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Research Article

Nonperiodic Damped Vibration Systems with Asymptotically Quadratic Terms at Infinity: Infinitely Many Homoclinic Orbits

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We study a class of nonperiodic damped vibration systems with asymptotically quadratic terms at infinity. We obtain infinitely many nontrivial homoclinic orbits by a variant fountain theorem developed recently by Zou. To the best of our knowledge, there is no result published concerning the existence (or multiplicity) of nontrivial homoclinic orbits for this class of non-periodic damped vibration systems with asymptotically quadratic terms at infinity.

1. Introduction and Main Results

In the end of 19th century, Poincaré recognized the importance of homoclinic orbits for dynamical systems. Since then the existence and multiplicity of homoclinic solutions have become one of the most important problems in the research of dynamical systems. In this paper, we consider the following *nonperiodic damped vibration system* (NDVS):

$$\ddot{u}(t) + M\dot{u}(t) - L(t)u(t) + H_u(t, u(t)) = 0, \quad t \in \mathbb{R}, \quad (1)$$

where M is an antisymmetric $N \times N$ constant matrix, $L(t) \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a symmetric matrix, $H(t,u) \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $H_u(t,u)$ denotes its gradient with respect to the u variable. We say that a solution $u(t)$ of (1) is homoclinic (to 0) if $u(t) \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that

$$u(t) \longrightarrow 0, \qquad \dot{u}(t) \longrightarrow 0 \quad \text{as } |t| \to \infty. \quad (2)$$

If $u(t) \not\equiv 0$, then $u(t)$ is called a nontrivial homoclinic solution.

If $M = 0$ (zero matrix), then (1) reduces to the following second-order Hamiltonian system:

$$\ddot{u}(t) - L(t)u(t) + H_u(t, u(t)) = 0, \quad t \in \mathbb{R}, \quad (3)$$

which is a classical equation which can describe many mechanical systems, such as a pendulum. In the past decades, the existence and multiplicity of periodic solutions and homoclinic orbits for (3) have been studied by many authors via variational methods; see [1–17] and the references therein. The periodic assumptions are very important in the study of homoclinic orbits for (3) since periodicity is used to control the lack of compactness due to the fact that (3) is set on all $\mathbb{R}$.

Nonperiodic problems are quite different from the ones described in periodic cases. Rabinowitz and Tanaka [10] introduced a type of coercivity condition on the matrix $L(t)$:

$$l(t) := \inf_{|u|=1} (L(t)u, u) \longrightarrow +\infty \quad \text{as } |t| \longrightarrow \infty \quad (4)$$

and obtained the existence of homoclinic orbit for nonperiodic (3) under the usual Ambrosetti-Rabinowitz (AR) superquadratic condition:

$$0 < \mu H(t,u) \le (H_u(t,u), u), \quad \forall t \in \mathbb{R},\ \forall u \in \mathbb{R}^N \setminus \{0\}, \quad (5)$$

where $\mu > 2$ is a constant, $(\cdot,\cdot)$ denotes the standard inner product in $\mathbb{R}^N$, and the associated norm is denoted by $|\cdot|$.

As usual, we say that H satisfies the *subquadratic* (or *superquadratic*) growth condition at infinity if

$$\lim_{|u|\to\infty} \frac{H(t,u)}{|u|^2} = 0 \qquad \left(\text{or } \lim_{|u|\to\infty} \frac{H(t,u)}{|u|^2} = +\infty\right). \quad (6)$$

If $M \neq 0$, that is, the damped vibration system (1), there are only a few authors who have studied homoclinic orbits of the NDVS (1), see [18–23]. Zhu [18] considered the *periodic* case of (1) (i.e., $L(t)$ and $H(t,u)$ are T-periodic in t with $T > 0$) and obtained the existence of nontrivial homoclinic solutions of (1). The authors [19–23] considered the *nonperiodic* case of (1): Zhang and Yuan [19] obtained the existence of at least one homoclinic orbit for (1) when H satisfies the *subquadratic* condition at infinity by using a standard minimizing argument. By a symmetric mountain pass theorem and a generalized mountain pass theorem, Wu and Zhang [20] obtained the existence and multiplicity of homoclinic orbits for (1) when H satisfies the local (AR) *superquadratic* growth condition:

$$0 < \mu H(t,u) \le (H_u(t,u), u), \quad \forall t \in \mathbb{R},\ \forall |u| \ge r, \tag{7}$$

where $\mu > 2$ and $r > 0$ are two constants. We should *notice* that the matrix $L(t)$ in (1) is required to satisfy condition (4) in the Previously mentioned two papers [19, 20]. Later, Sun et al. [21] obtained the existence of at least one homoclinic orbit for (1) when H satisfies the *superquadratic* condition at infinity by using the following conditions which are weaker than condition (4).

(L_1) There exists a constant $\beta > 1$ such that

$$\operatorname{meas}\left\{t \in \mathbb{R} : |t|^{-\beta} L(t) < bI_N\right\} < +\infty, \quad \forall b > 0. \tag{8}$$

(L_2) There exists a constant $\gamma \ge 0$ such that

$$l(t) := \inf_{|u|=1} (L(t)u, u) \ge -\gamma, \quad \forall t \in \mathbb{R}. \tag{9}$$

Recently, by using conditions (L_1) and (L_2), Chen [22, 23] obtained *infinitely many* nontrivial homoclinic orbits of (1) when H satisfies the *subquadratic* [22] (or *superquadratic* [23]) growth condition at infinity. In fact, conditions (L_1) and (L_2) are first used in [14]. As mentioned in [21], there are some matrix-valued functions $L(t)$ satisfying (L_1) and (L_2) but not satisfying (4). For example, $L(t) := (t^4 \sin^2 t + 1) I_N$. That is, conditions (L_1) and (L_2) are weaker than condition (4).

Remark 1. To the best of our knowledge, there is no result published concerning the existence (or multiplicity) of nontrivial homoclinic orbits for the NDVS (1) when H satisfies the *asymptotically quadratic* condition at infinity (see the following condition (H_3)).

Let $\widetilde{H}(t,u) := H(t,u) - (1/2)(H_u(t,u), u)$. We assume the following.

(H_1) There are constants $\mu \in (1,2)$ and $c_1, c_2, c_3 > 0$ such that

$$c_3 |u|^\mu \le |H(t,u)| \le c_1 |u|, \quad \forall t \in \mathbb{R},\ |u| \le c_2. \tag{10}$$

(H_2) $H(t,u) \ge (1/2)(H_u(t,u), u) \ge 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$.

(H_3) $\lim_{|u| \to \infty}((H(t,u))/|u|^2) = V(t)$ uniformly in t, where $0 < \inf_{t\in\mathbb{R}} V(t) \le \sup_{t\in\mathbb{R}} V(t) < +\infty$.

(H_4) $\widetilde{H}(t,u) \to +\infty$ as $|u| \to \infty$ and

$$\limsup_{|u|\to 0} \frac{|H_u(t,u)|^{\mu/(\mu-1)}}{\widetilde{H}(t,u)} = P(t) \text{ uniformly in } t, \quad |P(t)| < \infty. \tag{11}$$

We should mention that the coercive-type assumption (see (H_4)) of the function $\widetilde{H}$ was first observed and used by Costa and Magalhães [24].

Now, our main result reads as follows.

Theorem 2. *If (L_1), (L_2), (H_1)–(H_4), and $H(t,u)$ are even in u hold, then* (1) *possesses infinitely many nontrivial homoclinic orbits.*

Example 3. Let

$$H(t,u) := V(t)|u|^2 + |u|^\mu, \quad \mu \in (1,2), \tag{12}$$

where $V(t)$ is defined in (H_3). It is not hard to check that it satisfies conditions (H_1)–(H_4) with $P(t) = (2[2V(t) + \mu]^{\mu/(\mu-1)})/(2-\mu)$ in (H_4).

The rest of our paper is organized as follows. In Section 2, we establish the variational framework associated with (1) and give some preliminary lemmas, which are useful in the proof of our result, and then we give the detailed proof of our main result.

2. Variational Frameworks and the Proof of Our Main Result

In this section, we always assume that (L_1), (L_2), (H_1)–(H_4), and $H(t,u)$ are even in u hold.

In the following, we will use $\|\cdot\|_p$ to denote the norm of $L^p(\mathbb{R}, \mathbb{R}^N)$ for any $p \in [1, \infty]$. Let $E := H^1(\mathbb{R}, \mathbb{R}^N)$ be a Hilbert space with the inner product and the norm given, respectively, by

$$\langle u, v\rangle_E = \int_{\mathbb{R}} [(\dot{u}(t), \dot{v}(t)) + (u(t), v(t))]\, dt, \qquad \|u\|_E = \langle u, u\rangle_E^{1/2}, \quad \forall u, v \in E. \tag{13}$$

It is well known that E is continuously embedded in $L^p(\mathbb{R}, \mathbb{R}^N)$ for $p \in [2, \infty)$. We define an operator $J : E \to E$ by

$$(Ju, v) := \int_{\mathbb{R}} (Mu(t), \dot{v}(t))\, dt, \quad \forall u, v \in E. \tag{14}$$

Since M is an antisymmetric $N \times N$ constant matrix, J is self-adjoint on E. Moreover, we denote by χ the self-adjoint extension of the operator $-d^2/dt^2 + L(t) + J$ with the domain $\mathcal{D}(\chi) \subset L^2(\mathbb{R}, \mathbb{R}^N)$.

Let $W := \mathcal{D}(|\chi|^{1/2})$, the domain of $|\chi|^{1/2}$. We define, respectively, on W the inner product and the norm

$$\langle u, v\rangle_W := \left(|\chi|^{1/2} u, |\chi|^{1/2} v\right)_2 + (u, v)_2, \qquad \|u\|_W = \langle u, u\rangle_W^{1/2}, \tag{15}$$

where $(\cdot,\cdot)_2$ denotes the inner product in $L^2(\mathbb{R}, \mathbb{R}^N)$.

Lemma 4 ([21], Lemma 4). *If conditions (L_1) and (L_2) hold, then W is compactly embedded into $L^p(\mathbb{R},\mathbb{R}^N)$ for all $1 \le p \le +\infty$.*

By Lemma 4, it is easy to prove that the spectrum $\sigma(\chi)$ has a sequence of eigenvalues (counted with their multiplicities)

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \longrightarrow \infty, \tag{16}$$

and the corresponding system of eigenfunctions $\{e_k : k \in \mathbb{N}\}$ $(\chi e_k = \lambda_k e_k)$ forms an orthogonal basis in $L^2(\mathbb{R},\mathbb{R}^N)$. Let

$$\begin{gathered} k_1 := \sharp\left\{j : \lambda_j < 0\right\}, \qquad k_0 := \sharp\left\{j : \lambda_j = 0\right\}, \\ k_2 := k_0 + k_1, \\ W^- := \operatorname{span}\left\{e_1,\ldots,e_{k_1}\right\}, \\ W^0 := \operatorname{span}\left\{e_{k_1+1},\ldots,e_{k_2}\right\}, \\ W^+ := \operatorname{cl}_W\left(\operatorname{span}\left\{e_{k_2+1},\ldots\right\}\right). \end{gathered} \tag{17}$$

Then, one has the orthogonal decomposition

$$W = W^- \oplus W^0 \oplus W^+ \tag{18}$$

with respect to the inner product $\langle\cdot,\cdot\rangle_W$.

Now, we introduce, respectively, on W the following new inner product and norm:

$$\begin{gathered} \langle u,v\rangle := \left(u^0,v^0\right)_2 + \left(|\chi|^{1/2}u, |\chi|^{1/2}v\right)_2, \\ \|u\| = \langle u,u\rangle^{1/2}, \end{gathered} \tag{19}$$

where $u,v \in W = W^- \oplus W^0 \oplus W^+$ with $u = u^- + u^0 + u^+$, and $v = v^- + v^0 + v^+$. Clearly, the two norms $\|\cdot\|$ and $\|\cdot\|_W$ are equivalent (see [3]), and the decomposition $W = W^- \oplus W^0 \oplus W^+$ is also orthogonal with respect to both inner products $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)_2$.

For problem (1), we consider the following functional:

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\int_{\mathbb{R}} \left[|\dot u(t)|^2 + (Mu(t),\dot u(t)) + (L(t)u(t),u(t))\right]dt \\ &\quad - \int_{\mathbb{R}} H(t,u)\,dt, \quad u \in W. \end{aligned} \tag{20}$$

Then, Φ can be rewritten as

$$\begin{gathered} \Phi(u) = \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\left\|u^-\right\|^2 - \int_{\mathbb{R}} H(t,u)\,dt, \\ u = u^- + u^0 + u^+ \in W. \end{gathered} \tag{21}$$

Let $I(u) := \int_{\mathbb{R}} H(t,u)dt$. By the assumptions of H, we know that $\Phi, I \in C^1(W,\mathbb{R})$ and the derivatives are given by

$$\begin{gathered} I'(u)v = \int_{\mathbb{R}} \left(H_u(t,u),v\right)dt, \\ \Phi'(u)v = \langle u^+,v^+\rangle - \langle u^-,v^-\rangle - I'(u)v, \end{gathered} \tag{22}$$

for any $u,v \in W = W^- \oplus W^0 \oplus W^+$ with $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$. By the discussion of [25], the (weak) solutions of system (1) are the critical points of the C^1 functional $\Phi : W \to \mathbb{R}$. Moreover, it is easy to verify that if $u \not\equiv 0$ is a solution of (1), then $u(t) \to 0$ and $\dot u(t) \to 0$ as $|t| \to \infty$ (see Lemma 3.1 in [26]).

Let W be a Banach space with the norm $\|\cdot\|$ and $W := \overline{\bigoplus_{m\in\mathbb{N}} X_m}$ with $\dim X_m < \infty$ for any $m \in \mathbb{N}$. Set

$$Y_k := \bigoplus_{m=1}^{k} X_m, \qquad Z_k := \overline{\bigoplus_{m=k}^{\infty} X_m}. \tag{23}$$

Consider the following C^1-functional $\Phi_\lambda : W \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1,2]. \tag{24}$$

To continue the discussion, we give the following variant fountain theorem.

Lemma 5 (see [27]). *Assume that the functional Φ_λ defined previously satisfies*

(T_1) *Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$, and*

$$\Phi_\lambda(-u) = \Phi_\lambda(u) \quad \forall(\lambda,u) \in [1,2]\times W; \tag{25}$$

(T_2) *$B(u) \ge 0$ for all $u \in W$ and $B(u) \to +\infty$ as $\|u\| \to \infty$ on any finite-dimensional subspace of W;*

(T_3) *there exist $\rho_k > r_k > 0$ such that*

$$\begin{aligned} \alpha_k(\lambda) &:= \inf_{u\in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) \ge 0 > \beta_k(\lambda) \\ &:= \max_{u\in Y_k,\|u\|=r_k} \Phi_\lambda(u), \quad \forall\lambda \in [1,2], \end{aligned} \tag{26}$$

$$\begin{aligned} \xi_k(\lambda) &:= \inf_{u\in Z_k,\|u\|\le\rho_k} \Phi_\lambda(u) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \\ &\qquad \text{uniformly for } \lambda \in [1,2]. \end{aligned} \tag{27}$$

Then, there exist $0 < \lambda_j \to 1$ and $u_{\lambda_j} \in Y_j$ such that

$$\begin{gathered} \Phi'_{\lambda_j}|_{Y_j}\left(u_{\lambda_j}\right) = 0, \\ \Phi_{\lambda_j}\left(u_{\lambda_j}\right) \to \eta_k \in \left[\xi_k(2), \beta_k(1)\right] \quad \text{as } j \to \infty. \end{gathered} \tag{28}$$

Particularly, if $\{u_{\lambda_j}\}$ has a convergent subsequence for every k, then Φ_1 has infinitely many nontrivial critical points $\{u_k\} \subset W \setminus \{0\}$ satisfying $\Phi_1(u_k) \to 0^-$ as $k \to \infty$.

For $m \in \mathbb{N}$, let $X_m := \mathbb{R}e_m$ (the sequence $\{e_m\}$ is defined in Section 2 just below Lemma 4); then Z_k and Y_k can be defined as before. In order to apply the previously mentioned variant fountain theorem to prove our main result, we define the functionals A, B, and Φ_λ on W by

$$A(u) := \frac{1}{2}\|u^+\|^2, \qquad B(u) := \frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} H(t,u)\,dt,$$

$$\begin{aligned}\Phi_\lambda(u) &:= A(u) - \lambda B(u)\\ &= \frac{1}{2}\|u^+\|^2 - \lambda\left(\frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} H(t,u)\,dt\right)\end{aligned} \tag{29}$$

for all $u = u^0 + u^- + u^+ \in W = W^0 \oplus W^- \oplus W^+$ and $\lambda \in [1,2]$. Obviously, $\Phi_\lambda \in C^1(W,\mathbb{R})$ for all $\lambda \in [1,2]$.

Next, we will prove that conditions (T_2) and (T_3) of Lemma 5 hold, that is, the following two lemmas.

Lemma 6. *$B(u) \geq 0$ for all $u \in W$ and $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite-dimensional subspace of W.*

Proof. Obviously, condition (H_2) and the definition of B imply that $B(u) \geq 0$ for all $u \in W$. We claim that for any finite-dimensional subspace $X \subset W$, there exists a constant $\epsilon > 0$ such that

$$m(\{t \in \mathbb{R} : |u| \geq \epsilon\|u\|\}) \geq \epsilon, \qquad \forall u \in X \setminus \{0\}, \tag{30}$$

where $m(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}$. In fact, the detailed proof of (30) has been given by Chen (Lemma 2.3 in [22]).

For the ϵ given in (30), let

$$\Lambda_u := \{t \in \mathbb{R} : |u| \geq \epsilon\|u\|\}, \quad \forall u \in X \setminus \{0\}. \tag{31}$$

Then, by (30),

$$m(\Lambda_u) \geq \epsilon, \quad \forall u \in X \setminus \{0\}. \tag{32}$$

By (H_3), there exist constants $R_1, R_2 > 0$ such that

$$H(t,u) \geq R_1|u|^2, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N \quad \text{with } |u| \geq R_2. \tag{33}$$

The definition of Λ_u implies that for any $u \in X$ with $\|u\| \geq R_2/\epsilon$ there holds

$$|u| \geq R_2, \quad \forall t \in \Lambda_u. \tag{34}$$

Combining (H_2), (32)–(34), and the definition of Λ_u, for any $u \in X$ with $\|u\| \geq R_2/\epsilon$, we have

$$\begin{aligned}B(u) &= \frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} H(t,u)\,dt\\ &\geq \int_{\Lambda_u} H(t,u)\,dt\\ &\geq \int_{\Lambda_u} R_1|u|^2 dt\\ &\geq R_1\epsilon^2\|u\|^2 \cdot m(\Lambda_u)\\ &\geq R_1\epsilon^3\|u\|^2.\end{aligned} \tag{35}$$

It implies that $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite-dimensional subspace $X \subset W$. The proof is finished. □

Lemma 7. *There exist a positive integer l_0 and two sequences $0 < r_k < \rho_k \to 0$ as $k \to \infty$ such that*

$$\alpha_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) > 0, \quad \forall k \geq l_0, \tag{36}$$

$$\xi_k(\lambda) := \inf_{u\in Z_k, \|u\|\leq\rho_k} \Phi_\lambda(u) \longrightarrow 0 \quad \text{as } k \to \infty \quad \text{uniformly for } \lambda \in [1,2], \tag{37}$$

$$\beta_k(\lambda) := \max_{u\in Y_k, \|u\|=r_k} \Phi_\lambda(u) < 0, \quad \forall k \in \mathbb{N}, \tag{38}$$

where $Y_k = \bigoplus_{m=1}^{k} X_m$ and $Z_k = \overline{\bigoplus_{m=k}^{\infty} X_m}$ for all $k \in \mathbb{N}$.

Proof. (a) First, we show that (36) holds. Note that $Z_k \subset W^+$ for all $k \geq k_2 + 1$, where k_2 is the integer defined in (17) just below Lemma 4. By Lemma 4, there is a constant $\varepsilon_0 > 0$ such that $\|u\|_\infty \leq \varepsilon_0\|u\|$ for any $u \in W$. It follows that for any $u \in W$ with $\|u\| \leq c_2/\varepsilon_0$ there holds

$$|u| \leq \|u\|_\infty \leq c_2, \tag{39}$$

where c_2 is the constant in (H_1). It follows from (H_1) and the definition of Φ_λ that for any $k \geq k_2 + 1$ and $u \in Z_k$ with $\|u\| \leq c_2/\varepsilon_0$ there holds

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - 2\int_{\mathbb{R}} H(t,u)\,dt\\ &\geq \frac{1}{2}\|u\|^2 - 2c_1\|u\|_1, \quad \forall \lambda \in [1,2].\end{aligned} \tag{40}$$

Let

$$l_k := \sup_{u\in Z_k\setminus\{0\}} \frac{\|u\|_1}{\|u\|}, \quad \forall k \in \mathbb{N}. \tag{41}$$

Then

$$l_k \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \tag{42}$$

by Lemma 4 and the Rellich embedding theorem (see [28]). Consequently, (40) and (41) imply that

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u\|^2 - 2c_1 l_k\|u\| \tag{43}$$

for any $k \geq k_2 + 1$ and $u \in Z_k$ with $\|u\| \leq c_2/\varepsilon_0$. For any $k \in \mathbb{N}$, let

$$\rho_k := 8c_1 l_k. \tag{44}$$

Then, by (42), we have

$$0 < \rho_k \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{45}$$

Evidently, (45) implies that there exists a positive integer $l_0 > k_2 + 1$ such that

$$\rho_k \leq \frac{c_2}{\varepsilon_0}, \quad \forall k \geq l_0. \tag{46}$$

(43) together with (44) and (46) implies that

$$\alpha_k(\lambda) := \inf_{u\in Z_k,\|u\|=\rho_k} \Phi_\lambda(u) \geq \frac{\rho_k^2}{2} - \frac{\rho_k^2}{4} = \frac{\rho_k^2}{4} > 0, \quad \forall k \geq l_0. \tag{47}$$

That is, (36) holds.

(b) Second, we show that (37) holds. By (43), for any $k \geq l_0$ and $u \in Z_k$ with $\|u\| \leq \rho_k$, we have

$$\Phi_\lambda(u) \geq -2c_1 l_k \rho_k. \tag{48}$$

Observing that $\Phi_\lambda(0) = 0$ by (H_1), thus

$$0 \geq \inf_{u\in Z_k,\|u\|\leq\rho_k} \Phi_\lambda(u) \geq -2c_1 l_k \rho_k, \quad \forall k \geq l_0, \tag{49}$$

which together with (42) and (45) implies that

$$\xi_k(\lambda) := \inf_{u\in Z_k,\|u\|\leq\rho_k} \Phi_\lambda(u) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \quad \text{uniformly for } \lambda \in [1,2]. \tag{50}$$

That is, (37) holds.

(c) Last, we show that (38) holds. For any $k \in \mathbb{N}$ and $u \in Y_k$ with $\|u\| \leq c_2/\varepsilon_0$ (ε_0 is the constant above (39)), similar to (39), we have

$$|u| \leq c_2. \tag{51}$$

Therefore, by (51) and (H_1), for any $k \in \mathbb{N}$ and $u \in Y_k$ with $\|u\| \leq c_2/\varepsilon_0$, we have

$$\begin{aligned}\Phi_\lambda(u) &\leq \frac{1}{2}\|u^+\|^2 - \int_{\mathbb{R}} H(t,u)\,dt \\ &\leq \frac{1}{2}\|u\|^2 - c_3\|u\|_\mu^\mu \\ &\leq \frac{1}{2}\|u\|^2 - C_k\|u\|^\mu, \quad \forall \lambda \in [1,2],\end{aligned} \tag{52}$$

where the last inequality follows by the equivalence of the two norms $\|\cdot\|_\mu$ and $\|\cdot\|$ on finite dimensional space Y_k, and $C_k > 0$ is a constant depending on Y_k. For any $k \in \mathbb{N}$, if we choose

$$0 < r_k < \min\left\{\rho_k, C_k^{1/(2-\mu)}, \frac{c_2}{\varepsilon_0}\right\}, \tag{53}$$

Then, by (52), direct computation shows that

$$\beta_k(\lambda) := \max_{u\in Y_k,\|u\|=r_k} \Phi_\lambda(u) \leq -\frac{r_k^2}{2} < 0, \quad \forall k \in \mathbb{N}. \tag{54}$$

That is, (38) holds.

Therefore, the proof is finished by (a), (b), and (c). □

Proof of Theorem 2. By the assumptions of H and the definition of Φ_λ, we easily get that Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Note that $H(t,-u) = H(t,u)$, so we have $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda,u) \in [1,2]\times W$. Thus, the condition (T_1) of Lemma 5 holds. Lemma 6 shows that the condition (T_2) of Lemma 5 holds. Lemma 7 implies that the condition (T_3) of Lemma 5 holds for all $k \geq l_0$, where l_0 is given in Lemma 7. Therefore, by Lemma 5, for each $k \geq l_0$, there exist $0 < \lambda_j \to 1$, $u_{\lambda_j} \in Y_j$ such that

$$\begin{gathered}\Phi'_{\lambda_j}|_{Y_j}\left(u_{\lambda_j}\right) = 0, \\ \Phi_{\lambda_j}\left(u_{\lambda_j}\right) \longrightarrow \eta_k \in [\xi_k(2), \beta_k(1)] \quad \text{as } j \longrightarrow \infty.\end{gathered} \tag{55}$$

Next, we only need to prove the following two claims to complete the proof of Theorem 2. □

Claim 1. $\{u_{\lambda_j}\}$ is bounded in W.

Proof of Claim 1. By (55), we have

$$\frac{(1/2)\,\Phi'{}_{\lambda_j}|_{Y_j}\left(u_{\lambda_j}\right)u_{\lambda_j} - \Phi_{\lambda_j}\left(u_{\lambda_j}\right)}{\lambda_j} \leq C_1 \tag{56}$$

for some constant $C_1 > 0$. It follows from the definitions of Φ_{λ_j} and $\widetilde{H}$ that

$$\int_{\mathbb{R}} \widetilde{H}\left(t,u_{\lambda_j}\right)dt = \frac{(1/2)\,\Phi'_{\lambda_j}|_{Y_j}\left(u_{\lambda_j}\right)u_{\lambda_j} - \Phi_{\lambda_j}\left(u_{\lambda_j}\right)}{\lambda_j} \leq C_1. \tag{57}$$

Since (H_4) implies $\int_{\mathbb{R}} \widetilde{H}(t,u)dt \to +\infty$ as $|u| \to +\infty$, it follows from (57) that

$$\left|u_{\lambda_j}\right| \leq C_2, \quad \forall j \in \mathbb{N} \tag{58}$$

for some constant $C_2 > 0$. Note that $H(t,u) \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$; it follows from (H_4) that there is a constant $C_3 > 0$ such that

$$\left|H_u\left(t,u_{\lambda_j}\right)\right|^{\mu/(\mu-1)} \leq C_3\widetilde{H}\left(t,u_{\lambda_j}\right), \quad t \in \mathbb{R},\ \left|u_{\lambda_j}\right| \leq C_2. \tag{59}$$

Thus, by (57)–(59), $\Phi'_{\lambda_j}|_{Y_j}(u_{\lambda_j})u^+_{\lambda_j} = 0$, Hölder's inequality, and Lemma 4,

$$\begin{aligned}\left\|u^+_{\lambda_j}\right\|^2 &= \lambda_j \int_{\mathbb{R}} \left(H_u\left(t,u_{\lambda_j}\right), u^+_{\lambda_j}\right)dt \\ &\leq \lambda_j\Big(\int_{\mathbb{R}} \left|H_u\big(t,u_{\lambda_j}\big)\right|^{\mu/(\mu-1)}dt\Big)^{(\mu-1)/\mu}\Big(\int_{\mathbb{R}}\left|u^+_{\lambda_j}\right|^\mu dt\Big)^{1/\mu} \\ &\leq C_4\Big(\int_{\mathbb{R}} C_3\widetilde{H}\left(t,u_{\lambda_j}\right)dt\Big)^{(\mu-1)/\mu}\left\|u^+_{\lambda_j}\right\| \\ &\leq C_5\left\|u^+_{\lambda_j}\right\|\end{aligned} \tag{60}$$

for some positive constant C_4 and C_5. It implies that $\|u^+_{\lambda_j}\| \le C_5$. On the other hand, (H_2) and $\Phi'_{\lambda_j}|_{Y_j}(u_{\lambda_j})u_{\lambda_j} = 0$ imply that

$$\left\|u^+_{\lambda_j}\right\|^2 - \lambda_j\left\|u^-_{\lambda_j}\right\|^2 = \lambda_j \int_{\mathbb{R}} \left(H_u\left(t, u_{\lambda_j}\right), u_{\lambda_j}\right) dt \ge 0; \tag{61}$$

that is,

$$\lambda_j\left\|u^-_{\lambda_j}\right\|^2 \le \left\|u^+_{\lambda_j}\right\|^2. \tag{62}$$

It follows from $\left\|u^+_{\lambda_j}\right\| \le C_5$ that $\{u_{\lambda_j}\}$ is bounded in W. Therefore, Claim 1 is true. □

Claim 2. $\{u_{\lambda_j}\}$ has a strongly convergent subsequence in W.

Proof of Claim 2. Note that $\dim(W^0 \oplus W^-) < \infty$. By Claim 1, without loss of generality, we may assume that

$$u^-_{\lambda_j} \longrightarrow u^-, \qquad u^0_{\lambda_j} \longrightarrow u^0, \qquad u^+_{\lambda_j} \rightharpoonup u^+, \qquad u_{\lambda_j} \rightharpoonup u \quad \text{as } j \longrightarrow \infty \tag{63}$$

for some $u = u^0 + u^- + u^+ \in W = W^0 \oplus W^- \oplus W^+$. By virtue of the Riesz Representation Theorem, $\Phi'_{\lambda_j}|_{Y_j} : Y_j \to Y_j^*$ and $I' : W \to W^*$ can be viewed as $\Phi'_{\lambda_j}|_{Y_j} : Y_j \to Y_j$ and $I' : W \to W$, respectively, where Y_j^* and W^* are the dual spaces of Y_j and W, respectively. Note that

$$0 = \Phi'_{\lambda_j}|_{Y_j}\left(u_{\lambda_j}\right) = u^+_{\lambda_j} - \lambda_j\left[u^-_{\lambda_j} + \chi_j I'\left(u_{\lambda_j}\right)\right], \quad \forall j \in \mathbb{N}, \tag{64}$$

where $\chi_j : W \to Y_j$ is the orthogonal projection for all $j \in \mathbb{N}$; that is,

$$u^+_{\lambda_j} = \lambda_j\left[u^-_{\lambda_j} + \chi_j I'\left(u_{\lambda_j}\right)\right], \quad \forall j \in \mathbb{N}. \tag{65}$$

By the assumptions of H and the standard argument (see [29, 30]), we know $I' : W \to W^*$ is compact. Therefore, $I' : W \to W$ is also compact. Due to the compactness of I' and (63), the right-hand side of (65) converges strongly in W and hence $u^+_{\lambda_j} \to u^+$ in W. Combining this with (63), we have

$$u_{\lambda_j} \longrightarrow u \text{ in } W, \quad j \longrightarrow \infty. \tag{66}$$

Therefore, Claim 2 is true.

Now, from the last assertion of Lemma 5, we know that $\Phi = \Phi_1$ has infinitely many nontrivial critical points. Therefore, (1) possesses infinitely many nontrivial homoclinic orbits. □

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