Ann. Funct. Anal. 8 (2017), no. 3, 303–313
http://dx.doi.org/10.1215/20088752-0000015X
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

RATE OF APPROXIMATION BY q-DURRMEYER OPERATORS IN $L_p([0,1])$, $1 \leq p \leq \infty$

ASHA RAM GAIROLA,[1] KARUNESH KUMAR SINGH,[2] and VISHNU NARAYAN MISHRA[3,4*]

Communicated by K. S. Berenhaut

Abstract. We obtain global rates of approximation by q-Durrmeyer operators $D_{n,q}(f;x)$ for the functions in the class $L_p([0,1]), 1 \leq p \leq \infty$. First, rates of approximation in terms of the norms of f and f' and in terms of the ordinary modulus of smoothness are obtained. Subsequently, we obtain rates of approximation in terms of the generalized modulus of smoothness $\omega_\varphi(f,\delta)$.

1. Introduction and preliminaries

Using the generalized weights $p_{n,k}(q;x) = q^{\frac{k(k-1)}{2}} x^k \prod_{j=0}^{n-k-1}(1-q^j x)$, Gupta [6] proposed the q-Durrmeyer operators

$$D_{n,q}(f;x) := [n+1]_q \sum_{k=0}^{n} q^{-k} p_{n,k}(q;x) \int_0^1 f(t) p_{n,k}(q;qt)\, d_q t$$

for the functions f in the class $C[0,1]$, and he established error estimates in uniform norm by the operators $D_{n,q}(f)$ in terms of the usual modulus of continuity. Moreover, in [6] it was established that the operator sequence $D_{n,q}(f)$ converges uniformly to f if and only if the sequence (q_n) in $(0,1)$ is such that $q_n \to 1^-$. With this condition on (q_n), the operators $D_{n,q}(f)$ reduce to the Bernstein–Durrmeyer polynomials which have been extensively studied (see [3]). Some interesting results can be seen in [2], [4], [8], and [11].

Copyright 2017 by the Tusi Mathematical Research Group.
Received Jul. 1, 2016; Accepted Oct. 4, 2016.
First published online Apr. 4, 2017.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 41A3; Secondary 41A25, 11N80.
Keywords. q-Durrmeyer operators, rate of convergence, q-integers.

We prove approximation theorems corresponding to the operators sequence $D_{n,q}(f)$ for the functions f in $L_p([0,1]), 1 \le p \le \infty$. The space $L_p([0,1])$ consists of the pth integrable functions on $[0,1]$ for $1 \le p < \infty$. If $p = \infty$, then we assume that $f \in C[0,1]$.

The following theorem provides error estimates for twice-differentiable functions in terms of the norms of f and f'' for $1 < p < \infty$. The method used cannot be applied for $p = 1$ or for $p = \infty$; however, the result is of independent interest.

Theorem 1.1. *Let $1 < p < \infty$. If $f, f', f'' \in L_p([0,1])$, then*

$$\left\| D_{n,q}(f,x) - f(x) \right\|_{L_p([0,1])} \le C(p,q) \frac{1}{[n+2]_q} \left(\|f\|_{L_p([0,1])} + \|f''\|_{L_p([0,1])} \right),$$

where $C(p,q)$ depends on p and q and is free from n.

In order to prove the theorem for the case $p = 1$, we may proceed along the lines of Theorem 9.5.3 of [1]. However, the method of [1] involves cumbersome calculations and gives results in a smaller subinterval of $[0,1]$. We make use of intermediate functions such as the Steklov mean $f_{2,\delta}$ corresponding to f. The Steklov means are remarkably smooth functions and their application provides a shorter proof. We have the following theorem.

Theorem 1.2. *Let $1 \le p < \infty$, and let $f \in L_p([0,1])$. Then*

$$\left\| D_{n,q}(f,x) - f(x) \right\|_{L_p([0,1])} \le C(p,q) \omega_2 \left(f, [n+2]_q^{-1/2}, [0,1] \right)_{L_p([0,1])}.$$

Finally, the following is our main theorem in $L_p([0,1])$ that provides global error estimates as a combination of the Ditzian–Totik modulus of smoothness and the function norm.

Theorem 1.3. *Let $f \in L_p([0,1])$ be arbitrary, with $1 \le p \le \infty$. Then*

$$\begin{aligned} &\left\| D_{n,q}(f,x) - f \right\|_{L_p([0,1])} \\ &\quad \le C(p,q) \left(\omega_\varphi^2 \left(f, [n+2]_q^{-1/2} \right)_{L_p([0,1])} + [n+2]_q^{-1} \|f\|_{L_p([0,1])} \right). \end{aligned}$$

2. Definitions and notation

Definition 2.1. Let $h > 0$, let $f \in L_p([0,1])$, and let $1 \le p < \infty$ ($f \in C([0,1])$ in the case $p = \infty$). The second-order modulus of smoothness with step δ for the function f is defined as

$$\omega_2 \left(f, \delta, [0,1] \right)_p := \begin{cases} \sup_{0<|h|\le\delta} (\int_0^1 |\Delta_h^2 f(x)|^p \, dx)^{1/p}, & 1 \le p < \infty, \\ \sup_{0<|h|\le\delta} |\Delta_h^2 f(x)|, & p = \infty, \end{cases}$$

where $\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$.

Definition 2.2. For $f \in L_p([0,1]), 1 \le p \le \infty$, and $\varphi(x) = \sqrt{x(1-x)}, x \in [0,1]$, the Ditzian–Totik second-order modulus of smoothness (see [1, p. 8]) is given by

$$\omega_\varphi^2(f,\delta)_p := \sup_{0<h\le\delta} \left\| f\left(x + h\varphi(x)\right) - 2f(x) + f\left(x - h\varphi(x)\right) \right\|_{L_p([0,1])}.$$

Definition 2.3. The K-functional corresponding to $\omega_\varphi^2(f,\delta)_p$ is defined by

$$K_{2,\varphi}(f,\delta)_p := \inf_{g\in W_p^2}\big\{\|f-g\|_{L_p([0,1])} + \delta\|\varphi^2 g''\|_{L_p([0,1])}\big\},$$
$$\overline{K}_{2,\varphi}(f,\delta)_p := \inf_{g\in W_p^2}\big\{\|f-g\|_{L_p([0,1])} + \delta\|\varphi^2 g''\|_{L_p([0,1])} + \delta^2\|g''\|_{L_p([0,1])}\big\},$$

where

$$W_p^2 = \big\{g \in L_p\big([0,1]\big) : g' \in AC_{\mathrm{loc}}[0,1], \|\varphi^2 g''\|_{L_p([0,1])} < \infty\big\},$$

and $g' \in AC_{\mathrm{loc}}[0,1]$ means that g is differentiable such that g' is absolutely continuous in every interval $[a,b] \subset [0,1]$.

It is known (see [1, p. 11]) that $\omega_\varphi^2(f,\delta)_p, K_{2,\varphi}(f,\delta)_p$ and $\overline{K}_{2,\varphi}(f,\delta)_p$ are equivalent; that is, there exists an absolute constant $C>0$ such that

$$C^{-1}\omega_\varphi^2(f,\sqrt{\delta})_p \le K_{2,\varphi}(f,\delta)_p \le C\omega_\varphi^2(f,\sqrt{\delta})_p, \tag{2.1}$$
$$C^{-1}\omega_\varphi^2(f,\sqrt{\delta})_p \le \overline{K}_{2,\varphi}(f,\delta)_p \le C\omega_\varphi^2(f,\sqrt{\delta})_p, \tag{2.2}$$

where C is a positive constant independent of n and x, not necessarily the same in different cases.

From [7] we borrow some definitions and results on q-calculus in order to make article self-contained.

Definition 2.4. Let $n \in \mathbb{N}$. For $0<q<1$, the nonnegative q-integer $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1},$$

and $[n]_q = n$, for $q=1$. The q-factorial $[n]_q!$ is given by

$$[n]_q! := \begin{cases} \prod_{j=0}^{n-1}[n-j]_q, & n = 1,2,\ldots, \\ 1, & n=0. \end{cases}$$

Definition 2.5. The q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are defined by the quotient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{[n]_q!}{[k]_q![n-k]_q!}, & 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.6. The q-rising product $(a+b)_q^n$ is defined by

$$(a+b)_q^n := \prod_{s=0}^{n-1}(a+q^s b).$$

Definition 2.7. Let $a>0$. The q-Jackson integral is given by the sum

$$\int_0^a f(x)\,d_q x := (1-q)a\sum_{r=0}^{\infty} f(aq^r)q^r.$$

The integral exists whenever the series on the right-hand side is absolutely convergent. For more information and properties of the q-integers, we refer the reader to [7].

Definition 2.8. Let $1 \le p < \infty$, let $f \in L_p([0,1])$ ($f \in C[0,1]$ in the case $p = \infty$), and let $[a_1, b_1] \subset [0,1]$. Then, for sufficiently small $\delta > 0$, the Steklov mean $f_{\delta,2}$ of second order corresponding to f is defined as follows:

$$f_{\delta,2}(t) := \delta^{-2} \int_{-\delta/2}^{\delta/2} \int_{-\delta/2}^{\delta/2} \big(f(t) - \Delta^2_{t_1+t_2} f(t) \big) \, dt_1 \, dt_2, \quad t \in [a_1, b_1].$$

Definition 2.9. Let $f \in L_p([a,b])$, with $1 \le p < \infty$. Then the Hardy–Littlewood majorant (see [12, p. 244]) $\theta(x; f)$ of the function f is defined as

$$\theta(x; f) := \sup_{\xi \ne x} \frac{1}{\xi - x} \int_x^{\xi} f(t) \, dt \quad (a \le \xi \le b).$$

3. Preliminary results

Theorem 3.1 (see [6]). *We have*

$$D_{n,q}(1; x) = 1, \qquad D_{n,q}(t; x) = \frac{1 + q[n]_q x}{[n+2]_q},$$
$$D_{n,q}(t^2; x) = \frac{q^3 x^2 [n]_q([n]_q - 1) + (1+q)^2 q x [n]_q + 1 + q}{[n+3]_q [n+2]_q}.$$

Lemma 3.2. *Let $r \in \mathbb{N} \cup \{0\}$, with $1 \le p \le \infty$. For the functions $\mu_r^n(x) = D_{n,q}((t-x)^r; x)$, we have*

$$\|\mu_1^n\|_{L_p([0,1])} \le C(p,q) \frac{1}{[n+2]_q},$$
$$\|\mu_2^n\|_{L_p([0,1])} \le C(p,q) \frac{1}{[n+2]_q} \|\delta_n^2\|_{L_p([0,1])} \le C(p,q) \frac{1}{[n+2]_q} \|\varphi^2\|_{L_p([0,1])},$$

where $\delta_n^2(x) = \max\{\varphi^2(x), \frac{1}{[n+3]_q}\}$.

Proof. The first estimate follows from straightforward calculations. For the second inequality, we have

$$\begin{aligned}
\mu_2^n(x) &= \frac{(q^4[n]_q[n-1]_q - 2q[n]_q[n+3]_q + [n+2]_q[n+3]_q)x^2}{[n+2]_q[n+3]_q} \\
&\quad + \frac{([2]_q^2 q[n]_q - 2[n+3]_q)x + [2]_q}{[n+2]_q[n+3]_q} \\
&= \frac{(2q[n]_q[n+3]_q - q^4[n]_q[n-1]_q - [n+2]_q[n+3]_q)\varphi^2(x)}{[n+2]_q[n+3]_q} \\
&\quad + \frac{([2]_q^2 q[n] - 2[n+3]_q + q^4[n][n-1] - 2q[n][n+3]_q + [n+2]_q[n+3]_q)x + [2]_q}{[n+2]_q[n+3]_q} \\
&\le \frac{(2q[n][n+3]_q - q^4[n][n-1] - [n+2]_q[n+3]_q)\varphi^2(x) + [2]_q}{[n+2]_q[n+3]_q}.
\end{aligned}$$

From easy calculations it follows that

$$\mu_2^n(x) \le C(q) \begin{cases} \varphi^2(x), & x \in E_n^c, \\ \frac{1}{[n+3]_q}, & x \in E_n, \end{cases}$$

where $E_n = [0, \frac{1}{[n+3]_q}]$.

Now, the proof is completed by taking the norm. □

Lemma 3.3. *For the function $f_{\delta,2}$, we have*

(a) *$f'_{\delta,2}$ and $f''_{\delta,2}$ exist over $[a_1, b_1]$,*
(b) $\|f_{\delta,2}^{(r)}\|_{L_p([a_1,b_1])} \le C\delta^{-r}\omega_2(f,\delta,[a,b])_{L_p([0,1])}, r = 1, 2,$
(c) $\|f - f_{\delta,2}\|_{L_p([a_1,b_1])} \le C\omega_2(f,\delta,[0,1])_{L_p([0,1])},$
(d) $\|f_{\delta,2}\|_{L_p([a_1,b_1])} \le C\|f\|_{L_p([0,1])},$
(e) $\|f''_{\delta,2}\|_{L_p([a_1,b_1])} \le C\delta^{-2}\|f\|_{L_p([0,1])},$

where the constants C are independent of f and δ but can depend on r and $a < a_1 < b_1 < b$.

Proof. Following Timan [10, pp. 163–165], the proof of the lemma easily follows, and hence the details are omitted. □

Remark 3.4. It follows from the definition of the function $f_{\delta,2}$ that $f_{\delta,2}^{(r)} \in L_p([0,1])$, $r = 0, 1, 2$.

Lemma 3.5 (see [12])**.** *If $1 < p < \infty$, $f \in L_p([0,a])$, then $\theta(x;f) \in L_p([0,a])$ and*

$$\big\|\theta(x;f)\big\|_{L_p([0,a])} \le 2^{1/p}\frac{p}{p-1}\|f\|_{L_p([0,a])}.$$

Lemma 3.6. *If $f \in L_p([0,1]), 1 \le p \le \infty$, then $D_{n,q}f$ is a contraction on $L_p([0,1])$.*

Proof. We prove that

$$\|D_{n,q}f\|_{L_p([0,1])} \le \|f\|_{L_p([0,1])}.$$

First, we prove the lemma for $p = \infty$. It is easy to show that $p_{nk}(qt) \in L_1([0,1])$ since $\int_0^1 p_{nk}(qt)\,d_qt = \frac{q^k}{[n+1]_q}$. From Hölder's inequality it follows that

$$\begin{aligned} \int_0^1 \big|f(t)\big|p_{nk}(q;qt)\,d_qt &\le \|f\|_{L_\infty([0,1])}\|p_{nk}\|_{L_1([0,1])} \\ &\le \|f\|_{L_\infty([0,1])}\frac{q^k}{[n+1]_q}. \end{aligned} \tag{3.1}$$

It is easily established from straightforward calculations that $\int_0^1 \sum_{k=0}^n p_{nk}(q;x)\,dx = 1$. Hence, the definition of $D_{n,q}(f)$ and equation (3.1) enable us to write

$$\begin{aligned} \big\|D_{n,k}(f,x)\big\|_1 &\le [n+1]_q \sum_{k=0}^n q^{-k}\int_0^1 p_{nk}(q;x)\,dx \int_0^1 \big|f(t)\big|p_{nk}(q;qt)\,d_qt \\ &\le \|f\|_{L_\infty([0,1])}. \end{aligned}$$

Since $D_{n,k}(f,x)$ is a polynomial of degree n in x, we have $\|D_{n,k}(f,x)\|_{L_\infty([0,1])} \leq \|D_{n,k}(f,x)\|_{L_1([0,1])}$. So then

$$\big\|D_{n,k}(f,x)\big\|_{L_\infty([0,1])} \leq \|f\|_{L_\infty([0,1])}.$$

Next, we consider $p=1$. From (3.6) and the order relation in norms, it follows that

$$\int_0^1 \big|f(t)\big| p_{nk}(q;qt)\,d_q t \leq \|f\|_{L_1([0,1])}\|p_{nk}\|_{L_\infty([0,1])} \leq \|f\|_{L_1([0,1])}\|p_{nk}\|_{L_1([0,1])}.$$

The inequality is now established along the lines parallel to the case $p=\infty$. Finally, in view of the Riesz–Thorin interpolation theorem, the proof is established for $1 \leq p \leq \infty$. □

Lemma 3.7. *If $g \in W_p^2, 1 \leq p \leq \infty$, then*

$$\Big\|D_{n,q}\Big(\int_x^t (t-v)g''(v)\,dv, x\Big)\Big\|_{L_p([0,1])} \leq \frac{C}{[n+2]_q^{1/2}}\|\varphi^2 g''\|_{L_p([0,1])}.$$

Proof. (a) *Case $p=1$.* We know that (see [1, p. 141])

$$\Big|\int_x^t (t-v)g''(v)\,dv\Big| \leq \frac{|t-x|}{\varphi^2(x)}\Big|\int_x^t \varphi^2(v)\big|g''(v)\big|\,dv\Big|.$$

The above inequality, the fact that $D_{n,q}$ is a contraction for $p \geq 1$, and the inequality $|\int_x^t \varphi^2(v)|g''(v)|\,dv| \leq |\int_0^1 \varphi^2(v)|g''(v)|\,dv| = \|\varphi^2 g''\|_{L_1([0,1])}$ together imply

$$\begin{aligned}
&\Big\|D_{n,q}\Big(\int_x^t (t-v)g''(v)\,dv, x\Big)\Big\|_{L_p([0,1])} \\
&\quad\leq \int_0^1 \Big|D_{n,q}\Big(\frac{|t-x|}{\varphi^2(x)}\Big|\int_x^t \varphi^2(v)\big|g''(v)\big|\,dv\Big|, x\Big)\Big|\,dx \\
&\quad\leq \|\varphi^2 g''\|_{L_1[0,1]}\int_0^1 \varphi^{-2}(x)\big|D_{n,q}\big(|t-x|, x\big)\big|\,dx \\
&\quad\leq \frac{C}{[n+2]_q^{1/2}}\|\varphi^2 g''\|_{L_1([0,1])}.
\end{aligned}$$

(b) *Case $p=\infty$.* We have

$$\begin{aligned}
&\Big\|D_{n,q}\Big(\int_x^t (t-v)g''(v)\,dv, x\Big)\Big\|_{L_\infty([0,1])} \\
&\quad\leq \|\varphi^2 g''\|_{L_\infty([0,1])}\Big\|\frac{1}{\varphi^2(x)}D_{n,q}\big((t-x)^2, x\big)\Big\|_{L_\infty([0,1])} \\
&\quad\leq \frac{1}{[n+2]_q}\|\varphi^2 g''\|_{L_\infty([0,1])},
\end{aligned}$$

where we have used Lemma 3.2. An application of the Riesz–Thorin interpolation theorem (see [9, p. 231]) enables us to claim the result for $1 \leq p \leq \infty$. □

4. Proof of the theorems

Proof of Theorem 1.1. Using the finite form representation

$$f(t)-f(x)=(t-x)f'(x)+\frac{1}{2}(t-x)^2f''(\xi)$$

and the linearity of $D_{n,q}(f)$, we can write

$$\begin{aligned}D_{n,q}(f,x)-f(x)&=D_{n,q}(t-x,x)f'(x)+\frac{1}{2}D_{n,q}\big((t-x)^2f''(\xi),x\big)\\&=A_1+A_2,\quad\text{say,}\\|A_1|&=\big|f'(x)\big|\Big|\frac{1+[n]qx}{[n+2]_q}-x\Big|=\big|f'(x)\big|\Big|\frac{1-(1+q^{n+1})x}{[n+2]_q}\Big|\\&\le\frac{1}{[n+2]_q}\big|f'(x)\big|\le C(p,q)\frac{\|f'\|_{L_p([0,1])}}{[n+2]_q}.\end{aligned}$$

By using [5], we can write

$$\|A_1\|_{L_p([0,1])}\le C(p,q)\frac{1}{[n+2]_q}\big(\|f\|_{L_p([0,1])}+\|f''\|_{L_p([0,1])}\big).$$

Next,

$$\begin{aligned}|A_2|&\le[n+1]_q\sum_{k=0}^{n}q^{-k}p_{nk}(q;x)\int_0^1\Big|\int_x^t(t-u)f''(u)\,du\Big|p_{nk}(q;qt)\,d_qt\\&\le[n+1]_q\sum_{k=0}^{n}q^{-k}p_{nk}(q;x)\int_0^1(t-x)^2\big|\theta(x;f'')\big|p_{nk}(q;qt)\,d_qt\\&\le C(p,q)\big|\theta(x;f'')D_{n,q}\big((t-x)^2,x\big)\big|\\&\le C(p,q)\frac{1}{[n+2]_q}\big|\theta(x;f'')\big|.\end{aligned}$$

Here $\theta(x;f'')$ is the Hardy–Littlewood majorant of $f''(x)$.

Using Lemma 3.5, we write that

$$\|A_2\|_p\le C(p,q)\frac{1}{[n+2]_q}\big\|\theta(x;f'')\big\|_{L_p([0,1])}\le C(p,q)\frac{1}{[n+2]_q}\|f''\|_{L_p([0,1])}.$$

Finally, we have for $p>1$

$$\big\|D_{n,q}(f,x)-f(x)\big\|_p\le C(p,q)\frac{1}{[n+2]_q}\big(\|f\|_{L_p([0,1])}+\|f''\|_{L_p([0,1])}\big).\qquad\square$$

Proof of Theorem 1.2. We write, say,

$$\begin{aligned}&D_{n,q}(f,x)-f(x)\\&\quad=D_{n,q}(f-g,x)+\big(D_{n,q}(f_{\delta,2},x)-f_{\delta,2}(x)\big)+\big(f_{\delta,2}(x)-f(x)\big)\\&\quad=E_1+E_2+E_3.\end{aligned}$$

By Lemma 3.6,

$$\|E_1\|_{L_p([0,1])},\|E_3\|_{L_p([0,1])}\le C(p,q)\,\|f-f_{\delta,2}\|_{L_p([0,1])}.\tag{4.1}$$

An application of Theorem 1.1 enables us to write

$$\begin{aligned}\|E_2\|_{L_p([0,1])} &= \left\|\left(D_{n,q}(f_{\delta,2},x) - f_{\delta,2}(x)\right)\right\|_{L_p([0,1])}\\ &= C(p,q)\,\frac{1}{[n+2]_q}\left(\|f_{\delta,2}\|_{L_p([0,1])} + \|f''_{\delta,2}\|_{L_p([0,1])}\right).\end{aligned} \tag{4.2}$$

The result now follows from (4.1), (4.2), Lemma 3.3, and the equivalence of $K_2(f,\delta)_p$ and $\omega_2(f,\delta^{1/2})_p$. □

Proof of Theorem 1.3. Let $g \in W_p^2$. Using Lemma 3.2 and the smoothness of g, we can write

$$\begin{aligned}&\left|D_{n,q}(g,x) - g(x)\right|\\ &\quad\le \left|g'(x)\right|\left|D_{n,q}\left((t-x),x\right)\right| + \left|D_{n,q}\left(\int_x^t (t-v)g''(v)\,dv, x\right)\right|\\ &\quad\le \frac{C(q)}{[n+2]_q}\left|g'(x)\right| + \left|D_{n,q}\left(\int_x^t (t-v)g''(v)\,dv, x\right)\right|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|D_{n,q}g - g\|_{L_p([0,1])}\\ &\quad\le \frac{C}{[n+2]_q}\|g'\|_{L_p([0,1])} + \left\|D_{n,q}\left(\int_x^t (t-v)g''(v)\,dv, x\right)\right\|_{L_p([0,1])}.\end{aligned} \tag{4.3}$$

From [1, p. 135], it follows that

$$\|g'\|_{L_p([0,1])} \le C\left(\|\varphi^2 g''\|_{L_p([0,1])} + \|g\|_{L_p([0,1])}\right), \quad 1 \le p < \infty. \tag{4.4}$$

Using (4.4), Lemma 3.7, and (4.3), we get

$$\left\|D_{n,q}(g) - g\right\|_{L_p([0,1])} \le \frac{C}{[n+2]_q}\left(\|\varphi^2 g''\|_{L_p([0,1])} + \|g\|_{L_p([0,1])}\right). \tag{4.5}$$

The definition of $K_{2,\varphi}(f,[n+2]_q^{-1})_p$ implies that there exists a function $g_n \in W_p^2$ such that

$$\|f - g_n\|_p \le 2K_{2,\varphi}\left(f,[n+2]_q^{-1}\right)_p, \tag{4.6}$$

$$\|\varphi^2 g_n''\|_{L_p([0,1])} \le 2[n+2]_q^{-1} K_{2,\varphi}\left(f,[n+2]_q^{-1}\right)_p. \tag{4.7}$$

Making use of (4.5)–(4.7) and (2.1), we obtain

$$\begin{aligned}&\left\|D_{n,q}(g_n) - g_n\right\|_{L_p([0,1])}\\ &\quad\le \frac{C(p,q)}{[n+2]_q}\left(\|\varphi^2 g_n''\|_{L_p([0,1])} + \|g_n\|_{L_p([0,1])}\right)\\ &\quad\le \frac{C}{[n+2]_q}\left(\|\varphi^2 g_n''\|_{L_p([0,1])} + \|f-g_n\|_{L_p([0,1])} + \|g_n\|_{L_p([0,1])}\right)\\ &\quad\le C\left(K_{2,\varphi}\left(f,[n+2]_q{}^{-1}\right)_p + [n+2]_q{}^{-1}\|f\|_{L_p([0,1])}\right)\\ &\quad\le C\left(\omega_\varphi^2\left(f,[n+2]_q{}^{-1/2}\right)_p + [n+2]_q{}^{-1}\|f\|_{L_p([0,1])}\right).\end{aligned}$$

Let $[\sqrt{n}]$ be the integer part of n. By Theorem 7.3.1 in [1, p. 84], we can choose g_n to be $Q_{[\sqrt{n}]}$, the polynomial of best approximation of degree $[\sqrt{n}]$ in $L_p([0,1])$. Therefore,

$$\begin{aligned}&\big\|D_{n,q}(Q_{[\sqrt{n}]}) - Q_{[\sqrt{n}]}\big\|_{L_p([0,1])}\\ &\quad\le C(p,q)\big(\omega_\varphi^2\big(f,[n+2]_q{}^{-1/2}\big)_p + [n+2]_q{}^{-1}\|f\|_{L_p([0,1])}\big).\end{aligned}\tag{4.8}$$

Using Lemmas 2.1 and 3.6, along with (4.6) and (4.8), we obtain

$$\begin{aligned}&\big\|D_{n,q}(f) - f\big\|_{L_p([0,1])}\\ &\quad\le \big\|D_{n,q}(f-Q_{[\sqrt{n}]}) - (f-Q_{[\sqrt{n}]})\big\|_{L_p([0,1])} + \|D_{n,q}Q_{[\sqrt{n}]} - Q_{[\sqrt{n}]}\|_{L_p([0,1])}\\ &\quad\le C(p,q)\|f-Q_{[\sqrt{n}]}\|_{L_p([0,1])} + \|D_{n,q}Q_{[\sqrt{n}]} - Q_{[\sqrt{n}]}\|_{L_p([0,1])}\\ &\quad\le C(p,q)\big(\omega_\varphi^2\big(f,[n+2]_q^{-1/2}\big)_p + [n+2]_q^{-1}\|f\|_{L_p([0,1])}\big).\end{aligned}$$

Let the operator $\overline{D}_{n,q}(f)$ be defined as

$$\overline{D}_{n,q}(f,x) := D_{n,q}(f,x) - f\big(\zeta_{q,n}(x)\big) + f(x),\tag{4.9}$$

where $\zeta_{q,n}(x) = \frac{1}{[n+2]_q} + \frac{q[n]_q}{[n+2]_q}x$.

By Lemmas 3.2 and 3.6, $\overline{D}_{n,q}(f,x) = 1$ and

$$\big|\overline{D}_{n,q}(f,x)\big| \le \big|D_{n,q}(f,x)\big| + \big|f\big(\zeta_{q,n}(x)\big)\big| + \big|f(x)\big| \le 3\|f\|_\infty.\tag{4.10}$$

Since $D_{n,q}(t,x) = \zeta_{q,n}(x)$, $\overline{D}_{n,q}(f,x) = x$. Now, making use of Taylor's formula for $g \in W_\infty^2$, we write

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-v)g''(v)\,dv.\tag{4.11}$$

Hence, (4.9) and (4.11) together imply

$$\begin{aligned}&\overline{D}_{n,q}(g,x)\\ &\quad= g(x) + \overline{D}_{n,q}\Big(\int_x^t (t-v)g''(v)\,dv, x\Big)\\ &\quad= g(x) + D_{n,q}\Big(\int_x^t (t-v)g''(v)\,dv, x\Big) - \int_x^{\zeta_{q,n}(x)} \big(\zeta_{q,n}(x)-v\big)g''(v)\,dv.\end{aligned}\tag{4.12}$$

On the other hand,

$$\begin{aligned}\big|\zeta_{q,n}(x) - x\big| &= \Big|\frac{1}{[n+2]_q} + \frac{(q[n]-[n+2]_q)}{[n+2]_q}x\Big|\\ &\le \Big|\frac{1-(1+q^{n+1})x}{[n+2]_q}\Big|\\ &\le \frac{1}{[n+2]_q}.\end{aligned}\tag{4.13}$$

In view of the concavity of $\delta_{q,n}^2$ on $[0,1]$, there follows

$$\frac{|t-v|}{\delta_{q,n}^2(v)} \leq \frac{|t-x|}{\delta_{q,n}^2(x)}. \tag{4.14}$$

From $\delta_{q,n}^2(x) \leq (\varphi^2(x)+[n+2]_q^{-1})$, (4.12), (4.14), and Lemma 3.6, it follows that

$$\begin{aligned}
&\left|\overline{D}_{n,q}(g,x)-g(x)\right| \\
&\quad \leq D_{n,q}\Big(|t-x|\delta_{q,n}^{-2}(x)\Big|\int_x^t \delta_{q,n}^2(v)\left|g''(v)\right| dv\Big|, x\Big)\big(\zeta_{q,n}(x)-x\big)^2\|g''\|_{L_\infty([0,1])} \\
&\quad \leq \delta_{q,n}^{-2}(x)D_{n,q}\big((t-x)^2,x\big)\|\delta_{q,n}^2 g''\|_{L_\infty([0,1])}+\frac{1}{[n+2]_q^2}\|g''\|_{L_\infty([0,1])} \\
&\quad \leq \frac{C(p,q)}{[n+2]_q}\|\delta_{q,n}^2 g''\|_{L_\infty([0,1])}+\frac{1}{[n+2]_q^2}\|g''\|_{L_\infty([0,1])}.
\end{aligned} \tag{4.15}$$

Using (4.9), (4.10), (4.13), and (4.15), respectively, we get

$$\begin{aligned}
&\left|D_{n,q}(f,x)-f(x)\right| \\
&\quad \leq \left|\overline{D}_{n,q}(f,x)-f(x)\right|+\left|f\big(\zeta_{q,n}(x)\big)-f(x)\right| \\
&\quad \leq \left|\overline{D}_{n,q}(f-g,x)\right|+\left|\overline{D}_{n,q}(g,x)-g(x)\right| \\
&\qquad +\left|g(x)-f(x)\right|+\left|f\big(\zeta_{q,n}(x)\big)-f(x)\right| \\
&\quad \leq \|f-g\|_{L_\infty([0,1])}+\frac{C}{[n+2]_q}\|\delta_{q,n}^2 g''\|_{L_\infty([0,1])} \\
&\qquad +\frac{1}{[n+2]_q^2}\|g''\|_{L_\infty([0,1])}+\omega\big(f,\left|f\big(\zeta_{q,n}(x)\big)-f(x)\right|\big) \\
&\quad \leq C\Big(\|f-g\|_{L_\infty([0,1])}+\frac{1}{[n+2]_q}\|\delta_{q,n}^2 g''\|_{L_\infty([0,1])} \\
&\qquad +\frac{1}{[n+2]_q^2}\|g''\|_{L_\infty([0,1])}\Big)+\omega\Big(f,\frac{1}{2}[n+2]_q^{-1}\Big).
\end{aligned}$$

Finally, taking the infimum on the right-hand side over all $g \in W_\infty^2$, and using the equivalence between $\overline{K}_{2,\varphi}(f,[n+2]_q^{-1})_\infty$ and $\omega_\varphi^2(f,[n+2]_q^{-1/2})_\infty$, the required result follows. $\square$

Remark 4.1. In the proofs of Theorems 1.1–1.3, the condition $q_n \to 1^-$ is not required. However, in order to have $D_{n,q}(f,x)$ an approximation process in $L_p([0,1])$ the condition $q_n \to 1^-$ is essential, since, for a fixed q, the limit $\lim_n \frac{1}{[n+2]_q^{1/2}} = +\sqrt{1-q}$ is obtained. Consequently, the right-hand sides in Theorems 1.1–1.3 do not tend to zero as $n \to \infty$ as required for the uniform convergence of $D_{n,q}(f)$.

Remark 4.2. The approximation theorem (Theorem 1.3) for the weighted Ditzian–Totik modulus of smoothness, $\omega_{\varphi^\lambda}(f,x), 0 \leq \lambda \leq 1$, could not be proved by the authors of the present article. This can be considered an open problem.

Acknowledgments. The authors would like to thank the referee for his/her valuable suggestions which led to a better presentation of this article.

References

1. Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer Ser. Comput. Math. **9**, Springer, New York, 1987. Zbl 0666.41001. MR0914149. DOI 10.1007/978-1-4612-4778-4. 304, 305, 308, 310, 311
2. A. R. Gairola, Deepmala, and L. N. Mishra, *On the q-derivatives of certain linear positive operators*, preprint, to appear in Iran J. Sci. Technol. Trans. A Sci. 303
3. A. R. Gairola, Deepmala, and L. N. Mishra, *Rate of approximation by finite iterates of q-Durrmeyer operators*, Proc. Nat. Acad. Sci. India Sect. A **86** (2016), no. 2, 229–234. MR3498609. DOI 10.1007/s40010-016-0267-z. 303
4. R. B. Gandhi, Deepmala, and V. N. Mishra, *Local and global results for modified Szász-Mirakjan operators*, Math. Methods Appl. Sci. **2016**. DOI 10.1002/mma.4171. 303
5. S. Goldberg and A. Meir, *Minimum moduli of ordinary differential operators*, Proc. Lond. Math. Soc. (3) **23** (1971), 1–15. Zbl 0216.17003. MR0300145. DOI 10.1112/plms/s3-23.1.1. 309
6. V. Gupta, *Some approximation properties of q-Durrmeyer operators*, Appl. Math. Comput. **197** (2008), no. 1, 172–178. Zbl 1142.41008. MR2396302. DOI 10.1016/j.amc.2007.07.056. 303, 306
7. V. G. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer, New York, 2002. Zbl 0986.05001. MR1865777. 305
8. V. N. Mishra, K. Khatri, L. N. Mishra, and Deepmala, *Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators*, J. Inequal. Appl. **2013**, no. 586. Zbl 1295.41013. MR3213006. 303
9. G. O. Okikiolu, *Aspects of the Theory of Bounded Integral Operators in L^p-Spaces*, Academic Press, London, 1971. Zbl 0219.44002. MR0445237. 308
10. A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Dover, New York, 1994. Zbl 0117.29001. MR1262128. 307
11. A. Wafi and N. Rao, *Approximation properties by generalized Baskakov-Kantorovich-Stancu type operators*, to appear in Appl. Math. Inf. Sci., preprint, arXiv:1508.00999v1 [math.NA]. 303
12. A. Zygmund, *Trigonometric Series*, Dover, New York, 1955. Zbl 0065.05604. MR0072976. 306, 307

[1]Department of Mathematics, Doon University, Dehradun 248001, India.
E-mail address: ashagairola@gmail.com

[2]Government Polytechnic, Rampur 244901, Uttar Pradesh, India.
E-mail address: kks_iitr@gmail.com

[3]Applied Mathematics and Humanities Department, Sardar Vallabhbhai National Institute of Technology, Ichchhanath Mahadev Dumas Road, Surat 395007, Gujarat, India

[4]L. 1627 Awadh Puri Colony Beniganj, Phase-III, Opposite - Industrial Training Institute (I.T.I.), Ayodhya Main Road Faizabad 224001, Uttar Pradesh, India.
E-mail address: vishnunarayanmishra@gmail.com; vishnu_narayanmishra@yahoo.co.in