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Research Article

On General Integral Operator of Analytic Functions

Nasser Alkasbi and Maslina Darus

School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia

Correspondence should be addressed to Maslina Darus; maslina@ukm.my

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Let E_β be the integral operator defined by $E_\beta(z) = [\beta \int_0^z t^{\beta-1}(f_1'(t))^{\alpha_1}(f_1(t)/t)^{\gamma_1} P_1^{\zeta_1(t)} \cdots (f_n'(t))^{\alpha_n}(f_n(t)/t)^{\gamma_n} P_n^{\zeta_n(t)} dt]^{1/\beta}$, where each of the functions f_i and P_i is, respectively, analytic functions and functions with positive real part defined in the open unit disk for all $i = 1, \ldots, n$. The object of this paper is to obtain several univalence conditions for this integral operator. Our main results contain some interesting corollaries as special cases.

1. Introduction and Definitions

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathscr{C} : |z| < 1\} : S = \{f \in A : f \text{ is univalent in } U\}$. Also, let $\mathscr{P}$ be the class of all functions which are analytic in U and satisfy $P(0) = 1$, $\mathfrak{R}\{P(z)\} > 0$. Frasin and Darus [1] defined the family $B(\delta)$, $0 \le \delta < 1$, so that it consists of functions $f \in A$ satisfying the condition

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1 - \delta \quad (z \in U). \tag{2}$$

In this paper, we obtain new sufficient conditions for the univalence of the general integral operator $E_\beta(z)$ defined by

$$E_\beta(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{i=0}^{n} \left(f_i'(t)\right)^{\alpha_i} \left(\frac{f_i(t)}{t}\right)^{\gamma_i} P_i^{\zeta_i(t)} dt\right]^{1/\beta}, \tag{3}$$

where $\beta \in \mathscr{C}$, $\alpha_i, \gamma_i, \zeta_i \in \mathscr{C}$, $f_i \in A$, and $P_i \in \mathscr{P}$ for all $i = 1, 2, 3, \ldots n$.

Here and throughout in the sequel, every multivalued functions is taken with the principal branch.

Remark 1. Note that the integral operator E_β generalizes the following operators introduced and studied by several authors as follows.

(i) For $\alpha_i = 0$, where $i = 1, \ldots, n$, we obtain the integral operator

$$N_\beta(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{i=0}^{n} \left(\frac{f_i(t)}{t}\right)^{\gamma_i} P_i^{\zeta_i(t)} dt\right]^{1/\beta}, \tag{4}$$

introduced and studied by Frasin [2].

(ii) For $\alpha_i = \gamma_i = 0$, $i = 1, \ldots, n$, we obtain the integral operator

$$\mathscr{F}_\beta^{\zeta}(P_i)(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{i=0}^{n} P_i^{\zeta_i(t)} dt\right]^{1/\beta}, \tag{5}$$

introduced and studied by Frasin [3].

(iii) For $\zeta_i = 0, i = 1, \ldots, n$, we obtain the integral operator

$$\mathscr{F}_\beta^{\alpha_i,\gamma_i}(f_1, f_2, \ldots, f_n)(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{i=0}^{n} \left(f_i'(t)\right)^{\alpha_i} \left(\frac{f_i(t)}{t}\right)^{\gamma_i} dt\right]^{1/\beta}, \tag{6}$$

introduced and studied by Frasin [4].

(iv) For $\zeta_i = 0$, $\beta = 1$, and $i = 1, \ldots, n$, we obtain the integral operator

$$\mathcal{F}^{\alpha_i,\gamma_i}(f_1, f_2, \ldots, f_n)(z) = \int_0^z \prod_{i=0}^{n} \left(f_i'(t)\right)^{\alpha_i} \left(\frac{f_i(t)}{t}\right)^{\gamma_i} dt, \tag{7}$$

introduced and studied by Frasin [5].

(v) For $\zeta_i = \alpha_i = 0$ and $i = 1, \ldots, n$, we obtain the integral operator

$$\mathcal{F}_{\gamma}^{\beta_i}(f_1, f_2, \ldots, f_n)(z) = \left[\beta \int_0^z t^{\beta-1} \left(\frac{f_1(t)}{t}\right)^{\gamma_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\gamma_n} dt\right]^{1/\beta}, \tag{8}$$

introduced and studied by D. Breaz and N. Breaz [6].

(vi) For $\beta = 1$, $\zeta_i = \alpha_i = 0$, and $i = 1, \ldots, n$, we obtain the integral operator

$$F(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\gamma_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\gamma_n} dt, \tag{9}$$

introduced and studied by D. Breaz and N. Breaz [6].

(vii) For $\beta = 1$, $\zeta_i = \gamma_i = 0$, and $i = 1, \ldots, n$, we obtain the integral operator

$$F_{\alpha_1,\ldots,\alpha_n}(z) = \int_0^z \left(f_1'(t)\right)^{\alpha_1} \cdots \left(f_n'(t)\right)^{\alpha_n} dt, \tag{10}$$

introduced and studied by Breaz et al. [7].

(viii) For $\beta = 1$, $n = 1$, $\gamma_1 = \gamma$, and $\alpha_i = \zeta_i = 0$, we obtain the integral operator

$$F_{\alpha,\gamma}(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\gamma} \left(f'(t)\right)^{\alpha} dt, \tag{11}$$

studied in [8].

(ix) For $\beta = 1$, $n = 1$, $\gamma_1 = \gamma$, and $\alpha_i = \zeta_i = 0$, we obtain the integral operator

$$F_{\gamma}(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\gamma} dt \tag{12}$$

studied in [9]. In particular, for $\gamma = 1$, we obtain Alexander integral operator which was introduced in [10] as follows

$$F(z) = \int_0^z \frac{f(t)}{t} dt. \tag{13}$$

(x) For $\beta = 1$, $n = 1$, $\alpha_i = \alpha$, and $\gamma_i = \zeta_i = 0$, we obtain the integral operator

$$G(z) = \int_0^z \left(f'(t)\right)^{\alpha} dt, \tag{14}$$

studied in [11].

In order to derive our main results, we have to recall here the following lemmas.

Lemma 2 (see [12]). *Let $\eta \in \mathscr{C}$ with $\Re(\eta) > 0$. If $h \in A$ satisfies*

$$\frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)} \left|\frac{zh''(z)}{h'(z)}\right| \leq 1 \tag{15}$$

for all $z \in U$, then the integral operator

$$F_{\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \left(f'(t)\right) dt\right]^{1/\beta} \tag{16}$$

is in the class $\mathcal{S}$.

Lemma 3 (see [13]). *Let $\beta \in \mathscr{C}$ with $\Re(\beta) > 0$, $c \in \mathscr{C}$ with $|c| \leq 1$, $c \neq -1$. If $h \in A$ satisfies*

$$\left|c|z|^{2\beta} + \left(1 + |z|^{2\beta}\right) \frac{zh''(z)}{\beta h'(z)}\right| \leq 1 \tag{17}$$

for all $z \in U$, then the integral operator $F_{\beta}(z)$ defined by (16) is in the class $\mathcal{S}$.

Lemma 4 (see [14]). *If $P(z) \in \mathscr{P}$, then*

$$\left|\frac{zP''(z)}{\beta P'(z)}\right| \leq \frac{2|z|}{1 - |z|^2}. \tag{18}$$

Lemma 5 (see [9]). *If $f(z) \in B(\delta)$, then*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{(1-\delta)(1+|z|)}{1-|z|}. \tag{19}$$

When $\delta = 0$, so $f \in \mathcal{S}$.

Lemma 6 (see [9]). *If $f(z) \in B(\delta)$, then*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{(1-\delta)(2+|z|)}{1-|z|}. \tag{20}$$

Also we need the following general Schwarz lemma.

Lemma 7 (see [15]). *Let the function f be regular in the disk $U_R = \{z : |z| < R\}$, with $|M| < R$ for fixed M. If $f(z)$ has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \quad (z \in U_R). \tag{21}$$

The equality holds only if

$$f(z) = e^{i\theta}\left(\frac{M}{R^m}\right) z^m, \tag{22}$$

where θ is constant.

Lemma 8 (see [16]). *If $f \in A$, then*

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| \leq \frac{5}{4}\left|\frac{zf'(z)}{f(z)}\right|. \tag{23}$$

2. Univalence Conditions for the Operator E_β

We first prove the following theorem.

Theorem 9. *Let $f \in B(\delta_i)$, $0 \le \delta_i < 1$, $P_i(z) \in \mathcal{P}$ for all $i = 1, \ldots, n$ and $\eta \in \mathcal{C}$ with $\Re(\eta) = a > 0$. If*

$$\sum_{i=1}^{n} [(3|\alpha_i| + 2|\gamma_i|)(1-\delta_i) + 2|\zeta_i|] < \min\left\{a; \frac{1}{2}\right\}, \quad (24)$$

then the integral operator E_β defined by (3) *is in the class $\mathcal{S}$.*

Proof. Define the regular function $h(z)$ by

$$h(z) = \int_0^z \prod_{i=0}^{n} \left[(f_i'(t))^{\alpha_i} \left(\frac{f_i(t)}{t}\right)^{\gamma_i} P_i^{\zeta_i(t)} \right] dt. \quad (25)$$

Then it is easy to see that

$$h'(z) = \prod_{i=0}^{n} \left[(f_i'(t))^{\alpha_i} \left(\frac{f_i(t)}{t}\right)^{\gamma_i} P_i^{\zeta_i(t)} \right], \quad (26)$$

and $h(0) = 1 - h'(0) = 0$. Differentiating both sides of (26) logarithmically, we obtain

$$\begin{aligned} \frac{zh''(z)}{h'(z)} &= \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i''(z)}{f_i'(z)}\right) + \sum_{i=1}^{n} \gamma_i \left(\frac{zf_i'(z)}{f_i(z)} - 1\right) \\ &\quad + \sum_{i=1}^{n} \zeta_i \left(\frac{zP_i'(z)}{P_i(z)}\right). \end{aligned} \quad (27)$$

Thus, we have

$$\begin{aligned} \left|\frac{zh''(z)}{h'(z)}\right| &\le \sum_{i=1}^{n} |\alpha_i| \left|\frac{zf_i''(z)}{f_i'(z)}\right| + \sum_{i=1}^{n} |\gamma_i| \left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \\ &\quad + \sum_{i=1}^{n} |\zeta_i| \left|\frac{zP_i'(z)}{P_i(z)}\right|. \end{aligned} \quad (28)$$

Since $f_i \in B(\delta_i)$, $P_i(z) \in \mathcal{P}$ for all $i = 1, \ldots, n$, from (28), (18), (19), and (20), we obtain

$$\begin{aligned} &\left|\frac{zh''(z)}{h'(z)}\right| \\ &\quad \le \sum_{i=1}^{n} |\alpha_i| (1-\delta_i) \left(\frac{2+|z|}{1-|z|}\right) \\ &\qquad + \sum_{i=1}^{n} |\gamma_i| (1-\delta_i) \left(\frac{1+|z|}{1-|z|}\right) + \sum_{i=1}^{n} |\zeta_i| \left(\frac{2|z|}{1-|z|^2}\right) \\ &\quad \le \sum_{i=1}^{n} |\alpha_i| (1-\delta_i) \left(\frac{3}{1-|z|}\right) \\ &\qquad + \sum_{i=1}^{n} |\gamma_i| (1-\delta_i) \left(\frac{2}{1-|z|}\right) + \sum_{i=1}^{n} |\zeta_i| \left(\frac{2}{1-|z|}\right). \end{aligned} \quad (29)$$

Multiplying both sides of (29) by $(1-|z|^{2\Re(\eta)})/\Re(\eta)$, we get

$$\begin{aligned} &\frac{1-|z|^{2\Re(\eta)}}{\Re(\eta)} \left|\frac{zh''(z)}{h'(z)}\right| \\ &\quad \le \frac{1-|z|^{2\Re(\eta)}}{(1-|z|)\Re(\eta)} \\ &\qquad \times \sum_{i=1}^{n} [(3|\alpha_i| + 2|\gamma_i|)(1-\delta_i) + 2|\zeta_i|], \end{aligned} \quad (30)$$

for all $z \in U$.

Let us denote $|z| = x$, $x \in [0,1)$, $\Re(\eta) = a > 0$, and $\Phi(x) = (1-x^{2a})/(1-x)$. It is easy to prove that

$$\Phi(x) \le \begin{cases} 1, & 0 < a < 1, \\ 2a, & \dfrac{1}{2} < a < \infty. \end{cases} \quad (31)$$

From (31), (30), and the hypotheses (24), we have

$$\begin{aligned} &\frac{1-|z|^{2a}}{a} \left|\frac{zh''(z)}{h'(z)}\right| \\ &\le \begin{cases} \dfrac{1}{a} \displaystyle\sum_{i=1}^{n} [(3|\alpha_i| + 2|\gamma_i|)(1-\delta_i) + 2|\zeta_i|], & 0 < a < 1, \\ 2 \displaystyle\sum_{i=1}^{n} [(3|\alpha_i| + 2|\gamma_i|)(1-\delta_i) + 2|\zeta_i|], & \dfrac{1}{2} < a < \infty \end{cases} \\ &\le 1, \end{aligned} \quad (32)$$

for all $z \in U$. Applying Lemma 2 for the function $h(z)$, we prove that $E_\beta(z) \in \mathcal{S}$. □

Letting $n = 1$, $\delta_1 = \delta$, $\alpha_1 = \alpha$, $\gamma_1 = \gamma$, and $\zeta_1 = \zeta$ in Theorem 9, we obtain the following corollary.

Corollary 10. *Let $f \in B(\delta)$, $0 \le \delta < 1$, $P(z) \in \mathcal{P}$, and all $\eta, \gamma, \zeta \in \mathcal{C}$ with $\Re(\eta) = a > 0$. If*

$$(3|\alpha| + 2|\gamma|)(1-\delta) + 2|\zeta| < \min\left\{a; \frac{1}{2}\right\}, \quad (33)$$

and then the integral operator $N_\beta^{\alpha,\gamma,\zeta}$ defined by

$$N_\beta^{\alpha,\gamma,\zeta}(z) = \left[\beta \int_0^z t^{\beta-1} (f'(t))^{\alpha} \left(\frac{f(t)}{t}\right)^{\gamma} P^{\zeta}(t)\, dt\right]^{1/\beta} \quad (34)$$

is in the class $\mathcal{S}$.

If we set $\delta = 0$ in Corollary 10, we have the following.

Corollary 11. *Let $f \in \mathcal{S}$, $P(z) \in \mathcal{P}$ and all $\eta, \gamma, \zeta \in \mathcal{C}$ with $\Re(\eta) = a > 0$. If*

$$3|\alpha| + 2|\gamma| + 2|\zeta| < \min\left\{a; \frac{1}{2}\right\}, \quad (35)$$

then the integral operator $N_\beta^{\alpha,\gamma,\zeta}$ defined by (34) *is in the class $\mathcal{S}$.*

Next, we prove the following theorem.

Theorem 12. *Let $\alpha_i, \gamma_i, \zeta_i \in \mathscr{C}$ for all $i = 1, \ldots, n$ and each $f_i \in A$ satisfies condition* (15) *with $\Re(f_i(z)/z) > 0$ and*

$$\sum_{i=1}^{n}\left[29\left|\alpha_i\right| + 16\left|\gamma_i + \zeta_i\right|\right] \le \begin{cases} 4\Re(\eta), & \text{if } \Re(\eta) \in (0,1), \\ 4, & \text{if } \Re(\eta) \in [1,\infty) \end{cases} \tag{36}$$

and then, for any complex number σ with $\Re(\sigma) \ge \Re(\eta) > 0$, the integral operator N_β defined by (4) *is in the class $\mathcal{S}$.*

Proof. Suppose that $\Re(f_i(z)/z)$ for all $i = 1, \ldots, n$. Thus we have

$$\frac{f_i(z)}{z} = P_i(z), \tag{37}$$

where $P_i \in \mathscr{P}$ for all $i = 1, \ldots, n$. Differentiating both sides of (37) logarithmically, we obtain

$$\frac{zf_i'(z)}{f_i z} - 1 = \frac{zP_i'(z)}{P_i(z)}. \tag{38}$$

Define the regular function $h(z)$ as in (25). Thus from (27) we have

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)}\right) + \sum_{i=1}^{n}\left(\gamma_i + \zeta_i\right)\left(\frac{zf_i'(z)}{f_i(z)} - 1\right) \tag{39}$$

and so

$$\begin{aligned}\frac{zh''(z)}{h'(z)} &= \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)} + 1\right) - \sum_{n=i}^{0}\alpha_i \\ &\quad + \sum_{i=1}^{n}\left(\gamma_i + \zeta_i\right)\left(\frac{zf_i'(z)}{f_i(z)} - 1\right).\end{aligned} \tag{40}$$

From Lemma 8, it follows that

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right| &\le \frac{5}{4}\sum_{i=1}^{n}\left|\alpha_i\right|\left|\frac{zf_i'(z)}{f_i(z)}\right| \\ &\quad + \sum_{i=1}^{n}\left|\gamma_i + \zeta_i\right|\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + \sum_{n=i}^{0}\left|\alpha_i\right| \\ &\le \frac{5}{4}\sum_{i=1}^{n}\left|\alpha_i\right|\left[\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + 1\right] \\ &\quad + \sum_{i=1}^{n}\left|\gamma_i + \zeta_i\right|\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + \sum_{n=i}^{0}\left|\alpha_i\right| \\ &\le \sum_{i=1}^{n}\left[\left(\frac{5}{4}\left|\alpha_i\right| + \left|\gamma_i + \zeta_i\right|\right)\left|\frac{zf_i'(z)}{f_i(z)} - 1\right|\right] \\ &\quad + \frac{9}{4}\sum_{i=1}^{n}\left|\alpha_i\right| \\ &\le \sum_{i=1}^{n}\left[\left(\frac{5}{4}\left|\alpha_i\right| + \left|\gamma_i + \zeta_i\right|\right)\left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right)\right] \\ &\quad + \frac{9}{4}\sum_{i=1}^{n}\left|\alpha_i\right|.\end{aligned} \tag{41}$$

Multiplying both sides of (41) by $(1 - |z|^{2\Re(\eta)})/\Re(\eta)$, from Lemma 5 with $\delta = 0$, we get

$$\begin{aligned}&\frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)}\left|\frac{zh''(z)}{h'(z)}\right| \\ &\quad\le \frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)}\sum_{i=1}^{n}\left(\frac{5}{4}\left|\alpha_i\right| + \left|\gamma_i + \zeta_i\right|\right)\left(\frac{2}{1 - |z|}\right) \\ &\qquad + \frac{9\left(1 - |z|^{2\Re(\eta)}\right)\sum_{i=1}^{n}\left|\alpha_i\right|}{4\Re(\eta)}.\end{aligned} \tag{42}$$

Suppose that $\Re(\eta) \in (0,1)$. Define a function $\Phi : (0,1) \to R$ by

$$\Phi(x) = 1 - a^{2a} \quad (0 < a < 1). \tag{43}$$

Then Φ is an increasing function and consequently, for $|z| = a$; $z \in U$, we obtain

$$1 - |z|^{2\Re(\eta)} < 1 - |z|^2. \tag{44}$$

We thus find from (42) and (44) that

$$\begin{aligned}\frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)}\left|\frac{zh''(z)}{h'(z)}\right| &\le \frac{\sum_{i=1}^{n}\left(5\left|\alpha_i\right| + 4\left|\gamma_i + \zeta_i\right|\right)}{\Re(\eta)} \\ &\quad + \frac{9\sum_{i=1}^{n}\left|\alpha_i\right|}{4\Re(\eta)} \\ &= \frac{\sum_{i=1}^{n}\left(29\left|\alpha_i\right| + 19\left|\gamma_i + \zeta_i\right|\right)}{4\Re(\eta)}.\end{aligned} \tag{45}$$

Using the hypotheses (36) for $\Re(\eta) \in (0,1)$, we readily get

$$\frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)}\left|\frac{zh''(z)}{h'(z)}\right| \le 1. \tag{46}$$

Now if $\Re(\eta) \in [1,\infty)$, we define a function $\Psi : [1,\infty) \to R$ by

$$\Psi(x) = \frac{1 - a^{2x}}{x}, \quad (0 < a < 1). \tag{47}$$

We observe that the function Ψ is decreasing and consequently, for $|z| = a$; $z \in U$, we have

$$\frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)} < 1 - |z|^2, \tag{48}$$

for all $z \in U$. It follows from (40) and (42) that

$$\frac{1 - |z|^{2\Re(\eta)}}{\Re(\eta)}\left|\frac{zh''(z)}{h'(z)}\right| \le \sum_{i=1}^{n}\left(\frac{29}{4}\left|\alpha_i\right| + 4\left|\gamma_i + \zeta_i\right|\right). \tag{49}$$

Using once again the hypotheses (36) when $\Re(\eta) \in [1,\infty)$, we easily get

$$\frac{1-|z|^{2\Re(\eta)}}{\Re(\eta)}\left|\frac{zh''(z)}{h'(z)}\right| \le 1. \tag{50}$$

Finally by applying Lemma 2, we conclude that the integral operator N_β defined by (4) is in the class $\mathcal{S}$. □

Letting $n = 1$, $\delta_1 = \delta$, $\alpha_1 = \alpha$, $\gamma_1 = \gamma$, and $\zeta_1 = \zeta$ in Theorem 12, we obtain the following corollary.

Corollary 13. *Let $\alpha, \gamma, \zeta \in \mathscr{C}$, and $f \in A$ satisfies condition (15). If*

$$29|\alpha| + 16|\gamma+\zeta| \le \begin{cases} 4\Re(\eta), & \text{if } \Re(\eta) \in (0,1), \\ 4, & \text{if } \Re(\eta) \in [1,\infty) \end{cases} \tag{51}$$

then, for any complex number σ with $\Re(\sigma) \ge \Re(\eta) > 0$, the integral operator $N_\beta^{\alpha,\gamma,\zeta}$ defined by (4) is in the class $\mathcal{S}$.

Using Lemma 3, we derive the following theorem.

Theorem 14. *Let $\alpha_i, \gamma_i, \zeta_i, \beta \in \mathscr{C}$ for all $i = 1, \ldots, n$, $\Re(\beta) > 0$, $c \in \mathscr{C}(|c| \le 1)$ and each $f_i \in A$ satisfies condition (15). If*

$$\sum_{i=1}^{n}\left[29|\alpha_i| + 16|\gamma_i+\zeta_i|\right] \le \begin{cases} 4\beta(1-|c|), & \text{if } \Re(\eta) \in (0,1), \\ 4(1-|c|), & \text{if } \Re(\eta) \in [1,\infty) \end{cases} \tag{52}$$

then, for any complex number σ with $\Re(\sigma) \ge \Re(\eta) > 0$, the integral operator N_β defined by (4) is in the class $\mathcal{S}$.

Proof. From (40), we have

$$\begin{aligned} &\left|c|z|^{2\beta} + \left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| \\ &= \left|c|z|^{2\beta} + \frac{1-|z|^2\beta}{\beta}\right. \\ &\qquad \times\left[\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)}+1\right) - \sum_{i=1}^{n}\alpha_i\right. \\ &\qquad\qquad \left.\left.+\sum_{i=1}^{n}(\gamma_i+\zeta_i)\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right]\right| \\ &\le |c| + \frac{\left|1-|z|^2\right|}{|\beta|} \\ &\quad \times\left[\sum_{i=1}^{n}\left(\frac{5}{4}|\alpha_i| + |\gamma_i+\zeta_i|\right)\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right) + \frac{9}{4}\sum_{i=1}^{n}|\alpha_i|\right] \\ &\le |c| + \frac{\left|1-|z|^2\right|}{|\beta|} \\ &\quad \times\left[\sum_{i=1}^{n}\left(\frac{5}{4}|\alpha_i| + |\gamma_i+\zeta_i|\right)\left(\frac{2}{1-|z|}\right) + \frac{9}{4}\sum_{i=1}^{n}|\alpha_i|\right]. \end{aligned} \tag{53}$$

Suppose that $\beta \in (0,1)$. Define a function $\Phi : (0,1) \to R$ by

$$\Phi(x) = 1 - a^{2x} \quad (0 < a < 1). \tag{54}$$

Then Φ is an increasing function and consequently for $|z| = a$; $z \in U$, we obtain

$$1 - |z|^{2\beta} < 1 - |z|^2. \tag{55}$$

We thus find from (53) that

$$\left|c|z|^{2\beta} + \left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| \le \frac{\sum_{i=1}^{n}\left(29|\alpha_i| + 16|\gamma_i+\zeta_i|\right)}{4|\beta|}. \tag{56}$$

Using the hypotheses (52) for $\beta \in (0,1)$, we readily get

$$\left|c|z|^{2\beta} + \left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| \le 1. \tag{57}$$

Now if $\beta \in [1,\infty)$, we define a function $\Psi : [1,\infty) \to R$ by

$$\Psi(x) = \frac{1-a^{2x}}{x}, \quad (0 < a < 1). \tag{58}$$

We observe that the function Ψ is decreasing and consequently for $|z| = a$; $z \in U$, and using once again the hypotheses (36) when $\Re(\eta) \in [1,\infty)$, we easily get

$$\left|c|z|^{2\beta} + \left(1-|z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| \le 1. \tag{59}$$

Finally, by applying Lemma 3, we conclude that $N_\beta \in \mathcal{S}$. □

Conflict of Interests

The authors declare that they have no conflict interests.

Authors' Contribution

The first author is currently a Ph.D. student under supervision of the second author and jointly worked on deriving the results. All authors read and approved the paper.

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