)\big)^2\mu(\mathrm{d}x).\end{aligned}$$

Write

$$F_-(x)=\big(F(x)-F(a)\big)1_{\{x<a\}},\qquad F_+(x)=\big(F(x)-F(a)\big)1_{\{x>a\}}.$$

Then $F_-\in\mathcal{F}_{]-\infty,a[}$ and $F_+\in\mathcal{F}_{]a,\infty[}$. Hence we can apply Proposition 2.6 for both $\mathtt{G}^-=]-\infty,a[$, $\mathtt{G}^+=]a,\infty[$. Denote

$$\mathcal{E}_{]-\infty,a[}=\mathcal{E}_-\quad\text{and}\quad\mathcal{E}_{]a,+\infty[}=\mathcal{E}_+$$

and

$$\Phi_-(u) = \int_{[0,\infty[} \xi^{-(l+1)}\,\mathrm{d}\big(E_\xi^{\mathbb{G}^-}u,u\big), \quad \text{and} \quad \Phi_+(u) = \int_{[0,\infty[} \xi^{-(l+1)}\,\mathrm{d}\big(E_\xi^{\mathbb{G}^+}u,u\big).$$

Then, with $p = \frac{l+2}{l+1}$ and $q = l+2$,

$$\begin{aligned}
&\int_{-\infty}^{a}\big(F(x)-F(a)\big)^2\mu(\mathrm{d}x) + \int_a^{+\infty}\big(F(x)-F(a)\big)^2\mu(\mathrm{d}x) \\
&\quad\le \mathcal{E}_-^{1/p}(F_-)\Phi_-^{1/q}(F_-) + \mathcal{E}_+^{1/p}(F_+)\Phi_+^{1/q}(F_+) \\
&\quad= \Phi_-^{1/q}(F_-)\left(\int_{-\infty}^{a}\left(\frac{\mathrm{d}F}{\mathrm{d}S}\right)^2(t)\,\mathrm{d}S(t)\right)^{1/p} \\
&\qquad+ \Phi_+^{1/q}(F_+)\left(\int_a^{+\infty}\left(\frac{\mathrm{d}F}{\mathrm{d}S}\right)^2(t)\,\mathrm{d}S(t)\right)^{1/p} \\
&\quad\le \big(\Phi_-^{1/q}(F_-) + \Phi_+^{1/q}(F_+)\big)\left(\int_{-\infty}^{+\infty}\left(\frac{\mathrm{d}F}{\mathrm{d}S}\right)^2(t)\,\mathrm{d}S(t)\right)^{1/p} \\
&\quad= \mathcal{E}^{1/p}(F)\Phi_a^{1/q}(F),
\end{aligned}$$

where

$$\Phi_a(F) = \big(\Phi_-^{1/q}(F_-) + \Phi_+^{1/q}(F_+)\big)^q.$$

The above result holds for any $a \in \mathbb{R}$. Hence we can put

$$\Phi(F) = \sup_{t\ge 0}\inf_{a\in\mathbb{R}} \Phi_a(P_tF). \tag{3.6}$$

Then $\Phi(cF) = c^2\Phi(F)$ and $\Phi(P_tF) \le \Phi(F)$ are trivially satisfied. It remains to show that under the conditions of the theorem, Φ satisfies (3.4). In virtue of Theorem 2.2,

$$\begin{aligned}
\Phi_-(F_-) &= \frac{1}{\Gamma(l+1)}\int_{\mathbb{R}} 1_{]-\infty,a[}(x)\big(F(x)-F(a)\big) \times \mathbb{E}_x\int_0^{T_a} s^l \times \big(F(X_s)-F(a)\big)\,\mathrm{d}s\,\mu(\mathrm{d}x) \\
&\le \frac{4}{\Gamma(l+2)}\big\|F-\mu(F)\big\|_\infty^2 \int_{\mathbb{R}} 1_{]-\infty,a[}(x)\mathbb{E}_x T_a^{l+1}\mu(\mathrm{d}x).
\end{aligned} \tag{3.7}$$

In the same way,

$$\Phi_+(F_+) \le \frac{4}{\Gamma(l+2)}\big\|F-\mu(F)\big\|_\infty^2 \int_{\mathbb{R}} 1_{]a,\infty[}(x)\mathbb{E}_x T_a^{l+1}\mu(\mathrm{d}x),$$

and thus, with $q = l+2$,

$$\Phi_a(F) \le 2^q\frac{4}{\Gamma(l+2)}\|F-\mu(F)\|_\infty^2\mathbb{E}_\mu T_a^{l+1}.$$

We deduce, using $\|P_tF\|_\infty \le \|F\|_\infty$ and $\mu(P_tF) = \mu(F)$ that

$$\begin{aligned}
\inf_a \Phi_a(P_tF) &\le 2^q\frac{4}{\Gamma(l+2)}\big\|P_tF-\mu(P_tF)\big\|_\infty^2 \inf_{a\in\mathbb{R}}\mathbb{E}_\mu T_a^{l+1} \\
&\le C\big\|F-\mu(F)\big\|_\infty^2,
\end{aligned}$$

where

$$C = 2^{l+2} \frac{4}{\Gamma(l+2)} \inf_{a \in \mathbb{R}} \mathbb{E}_\mu T_a^{l+1},$$

and this implies (3.4). □

Remark 4. *If the Nash inequality* (3.3) *holds with a functional Φ satisfying the properties of Theorem* 3.2, *then Liggett* [9], *Theorem* 2.2, *shows that*

$$\left\| P_t F - \mu(F) \right\|_2^2 \le C \frac{\Phi(F)}{t^{l+1}}, \quad F \in \mathcal{F}.$$

Hence under the assumption of integrability of $l+1$-moments of hitting times we obtain a polynomial decay of the transition semigroup P_t of X at the same rate $t^{-(l+1)}$.

3.2. *Polynomial moments $\Longrightarrow$ Nash inequality. General diffusion case*

In this section we come back to the general conditions of Section 2 and consider the μ-symmetric Hunt process X on the LCCB space E such that $\mu(E) = 1$, with semigroup $(P_t)_{t \ge 0}$ and associated Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $\mathbb{L}^2(\mu)$.

Assumption 3.3. *Assume that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular and admits a carré du champ Γ.*

Following Bouleau and Hirsch [1], Proposition 4.1.3, this means that there exists a unique positive symmetric and continuous bilinear form from $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ into $\mathbb{L}^1(\mu)$, denoted by Γ and called *the carré du champ* operator, such that $\forall f, g, h \in \mathcal{D}(\mathcal{E}) \cap \mathbb{L}^\infty$,

$$\mathcal{E}(fh, g) + \mathcal{E}(gh, f) - \mathcal{E}(h, fg) = \int h \Gamma(f, g) \, \mathrm{d}\mu. \tag{3.8}$$

Assumption 3.4. *Assume that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is local.*

In this case, by [1], Proposition 6.1.1,

$$\forall f \in \mathcal{D}(\mathcal{E}), \quad \mathcal{E}(f, f) = \frac{1}{2} \int_{\mathrm{E}} \Gamma(f, f) \, \mathrm{d}\mu.$$

Note that the locality of the form is equivalent to assume that the process X is a diffusion process, in the sense that X has a.s. continuous trajectories, see Theorem 4.5.1 of [8].

Assumption 3.5. *Assume that $\mathcal{E}$ is recurrent*, i.e. $1 \in \mathcal{D}(\mathcal{E})$ *and* $\mathcal{E}(1, 1) = 0$.

Recall the definition of the spaces $\mathbb{D}_p$, $p \ge 2$, similar to the definition of Sobolev spaces, ([1], Definition 6.2.1):

$$\mathbb{D}_p = \left\{ f \in \mathcal{D}(\mathcal{E}) \cap \mathbb{L}^p; \Gamma(f, f)^{1/2} \in \mathbb{L}^p \right\}$$

and, for $f \in \mathbb{D}_p$,

$$\|f\|_{\mathbb{D}_p} = \|f\|_{\mathbb{L}^p} + \|\Gamma(f, f)^{1/2}\|_{\mathbb{L}^p}.$$

The following proposition is proved in [1] (Proposition 6.2.3).

Proposition 3.6. *Let $p, q, r \ge 2$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then*

$$f \in \mathbb{D}_p \quad \text{and} \quad g \in \mathbb{D}_q \quad \Longrightarrow \quad fg \in \mathbb{D}_r \quad \text{and} \quad \|fg\|_{\mathbb{D}_r} \le \|f\|_{\mathbb{D}_p} \|g\|_{\mathbb{D}_q}.$$

For any set $\mathtt{G}$ and any $r > 0$ we set $\mathtt{G}_r = \{x \in \mathtt{E}\colon \operatorname{dist}(x, \mathtt{G}) < r\}$. Under Assumptions 3.3, 3.4 and 3.5 the following theorem holds:

Theorem 3.7. *Let $l > 0$. Suppose there exists an open subset $\mathtt{G} \subset \mathtt{E}$ and $r > 0$ such that $\mu(\mathtt{G}_r \setminus \bar{\mathtt{G}}) > 0$ and such that the following conditions are satisfied.*

1. *For $A \in \{\mathtt{G}_r, \bar{\mathtt{G}}^c\}$,*

$$\mathbb{E}_\mu \tau_A^{l+1} < \infty.$$

2. *μ satisfies a local Poincaré inequality in restriction to $\mathtt{G}_r \setminus \bar{\mathtt{G}}$, i.e.*

$$\int_{\mathtt{G}_r \setminus \bar{\mathtt{G}}} f^2 \,\mathrm{d}\mu \le C_P(\mathtt{G}, r) \int_{\mathtt{G}_r \setminus \bar{\mathtt{G}}} \Gamma(f, f) \,\mathrm{d}\mu$$

for all $f \in \mathcal{D}(\mathcal{E})$ such that $\int_{\mathtt{G}_r \setminus \bar{\mathtt{G}}} f \,\mathrm{d}\mu = 0$.

3. *There exists a regularized indicator function $u \in \mathcal{D}(\mathcal{E})$ associated to the sets $\mathtt{G}$ and $\mathtt{G}_r$ such that $0 \le u \le 1$, $u \equiv 1$ on $\bar{\mathtt{G}}$, $u \equiv 0$ on $\mathtt{G}_r^c$, which verifies*

$$C(u, r) := \|\Gamma(u, u)\|_\infty < \infty. \tag{3.9}$$

Then there exists a functional $\Phi : \mathbb{L}^2(\mu) \to [0, \infty]$ such that the following Nash inequality holds.

For any $f \in \mathcal{D}(\mathcal{E})$ with $\mu(f) = 0$,

$$\mu\big(f^2\big) \le \mathcal{E}^{1/p}(f, f)\Phi^{1/q}(f), \tag{3.10}$$

where $p = (l+2)/(l+1)$ and $q = l+2$. Φ satisfies $\Phi(af) = a^2\Phi(f)$ for all $a \in \mathbb{R}$ and $\Phi(P_t f) \le \Phi(f)$ for all $t \ge 0$, $f \in \mathbb{L}^2(\mu)$. Moreover,

$$\Phi(f) \le 2^{q/p}\big[1 + C(u, r)C_P(\mathtt{G}, r)\big]^{q/p} \frac{1}{\Gamma(l+2)} \times \operatorname{Osc}(f)^2\big[\big(\mathbb{E}_\mu \tau_{\mathtt{G}_r}^{l+1}\big)^{1/q} + \big(\mathbb{E}_\mu \tau_{\bar{\mathtt{G}}^c}^{l+1}\big)^{1/q}\big]^q. \tag{3.11}$$

Proof. Let $f \in \mathcal{D}(\mathcal{E})$ with $\mu(f) = 0$. Let

$$c = \frac{1}{\mu(\mathtt{G}_r \setminus \bar{\mathtt{G}})} \int_{\mathtt{G}_r \setminus \bar{\mathtt{G}}} f \,\mathrm{d}\mu.$$

The use of this constant will become clear in formula (3.16) later. By the variational definition of the variance, we have that

$$\int f^2(x)\mu(\mathrm{d}x) \le \int \big(f(x) - c\big)^2 \mu(\mathrm{d}x).$$

Denote $\tilde{f} = f - c$ and let u be the regularized indicator of $\mathtt{G}$. Write $\tilde{f} = \tilde{f}u + \tilde{f}(1-u)$. Using Proposition 3.6, since $u \in \mathbb{D}_\infty$ and $\tilde{f} \in \mathbb{D}_2 = \mathcal{D}(\mathcal{E})$, we have $\tilde{f}u \in \mathcal{D}(\mathcal{E})$. Hence both $\tilde{f}u$ and $\tilde{f}(1-u)$ belong to $\mathcal{D}(\mathcal{E})$. Now we can write

$$\begin{aligned}
\int_{\mathtt{E}} \big(\tilde{f}(x)\big)^2 \mu(\mathrm{d}x) &= \int_{\mathtt{E}} \big(\tilde{f}u + \tilde{f}(1-u)\big)^2(x)\mu(\mathrm{d}x) \\
&\le 2\left[\int_{\mathtt{E}} (\tilde{f}u)^2(x)\mu(\mathrm{d}x) + \int_{\mathtt{E}} \big(\tilde{f}(1-u)\big)^2(x)\mu(\mathrm{d}x)\right] \\
&= 2\left[\int_{\mathtt{G}_r} (\tilde{f}u)^2(x)\mu(\mathrm{d}x) + \int_{\bar{\mathtt{G}}^c} \big(\tilde{f}(1-u)\big)^2(x)\mu(\mathrm{d}x)\right].
\end{aligned}$$

In a first step, we want to apply the Nash inequality on G_r. For that sake, note that $\tilde{f}u \in \mathcal{D}(\mathcal{E})$ and its quasicontinuous modification is zero on G_r^c. (For the definition of quasicontinuity, we refer the reader to Chapter 2.1 of [8].) Hence by (4.3.1) of [8], $\tilde{f}u \in \mathcal{D}(\mathcal{E}_{\mathsf{G}_r})$. Therefore, we introduce

$$\Phi^{\mathsf{G}_r}(\tilde{f}) = \int_{[0,\infty[} \xi^{-(l+1)}\,\mathrm{d}\bigl(E_\xi^{\mathsf{G}_r}\tilde{f}u, \tilde{f}u\bigr)$$

and obtain, applying Proposition 2.6,

$$\begin{aligned}\int_{\mathsf{G}_r} (\tilde{f}u)^2(x)\mu(\mathrm{d}x) &\le \bigl[\mathcal{E}_{\mathsf{G}_r}(\tilde{f}u, \tilde{f}u)\bigr]^{1/p}\bigl[\Phi^{\mathsf{G}_r}(\tilde{f})\bigr]^{1/q} \\ &\le \mathcal{E}^{1/p}(\tilde{f}u, \tilde{f}u)\bigl[\Phi^{\mathsf{G}_r}(\tilde{f})\bigr]^{1/q},\end{aligned} \tag{3.12}$$

where the first inequality follows from Proposition 2.6, and the second since $\mathcal{E}_{\mathsf{G}_r}$ is just the restriction of the Dirichlet form $\mathcal{E}$ to $\mathcal{F}_{\mathsf{G}_r}$.

In the same way, $\tilde{f}(1-u) \in \mathcal{D}(\mathcal{E}_{\bar{\mathsf{G}}^c})$. Introducing

$$\Phi^{\bar{\mathsf{G}}^c}(\tilde{f}) = \int_{[0,\infty[} \xi^{-(l+1)}\,\mathrm{d}\bigl(E_\xi^{\bar{\mathsf{G}}^c}\tilde{f}(1-u), \tilde{f}(1-u)\bigr),$$

we obtain

$$\int_{\bar{\mathsf{G}}^c} \bigl(\tilde{f}(1-u)\bigr)^2(x)\mu(\mathrm{d}x) \le \mathcal{E}^{1/p}\bigl(\tilde{f}(1-u), \tilde{f}(1-u)\bigr)\bigl[\Phi^{\bar{\mathsf{G}}^c}(\tilde{f})\bigr]^{1/q}. \tag{3.13}$$

In order to control $\mathcal{E}(\tilde{f}u, \tilde{f}u)$ and $\mathcal{E}(\tilde{f}(1-u), \tilde{f}(1-u))$, we use (Proposition 6.2.3 of [1] and Cauchy–Schwarz) that

$$\Gamma(\tilde{f}u, \tilde{f}u) \le 2\bigl(\Gamma(\tilde{f}, \tilde{f}) + \tilde{f}^2\Gamma(u,u)\bigr). \tag{3.14}$$

By the locality of the form, it is classical to show that $\Gamma(\tilde{f}, \tilde{f}) = \Gamma(f, f)$ and $\Gamma(u,u) = 0$ on $\bar{\mathsf{G}}$ and G_r^c. Hence,

$$\begin{aligned}\mathcal{E}(\tilde{f}u, \tilde{f}u) &= \frac{1}{2}\int_{\mathsf{G}_r} \Gamma(\tilde{f}u, \tilde{f}u)\,\mathrm{d}\mu \\ &\le \int_{\mathsf{E}} \Gamma(f, f)\,\mathrm{d}\mu + \int_{\mathsf{G}_r\setminus\bar{\mathsf{G}}} \tilde{f}^2(x)\Gamma(u,u)(x)\mu(\mathrm{d}x) \\ &\le \int_{\mathsf{E}} \Gamma(f, f)\mu(\mathrm{d}x) + C(u,r)\int_{\mathsf{G}_r\setminus\bar{\mathsf{G}}} \tilde{f}^2(x)\mu(\mathrm{d}x),\end{aligned}$$

which implies that

$$\mathcal{E}(\tilde{f}u, \tilde{f}u) \le 2\mathcal{E}(f, f) + C(u,r)\int_{\mathsf{G}_r\setminus\bar{\mathsf{G}}} \tilde{f}^2(x)\mu(\mathrm{d}x).$$

The role of u and $1-u$ being symmetric, we get in the same way

$$\mathcal{E}\bigl(\tilde{f}(1-u), \tilde{f}(1-u)\bigr) \le 2\mathcal{E}(f, f) + C(u,r)\int_{\mathsf{G}_r\setminus\bar{\mathsf{G}}} \tilde{f}^2(x)\mu(\mathrm{d}x),$$

with the same constant $C(u,r)$. Putting things together, we conclude that

$$\int_{\mathsf{E}} f^2(x)\mu(dx) \le \left(2\mathcal{E}(f, f) + C(u,r)\int_{\mathsf{G}_r\setminus\bar{\mathsf{G}}} \tilde{f}^2(x)\mu(\mathrm{d}x)\right)^{1/p}\Psi^{1/q}(f), \tag{3.15}$$

where

$$\Psi(f) = \left(\left[\Phi^{\mathtt{G}_r}(\tilde{f})\right]^{1/q} + \left[\Phi^{\bar{\mathtt{G}}^c}(\tilde{f})\right]^{1/q}\right)^q.$$

It remains to treat the term

$$\int_{\mathtt{G}_r \setminus \bar{\mathtt{G}}} \tilde{f}^2(x)\mu(\mathrm{d}x).$$

It is here that we need the fact that $\int_{\mathtt{G}_r \setminus \bar{\mathtt{G}}} \tilde{f}(x)\mu(\mathrm{d}x) = 0$, by definition of the constant c. Now we can apply the local Poincaré inequality in order to deduce that

$$\int_{\mathtt{G}_r \setminus \bar{\mathtt{G}}} \tilde{f}^2(x)\mu(\mathrm{d}x) \le C_P(\mathtt{G}, r) \int_{\mathtt{G}_r \setminus \bar{\mathtt{G}}} \Gamma(f, f)\,\mathrm{d}\mu. \tag{3.16}$$

Coming back to (3.15) we conclude that

$$\int_{\mathtt{E}} f^2(x)\mu(\mathrm{d}x) \le \left[2 + 2C(u, r)C_P(\mathtt{G}, r)\right]^{1/p} \mathcal{E}(f, f)^{1/p} \Psi^{1/q}(f).$$

Putting

$$\Phi(f) := \left[2 + 2C(u, r)C_P(\mathtt{G}, r)\right]^{q/p} \sup_{t \ge 0} \Psi(P_t f), \tag{3.17}$$

the result now follows, provided we show that Φ satisfies the desired properties. It is evident that $\Phi(cf) = c^2\Phi(f)$ and $\Phi(P_t f) \le \Phi(f)$.

In virtue of Theorem 2.2,

$$\Phi^{\mathtt{G}_r}(\tilde{f}) = \frac{1}{\Gamma(l+1)} \int_{\mathtt{G}_r} \tilde{f}(x)u(x) \times \mathbb{E}_x \int_0^{\tau_{\mathtt{G}_r}} s^l \times (\tilde{f}u)(X_s)\,\mathrm{d}s\,\mu(\mathrm{d}x).$$

This implies that for bounded f, since $0 \le u(\cdot) \le 1$, and by definition of the constant c,

$$\Phi^{\mathtt{G}_r}(\tilde{f}) \le \frac{1}{\Gamma(l+2)} \operatorname{Osc}(f)^2 \mathbb{E}_\mu \tau_{\mathtt{G}_r}^{l+1}.$$

In the same way,

$$\Phi^{\bar{\mathtt{G}}^c}(\tilde{f}) \le \frac{1}{\Gamma(l+2)} \operatorname{Osc}(f)^2 \mathbb{E}_\mu \tau_{\bar{\mathtt{G}}^c}^{l+1},$$

since $\|f - c\|_\infty \le \operatorname{Osc}(f)$. Observing that $\operatorname{Osc}(P_t f) \le \operatorname{Osc}(f)$ concludes our proof. □

Remark 5.

1. *Sometimes it is convenient to replace Condition* 2 *of Theorem* 3.7, *assuming that a local Poincaré inequality holds*, *by the following weaker condition. Suppose that* $\mathtt{E} = \mathbb{R}^d$ *and let* λ^d *be Lebesgue's measure on* $\mathbb{R}^d$. *We suppose that* $\mathtt{G}$ *and* $\mathtt{G}_r$ *are relatively compact*, *that* $\lambda^d(\partial\bar{\mathtt{G}}) = \lambda^d(\partial\mathtt{G}_r) = 0$ *and that* $\mu \sim \lambda^d$ *with locally bounded Radon–Nikodym densities* $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda^d}$ *and* $\frac{\mathrm{d}\lambda^d}{\mathrm{d}\mu}$. *Then it is sufficient to assume the following weaker condition*:

   There exists $\Omega \subset \mathbb{R}^d$ *a smooth bounded open connected domain such that* $\overline{\mathtt{G}_r} \setminus \mathtt{G} \subset \Omega$ *and*

$$\int_{\mathtt{G}_r \setminus \bar{\mathtt{G}}} f^2\,\mathrm{d}\lambda^d \le C_P \int_\Omega \Gamma(f, f)\,\mathrm{d}\lambda^d, \tag{3.18}$$

   for all $f \in \mathcal{D}(\mathcal{E})$ *such that* $\int_{\mathtt{G}_r \setminus \bar{\mathtt{G}}} f\,\mathrm{d}\lambda^d = 0$.

2. *Condition* (3.18) *is not very restrictive and follows from non-degeneracy of the diffusion, for example if Hörmander's condition is satisfied. Indeed, Wang* [17], *Lemma* 2.3, *shows that* (3.18) *holds in the following case. Take a smooth function* V *such that* $\int \mathrm{e}^V \, \mathrm{d}\lambda^d = 1$. *Let* $Y_i, i = 1, \ldots, n$, *be a family of smooth bounded vector fields satisfying the Hörmander condition. Consider*

$$A = \sum_{i=1}^{n} \bigl(Y_i^2 + (\mathrm{div}_\mu Y_i) Y_i\bigr),$$

where $\mathrm{div}_\mu Y_i = \sum_{k=1}^d Y_i^k \, \partial_k V + \frac{\partial Y_i^k}{\partial x_k}$. *Then* (3.18) *holds for any relatively compact open set* G *and any finite* $r > 0$.

3.3. *Example*

We continue the discussion of Remark 5 and give an example of a process X defined in terms of its generator A which satisfies all conditions needed for Theorem 3.7. We take $\mathsf{E} = \mathbb{R}^d$, a smooth function V such that $\int \mathrm{e}^V \, \mathrm{d}\lambda^d = 1$ and put $\mu = \mathrm{e}^V \lambda^d$. Then we define an operator L on $\mathcal{D}(L) = C_c^\infty(\mathbb{R}^d)$ by

$$Lf = \frac{1}{2} \Delta f + \frac{1}{2} \nabla V \nabla f.$$

This operator L defined on $\mathcal{D}(L)$ is symmetric in $\mathbb{L}^2(\mu)$. Hence we can define for all $f, g \in \mathcal{D}(L)$,

$$\mathcal{E}(f, g) = -\int g L f \, \mathrm{d}\mu.$$

It is classical to show, see e.g. Example 1.3.4 of Bouleau and Hirsch [1], that $\mathcal{E}$ is closable. Let us denote $(\overline{\mathcal{E}}, \overline{\mathcal{D}})$ the closure of $(\mathcal{E}, \mathcal{D}(L))$ and let A be the generator of $\overline{\mathcal{E}}$. Then $-A$ is a positive self-adjoint extension of $-L$, called the Friedrichs extension of L. It is standard to show that $(\overline{\mathcal{E}}, \overline{\mathcal{D}})$ is a Dirichlet form for which Condition 3 of Theorem 3.7 is trivially satisfied.

When identifying with the classical form

$$Lf = \frac{1}{2} \sum_{j,k} a_{j,k} \, \partial_j \, \partial_k f + \sum_k b_k \, \partial_k f$$

we find $b = \frac{1}{2} \nabla V$ and $a \equiv I$ (the identity matrix). Thus, being in the uniform elliptic case, Condition 2 is satisfied for any open relatively compact set G, see e.g. Wang [17].

Finally, concerning the moment condition 1, suppose that for some $r > \frac{d}{2} + 1 + l$, and $M > 0$,

$$V(x) = -2r \ln |x|, \quad |x| > M.$$

Then Veretennikov's condition (see Veretennikov [15])

$$\bigl\langle b(x), x/|x| \bigr\rangle \le -r/|x|, \quad |x| \ge M,$$

is fulfilled. Under this condition for $K > M$, $\tau = \inf\{t \ge 0; |X_t| \le K\}$, any $x \in \mathbb{R}^d$ and all $\varepsilon \in]0, 2r - d - 2l - 2[$, $\mathbb{E}_x \tau^{l+1} \le C(1 + |x|^{2l+2+\varepsilon})$, ([15], Theorem 3) and $\mathbb{E}_\mu \tau^{l+1} < \infty$. As a consequence, if we define

$$\mathsf{G} := \{x \colon |x| < K\},$$

then Condition 1 is satisfied for $A = \bar{\mathsf{G}}^c$. For $A = \mathsf{G}_r = \{x \colon |x| < K + r\}$, Condition 1 is always satisfied, since G_r is bounded and the diffusion uniformly elliptic, which implies that τ_{G_r} possesses exponential moments, see Friedman [7]. As a consequence, the Nash inequality holds for all $p > p^*$ where

$$p^* = 1 + \frac{1}{r - d/2}.$$

On the other hand, Theorem 3 of Balaji and Ramasubramanian [2], with $A(x)=1$, $B(x)=d$, $C(x)=-2r$ shows that for all $p>r-d/2+1$ and $|x|>K$, $\mathbb{E}_x\tau^p=\infty$. This shows clearly, that we are not in the case where the Poincaré inequality holds. Moreover, the Nash inequality does not hold any more for all $p<p_*$, see Theorem 4.1 below, where

$$p_*=1+\frac{1}{r-d/2+1}.$$

4. Polynomial moments under Nash inequality

In Section 3, we have shown that for diffusions, the existence of polynomial moments of hitting times implies the Nash inequality.

We now address the inverse question: Does Nash inequality imply the existence of moments? The answer is yes, at least if the functional Φ satisfies (3.4).

All statements of this section hold true under the general conditions of Section 2, for a conservative Hunt process which is μ-symmetric, with μ a probability measure. Let $l>0$.

Theorem 4.1. *Suppose that Nash inequality holds with $p=\frac{l+2}{l+1}$ and with Φ such that (3.4) holds. Then for all $\varepsilon>0$ and for any open set G such that $\mu(\mathsf{G}^c)>0$,*

$$\mathbb{E}_\mu\tau_{\mathsf{G}}^{l+1-\varepsilon}<\infty.$$

The idea of the proof is not new and follows ideas exposed in Section 3 of Cattiaux and Guillin [3].

In the following, C denotes a constant that might change from occurrence to occurrence. For integrable f we write $\tilde f=f-\mu(f)$. By [9], we know that under the conditions of Theorem 4.1,

$$\operatorname{Var}_\mu(P_tf)\le Ct^{-(l+1)}\Phi(f)\le Ct^{-(l+1)}\operatorname{Osc}(f)^2.$$

This implies that the stationary process X_t under $\mathbb{P}_\mu$ is strongly mixing, and by symmetry, its mixing coefficient is bounded by

$$\alpha(t)\le C\left(\frac{t}{2}\right)^{-(l+1)}=Ct^{-(l+1)}.$$

The main step of the proof of Theorem 4.1 is the following deviation inequality.

Proposition 4.2. *Fix $t\ge1$ and let V be such that $\|V\|_\infty=1$. Then for any $\lambda>0$,*

$$\mathbb{P}_\mu\left(\left|\frac{1}{t}\int_0^t V(X_s)\,\mathrm{d}s-\mu(V)\right|\ge4\lambda\right)\le C\bigl[\lambda^{-(l+2)}\vee\lambda^{-2(l+1)}\bigr]t^{-(l+1)}.$$

Proof. We mimic the proof of Proposition 4.5 of Cattiaux and Guillin [3], by making use of moment bounds for sums of strongly mixing sequences obtained by Rio [13]. Let $n=[t]$ be the integer part of t, $n\ge1$, since $t\ge1$. Then

$$\int_0^t\tilde V(X_s)\,\mathrm{d}s=\sum_{k=1}^n Y_k,\quad \text{where } Y_k=\int_{(k-1)t/n}^{kt/n}\tilde V(X_s)\,\mathrm{d}s.$$

(Y_j) is a $\mathbb{P}_\mu$-stationary sequence of strongly mixing centered random variables satisfying $|Y_j|\le2\frac{t}{n}$, with mixing coefficients $\bar\alpha(k)$, $k\ge0$, which can be upper bounded for all $k\ge2$ by

$$\bar\alpha(k)=\alpha\left((k-1)\frac{t}{n}\right)\le C(k-1)^{-(l+1)}\left(\frac{t}{n}\right)^{-(l+1)}\le C(k-1)^{-(l+1)}. \tag{4.1}$$

Here, in the last inequality we have used that $t/n \geq 1$. We write $\tilde{Y}_j = Y_j/(2t/n)$ and apply the inequality (6.19b) of [13] to $S_n = \sum_{k=1}^n \tilde{Y}_k$, with $a = l+1$. So we obtain for any $r \geq 1$,

$$\mathbb{P}_\mu\left(\left|\frac{1}{t}\int_0^t V(X_s)\,\mathrm{d}s - \mu(V)\right| \geq 4\lambda\right) = \mathbb{P}_\mu\left(\left|\sum_{k=1}^n Y_k\right| \geq 4\lambda t\right)$$

$$= \mathbb{P}_\mu\big(|S_n| \geq 4(\lambda n/2)\big) \leq 4\left(1 + \frac{\lambda^2 n^2}{4r s_n^2}\right)^{-r/2} + 2ncr^{-1}\left(\frac{4r}{\lambda n}\right)^{l+2},$$

where $s_n^2 = \sum_{i=1}^n \sum_{j=1}^n |\operatorname{Cov}(\tilde{Y}_i, \tilde{Y}_j)|$. Using Corollaire 1.1 of [13] we can control this sum of covariances as follows

$$s_n^2 \leq 4\sum_{k=1}^n \int_0^1 \big[\alpha^{-1}(u) \wedge n\big] Q_k^2(u)\,\mathrm{d}u, \quad \text{where}$$

$$a^{-1}(u) = \inf\{k \in \mathbb{N}; \bar{\alpha}(k) \leq u\} = \sum_{i \geq 0} \mathbf{1}_{\bar{\alpha}(i) > u}, \qquad a^{-1}(u) \wedge n = \sum_{i=0}^{n-1} \mathbf{1}_{\bar{\alpha}(i) > u},$$

and where $Q_k(u)$ is the inverse function of $H_{\tilde{Y}_k}(t) = \mathbb{P}(|\tilde{Y}_k| > t)$. Since $|\tilde{Y}_k| \leq 1$ for all $k \leq n$, $Q_k^2(u) \leq 1$, and thus (see [13], p. 15),

$$s_n^2 \leq 4n \int_0^1 \big[a^{-1}(u) \wedge n\big]\,\mathrm{d}u \leq 4n \sum_{i=0}^\infty \bar{\alpha}(i).$$

Since $l > 0$, this last series converges (compare to (4.1)), and we obtain $s_n^2 \leq Cn$ for some constant $C > 0$. As a consequence,

$$\mathbb{P}_\mu\left(\left|\frac{1}{t}\int_0^t V(X_s)\,\mathrm{d}s - \mu(V)\right| \geq 4\lambda\right) \leq 4\left(1 + \frac{\lambda^2 n^2}{4Crn}\right)^{-r/2} + 2ncr^{-1}\left(\frac{4r}{\lambda n}\right)^{l+2}$$

$$\leq 4\left(\frac{\lambda^2 n}{4Cr}\right)^{-r/2} + 2cr^{-1}\left(\frac{4r}{\lambda}\right)^{l+2} n^{-(l+1)}.$$

Finally we choose $r = 2(l+1)$ and use that

$$n^{-(l+1)} = \left(\frac{t}{n}\right)^{l+1} t^{-(l+1)} \leq 2^{l+1} t^{-(l+1)},$$

where we have used that $t/n \leq 2$, which follows from $n \geq 1$, since $t \geq 1$. Thus we get the result. □

Proof of Theorem 4.1. We apply the above deviation inequality with $V = 1_{\mathrm{G}^c}$ and use that

$$\{\tau_{\mathrm{G}} > t\} \subset \left\{\frac{1}{t}\int_0^t V(X_s)\,\mathrm{d}s = 0\right\} \subset \left\{\frac{1}{t}\left|\int_0^t \tilde{V}(X_s)\,\mathrm{d}s\right| \geq \mu\big(\mathrm{G}^c\big)\right\}.$$

Hence

$$\int \mu(\mathrm{d}x)\mathbb{P}_x(\tau_{\mathrm{G}} > t) \leq \mathbb{P}_\mu\left(\frac{1}{t}\left|\int_0^t \tilde{V}(X_s)\,\mathrm{d}s\right| \geq \mu\big(\mathrm{G}^c\big)\right) \leq Ct^{-(l+1)},$$

whence for every $\varepsilon > 0$ "small," $\mathbb{E}_\mu \tau_{\mathrm{G}}^{l+1-\varepsilon} < \infty$. □

Acknowledgements

The authors thank Simon Holbach for many useful remarks and careful reading of the manuscript.

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