Research Article

Normal Families of Meromorphic Functions Concerning Higher Derivative and Shared Values

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We study the normal families related to a Hayman conjecture of higher derivative and concerning shared values and get two normal criteria. Our results improve the related theorems which were obtained independently, respectively by Fang and Yuan (2001), Yuan et al. ((2011) and (2012)), Wang et al. (2011), and Qiu et al. (2012). Meanwhile, some examples are given to show the sharpness of our results.

1. Introduction and Main Results

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in a domain $D \subseteq \mathbb{C}$, and let a be a finite complex value. We say that f and g share a CM (or IM) in D provided that $f - a$ and $g - a$ have the same zeros counting (or ignoring) multiplicity in D. When $a = \infty$, the zeros of $f - a$ mean the poles of f (see [1]). It is assumed that the reader is familiar with the standard notations and the basic results of Nevanlinna's value-distribution theory ([1] or [2–4]).

It is very interesting to find normality criteria from the point of view of shared values. In this area, Schwick [5] first proved an interesting result that a family of meromorphic functions in a domain is normal in which every function shares three distinct finite complex numbers with their first derivative. And later, more results about normality criteria concerning shared values have emerged; for instance, see [6–8]. In recent years, this subject has attracted the attention of many researchers worldwide.

We now first introduce a normality criterion related to a Hayman normal conjecture [9].

Theorem 1. *Let $\mathscr{F}$ be a family of holomorphic (meromorphic) functions defined in a domain, $n \in \mathbb{N}$, $a \neq 0$, and $b \in \mathbb{C}$. If $f'(z) + af^n(z) - b$ does not vanish in D for each function $f(z) \in \mathscr{F}$ and $n \geq 2$ ($n \geq 3$), then $\mathscr{F}$ is normal in D.*

The results for the holomorphic case are due to Drasin [10] for $n \geq 3$, Pang [11] for $n = 3$, Chen and Fang [12] for $n = 2$, Ye [13] for $n = 2$, and Chen and Gu [14] for the generalized result with a and b replaced by meromorphic functions. The results for the meromorphic case are due to Li [15], Li [16], and Langley [17] for $n \geq 5$, Pang [11] for $n = 4$, Chen and Fang [12] for $n = 3$, and Zalcman [18] for $n = 3$, obtained independently.

When $n = 2$ and $\mathscr{F}$ is meromorphic, Theorem 1 is not valid in general. Fang and Yuan [19] gave an example to show this and got a special result below.

Example 2. The family of meromorphic functions $\mathscr{F} = \{f_j(z) = jz/(\sqrt{j}z - 1)^2 : j = 1, 2, \ldots, \}$ is not normal in $D = \{z : |z| < 1\}$. This is deduced by $f_j^{\#}(0) = j \to \infty$, as $j \to \infty$ and Marty's criterion [2], although for any $f_j(z) \in \mathscr{F}$, $f_j' + f_j^2 = j(\sqrt{j}z - 1)^{-4} \neq 0$.

Here $f^{\#}(\xi)$ denotes the spherical derivative

$$f^{\#}(\xi) = \frac{\left| f'(\xi) \right|}{1 + \left| f(\xi) \right|^2}. \tag{1}$$

Theorem 3. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, and $a \neq 0$, $b \in \mathbb{C}$. If $f'(z) + af^2(z) - b$ does not*

vanish in D and the poles of $f(z)$ are of multiplicity ≥ 3 for each $f(z) \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.

In 2008, by the ideas of shared values, Zhang [8] proved the following.

Theorem 4. *Let $\mathcal{F}$ be a family of meromorphic (holomorphic) functions in D, let n be a positive integer and let a, b be two finite complex numbers such that $a \neq 0$. If $n \geq 4$ ($n \geq 2$) and for every pair of functions f and g in $\mathcal{F}$, $f' - af^n$ and $g' - ag^n$ share the value b IM, then $\mathcal{F}$ is normal in D.*

Example 5 (see [8]). The family of meromorphic functions $\mathcal{F} = \{f_j(z) = 1/\sqrt{j}(z - 1/j) : j = 1, 2, \ldots, \}$ is not normal in $D = \{z : |z| < 1\}$. Obviously $f_j' - f_j^3 = -z/\sqrt{j}(z - 1/j)^3$. So for each pair of m, j, $f_j' - f_j^3$ and $f_m' - f_m^3$ share the value 0 in D, but $\mathcal{F}$ is not normal at the point $z = 0$, since $f_j^\#(0) = 2(\sqrt{j})^3/(1 + j) \to \infty$, as $j \to \infty$.

Remark 6. Example 5 shows that Theorem 4 is not valid when $n = 3$, and the condition $n = 4$ is best possible for meromorphic case.

In 2011, Yuan et al. [20] and Wang et al. [21] proved the following theorems, independently, respectively.

Theorem 7 (see [20, 21]). *Let $\mathcal{F}$ be a family of meromorphic functions in D and a, b two finite complex numbers such that $a \neq 0$. Suppose that each $f \in \mathcal{F}$ has no simple pole. If $f' - af^3$ and $g' - ag^3$ share the value b IM for every pair of functions f and g in $\mathcal{F}$, then $\mathcal{F}$ is normal in D.*

Theorem 8 (see [20]). *Let $\mathcal{F}$ be a family of meromorphic functions in D and a and b two finite complex numbers such that $a \neq 0$. Suppose that each $f \in \mathcal{F}$ admits zeros of multiple and the poles of multiplicity at least 3. If $f' - af^2$ and $g' - ag^2$ share the value b IM for every pair of functions f and g in $\mathcal{F}$, then $\mathcal{F}$ is normal in D.*

Lately, Yuan et al. [22] and Qiu et al. [23] studied this result, independently, respectively, in which the derivative f' was replaced by kth derivative $f^{(k)}$, and they got the following results.

Theorem 9 (see [22]). *Let $\mathcal{F}$ be a family of meromorphic functions in D, and let $k(\geq 2)$ and $n(\geq k + 2)$ be two positive integers. Let $a(\neq 0)$ and b be two finite complex numbers. If*

(i) *$f^{(k)} - af^n$ and $g^{(k)} - ag^n$ share b IM in D for every pair of functions f and g in $\mathcal{F}$,*

(ii) *f has no simple pole and no zero of multiplicity less than k in D for every function $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

Theorem 10 (see [22, 23]). *Let $\mathcal{F}$ be a family of meromorphic functions in D, $k(\geq 2)$ and $n(\geq k + 3)$ be two positive integers. Let $a(\neq 0)$ and b be two finite complex numbers. If*

(i) *$f^{(k)} - af^n$ and $g^{(k)} - ag^n$ share b IM in D for every pair of functions f and g in $\mathcal{F}$,*

(ii) *f has no zero of multiplicity less than k in D for every function $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

It is natural to ask whether the condition $n \geq k + 2$ or $n \geq k + 3$ in the previous theorems can be reduced. In this paper, we study this problem and get the following results.

Theorem 11 (main theorem). *Let D be a domain in $\mathbb{C}$ and let $\mathcal{F}$ be a family of meromorphic functions in D. Let $k, n, d \in \mathbb{N}^+, n \geq 3, d \geq (k + 1)/(n - 2)$ and let a, b be two finite complex numbers with $a \neq 0$. Suppose that every $f \in \mathcal{F}$ has all its zeros of multiplicity at least k and all its poles of multiplicity at least d. If $f^{(k)} - af^n$ and $g^{(k)} - ag^n$ share the value b IM for every pair of functions (f, g) of $\mathcal{F}$, then $\mathcal{F}$ is a normal family in D.*

Remark 12. When $n \geq k + 2$ or $n \geq k + 3$, we have $d = 1$ or 2, respectively. It follows that Theorem 11 generalizes Theorems 4, 7, 9, and 10.

Theorem 13 (main theorem). *Let D be a domain in $\mathbb{C}$ and let $\mathcal{F}$ be a family of meromorphic functions in D. Let $k \in \mathbb{N}^+$ and a, b be two finite complex numbers with $a \neq 0$. Suppose that every $f \in \mathcal{F}$ has all its zeros of multiplicity at least $k + 1$ and all its poles of multiplicity at least $k + 2$. If $f^{(k)} - af^2$ and $g^{(k)} - ag^2$ share the value b IM for every pair of functions (f, g) of $\mathcal{F}$, then $\mathcal{F}$ is a normal family in D.*

Remark 14. When $k = 1$, it follows that Theorem 13 generalizes Theorem 8.

Example 15 (see [23]). Let $n, k \geq 2$ be two positive integers and a a nonzero complex constant. The family of meromorphic functions is $\mathcal{F} = \{f_j(z) = jz^{k-1} : j = 1, 2, \ldots, \}$, $D = \{z : |z| < 1\}$. Obviously, for each pair of m, j, $f_j^{(k)} - af_j^n$ and $f_m^{(k)} - af_m^n$ share the value 0 in D, but $\mathcal{F}$ is not normal.

Example 16. Let k be a positive integer and a, b two nonzero complex constants such that $b_j = a/(-1)^k((2k)!/k!)j^k$. The family of meromorphic functions is $\mathcal{F} = \{f_j(z) = 1/(jz - b_j)^{k+1} : j = 1, 2, \ldots\}$, $D = \{z : |z| < 1\}$. Obviously, for each pair of m, j, $f_j^{(k)} - af_j^2$ and $f_m^{(k)} - af_m^2$ share the value 0 in D, but $\mathcal{F}$ is normal since $f^\#(0) \to \infty$ as $j \to \infty$.

Remark 17. Example 15 shows that the condition that $f(z)$ admits zeros of multiplicity at least k is best in Theorem 11. For the case $k = 1, n = 3$, Example 5 shows that the condition that $f(z)$ admits poles of multiplicity at least d is sharp in Theorem 11. For the case $n = 2$, Example 16 shows that the condition that $f(z)$ admits poles of multiplicity at least $k + 2$ is sharp in Theorem 13. For the case $k = 1, n = 2$, Example 2 shows that the condition that $f(z)$ admits zeros of multiplicity at least $k + 1$ in Theorem 13 is sharp.

2. Preliminary Lemmas

In order to prove our results, we need the following lemmas. The first is the extended version Zalcman's [24] concerning normal families.

Lemma 1 (see [25]). *Let $\mathscr{F}$ be a family of meromorphic functions on the unit disc satisfying all zeros of functions in $\mathscr{F}$ having multiplicity $\geq p$ and all poles of functions in $\mathscr{F}$ having multiplicity $\geq q$. Let α be a real number satisfying $-q < \alpha < p$. Then $\mathscr{F}$ is not normal at 0 if and only if there exist*

(a) *a number $0 < r < 1$;*
(b) *points z_n with $|z_n| < r$;*
(c) *functions $f_n \in \mathscr{F}$;*
(d) *positive numbers $\rho_n \to 0$*

such that $g_n(\zeta) := \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta)$ converges spherically uniformly on each compact subset of $\mathbb{C}$ to a nonconstant meromorphic function $g(\zeta)$, whose all zeros have multiplicity $\geq p$ and all poles have multiplicity $\geq q$ and order is at most 2.

Lemma 2. *Let $f(z)$ be a meromorphic function such that $f^{(k)}(z) \not\equiv 0$ and $c \in \mathbb{C} \setminus \{0\}$, $k, n, d \in \mathbb{N}^+$ with $n \geq 3$, $d \geq (k+1)/(n-2)$. If all zeros of f are of multiplicity at least k and all poles of f are of multiplicity at least d, then*

$$T(r, f) \leq \frac{1}{k} N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - cf^n}\right) + S(r, f), \quad (2)$$

where $S(r, f) = o(T(r, f))$, as $r \to \infty$, possibly outside a set with finite linear measure.

Proof. Set

$$\Phi(z) := \frac{f^{(k)}(z)}{cf^n(z)}. \quad (3)$$

Since $f^{(k)}(z) \not\equiv 0$, we have $\Phi(z) \not\equiv 0$. Thus,

$$f^n(z) = \frac{f^{(k)}(z)}{c\Phi(z)}. \quad (4)$$

Hence,

$$nm(r, f) = m(r, f^n) \leq m\left(r, \frac{f^{(k)}}{\Phi}\right) + \log^+ \frac{1}{|c|}$$
$$\leq m\left(r, \frac{1}{\Phi}\right) + m\left(r, f^{(k)}\right) + \log^+ \frac{1}{|c|}$$
$$\leq m\left(r, \frac{1}{\Phi}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + m(r, f) + \log^+ \frac{1}{|c|}. \quad (5)$$

So that

$$(n-1)m(r, f) \leq m\left(r, \frac{1}{\Phi}\right) + m\left(r, \frac{f^{(k)}}{f}\right) + \log^+ \frac{1}{|c|}. \quad (6)$$

On the other hand, (4) gives

$$nN(r, f) \leq N(r, f^n) = N\left(r, \frac{f^{(k)}}{\Phi}\right)$$
$$\leq N\left(r, f^{(k)}\right) + N\left(r, \frac{1}{\Phi}\right) - \overline{N}\left(r, \Phi = f^{(k)} = 0\right), \quad (7)$$

where $\overline{N}(r, \Phi = f^{(k)} = 0)$ denotes the counting function of zeros of both Φ and $f^{(k)}$.

We obtain

$$nN(r, f) \leq N(r, f) + k\overline{N}(r, f)$$
$$+ N\left(r, \frac{1}{\Phi}\right) - \overline{N}\left(r, \Phi = f^{(k)} = 0\right),$$

$$(n-1)N(r, f) \leq k\overline{N}(r, f)$$
$$+ N\left(r, \frac{1}{\Phi}\right) - \overline{N}\left(r, \Phi = f^{(k)} = 0\right). \quad (8)$$

By (4), we have

$$\overline{N}(r, \varphi) + \overline{N}\left(r, \frac{1}{\varphi}\right)$$
$$\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + \overline{N}\left(r, \Phi = f^{(k)} = 0\right). \quad (9)$$

From (6)~(9), we obtain

$$(n-1)T(r, f)$$
$$\leq k\overline{N}(r, f) + T\left(r, \frac{1}{\Phi}\right) - \overline{N}\left(r, \Phi = f^{(k)} = 0\right) + S(r, f)$$
$$\leq k\overline{N}(r, f) + T(r, \Phi) - \overline{N}\left(r, \Phi = f^{(k)} = 0\right) + S(r, f)$$
$$\leq k\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{\Phi}\right) + \overline{N}(r, \Phi) + \overline{N}\left(r, \frac{1}{\Phi - 1}\right)$$
$$- \overline{N}\left(r, \Phi = f^{(k)} = 0\right) + S(r, f)$$
$$\leq (k+1)\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right)$$
$$+ \overline{N}\left(r, \frac{1}{f^{(k)} - cf^n}\right) + S(r, f). \quad (10)$$

Since all zeros and poles of f are multiplicities at least k and d, we get

$$\overline{N}(r, f) \leq \frac{1}{d} N(r, f) \leq \frac{1}{d} T(r, f) \leq \frac{n-2}{k+1} T(r, f),$$
$$\overline{N}\left(r, \frac{1}{f}\right) \leq \frac{1}{k} N\left(r, \frac{1}{f}\right). \quad (11)$$

So that

$$T(r, f) \leq \frac{1}{k} N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - cf^n}\right) + S(r, f). \quad (12)$$

□

Lemma 3. *Let $f(z)$ be a transcendental meromorphic function such that $f^{(k)}(z) \not\equiv 0$. Let $k \in \mathbb{N}^+$ and $c \in \mathbb{C} \setminus \{0\}$. If all zeros of f are of multiplicity at least $k + 1$ and all poles of f are of multiplicity at least $k + 2$, then $f^{(k)} - cf^2$ has infinitely many zeros.*

Proof. Suppose that $f^{(k)} - cf^2$ has only finitely many zeros; then $N(r, 1/(f^{(k)} - cf^2)) = S(r, f)$. Clearly, an arbitrary zero of f is a zero of $f^{(k)} - cf^2$ since all zeros of f are of multiplicity at least $k + 1$; then we can deduce that f has only finite zeros, so $N(r, 1/f) = O(\log r) = S(r, f)$.

Set

$$\Phi(z) := \frac{f^{(k)}(z)}{cf^2(z)}. \tag{13}$$

Similarly, with the proof of Lemma 2, we can get

$$T(r, f) \leq (k+1)\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) \\ + \overline{N}\left(r, \frac{1}{f^{(k)} - cf^2}\right) + S(r, f). \tag{14}$$

Since all poles of f are multiplicities at least $k + 2$, we obtain

$$\overline{N}(r, f) \leq \frac{1}{k+2} N(r, f) \leq \frac{1}{k+2} T(r, f), \tag{15}$$

so that

$$T(r, f) \leq (k+2)\overline{N}\left(r, \frac{1}{f}\right) \\ + (k+2)\overline{N}\left(r, \frac{1}{f^{(k)} - cf^2}\right) + S(r, f) = S(r, f). \tag{16}$$

This is contradicting with the fact that f is transcendental.

Hence, Lemma 3 is proved completely. $\qquad\square$

Lemma 4. *Let $f(z)$ be a nonconstant rational function such that $f^{(k)}(z) \not\equiv 0$. Let $a \in \mathbb{C} \setminus \{0\}$, and $k, n, m \in \mathbb{N}^+$ with $n \geq 2$ and $m \geq (k+2)/(n-1)$. If $f \neq 0$ and all poles of f are of multiplicity at least m, then $f^{(k)} - af^n$ has at least two zeros.*

Proof. Suppose, to the contrary, that $f^{(k)} - af^n$ has at most one zero.

Since $f \neq 0$, we get that f is a rational but not a polynomial.

Case 1. If $f^{(k)} - af^n$ has only zero z_0 with multiplicity l, set

$$f(z) = \frac{A}{(z - z_1)^{\beta_1}(z - z_2)^{\beta_2} \cdots (z - z_t)^{\beta_t}}, \tag{17}$$

where A is a nonzero constant and $\beta_i \geq (k+2)/(n-1)$ $(i = 1, 2, \ldots, t)$.

For the sake of simplicity, we denote

$$\beta_1 + \beta_2 + \cdots + \beta_t = q. \tag{18}$$

From (17), we have

$$f^{(k)} = \frac{g(z)}{(z - z_1)^{\beta_1 + k}(z - z_2)^{\beta_2 + k} \cdots (z - z_t)^{\beta_t + k}}, \tag{19}$$

where $g(z)$ is a polynomial such that $\deg(g(z)) \leq k(t-1)$.

From (17) and (19), we get

$$f^{(k)} - af^n$$

$$= \frac{g(z)}{(z - z_1)^{\beta_1 + k}(z - z_2)^{\beta_2 + k} \cdots (z - z_t)^{\beta_t + k}}$$

$$\quad - \frac{aA^n}{(z - z_1)^{n\beta_1}(z - z_2)^{n\beta_2} \cdots (z - z_t)^{n\beta_t}} \tag{20}$$

$$= \left[g(z)(z - z_1)^{(n-1)\beta_1 - k}(z - z_2)^{(n-1)\beta_2 - k} \right.$$

$$\left. \cdots (z - z_t)^{(n-1)\beta_t - k} - aA^n \right]$$

$$\times \left((z - z_1)^{n\beta_1}(z - z_2)^{n\beta_2} \cdots (z - z_t)^{n\beta_t} \right)^{-1}.$$

By the assumption that $f^{(k)} - af^n$ has exactly one zero z_0 with multiply l, we have

$$f^{(k)} - af^n = \frac{C(z - z_0)^l}{(z - z_1)^{n\beta_1}(z - z_2)^{n\beta_2} \cdots (z - z_t)^{n\beta_t}}, \tag{21}$$

where C is a nonzero constant. Thus,

$$C(z - z_0)^l \equiv g(z)(z - z_1)^{(n-1)\beta_1 - k} \cdots (z - z_t)^{(n-1)\beta_t - k} - aA^n. \tag{22}$$

Differentiating (22), we obtain

$$Cl(z - z_0)^{l-1}$$

$$\equiv (z - z_1)^{(n-1)\beta_1 - k - 1} \cdots (z - z_t)^{(n-1)\beta_t - k - 1}$$

$$\times \left[g'(z)(z - z_1) \cdots (z - z_t) \right.$$

$$\left. + g(z) \sum_{i=1}^{t} ((n-1)\beta_i - k) \prod_{j=1, j \neq i}^{t} (z - z_j) \right]. \tag{23}$$

For the sake of simplicity, we denote

$$g_1(z) = Cl(z - z_0)^{l-1},$$

$$g_2(z) = (z - z_1)^{(n-1)\beta_1 - k - 1} \cdots (z - z_t)^{(n-1)\beta_t - k - 1}$$

$$\times \left[g'(z)(z - z_1) \cdots (z - z_t) \right.$$

$$\left. + g(z) \sum_{i=1}^{t} ((n-1)\beta_i - k) \prod_{j=1, j \neq i}^{t} (z - z_j) \right]. \tag{24}$$

Hence,

$$g_1(z) \equiv g_2(z). \tag{25}$$

Since $(n-1)\beta_i - k - 1 \geq 1$, we have $g_2(z_i) = 0$. But $g_1(z_i) \neq 0$ $(i = 1, 2, \ldots, t)$, a contradiction.

Case 2. If $f^{(k)} - af^n$ has no zeros, then $l = 0$ for (21). We have

$$f^{(k)} - af^n = \frac{C}{(z-z_1)^{n\beta_1}(z-z_2)^{n\beta_2}\cdots(z-z_t)^{n\beta_t}}, \quad (26)$$

where C is a nonzero constant. Thus,

$$C \equiv g(z)(z-z_1)^{(n-1)\beta_1-k}\cdots(z-z_t)^{(n-1)\beta_t-k} - aA^n,$$
$$g(z)(z-z_1)^{(n-1)\beta_1-k}\cdots(z-z_t)^{(n-1)\beta_t-k} \equiv C + aA^n. \quad (27)$$

Obviously, $g(z)(z-z_1)^{(n-1)\beta_1-k}\cdots(z-z_t)^{(n-1)\beta_t-k}$ is not a constant, a contradiction.

Lemma 4 is proved. □

Lemma 5. *Let $f(z)$ be a rational function and $a \in \mathbb{C}\setminus\{0\}$, and $k, n, m \in \mathbb{N}^+$ with $n \geq 2$ and $m \geq (k+2)/(n-1)$. If all zeros of f are of multiplicity at least $k+1$ and all poles of f are of multiplicity at least m, then $f^{(k)} - af^n$ has at least two distinct zeros.*

Proof. Suppose, to the contrary, that $f^{(k)} - af^n$ has at most one zero.

Case I. When f is a nonconstant polynomial, noting that all zeros of f have multiplicity at least $k+1$, we know that $f^{(k)} - af^n$ must have zeros. We claim that f has exactly one zero. Otherwise, we can get that $f^{(k)} - af^n$ has at least two zeros, which contradicts our assumption.

Set

$$f(z) = B(z-z_0)^s, \quad (28)$$

where $s \geq k+1$, B is a nonzero constant. Then

$$f^{(k)}(z) - af^n(z)$$
$$= B(z-z_0)^{s-k}$$
$$\times \left[s(s-1)\cdots(s-k+1) - a(z-z_0)^{(n-1)s+k}\right]. \quad (29)$$

Since $s - k \geq 1$, we obtain that $s(s-1)\cdots(s-k+1) - a(z-z_0)^{s+k}$ has at least one zero which is not z_0 from (29). Therefore, $f^{(k)} - af^n$ has at least two distinct zeros, a contradiction.

Case II. When f is rational but not a polynomial, we consider two cases.

Case 1. Suppose that $f^{(k)} - af^n$ has only zero z_0 with multiplicity at least l. If $f \neq 0$, by Lemma 4, we get a contradiction. So f has zeros, and then we can deduce that z_0 is the only zero of f. Otherwise, $f^{(k)} - af^n$ has at least two distinct zeros, a contradiction.

We set

$$f(z) = \frac{A(z-z_0)^s}{(z-z_1)^{\beta_1}(z-z_2)^{\beta_2}\cdots(z-z_t)^{\beta_t}}, \quad (30)$$

where A is a nonzero constant and $s \geq k+1$, $\beta_i \geq m \geq ((k+2)/(n-1))$ $(i = 1, 2, \ldots, t)$.

For the sake of simplicity, we denote

$$\beta_1 + \beta_2 + \cdots + \beta_t = q. \quad (31)$$

From (31), we have

$$f^{(k)} = \frac{A(z-z_0)^{s-k}g(z)}{(z-z_1)^{\beta_1+k}(z-z_2)^{\beta_2+k}\cdots(z-z_t)^{\beta_t+k}}, \quad (32)$$

where $g(z)$ is a polynomial with $\deg(g) \leq k(s+t+1)$.

From (30) and (32), we get

$$f^{(k)} - af^n$$

$$= \frac{A(z-z_0)^{s-k}g(z)}{(z-z_1)^{\beta_1+k}(z-z_2)^{\beta_2+k}\cdots(z-z_t)^{\beta_t+k}}$$
$$- \frac{A^n(z-z_0)^{ns}}{(z-z_1)^{n\beta_1}(z-z_2)^{n\beta_2}\cdots(z-z_t)^{n\beta_t}}$$
$$= \Big(A(z-z_0)^{s-k} \quad (33)$$
$$\times \Big[g(z)(z-z_1)^{(n-1)\beta_1-k}(z-z_2)^{(n-1)\beta_2-k}$$
$$\cdots(z-z_t)^{(n-1)\beta_t-k} - aA^{n-1}$$
$$\times(z-z_0)^{(n-1)s+k}\Big]\Big)$$
$$\times \Big((z-z_1)^{n\beta_1}(z-z_2)^{n\beta_2}\cdots(z-z_t)^{n\beta_t}\Big)^{-1}.$$

By assumption that $f^{(k)} - af^n$ has exactly one zero z_0 with multiplicity l, we have

$$f^{(k)} - af^n = \frac{B(z-z_0)^l}{(z-z_1)^{n\beta_1}(z-z_2)^{n\beta_2}\cdots(z-z_t)^{n\beta_t}}, \quad (34)$$

where B is a nonzero constant. Thus,

$$B(z-z_0)^l \equiv A(z-z_0)^{s-k}$$
$$\times \Big[g(z)(z-z_1)^{(n-1)\beta_1-k}\cdots(z-z_t)^{(n-1)\beta_t-k}$$
$$-aA^{n-1}(z-z_0)^{(n-1)s+k}\Big]. \quad (35)$$

Case 1.1. If $l > s-k$, from (35), we can deduce that z_0 is a zero of $(z-z_1)^{(n-1)\beta_1-k}\cdots(z-z_t)^{(n-1)\beta_t-k}$, a contradiction.

Case 1.2. If $l = s-k$, from (35), it follows that

$$g(z)(z-z_1)^{(n-1)\beta_1-k}\cdots(z-z_t)^{(n-1)\beta_t-k}$$
$$-aA^{n-1}(z-z_0)^{(n-1)s+k} \equiv \frac{B}{A}. \quad (36)$$

Differentiating (36), we have

$$(z - z_1)^{(n-1)\beta_1 - k - 1} \cdots (z - z_t)^{(n-1)\beta_t - k - 1}$$

$$\times \left[g'(z)(z - z_1) \cdots (z - z_t) \right.$$

$$\left. + g(z) \sum_{i=1}^{t} ((n-1)\beta_i - k) \prod_{j=1, j \neq i}^{t} (z - z_j) \right]$$

$$\equiv a((n-1)s + k) A^{n-1} (z - z_0)^{(n-1)s + k - 1}.$$

(37)

For the sake of simplicity, we denote

$$g_1(z) = (z - z_1)^{(n-1)\beta_1 - k - 1} \cdots (z - z_t)^{(n-1)\beta_t - k - 1}$$

$$\times \left[g'(z)(z - z_1) \cdots (z - z_t) \right.$$

$$\left. + g(z) \sum_{i=1}^{t} ((n-1)\beta_i - k) \prod_{j=1, j \neq i}^{t} (z - z_j) \right],$$

$$g_2(z) = a((n-1)s + k) A^{n-1} (z - z_0)^{(n-1)s + k - 1}.$$

(38)

Thus,

$$g_1(z) \equiv g_2(z). \tag{39}$$

Since $(n-1)\beta_i - k - 1 \geq 1$, we get $g_1(z_i) = 0$, but $g_2(z_i) \neq 0$ $(i = 1, 2, \ldots, t)$, a contradiction.

Case 2. If $f^{(k)} - af^n$ has no zeros, then $l = 0$ for (34). Similarly, as the proof of Case 1, we also have a contradiction.

The proof is completed. □

3. Proofs of Theorems

Proof. In Theorem 11, suppose that $\mathcal{F}$ is not normal in D. Then there exists at least one point z_0 such that $\mathcal{F}$ is not normal at the point z_0. Without loss of generality, we assume that $z_0 = 0$. By Lemma 1, there exist points $z_j \to 0$, positive numbers $\rho_j \to 0$, and functions $f_j \in \mathcal{F}$ such that

$$g_j(\xi) = \rho_j^{k/(n-1)} f_j(z_j + \rho_j \xi) \Longrightarrow g(\xi) \tag{40}$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function in $\mathbb{C}$ and whose poles and zeros are of multiplicity at least d and k, respectively. Moreover, the order of g is at most 2.

From (40), we know that

$$g_j^{(k)}(\xi) = \rho_j^{nk/(n-1)} f_j^{(k)}(z_j + \rho_j \xi) \Longrightarrow g^{(k)}(\xi), \tag{41}$$

$$g_j^{(k)}(\xi) - ag_j^n(\xi) - \rho_j^{nk/(n-1)} b$$

$$= \rho_j^{nk/(n-1)} \left(f_j^{(k)}(z_j + \rho_j \xi) - af_j^n(z_j + \rho_j \xi) - b \right)$$

$$\Longrightarrow g^{(k)}(\xi) - ag^n(\xi)$$

(42)

also locally uniformly with respect to the spherical metric.

Case 1. If $k = 1$, by Theorems 4 and 7, the Theorem 11 assumes.

Case 2. If $k = 2, n \geq 4$, by Theorems 9 and 10, the Theorem 11 assumes.

Case 3. If $k = 2, n = 3$ or $k \geq 3$, $3 \leq n \leq k + 1$ (If $n \geq k + 2$, by Theorems 9 and 10, we can get Theorem 11).

If $g^{(k)}(\xi) - ag^n(\xi) \equiv 0$, since all poles of g have multiplicity at least d, we have

$$nT(r, g) = T(r, g^n) = T(r, g^{(k)}) + O(1)$$

$$= m(r, g^{(k)}) + N(r, g^{(k)}) + O(1)$$

$$\leq m(r, g) + N(r, g) + k\overline{N}(r, g) + S(r, g) \quad (43)$$

$$\leq T(r, g) + \frac{k(n-2)}{k+1} T(r, g) + S(r, g)$$

$$\leq (n-1) T(r, g) + S(r, g).$$

Therefore, $g(\xi)$ is a constant, a contradiction. So $g^{(k)}(\xi) - ag^n(\xi) \not\equiv 0$.

By Lemma 2, we have

$$T(r, g) \leq \frac{1}{k} N\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^{(k)} - ag^n}\right) + S(r, g)$$

$$\leq \frac{1}{k} T\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^{(k)} - ag^n}\right) + S(r, g).$$

(44)

Then

$$\left(1 - \frac{1}{k}\right) T(r, g) \leq \overline{N}\left(r, \frac{1}{g^{(k)} - ag^n}\right) + S(r, g), \tag{45}$$

$$T(r, g) \leq \left(1 + \frac{1}{k-1}\right) \overline{N}\left(r, \frac{1}{g^{(k)} - ag^n}\right) + S(r, g). \tag{46}$$

If $g^{(k)}(\xi) - ag^n(\xi) \neq 0$, then (46) gives that $g(\xi)$ is also a constant. Hence, $g^{(k)}(\xi) - ag^n(\xi)$ is a nonconstant meromorphic function and has at least one zero.

Next we prove that $g^{(k)}(\xi) - ag^n(\xi)$ has just a unique zero. On the contrary, let ξ_0 and ξ_0^* be two distinct zeros of $g^{(k)}(\xi) - ag^n(\xi)$, and choose $\delta(> 0)$ small enough such that $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \phi$, where $D(\xi_0, \delta) = \{\xi : |\xi - \xi_0| < \delta\}$ and $D(\xi_0^*, \delta) = \{\xi : |\xi - \xi_0^*| < \delta\}$. From (42), by Hurwitz's theorem, there exist points $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ such that for sufficiently large j

$$f_j^{(k)}(z_j + \rho_j \xi_j) - af_j^n(z_j + \rho_j \xi_j) - b = 0,$$

$$f_j^{(k)}(z_j + \rho_j \xi_j^*) - af_j^n(z_j + \rho_j \xi_j^*) - b = 0.$$

(47)

By the hypothesis that for each pair of functions f and g in $\mathcal{F}$, $f^{(k)} - af^n$ and $g^{(k)} - ag^n$ share b in D, we know that for any positive integer m

$$f_m^{(k)}(z_j + \rho_j \xi_j) - af_m^n(z_j + \rho_j \xi_j) - b = 0,$$

$$f_m^{(k)}(z_j + \rho_j \xi_j^*) - af_m^n(z_j + \rho_j \xi_j^*) - b = 0.$$

(48)

Fix m, take $j \to \infty$, and note $z_j + \rho_j \xi_j \to 0$, $z_j + \rho_j \xi_j^* \to 0$; then $f_m^{(k)}(0) - a f_m^n(0) - b = 0$. Since the zeros of $f_m^{(k)} - a f_m^n - b$ have no accumulation point, so

$$z_j + \rho_j \xi_j = 0, \qquad z_j + \rho_j \xi_j^* = 0. \tag{49}$$

Hence, $\xi_j = -z_j / \rho_j$, $\xi_j^* = -z_j / \rho_j$. This contradicts $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ and $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \phi$. So $g^{(k)}(\xi) - a g^n(\xi)$ has just a unique zero, which can be denoted by ξ_0.

Noting that g has poles and zeros of multiplicities at least d and k, respectively, (46) deduces that $g(\xi)$ is a rational function with degree at most 2.

If $g(\xi)$ is a polynomial, noting that $\deg g \le 2$ and the multiplicities of zeros are at least k, we have $k = 2$ and $n = 3$. Hence, there exist ξ_1 and $c \ne 0$ such that $g(\xi) = c(\xi - \xi_1)^2$, and then $g''(\xi) - a g^3(\xi) = c(2 - ac^2(\xi - \xi_1)^6)$ has 6 distinct zeros, a contradiction.

Suppose that $g(\xi)$ is not a polynomial, $k = 2$ and $n = 3$. Then the multiplicities of poles of $g(\xi)$ are at least $(k+1)/(n-2) = 3$, which implies that $\deg g \ge 3$, a contradiction.

Suppose that $g(\xi)$ is not a polynomial, $k \ge 3$ and $3 \le n \le k + 1$; we distinguish two cases.

Case i. If $g(\xi)$ has zeros, since all zeros of $g(\xi)$ have multiplicity at least $k(\ge 3)$, it follows that $\deg g \ge 3$, a contradiction.

Case ii. If $g(\xi) \ne 0$, then (46) should be as follows

$$T(r, g) \le \overline{N}\left(r, \frac{1}{g^{(k)} - a g^n}\right) + S(r, g). \tag{50}$$

From (50), we can see that $g(\xi)$ is a rational function with degree at most 1. Since all poles of $g(\xi)$ have multiplicity at least $d(\ge (k+1)/(n-2) \ge (k+1)/(k-1) > 1)$, which gives that $\deg g \ge 2$, a contradiction.

This completes the proof of Theorem 11. $\qquad \square$

Proof. In Theorem 13, suppose that $\mathscr{F}$ is not normal in D. Then there exists at least one point z_0 such that $\mathscr{F}$ is not normal at the point z_0. Without loss of generality, we assume that $z_0 = 0$. By Lemma 1, there exist points $z_j \to 0$, positive numbers $\rho_j \to 0$, and functions $f_j \in \mathscr{F}$ such that

$$g_j(\xi) = \rho_j^k f_j(z_j + \rho_j \xi) \Longrightarrow g(\xi) \tag{51}$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function in $\mathbb{C}$ and whose poles and zeros are of multiplicity at least $k + 2$ and $k + 1$, respectively. Moreover, the order of g is at most 2.

From (51), we know

$$g_j^{(k)}(\xi) = \rho_j^{2k} f_j^{(k)}(z_j + \rho_j \xi) \Longrightarrow g^{(k)}(\xi), \tag{52}$$

$$\begin{aligned} g_j^{(k)}(\xi) - a g_j^2(\xi) - \rho_j^{2k} b \\ = \rho_j^{2k}\left(f_j^{(k)}(z_j + \rho_j \xi) - a f_j^2(z_j + \rho_j \xi) - b\right) \\ \Longrightarrow g^{(k)}(\xi) - a g^2(\xi) \end{aligned} \tag{53}$$

also locally uniformly with respect to the spherical metric.

If $g^{(k)}(\xi) - a g^2(\xi) \equiv 0$, since all poles of f have multiplicity at least $k + 2$, we can deduce that $g(\xi)$ is an entire function easily. Thus,

$$\begin{aligned} 2T(r, g) &= T(r, g^2) = T(r, g^{(k)}) + O(1) \\ &= m(r, g^{(k)}) + N(r, g^{(k)}) + O(1) \\ &\le m(r, g) + N(r, g) + k\overline{N}(r, g) + S(r, g) \\ &\le T(r, g) + S(r, g). \end{aligned} \tag{54}$$

Therefore, $g(\xi)$ is a constant, a contradiction. So $g^{(k)}(\xi) - a g^2(\xi) \not\equiv 0$. By Lemmas 3, 4, and 5, $g^{(k)}(\xi) - a g^2(\xi)$ has at least two distinct zeros. Proceeding as in the later proof of Theorem 11, we will get a contradiction. The proof is completed. $\qquad \square$

Similarly, as the proof of Theorem 11, when f is a holomorphic function, we can get the following theorem which has been gotten in [23].

Theorem 18 (see [23]). *Let D be a domain in $\mathbb{C}$ and let $\mathscr{F}$ be a family of holomorphic functions in D. Let $k, n \in \mathbb{N}^+$, $n \ge 2$ and let a, b be two finite complex numbers with $a \ne 0$. Suppose that every $f \in \mathscr{F}$ has all its zeros of multiplicity at least k. If $f^{(k)} - a f^n$ and $g^{(k)} - a g^n$ share the value b IM for every pair of functions (f, g) of $\mathscr{F}$, then $\mathscr{F}$ is a normal family in D.*

Acknowledgments

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