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Research Article

Some Intersections of the Weighted L^p-Spaces

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Let G be a locally compact group Ω an arbitrary family of the weight functions on G, and $1 \leq p < \infty$. The locally convex space $IL_p(G, \Omega)$ as a subspace of $\cap_{\omega\in\Omega} L^p(G, \omega)$ is defined. Also, some sufficient conditions for that space to be a Banach space are provided. Furthermore, for an arbitrary subset J of $[1, \infty)$ and a positive submultiplicative weight function ω on G, Banach subspace $IL_J(G, \omega)$ of $\cap_{p\in J} L^p(G, \omega)$ is introduced. Then some algebraic properties of $IL_J(G, \omega)$, as a Banach algebra under convolution product, are investigated.

1. Introduction

Throughout the paper, let G be a locally compact group with a fixed left Haar measure λ or dx. We call any Borel measurable function $\omega : G \to [0, \infty)$ a weight function. For $1 \leq p \leq \infty$, the weighted L^p-space $L^p(G, \omega)$ with respect to λ is the set of all complex valued measurable functions f on G such that $f\omega \in L^p(G)$, the usual Lebesgue space as defined in [1]. This space will be denoted by $\ell^p(G, \omega)$, when G is discrete. Two functions in $L^p(G, \omega)$ are considered equal if they are equal λ-almost everywhere on G. If $1 \leq p < \infty$, then $L^p(G, \omega)$ is a locally convex space endowed with the topology generated by the seminorms $\rho_\omega : L^p(G, \omega) \to \mathbb{R}$ defined by

$$\rho_\omega(f) = \|f\|_{p,\omega} = \left(\int_G |f(x)|^p \omega(x)^p d\lambda(x)\right)^{1/p}. \quad (1)$$

For λ-measurable functions f and g on G, the convolution multiplication is defined by

$$(f * g)(x) = \int_G f(y)\, g\left(y^{-1}x\right) d\lambda(y) \quad (x \in G) \quad (2)$$

at each point $x \in G$ for which this makes sense. Then $f * g$ is said to exist if $(f * g)(x)$ exists for almost all $x \in G$. Several authors have studied the convolution properties on the space $L^p(G)$ and $L^p(G, \omega)$, where ω is positive and submultiplicative. It has been shown in [2–4] that the convolution of elements in $L^p(G)$ and also $L^p(G, \omega)$ does not exist in general. If this is the case, then it is desirable to study the closedness of these spaces under the convolution. For related results on the subject related to $L^p(G)$ see also [5]. Also we refer to [3, 6–12] for the more general case of weighted L^p-spaces. Besides these significant issues, some authors considered and investigated the intersection of the L^p-spaces to each other and also together with other Banach spaces; for example, see [13–15].

It should be noted that weighted L^p-spaces and its intersections have been studied more completely years ago, especially in the decade of 1970. We first refer to the Ph.D. thesis of Feichtinger titled by "subconvolutive functions" for a survey, which also contains many invaluable information related to the weight functions. Moreover, we found a lot of invaluable results related to weighted L^p-spaces and also weight functions in many earlier publications. We refer to some of them such as [16–21]. Note that [17] (downloadable as [fe74] from http://www.univie.ac.at/nuhag-php/bibtex/) is a technical report which contains many remarkable results related to the weight functions. Moreover, we found more complete results related to distributions and weighted spaces in [22]. In fact, some of our results in the present work, have been inspired by the results given in [22].

Also recently we considered an arbitrary intersection of the L^p-spaces denoted by $\cap_{p\in J} L^p(G)$, where $J \subseteq [1, \infty]$. Then we introduced the subspace $IL_J(G)$ of $\cap_{p\in J} L^p(G)$ as

$$IL_J(G) = \left\{ f \in \bigcap_{p\in J} L^p(G) : \|f\|_J = \sup_{p\in J} \|f\|_p < \infty \right\} \quad (3)$$

and studied $IL_J(G)$ as a Banach algebra under convolution product, for the case where $1 \in J$; see [23].

The purpose of the present work is to generalize the results of [23] to the weighted case. We first give general information about the weight functions and collect most of the available results in a more concise way. Then for an arbitrary family Ω of the weight functions on G and $1 \leq p < \infty$, we introduce the subspace $IL_p(G, \Omega)$ of the locally convex space $L^p(G, \Omega) = \cap_{\omega\in\Omega} L^p(G, \omega)$. Moreover, we provide some sufficient conditions on G and also Ω to construct a norm on $IL_p(G, \Omega)$. Particularly, we show the deficiency of this space in taking a norm in the general case, with presenting some fundamental examples. The third section is assigned to the Lorentz spaces, which are suggested to us by the referee. We first give some preliminaries related to Lorentz spaces $L_{p,q}(G)$. Then for the case where p is fixed and q runs through $J \subseteq (0, \infty)$, we introduce $IL_{p,J}(G)$ as a subspace of $\cap_{q\in J} L_{p,q}(G)$. As the main result, we prove that $IL_{p,J}(G) = L_{p,m_J}(G)$, where $m_J = \inf\{q : q \in J\}$ is positive.

Stimulated by these results, in the last two sections, we assume Ω consists just of one positive and submultiplicative weight function ω. Then we introduce the Banach space $IL_p(G, \omega)$ and also the space $IL_J(G, \omega)$, where $J \subseteq [1, \infty)$, to imitate of the recent work of the authors [23]. Then we generalize the results of the third section in [23] to the space $IL_J(G, \omega)$. The last section is essentially devoted to $IL_J(G, \omega)$ as a Banach algebra under convolution product. We first show that $IL_J(G, \omega)$ is always an abstract Segal algebra with respect to $L^1(G, \omega)$. At the end we obtain some results on the amenability of $IL_J(G, \omega)$ and its second dual.

2. Weighted L^p-Algebra $L^p(G,\omega)$

Let G be a locally compact group and ω a weight function on G and $1 \leq p < \infty$. It is plain to verify that the function $\|\cdot\|_{p,\omega}$ defines a norm on $L^p(G, \omega)$ if and only if ω is almost everywhere positive on G. Due to the importance of this subject, most of the time the authors assume positivity in the general definition of a weight function. Thus in this and the last two sections, all weight functions are assumed to be positive. The present section is completely devoted to Banach space $L^p(G, \omega)$. In fact, some important results connected to the properties of convolution product on $L^p(G, \omega)$ are gathered. First we recall two important kinds of positive weight functions which play an essential role in this survey. We refer to [17, 24] and also [21] which contain many valuable information related to the weight functions.

(i) The weight function ω is called submultiplicative if for all $x, y \in G$

$$\omega(xy) \leq \omega(x)\,\omega(y). \tag{4}$$

The class of weights defining convolution algebras $L^1(G, \omega)$ admits a complete description, and it turns out that every weight is equivalent to a continuous function. Moreover, it should be noted that $L^1(G, \omega)$ is closed under convolution product if and only if ω is equivalent to a continuous submultiplicative weight function; see [19, 24, 25] for a full description. But the condition of submultiplicativity of ω is not a necessary condition, whenever $1 < p < \infty$; see [11, Example 2.1]. However, for all $1 \leq p < \infty$, on a discrete group, a weight function of any $\ell^p(G, \omega)$ that are Banach algebra is submultiplicative; indeed, for all $x, y \in G$

$$\omega(xy) = \left\|\delta_x * \delta_y\right\|_{p,\omega}^{1/p} \leq \left\|\delta_x\right\|_{p,\omega}^{1/p} \left\|\delta_y\right\|_{p,\omega}^{1/p} = \omega(x)\,\omega(y), \tag{5}$$

where δ_x is the Dirac measure at x.

(ii) The weight function ω is called of moderate growth if

$$\operatorname{ess\,sup}_{y\in G} \frac{\omega(xy)}{\omega(y)} < \infty, \tag{6}$$

for all $x \in G$. It is remarkable to note that if ω is of moderate growth, then inclusion (6) implies that

$$\operatorname{ess\,inf}_{y\in G} \frac{\omega(xy)}{\omega(y)} > 0. \tag{7}$$

Also the condition of moderate growth for ω is equivalent to the space $L^p(G, \omega)$, for all $1 \leq p < \infty$, being left translation-invariant.

2.1. Local Integrable Property of the Positive Weight Functions. Let G be a locally compact group and ω a positive weight function on G and $1 \leq p < \infty$. We say that ω is locally integrable if $\omega \in L^1(K)$, for all compact subsets K of G. This property is very vital in this research. Thus most of the authors take it as an assumption. However, this is redundant if $L^p(G, \omega)$ is a Banach algebra [11, Lemma 2.1]. It also should be emphasized that if ω is submultiplicative, then it is bounded and bounded away from zero on every compact subset of G [24, Proposition 1.16]. It follows that ω is obviously locally integrable. It is required for the progress to give some special properties of the class of locally integrable positive weight functions. Two important results created by the local integrability of ω are obtained in the following. See [11, 12, 19, 24] for more information.

(1) If ω is locally integrable, then

$$B_0(G) \subseteq L^p(G, \omega), \tag{8}$$

where $B_0(G)$ is the space of all bounded compactly supported functions on G. It implies that $B_0(G)$ is dense in $L^p(G, \omega)$ [11, Lemma 2.2]. Note that when ω is continuous, one can easily replace $C_{00}(G)$ rather than $B_0(G)$, where $C_{00}(G)$ is the space consisting of all continuous functions with compact support.

(2) An important result related to the weight functions has been proved in [19, Theorem 2.7]. Indeed, let ω be both of moderate growth and also locally integrable. Then ω is equivalent to a continuous weight function α; that is, for some constants C_1, C_2,

$$C_1 \leq \frac{\omega(x)}{\alpha(x)} \leq C_2 \tag{9}$$

locally almost everywhere on G. It follows that ω is bounded and bounded away from zero on every compact subset of G.

2.2. The Main Results. Let us recall the function $\Omega_\omega : G \to (0, \infty]$ on G for $1 < p \leq \infty$, from [7], as the following:

$$\Omega_\omega^p(x) = \int_G \left(\frac{\omega(x)}{\omega(y)\,\omega(y^{-1}x)} \right)^q d\lambda(y) \quad (x \in G), \tag{10}$$

where q is the exponential conjugate of p, defined by $1/p + 1/q = 1$. It is known that for $1 < p < \infty$, if $\Omega_\omega^p \in L^\infty(G)$, then $L^p(G,\omega)$ is closed under convolution; see [26] as a more general case and also [7, Theorem 2.2]. This result has also been pointed out in [10]. Furthermore, we asked about the converse of this result whenever ω is a submultiplicative weight function in [7]. It is noticeable to know that this conjecture has been rejected by the examples given by Kuznetsova in [10], for an arbitrary positive weight function. Also the conjecture is rejected in a simultaneous work with [7], for a suitable submultiplicative weight function; see [12, Theorem 1.1]. It is unlike the result [27, Corollary 4.10]. In fact, as it is shown by this counterexample, the claim given in [27, Corollary 4.10] is not true. However, in the following proposition we show that the conjecture holds for the case where $p = \infty$. Anyway, the conjecture given in [7] had been settled only for $1 < p < \infty$. It should be noted that one can extract this result from [28, Lemma 1] and also [17, page 12].

Proposition 1. *Let G be a locally compact group and ω a positive weight function on G. Then $L^\infty(G,\omega)$ is closed under convolution if and only if $\Omega_\omega^\infty \in L^\infty(G)$.*

Proof. First let $L^\infty(G,\omega)$ be closed under convolution. Since $1/\omega \in L^\infty(G,\omega)$, it follows that $1/\omega * 1/\omega \in L^\infty(G,\omega)$, and so the function

$$\Omega_\omega^\infty(x) = \int_G \frac{\omega(x)}{\omega(y)\,\omega(y^{-1}x)} d\lambda(y) \tag{11}$$

belongs to $L^\infty(G)$, clearly. For the converse, suppose that $\Omega_\omega^\infty \in L^\infty(G)$ and $f, g \in L^\infty(G,\omega)$. Then for each $x \in G$, the function θ defined by

$$y \longmapsto f(y)\,\omega(y)\,g\left(y^{-1}x\right)\omega\left(y^{-1}x\right) \tag{12}$$

belongs to $L^\infty(G)$ and so

$$\begin{aligned} |f * g(x)|\,\omega(x) &= \left| \int_G f(y)\,\omega(y)\,g\left(y^{-1}x\right)\omega\left(y^{-1}x\right) \right. \\ &\qquad \left. \times \frac{\omega(x)}{\omega(y)\,\omega(y^{-1}x)} d\lambda(y) \right| \\ &\leq \left|\Omega_\omega^\infty(x)\right| \|\theta\|_\infty \\ &\leq \|\Omega_\omega^\infty\|_\infty \|f\|_{\infty,\omega} \|g\|_{\infty,\omega}, \end{aligned} \tag{13}$$

almost everywhere on G. Consequently

$$\|f * g\|_{\infty,\omega} \leq \|\Omega_\omega^\infty\|_\infty \|f\|_{\infty,\omega} \|g\|_{\infty,\omega}, \tag{14}$$

and the result is obtained. □

Concisely, we indicate the results of almost all surveys done by the mathematicians in this field in the following remark.

Remarks. Let G be a locally compact group and ω a positive weight function on G and $1 \leq p \leq \infty$.

(i) $L^1(G,\omega)$ is closed under convolution if and only if ω is equivalent to a continuous submultiplicative weight function [11, Theorem 3.1].

(ii) If $\Omega_\omega^p \in L^\infty(G)$, for some $1 \leq p \leq \infty$, then the function $y \mapsto \omega(x)^q/\omega(y)^q\omega(y^{-1}x)^q$ belongs to $L^1(G)$, for almost everywhere $x \in G$. Since this function is positive, it follows that G is σ-compact.

(iii) If $2 < p < \infty$, ω is submultiplicative, and $f * g$ exists as a function for all $f, g \in L^p(G,\omega)$, then G is σ-compact [3, Theorem 2.5].

(iv) If $1 < p < \infty$, G is amenable, and $L^p(G,\omega)$ is closed under convolution, then G is σ-compact [12, Corollary 3.3].

(v) G is σ-compact if and only if for some $1 < p < \infty$, there exists a weight ω satisfying $\Omega_\omega^p \in L^\infty(G)$ [11, Theorem 1.1].

(vi) If $1 < p \leq 2$, then the σ-compactness of G is not in general a necessary condition for the closedness of $L^p(G,\omega)$ under convolution [11, Theorem 1.1 and Proposition 1.2].

(vii) If $1 < p < \infty$ and $\Omega_\omega^p \in L^\infty(G)$, then [7, Theorem 2.2] and also [10] imply that $L^p(G,\omega)$ is closed under convolution.

(viii) $L^\infty(G,\omega)$ is closed under convolution if and only if $\Omega_\omega^\infty \in L^\infty(G)$, as we proved in Proposition 1. Also [28, Lemma 1] and [17, page 12].

3. General Properties of Arbitrary Weight Functions

Let G be a locally compact group and $1 \leq p < \infty$. Take Ω to be an arbitrary family of the weight functions on G such that the function W defined as

$$W(x) = \sup_{\omega \in \Omega} \omega(x) \tag{15}$$

is finite everywhere on G. Then W is in fact a weight function on G. Set

$$L^p(G,\Omega) = \bigcap_{\omega \in \Omega} L^p(G,\omega). \tag{16}$$

We equip the space $L^p(G,\Omega)$ with the natural locally convex topology τ_Ω generated by the family of seminorms $\{\rho_\omega\}_{\omega\in\Omega}$, where

$$\rho_\omega : L^p(G,\Omega) \longrightarrow \mathbb{R}, \qquad f \longmapsto \rho_\omega(f) = \|f\|_{p,\omega}, \tag{17}$$

and ω runs through Ω. We will explain that the topology τ_Ω on $L^p(G,\Omega)$ is generated by the neighborhoods

$$V(f, \rho_\omega, \varepsilon) = \{g \in L^p(G,\Omega) : \rho_\omega(f - g) < \varepsilon\}, \tag{18}$$

where $f \in L^p(G,\Omega)$, ε is any positive real number, and $\omega \in \Omega$. In general, $L^p(G,\Omega)$ is not necessarily Hausdorff under τ_Ω.

In fact, a locally convex space is Hausdorff if and only if it has a separated family of seminorms; see [29] for full information about the locally convex vector spaces. Now, consider the following subset of $L^p(G,\Omega)$:

$$IL_p(G,\Omega)=\left\{f\in L^p(G,\Omega):\|f\|_{p,\Omega}=\sup_{\omega\in\Omega}\|f\|_{p,\omega}<\infty\right\}. \tag{19}$$

It is obvious that in general

$$L^p(G,W)\subseteq IL_p(G,\Omega)\subseteq L^p(G,\Omega), \tag{20}$$

and $\|f\|_{p,\Omega}\le\|f\|_{p,W}$, for each $f\in L^p(G,W)$. Also some elementary calculations show that if $\Omega=\{\omega_1,\dots,\omega_n\}$ is a finite set, then

$$IL_p(G,\Omega)=L^p(G,\Omega)=L^p(G,W), \tag{21}$$

and for each $f\in IL_p(G,\Omega)$,

$$\|f\|_{p,\Omega}\le\|f\|_{p,W}\le\sum_{i=1}^{n}\|f\|_{p,\omega_i}\le n\|f\|_{p,\Omega}. \tag{22}$$

The following example shows that the inclusions (20) can be proper.

Example 2. Take G to be the additive group of the real numbers $\mathbb{R}$ endowed with the discrete topology. Set $\Omega=\{\omega_x : x\in\mathbb{R}\}$, where $\omega_x=\chi_x$, the characteristic function of the set $\{x\}$. Obviously $L^p(\mathbb{R},\Omega)$ is the space of all complex valued functions on $\mathbb{R}$, which is in fact a locally convex space. Moreover, $IL_p(\mathbb{R},\Omega)=\ell^\infty(\mathbb{R})$ and $||f||_{p,\Omega}=||f||_\infty$. Also $\ell^p(\mathbb{R},W)=\ell^p(\mathbb{R})$. It follows that

$$\ell_p(\mathbb{R},W)\subsetneqq IL_p(\mathbb{R},\Omega)\subsetneqq L^p(\mathbb{R},\Omega). \tag{23}$$

The main purpose of this section is to provide some conditions for that $\|\cdot\|_{p,\Omega}$ acts as a norm function on $IL_p(G,\Omega)$. Although all of them are sufficient conditions and occur naturally in applications, they can be useful in their own right. Let us first turn the attention to the following example.

Example 3. Consider the additive group of real numbers $\mathbb{R}$ endowed with its standard topology, and let

$$\Omega=\{\omega_n=\chi_{[n,n+1]} : n\in\mathbb{N}\}. \tag{24}$$

Suppose that $f=\chi_{[0,1]}$. Thus $f\in L^p(\mathbb{R},\omega_n)$ for all $n\in\mathbb{N}$. Also

$$\|f\|_{p,\omega_n}=\int_{-\infty}^{\infty}|f(t)|^p\omega_n(t)^p d\lambda(t)=0, \tag{25}$$

and so

$$\|f\|_{p,\Omega}=\sup_{n\in\mathbb{N}}\|f\|_{p,\omega_n}=0, \tag{26}$$

whereas $f\neq 0$. It follows that $IL_p(\mathbb{R},\Omega)$ is not a normed space.

According to Example 3, $\|\cdot\|_{p,\Omega}$ may not be treated as a norm function, even for a countable set Ω of the weight functions. The following result shows that countability of Ω can be a sufficient condition for normability of $IL_p(G,\Omega)$, whenever W is positive almost everywhere on G.

Proposition 4. *Let G be a locally compact group and $1\le p<\infty$ and let Ω be a countable family of weight functions such that $W(x)>0$ almost everywhere on G. Then $(IL_p(G,\Omega),\|\cdot\|_{p,\Omega})$ is a normed space.*

Proof. Assume that $f\in IL_p(G,\Omega)$ and $||f||_{p,\Omega}=0$. It follows that $\|f\|_{p,\omega}=0$ and so $f\omega=0$ almost everywhere on G, for all $\omega\in\Omega$. Let $A=\{x : W(x)=0\}$, and for each $\omega\in\Omega$, $B_\omega=\{x : f(x)\omega(x)\neq 0\}$, and put $C=A\cup(\cup_{\omega\in\Omega}B_\omega)$. Since Ω is countable and $\lambda(B_\omega)=\lambda(A)=0$, then $\lambda(C)=0$. Now let $x\in G\setminus C$. Since $W(x)>0$, there exists at least one $\omega\in\Omega$ such that $\omega(x)>0$. It follows that $f(x)=0$. Consequently, $f=0$ almost everywhere on G and the result is obtained. □

In the following examples, we determine $IL_p(\mathbb{R},\Omega)$ for two families of the weight functions.

Example 5. Take $G=\mathbb{R}$, the additive group of real numbers endowed with the usual topology.

(1) Let $\omega_1=\delta_{(-\infty,0)}$, $\omega_2=\delta_{[0,\infty)}$, and $\Omega=\{\omega_1,\omega_2\}$. Since Ω is finite and also, for every $x\in\mathbb{R}$, $W(x)=1$, then

$$IL_p(\mathbb{R},\Omega)=L^p(\mathbb{R},W)=L^p(\mathbb{R},\Omega)=L^p(\mathbb{R}). \tag{27}$$

We explain this example in detail. For each $f\in L^p(\mathbb{R},\Omega)$, we have

$$\int_{-\infty}^{0}|f|^p<\infty,\qquad\int_{0}^{\infty}|f|^p<\infty, \tag{28}$$

and so $f\in L^p(\mathbb{R})$. It follows that $L^p(\mathbb{R},\Omega)=L^p(\mathbb{R})$. Now suppose that $f\in IL_p(\mathbb{R},\Omega)$. Then

$$\begin{aligned}\|f\|_{p,\Omega}&=\max\left\{\left(\int_0^\infty|f|^p\right)^{1/p},\left(\int_{-\infty}^0|f|^p\right)^{1/p}\right\}\\&\le\left(\int_{-\infty}^{\infty}|f|^p\right)^{1/p}\\&=\|f\|_p\\&\le 2\|f\|_{p,\Omega},\end{aligned} \tag{29}$$

and so

$$\frac{1}{2}\|f\|_p\le\|f\|_{p,\Omega}\le\|f\|_p. \tag{30}$$

(2) Let $\Omega=\{\omega_n : n\in\mathbb{Z}\}$, where $\omega_n=\chi_{(n,n+1]}$. Since, for each $x\in\mathbb{R}$, $W(x)=1$, it follows that $L^p(\mathbb{R},W)=L^p(\mathbb{R})$. Now let $f(x)\equiv 1$, the constant function of value 1 and $g(x)=[x]$, the bracket function on $\mathbb{R}$.

Then, $f \in IL_p(\mathbb{R}, \Omega)$ but $f \notin L^p(\mathbb{R})$. Also $g \in L^p(\mathbb{R}, \Omega)$ and since $||g||_{p,\omega_n} = n$, it follows that $g \notin IL_p(\mathbb{R}, \Omega)$. Therefore

$$L^p(\mathbb{R}) \subsetneqq IL_p(\mathbb{R}, \Omega) \subsetneqq L^p(\mathbb{R}, \Omega). \tag{31}$$

Moreover, $IL_p(\mathbb{R}, \Omega)$ is a normed space by Proposition 4.

It is clear that the existence of at least one positive weight in Ω is enough for normability of $IL_p(G, \Omega)$ with the topology induced by $\|\cdot\|_{p,\Omega}$. Such a condition is not imposed in the present section. Instead, we introduce a more delicate framework for Ω as the following.

Definition 6. Let G be a locally compact group and Ω a family of weight functions. Then Ω is called locally positive if for each $x \in G$ there exist $\omega_x \in \Omega$ and an open neighborhood U_x of x such that ω_x is positive on U_x.

It is obvious that if Ω consists of just one element ω, then local positivity of Ω is equivalent to the fact that ω is positive. In this situation, $IL_p(G, \Omega)$ is always a normed space under $\|\cdot\|_{p,\Omega}$.

Note that if Ω is locally positive, then $W(x) > 0$ for each $x \in G$. Thus the following result is obtained clearly from Proposition 4.

Corollary 7. *Let G be a locally compact group and $1 \leq p < \infty$ and let Ω be countable and locally positive. Then $(IL_p(G, \Omega), \|\cdot\|_{p,\Omega})$ is a normed space.*

We give another criterion for the normability of $IL_p(G, \Omega)$ under $\|\cdot\|_{p,\Omega}$ in the next result. It shows that countability of Ω can be removed in Corollary 7, in the case where G is σ-compact.

Proposition 8. *Let G be a σ-compact locally compact group $1 \leq p < \infty$ and let Ω be locally positive. Then $(IL_p(G, \Omega), \|\cdot\|_{p,\Omega})$ is a normed space.*

Proof. Suppose that $G = \cup_{n=1}^{\infty} G_n$, where G_n is a compact subset of G, for each $n \in \mathbb{N}$. Take $f \in IL_p(G, \Omega)$ such that $||f||_{p,\Omega} = 0$. Thus $||f||_{p,\omega} = 0$, for each $\omega \in \Omega$, and so $f\omega = 0$ almost everywhere on G. Hence, $f\omega = 0$ almost everywhere on G_n, for each $n \in \mathbb{N}$. By the local positivity of Ω, for each $x \in G_n$, there exists the neighborhood U_x of x and the weight function $\omega_x \in \Omega$ such that ω_x is positive on U_x. So $f = 0$ almost everywhere on U_x. Since G_n is compact, there exist the elements $x_1, \ldots, x_{k_n}$ of G_n such that $G_n \subseteq \cup_{i=1}^{k_n} U_{x_i}$. It follows that $f = 0$ almost everywhere on G_n. Therefore, $f = 0$ almost everywhere on G, and so the result is obtained. □

Nevertheless, in the following example we show that σ-compactness of G and also countability of Ω are not necessary conditions for normability of $IL_p(G, \Omega)$.

Example 9. Consider again the additive group of real numbers $\mathbb{R}$ endowed with the discrete topology, and let $\Omega = \{\omega_x = \chi_x : x \in \mathbb{R}\}$. Then Ω is clearly locally positive and uncountable. Suppose that $f \in IL_p(\mathbb{R}, \Omega)$ with $||f||_{p,\Omega} = 0$. Thus $||f||_{p,\omega_x} = 0$, for all $x \in \mathbb{R}$, and so $f = 0$ everywhere on $\mathbb{R}$. It follows that $\|\cdot\|_{p,\Omega}$ acts as a norm on $IL_p(\mathbb{R}, \Omega)$, whereas the group is not σ-compact.

As the final result in this section, we provide some sufficient conditions for $IL_p(G, \Omega)$ to be a Banach space under the norm $\|\cdot\|_{p,\Omega}$.

Theorem 10. *Let G be a locally compact group $1 \leq p < \infty$ and let Ω be a countable family of positive weight functions. Then $IL_p(G, \Omega)$ is a Banach space.*

Proof. By Corollary 7, $\|\cdot\|_{p,\Omega}$ is a norm on $IL_p(G, \Omega)$. To that end, let $\Omega = \{\omega_n : n \in \mathbb{N}\}$, and let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $IL_p(G, \Omega)$. Then for each $\omega_k \in \Omega$, $(f_n\omega_k)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(G)$. So there exists a net $\{g_k\} \subseteq L^p(G)$ such that $(g_k/\omega_k) \in L^p(G, \omega_k)$ and $\lim_{n\to\infty} \|f_n\omega_k - g_k\|_p = 0$, for each $k \in \mathbb{N}$. Therefore there exists a subnet $\{f_{n_{i_1}}\}$ of $(f_n)_{n\in\mathbb{N}}$ such that $f_{n_{i_1}}\omega_1 \to g_1$ in the pointwise sense, outside a measurable subset A_1 of G with $\lambda(A_1) = 0$. Continuallty, there exists a subnet $\{f_{n_{i_2}}\} \subseteq \{f_{n_{i_1}}\}$ such that $f_{n_{i_2}}\omega_2 \to g_2$ in the pointwise sense, outside a measurable subset A_2 of G with $\lambda(A_2) = 0$. Inductively for each k, there exists a subnet $\{f_{n_{i_k}}\}$ of $\{f_{n_{i_{k-1}}}\}$ such that $f_{n_{i_k}}\omega_k$ tends in the pointwise sense to g_k, outside a measurable subset A_k of G with $\lambda(A_k) = 0$. Set $A = \cup_{k=1}^{\infty} A_k$. It follows that for all $k, m \in \mathbb{N}$, $(g_k/\omega_k) = (g_m/\omega_m)$ outside the set A, and so all the functions (g_n/ω_n) are almost everywhere equivalent to a function g on G. Thus

$$g \in \bigcap_{k=1}^{\infty} L^p(G, \omega_k) = L^p(G, \Omega). \tag{32}$$

We prove that $\{f_n\}$ as a sequence in $IL_p(G, \Omega)$ converges to the function g. Since $\{f_n\}$ is a Cauchy sequence in $IL_p(G, \Omega)$, for each $\varepsilon > 0$, one can find a positive integer N such that for all $m, n \geq N$,

$$\|(f_n - f_m)\omega_k\|_p \leq \|f_n - f_m\|_{p,\Omega} < \frac{\varepsilon}{2}, \tag{33}$$

where $k \in \mathbb{N}$. It follows that

$$\|f_n\omega_k - g_k\|_p < \varepsilon, \tag{34}$$

and so for each $k \in \mathbb{N}$,

$$\|f_n\omega_k - g\omega_k\|_p < \varepsilon. \tag{35}$$

Choosing $n := N$, for each $k \in \mathbb{N}$,

$$\|g\omega_k\|_p \leq \|f_N\omega_k\|_p + \varepsilon \leq \|f_N\|_{p,\Omega} + \varepsilon. \tag{36}$$

Then,

$$\|g\|_{p,\Omega} \leq \|f_N\|_{p,\Omega} + \varepsilon < \infty. \tag{37}$$

Consequently $g \in IL_p(G, \Omega)$. Also inequality (35) implies that $\|f_n - g\|_{p,\Omega} \to 0$, and the proof is completed. □

Remarks. It is worth noting that we point out here to the paper of Beurling [16] in this field. This valuable work was introduced to us by the referee. Because of the value of this work, let us mention it again briefly. Let G be an abelian locally compact group and $1 < p < \infty$ and let Ω be a collection of the positive locally integrable weight functions on G. Also suppose that N is a function on Ω such that for each $\omega \in \Omega$, $N(\omega)$ takes a finite value and also satisfies the conditions (1.1) till (1.5) in [16]. Consider the subset Ω_0 of Ω consisting of all $\omega \in \Omega$ with $N(\omega) = 1$ (such a weight function is called normalized). For a fixed p, let $\omega' = 1/\omega^{p-1}$. Set

$$A^p = \bigcup_{\omega\in\Omega_0} L^p\left(G, \omega'\right),$$
$$B^q = \bigcap_{\omega\in\Omega_0} L^q\left(G, \omega\right), \tag{38}$$

where q is the exponential conjugate of p defined by $1/p + 1/q = 1$. Note that in [16], the definition of the norm functions $\|\cdot\|_{p,\omega'}$ and also $\|\cdot\|_{q,\omega}$ has been given in a slightly different form from the usual way. In fact for each $f \in L^p(G, \omega')$ and $g \in L^q(G, \omega)$,

$$\|f\|_{p,\omega'} = \left(\int_G |f(x)|^p \omega'(x)\, d\lambda(x)\right)^{1/p},$$
$$\|f\|_{q,\omega} = \left(\int_G |f(x)|^q \omega(x)\, d\lambda(x)\right)^{1/q}. \tag{39}$$

Now for each $F \in A^p$ and $G \in B^q$, let

$$\|F\|_{A^p} = \inf_{\omega\in\Omega_0} \|F\|_{p,\omega'}, \qquad \|G\|_{B^q} = \sup_{\omega\in\Omega_0} \|F\|_{q,\omega}. \tag{40}$$

Then $\|\cdot\|_{A^p}$ (resp., $\|\cdot\|_{B^q}$) acts as a norm function on A^p (resp., B^q). More importantly, by [16, Theorem 1], $(A^p, \|\cdot\|_{A^p})$ is a Banach algebra under convolution and $(B^q, \|\cdot\|_{B^q})$ is a Banach space which is the dual of A^p. Also, we refer to the examples given in [16, Section 2] for making these spaces more clear. Indeed, in these examples, the spaces A^p and B^q are investigated and characterized for some suitable classes of the weight functions on the Euclidean space $\mathbb{R}^n$.

4. Some Intersections of the Lorentz Spaces

In this section, we investigate the intersection of Lorentz spaces, which in fact was suggested to us by the referee. First we give some preliminaries and definitions that will be used throughout the section. See [30] for complete information in this field. Let f be a complex valued measurable function on G. For each $\alpha > 0$, let

$$d_f(\alpha) = \mu\left(\{x \in G : |f(x)| > \alpha\}\right). \tag{41}$$

The decreasing rearrangement of f is the function $f^* : [0, \infty) \to [0, \infty]$ defined by

$$f^*(t) = \inf\left\{s > 0 : d_f(s) \le t\right\}. \tag{42}$$

We adopt the convention $\inf \emptyset = \infty$, thus having $f^*(t) = \infty$ whenever $d_f(\alpha) > t$ for all $\alpha \ge 0$. By [30, Proposition 1.4.5] for each $0 < p < \infty$ we have

$$\int_G |f(x)|^p d\lambda(x) = \int_0^\infty f^*(t)^p dt, \tag{43}$$

where dt is the Lebesgue measure. For $0 < p, q < \infty$, define

$$\|f\|_{L_{p,q}} = \left(\int_0^\infty \left(t^{(1/p)} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q}. \tag{44}$$

The set of all f with $\|f\|_{p,q} < \infty$ is denoted by $L_{p,q}(G)$ and is called the Lorentz space with indices p and q. As in L^p-spaces, two functions in $L_{p,q}(G)$ are considered equal if they are equal to λ-almost everywhere on G. Note that (43) implies that $L_{p,p}(G) = L^p(G)$.

We recall here [30, Proposition 1.4.10] which is very useful in our main results.

Proposition 11. *Suppose $0 < p < \infty$ and $0 < q < r < \infty$. Then there exists a constant $C_{p,q,r}$ such that*

$$\|f\|_{L_{p,r}} \le C_{p,q,r} \|f\|_{L_{p,q}}, \tag{45}$$

$C_{p,q,r} = (q/p)^{1/q-1/r}$. In other words, $L_{p,q}(G)$ is a subspace of $L_{p,r}(G)$.

Now let $0 < p < \infty$ be fixed and J an arbitrary subset of $(0, \infty)$ with $m_J = \inf\{q : q \in J\}$. We introduce $IL_{p,J}(G)$ as a subset of $\cap_{q\in J} L_{p,q}(G)$ by

$$IL_{p,J}(G) = \left\{f \in \cap_{q\in J} L_{p,q}(G) : \|f\|_{L_{p,J}} = \sup_{q\in J} \|f\|_{L_{p,q}} < \infty\right\}. \tag{46}$$

The main result of the present section is provided in the following.

Theorem 12. *Let G be a locally compact group $0 < p < \infty$ and let J be an arbitrary subset of $(0, \infty)$ such that $m_J > 0$. Then $IL_{p,J}(G) = L_{p,m_J}(G)$, as two sets. Moreover, for each $f \in L_{p,m_J}(G)$,*

$$\|f\|_{L_{p,m_J}} \le \|f\|_{L_{p,J}} \le \max\left\{1, \left(\frac{m_J}{p}\right)^{1/m_J}\right\} \|f\|_{L_{p,m_J}}. \tag{47}$$

Proof. Proposition 11 implies that $L_{p,m_J}(G) \subseteq L_{p,q}(G)$, for each $q \in J$. Also for each $f \in L_{p,m_J}(G)$

$$\|f\|_{L_{p,q}} \le \left(\frac{m_J}{p}\right)^{1/m_J - 1/q} \|f\|_{L_{p,m_J}} \le \max\left\{1, \left(\frac{m_J}{p}\right)^{1/m_J}\right\} \|f\|_{L_{p,m_J}}. \tag{48}$$

It follows that $f \in IL_{p,J}(G)$ and

$$\|f\|_{L_{p,J}} \le \max\left\{1, \left(\frac{m_J}{p}\right)^{1/m_J}\right\} \|f\|_{L_{p,m_J}}. \tag{49}$$

Thus $L_{p,m_J}(G) \subseteq IL_{p,J}(G)$. Now we prove the reverse of the inclusion. If $m_J \in J$, then $IL_{p,J}(G) \subseteq L_{p,m_J}(G)$, obviously. Moreover $\|f\|_{L_{p,m_J}} \leq \|f\|_{L_{p,J}}$, for each $f \in IL_{p,J}(G)$. Now let $m_J \notin J$. Thus there is a sequence $(x_n)_{n\in\mathbb{N}}$ in J, converging to m_J. For each $f \in IL_{p,J}(G)$, Fatou's lemma implies that

$$\begin{aligned}\int_0^\infty \left(t^{1/p} f^*(t)\right)^{m_J} \frac{dt}{t} &= \int_0^\infty \liminf_n \left(t^{1/p} f^*(t)\right)^{x_n} \frac{dt}{t} \\ &\leq \liminf_n \int_0^\infty \left(t^{1/p} f^*(t)\right)^{x_n} \frac{dt}{t} \\ &= \liminf_n \|f\|_{L_{p,x_n}}^{x_n} \leq \liminf_n \|f\|_{L_{p,J}}^{x_n} \\ &= \|f\|_{L_{p,J}}^{m_J}.\end{aligned} \tag{50}$$

Consequently

$$\left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^{m_J} \frac{dt}{t}\right)^{1/m_J} \leq \|f\|_{L_{p,J}}. \tag{51}$$

It follows that $f \in L_{p,m_J}(G)$ and $\|f\|_{L_{p,m_J}} \leq \|f\|_{L_{p,J}}$, as claimed. □

By [30, Theorem 1.4.11], the spaces $L_{p,q}(G)$ are always quasi-Banach spaces (i.e., a complete quasi-normed space). Moreover, [30, Exercise 1.4.3] implies that $L_{p,q}(G)$ is a Banach space in the case where $1 < p < \infty$ and $1 \leq q < \infty$. We end this section with the following result, which is immediately obtained from Theorem 12 and [30, Exercise 1.4.3].

Corollary 13. *Let G be a locally compact group $1 < p < \infty$ and let J be an arbitrary subset of $[1, \infty)$. Then $(IL_{p,J}(G), \|\cdot\|_{L_{p,J}})$ is a Banach space.*

5. Introducing Some Intersection of Weighted L^p-Spaces

Let G be a locally compact group and $1 \leq p < \infty$ and let Ω consist of just one weight function ω that is submultiplicative and positive. Thus $L^p(G,\omega)$ is a Banach space under $\|\cdot\|_{p,\omega}$, as we mentioned in the first section. Moreover, its dual space is the Banach space $L^q(G, 1/\omega)$ under the duality

$$\langle f, g\rangle := \int_G f(x)\, g(x)\, d\lambda(x), \tag{52}$$

where $f \in L^p(G,\omega)$ and $g \in L^q(G, 1/\omega)$, and q is the exponential conjugate of p. The aim of this section is investigating an arbitrary intersection of the weighted L^p-spaces, where p runs through a subset J of $[1, \infty)$. It is performed in a similar way to the structure of $IL_J(G)$ and introduced in [23]. Since the dual of each L^p-space may be participated in $IL_J(G)$, then the structure of the dual of $L^p(G,\omega)$ leads us to include it in our definition. Also our expectation of the behavior of this space as a Banach algebra under convolution necessitates us to insert $L^1(G,\omega)$. We turn the attention to this fact that $L^1(G,\omega)$ is a Banach algebra under convolution whenever ω is submultiplicative. It justifies the assumption of submultiplicativity of ω. All these reasons justify that this space should be defined in a slightly different way from $IL_J(G)$. We first introduce the space $IL_p(G,\omega)$, where $1 \leq p < \infty$. Set

$$IL_p(G,\omega) = L^1(G,\omega) \cap L^p(G,\omega) \cap L^q\left(G, \frac{1}{\check{\omega}}\right), \tag{53}$$

where $\check{\omega}(x) = \omega(x^{-1})$, for each $x \in G$. Then the function $\|\cdot\|_{IL_p}$ defined by

$$\|f\|_{IL_p} = \max\left\{\|f\|_{1,\omega}, \|f\|_{p,\omega}, \|f\|_{q,1/\check{\omega}}\right\} \tag{54}$$

is clearly a norm on $IL_p(G,\omega)$. Furthermore, we have the next result that is in fact a partial case of the classical results on interpolation spaces; see [31].

Proposition 14. *Let G be a locally compact group and ω a positive weight function on G and $1 \leq p < \infty$. Then $IL_p(G,\omega)$ is a Banach space under $\|\cdot\|_{IL_p}$.*

Let us recall from [32, Theorem 1] that $L^1(G) \subseteq L^p(G)$ (resp., $L^p(G) \subseteq L^1(G)$), for some $1 < p \leq \infty$ if and only if G is discrete (resp., compact). Similar arguments can be applied to get the same consequences in the weighted case.

Proposition 15. *Let G be a discrete group and ω a positive submultiplicative weight function on G and $1 \leq p < \infty$. Then $IL_p(G,\omega) = \ell^1(G,\omega)$, as Banach spaces. Moreover, $\|f\|_{IL_p} = \|f\|_{1,\omega}$, for each $f \in \ell^1(G,\omega)$.*

Proof. Let $f \in \ell^1(G,\omega)$. If $p = 1$, since ω is submultiplicative, then for each $x \in G$ we have

$$\frac{|f(x)|}{\check{\omega}(x)} \leq |f(x)|\,\omega(x) \leq \sum_{y\in G} |f(y)|\,\omega(y) = \|f\|_{1,\omega}. \tag{55}$$

Thus $f \in \ell^\infty(G, 1/\check{\omega})$ and $\|f\|_{\infty,1/\check{\omega}} \leq \|f\|_{1,\omega}$, and consequently $\ell^1(G,\omega) \subseteq \ell^\infty(G, 1/\check{\omega})$. It follows that $IL_1(G,\omega) = \ell^1(G,\omega)$ and also $\|f\|_{IL_1} = \|f\|_{1,\omega}$. Now let $1 < p < \infty$. We first show that $f \in \ell^q(G, 1/\check{\omega})$. Again the submultiplicativity of ω yields that

$$\begin{aligned}\sum_{x\in G}\left(\frac{|f(x)|}{\check{\omega}(x)}\right)^q &\leq \sum_{x\in G} |f(x)|^q \omega(x)^q \\ &\leq \left(\sum_{x\in G} |f(x)|\,\omega(x)\right)^q = \|f\|_{1,\omega}^q < \infty.\end{aligned} \tag{56}$$

It follows that $f \in \ell^q(G, 1/\check{\omega})$, and so $\ell^1(G,\omega) \subseteq \ell^q(G, 1/\check{\omega})$. Also the explanation preceding the proposition implies that $\ell^1(G,\omega) \subseteq \ell^p(G,\omega)$. Consequently $IL_p(G,\omega) = \ell^1(G,\omega)$ and $\|f\|_{1,\omega} = \|f\|_{IL_p}$. □

Proposition 16. *Let G be a compact group and ω a positive submultiplicative weight function on G and $1 \leq p < \infty$. Then the following assertions hold.*

(i) *If* $1 \le p < q < \infty$, *then* $IL_p(G,\omega) = L^q(G, 1/\check{\omega})$, *as Banach spaces.*

(ii) *If* $1 < q \le p < \infty$, *then* $IL_p(G,\omega) = L^p(G,\omega)$, *as Banach spaces.*

Proof. (i) To get the result, it is sufficient to show that $L^q(G, 1/\check{\omega}) \subseteq L^p(G,\omega)$. First let $p = 1$ and $f \in L^\infty(G, 1/\check{\omega})$. Since $\omega^* = \omega\check{\omega}$ is also a positive submultiplicative weight on G, thus there is a positive constant M such that $\omega^*(x) \le M$, for each $x \in G$ [24, Proposition 1.16]. By normalizing Haar measure on G appropriately, we may assume that $\lambda(G) = 1$ and thus

$$\begin{aligned}\int_G |f(x)|\,\omega(x)\,dx &= \int_G \frac{|f(x)|}{\check{\omega}(x)}\omega^*(x)\,dx \\ &\le M\|f\|_{\infty,1/\check{\omega}} < \infty.\end{aligned} \tag{57}$$

It follows that $f \in L^1(G,\omega)$, and so $IL_1(G,\omega) = L_\infty(G, 1/\check{\omega})$. Also

$$\|f\|_{\infty,1/\check{\omega}} \le \|f\|_{IL_1} \le (1+M)\,\|f\|_{\infty,1/\check{\omega}}. \tag{58}$$

Now let $1 < p < q < \infty$ and $f \in L^q(G, 1/\check{\omega})$. By some easy calculations we have

$$\|f\|_{p,\omega} = \|f\omega\|_p \le M\left\|\frac{f}{\check{\omega}}\right\|_p \le M\left\|\frac{f}{\check{\omega}}\right\|_q = M\|f\|_{q,1/\check{\omega}} < \infty. \tag{59}$$

Thus, $f \in L^p(G,\omega)$, and so $L^q(G, 1/\check{\omega}) \subseteq L^p(G,\omega) \subseteq L^1(G,\omega)$. It follows that

$$IL_p(G,\omega) = L^q\left(G, \frac{1}{\check{\omega}}\right). \tag{60}$$

Also

$$\|f\|_{q,1/\check{\omega}} \le \|f\|_{IL_p} \le (M+1)\,\|f\|_{q,1/\check{\omega}}. \tag{61}$$

(ii) It is obtained by some similar arguments above that

$$L^p(G,\omega) \subseteq L^q\left(G, \frac{1}{\check{\omega}}\right) \subseteq L^1(G,\omega). \tag{62}$$

Also $\|f\|_{p,\omega} = \|f\|_{IL_p}$, for each $f \in L^p(G,\omega)$, and so the result is provided. □

Corollary 17. *Let G be a compact group and ω a positive submultiplicative weight function on G and $1 \le p < \infty$. Consider the following.*

(i) *If* $1 \le p < 2$, *then* $IL_p(G,\omega) = L^q(G, 1/\check{\omega})$

(ii) *If* $2 < p < \infty$, *then* $IL_p(G,\omega) = L^p(G,\omega)$.

(iii) $IL_2(G,\omega) = L^2(G, 1/\check{\omega}) = L^2(G,\omega)$.

5.1. The Banach Space $IL_J(G,\omega)$. For a locally compact group G and $J \subseteq [1,\infty)$, set

$$m_J = \inf\{p : p \in J\}, \qquad M_J = \sup\{p : p \in J\}. \tag{63}$$

Let ω be a positive submultiplicative weight function on G. Similarly to our recent work [23], we introduce $IL_J(G,\omega)$ by

$$IL_J(G,\omega) = \left\{ f \in \bigcap_{p\in J} IL_p(G,\omega) : \|f\|_J = \sup_{p\in J}\|f\|_{IL_p} < \infty \right\} \tag{64}$$

as a subspace of $\cap_{p\in J} IL_p(G,\omega)$. Then $\|\cdot\|_J$ is obviously a norm on $IL_J(G,\omega)$. The main purpose of the present section is describing the properties of $IL_J(G,\omega)$ as a Banach space under the norm function $\|\cdot\|_J$. We will discuss first Proposition 2.2 in [23] for $IL_J(G,\omega)$ that is in fact a partial usage of the Riesz convexity Theorem [33, Theorem 13.19].

Proposition 18. *Let G be a locally compact group and ω a positive submultiplicative weight function on G and $1 \le p < t < \infty$. Then*

$$\bigcap_{r\in[p,t]} IL_r(G,\omega) = IL_p(G,\omega) \cap IL_t(G,\omega) \tag{65}$$

and for each $f \in IL_p(G,\omega) \cap IL_t(G,\omega)$ and $p \le r \le t$, one has

$$\|f\|_{IL_r} \le \max\left\{\|f\|_{IL_p}, \|f\|_{IL_t}\right\}. \tag{66}$$

Proof. Let $f \in IL_p(G,\omega) \cap IL_t(G,\omega)$. Then

$$\begin{aligned} f\omega &\in L^1(G) \cap L^p(G) \cap L^t(G), \\ \frac{f}{\check{\omega}} &\in L^q(G) \cap L^{t/(t-1)}(G). \end{aligned} \tag{67}$$

Thus, [23, Proposition 2.2] implies that

$$f\omega \in L^1(G) \cap \left(\bigcap_{r\in[p,t]} L^r(G)\right), \qquad \frac{f}{\check{\omega}} \in \bigcap_{r\in[p,t]} L^{(r/(r-1))}(G). \tag{68}$$

Hence,

$$f \in \bigcap_{r\in[p,t]} IL_r(G,\omega), \tag{69}$$

and since $||f||_{r,\omega} \le \max\{||f||_{p,\omega}, ||f||_{t,\omega}\}$ and $||f||_{r/(r-1),1/\check{\omega}} \le \max\{||f||_{q,1/\check{\omega}}, ||f||_{t/(t-1),1/\check{\omega}}\}$, then

$$\begin{aligned}\|f\|_{IL_r} &\le \max\left\{\|f\|_{1,\omega}, \|f\|_{p,\omega}, \|f\|_{q,1/\check{\omega}}, \|f\|_{t,\omega}, \|f\|_{t/(t-1),1/\check{\omega}}\right\} \\ &= \max\left\{\|f\|_{IL_p}, \|f\|_{IL_t}\right\},\end{aligned} \tag{70}$$

and the proof is complete. □

The next proposition shows an intimate relation between the spaces $IL_p(G,\omega)$, whenever p runs in an arbitrary subset of $[1,\infty)$. The proof is immediate.

Proposition 19. *Let G be a locally compact group and J a subset of $[1,\infty)$. Then the following assertions hold.*

(1) *If $m_J, M_J \in J$, then $\cap_{p\in[m_J,M_J]} IL_p(G,\omega) = IL_{m_J}(G,\omega) \cap IL_{M_J}(G,\omega)$.*

(2) *If $m_J \in J$ and $M_J \notin J$, then $\cap_{p\in J} IL_p(G,\omega) = \cap_{p\in[m_J,M_J)} IL_p(G,\omega)$.*

(3) *If $M_J \in J$ and $m_J \notin J$, then $\cap_{p\in J} IL_p(G,\omega) = \cap_{p\in(m_J,M_J]} IL_p(G,\omega)$.*

(4) *If $M_J \notin J$ and $m_J \notin J$, then $\cap_{p\in J} IL_p(G,\omega) = \cap_{p\in(m_J,M_J)} IL_p(G,\omega)$.*

Using similar tools to the proof of [23, Lemma 3.1], it is obtained that $IL_J(G,\omega) \subseteq IL_{m_J}(G,\omega)$ and also $IL_J(G,\omega) \subseteq IL_{M_J}(G,\omega)$. It follows that [23, Theorem 3.2] is also valid for the weighted case. It is given in the next result.

Theorem 20. *Let G be a locally compact group and J a subset of $[1,\infty)$. Then*

$$\begin{aligned} IL_J(G,\omega) &= IL_{(m_J,M_J)}(G,\omega) = IL_{[m_J,M_J)}(G,\omega) \\ &= IL_{(m_J,M_J]}(G,\omega) = IL_{[m_J,M_J]}(G,\omega), \end{aligned} \tag{71}$$

and all are equal to $IL_{m_J}(G,\omega) \cap IL_{M_J}(G,\omega)$. Furthermore, $IL_J(G,\omega)$ is a Banach space under the following norm:

$$\|f\|_{IL_J} = \sup_{p\in J} \|f\|_{IL_p} = \max\left\{\|f\|_{IL_{m_J}}, \|f\|_{IL_{M_J}}\right\}. \tag{72}$$

Note. In [23, Example 2.4] and also the explanation after [23, Proposition 2.3] of our recent paper, there are four misprints. All four inclusions have been printed in reverse. We correct them as follows. Suppose that G is a locally compact group and $a, b \in [1,\infty]$. Then

$$\begin{aligned} \bigcap_{p\in[a,b]} L^p(G) &\subseteq \bigcap_{p\in[a,b)} L^p(G), \\ \bigcap_{p\in[a,b]} L^p(G) &\subseteq \bigcap_{p\in(a,b]} L^p(G). \end{aligned} \tag{73}$$

Also in the example,

$$\begin{aligned} \bigcap_{p\in[1,\infty]} L^p(\mathbb{R}) &\subsetneq \bigcap_{p\in[1,\infty)} L^p(\mathbb{R}), \\ \bigcap_{p\in[a,b]} L^p(\mathbb{R}) &\subsetneq \bigcap_{p\in(a,b]} L^p(\mathbb{R}). \end{aligned} \tag{74}$$

6. $IL_J(G,\omega)$ as a Banach Algebra under Convolution Product

Let G be a locally compact group and ω a positive submultiplicative weight function on G and $J \subseteq [1,\infty)$. It is appropriate to recall from the first section that $L^1(G,\omega)$ is a Banach algebra under convolution product if and only if ω is equivalent to a submultiplicative weight function. Furthermore, we provided some satisfactory results for closedness of $L^p(G,\omega)$ under convolution, in the case where $1 < p \le \infty$. According to these results, it also is noticeable to know that $IL_J(G,\omega)$ is always closed under convolution. It is provided in the next proposition.

Proposition 21. *Let G be a locally compact group and ω a positive submultiplicative weight function on G and $J \subseteq [1,\infty)$. Then $IL_J(G,\omega)$ is a Banach algebra under convolution product and norm $\|\cdot\|_J$.*

Proof. We first show that $IL_p(G,\omega)$ is a Banach algebra, for each $1 \le p < \infty$. If $p = 1$ and $f, g \in IL_1(G,\omega)$, then

$$\begin{aligned} \|f * g\|_{IL_1} &= \max\left\{\|f * g\|_{1,\omega}, \|f * g\|_{\infty,1/\check{\omega}}\right\} \\ &\le \|f\|_{1,\omega} \max\left\{\|g\|_{1,\omega}, \|g\|_{\infty,1/\check{\omega}}\right\} \\ &\le \|f\|_{IL_1} \|g\|_{IL_1}. \end{aligned} \tag{75}$$

Now let $1 < p < \infty$ and $f, g \in IL_p(G,\omega)$. Since ω is submultiplicative, $f * g \in L^p(G,\omega)$ by [11, Theorem 3.1], and so

$$\begin{aligned} \|f * g\|_{p,\omega} &\le \|f\|_{1,\omega} \|g\|_{p,\omega}, \\ \|f * g\|_{1,\omega} &\le \|f\|_{1,\omega} \|g\|_{1,\omega}. \end{aligned} \tag{76}$$

Also

$$\begin{aligned} \|f * g\|_{q,1/\check{\omega}}^q &= \int_G \left| \int_G \frac{f(y)\, g\left(y^{-1}x\right)}{\check{\omega}(x)} dy \right|^q dx \\ &\le \int_G \left| \left(f\omega * \frac{g}{\check{\omega}}\right)(x) \right|^q dx = \left\| f\omega * \frac{g}{\check{\omega}} \right\|_q^q \\ &\le \|f\|_{1,\omega}^q \|g\|_{q,1/\check{\omega}}^q. \end{aligned} \tag{77}$$

It follows that

$$\|f * g\|_{IL_p} \le \|f\|_{1,\omega} \|g\|_{IL_p} \le \|f\|_{IL_p} \|g\|_{IL_p}, \tag{78}$$

and so the result is obtained. Now let $f, g \in IL_J(G,\omega)$. Then the implication (78) implies that

$$\begin{aligned} \|f * g\|_{IL_J} &= \sup_{p\in J} \|f * g\|_{IL_p} \\ &\le \sup_{p\in J} \|f\|_{IL_p} \|g\|_{IL_p} \le \|f\|_{IL_J} \|g\|_{IL_J}, \end{aligned} \tag{79}$$

and the proof is complete. □

Proposition 21 leads us to study some algebraic properties of $IL_J(G,\omega)$. The particular object of study in this section is the amenability of $IL_J(G,\omega)$. First, we show that $IL_J(G,\omega)$ is always an abstract Segal algebra with respect to $L^1(G,\omega)$, which is interesting in its own right.

6.1. $IL_J(G,\omega)$ as an Abstract Segal Algebra. For the sake of completeness, we first repeat the basic definitions of abstract Segal algebras; see [34] for more details.

Let $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ be a Banach algebra. Then $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ is an abstract Segal algebra with respect to $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ if

(1) $\mathscr{B}$ is a dense left ideal in $\mathscr{A}$ and $\mathscr{B}$ is a Banach algebra with respect to $\|\cdot\|_{\mathscr{B}}$;

(2) there exists $M > 0$ such that $\|f\|_{\mathscr{A}} \leq M\|f\|_{\mathscr{B}}$, for each $f \in \mathscr{B}$;

(3) there exists $C > 0$ such that $\|fg\|_{\mathscr{B}} \leq C\|f\|_{\mathscr{A}}\|g\|_{\mathscr{B}}$, for each $f, g \in \mathscr{B}$.

Proposition 22. *Let G be a locally compact group and ω a positive submultiplicative weight function on G and $J \subseteq [1,\infty)$. Then $IL_J(G,\omega)$ is an abstract Segal algebra with respect to $L^1(G,\omega)$.*

Proof. We first get the result for $IL_p(G,\omega)$ whenever $1 \leq p < \infty$. Then one can easily prove this statement for $IL_J(G,\omega)$. Let $p = 1$ and $f \in IL_1(G,\omega)$ and $g \in L^1(G,\omega)$. Then for each $x \in G$

$$\left|\frac{g * f}{\check{\omega}}(x)\right| \leq \int_G |g(y)| \frac{\check{\omega}\left(y^{-1}x\right)}{\check{\omega}(x)} \frac{\left|f\left(y^{-1}x\right)\right|}{\check{\omega}\left(y^{-1}x\right)} dy \leq \left\|\frac{f}{\check{\omega}}\right\|_\infty \int_G |g(y)|\,\omega(y)\,dy = \|f\|_{\infty,1/\check{\omega}}\|g\|_{1,\omega}. \tag{80}$$

Thus $g * f \in L^\infty(G, 1/\check{\omega})$, and so $g * f \in IL_1(G,\omega)$. Hence $IL_1(G,\omega)$ is a left ideal in $L^1(G,\omega)$. Since ω is submultiplicative, then it is equivalent to a continuous function, and so

$$C_{00}(G) \subseteq IL_1(G,\omega) \subseteq L^1(G,\omega). \tag{81}$$

It follows that $IL_1(G,\omega)$ is dense in $L^1(G,\omega)$. Thus the first condition of the theory of abstract Segal algebras is satisfied. The second condition is clear. Also as we showed in the first paragraph of the proof, for each $g \in L^1(G,\omega)$ and $f \in IL_1(G,\omega)$,

$$\|g * f\|_{\infty,1/\check{\omega}} \leq \|f\|_{\infty,1/\check{\omega}}\|g\|_{1,\omega}. \tag{82}$$

Since ω is submultiplicative, then

$$\|g * f\|_{1,\omega} \leq \|g\|_{1,\omega}\|f\|_{1,\omega}. \tag{83}$$

Thus

$$\|g * f\|_{IL_1} \leq \|f\|_{IL_1}\|g\|_{1,\omega}, \tag{84}$$

and the third condition is also obtained. Now let $1 < p < \infty$. Similar arguments show that the first and the second conditions of the theory of abstract Segal algebras are satisfied. Moreover,

$$\|g * f\|_{IL_p} \leq \|f\|_{IL_p}\|g\|_{1,\omega}, \tag{85}$$

for all $f \in IL_p(G,\omega)$ and $g \in L^1(G,\omega)$. It follows that

$$\|g * f\|_{IL_J} \leq \|f\|_{IL_J}\|g\|_{1,\omega}, \tag{86}$$

and so the proof is completed. □

6.2. Amenability of $IL_J(G,\omega)$ and Its Second Dual. Let $\mathscr{A}$ be a Banach algebra and X a Banach $\mathscr{A}$-bimodule. A derivation is a linear map $D : \mathscr{A} \to X$ such that

$$D(ab) = aD(b) + D(a)b \quad (a, b \in \mathscr{A}). \tag{87}$$

A derivation D from $\mathscr{A}$ into X is inner if there is $\xi \in X$ such that

$$D(a) = a\xi - \xi a \quad (a \in \mathscr{A}). \tag{88}$$

The Banach algebra $\mathscr{A}$ is amenable if every continuous derivation $D : A \to X^*$ is inner for all Banach $\mathscr{A}$-bimodules X.

As a vital result, we first turn our attention to the fact that $IL_J(G,\omega)$ admits a bounded left approximate identity just when it is equal to $L^1(G,\omega)$. It is in fact a direct result due to Burnham [35], as the following.

Lemma 23. *Let $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ be an abstract Segal algebra with respect to $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ and $(e_\alpha)_{\alpha\in\Lambda}$ a left approximate identity of $\mathscr{B}$. If $\mathscr{B}$ is a proper subset of $\mathscr{A}$, then $(e_\alpha)_{\alpha\in\Lambda}$ is not bounded in the $\mathscr{B}$ norm.*

The next result is completely fulfilled from Proposition 22 and Lemma 23.

Proposition 24. *Let G be a locally compact group and ω a positive submultiplicative weight function on G and $J \subseteq [1,\infty)$. If $IL_J(G,\omega)$ possesses a bounded left approximate identity then $IL_J(G,\omega) = L^1(G,\omega)$, as Banach algebras.*

Theorem 25. *Let G be a locally compact group and ω a positive submultiplicative weight function on G and $J \subseteq [1,\infty)$. Then $IL_J(G,\omega)$ is amenable if and only if G is discrete and amenable and ω^* is bounded.*

Proof. First let $IL_J(G,\omega)$ be amenable. Then $IL_J(G,\omega)$ possesses a bounded approximate identity [36, Proposition 1.6], and by Proposition 24, $IL_J(G,\omega) = L^1(G,\omega)$ as Banach algebras. Thus $L^1(G,\omega)$ is amenable which implies that G is amenable and ω^* is bounded [37]. To that end, we show that G is discrete. If there exists $p \in J$ with $1 < p < \infty$, then $IL_J(G,\omega) = L^1(G,\omega)$ follows that $L^1(G,\omega) \subseteq L^p(G,\omega)$, and so G is discrete by the explanation before Proposition 15. In the case where $J = \{1\}$, note that $IL_1(G,\omega) = L^1(G,\omega)$ implies that $L^1(G,\omega) \subseteq L^\infty(G, 1/\check{\omega})$. By the boundedness of ω^*, we have $L^1(G) \subseteq L^\infty(G)$, and the discreteness of G is obtained by [32, Theorem 1]. Conversely, suppose that G is discrete and amenable and ω^* is bounded. By [36, Theorem 2.5], $\ell^1(G)$ is an amenable Banach algebra. Proposition 15 and also [37] yield that

$$IL_J(G,\omega) = \ell^1(G,\omega) = \ell^1(G) \tag{89}$$

as Banach algebra. Therefore, $IL_J(G,\omega)$ is an amenable Banach algebra. □

For every Banach algebra $\mathscr{A}$, there exist two (Arens) products $\square$ and $\lozenge$ on the second dual $\mathscr{A}^{**}$, extending the

product of $\mathcal{A}$. For further details on the properties of Arens products see the survey article [38]. We end this work with the next theorem which provides a necessary and sufficient condition for the amenability of $\mathcal{A}^{**}$.

Theorem 26. *Let G be a locally compact group and ω a positive submultiplicative weight function on G and $J \subseteq [1, \infty)$. Then the following statements are equivalent.*

(i) $IL_J(G,\omega)^{**}$ *is amenable.*

(ii) $L^1(G,\omega)^{**}$ *is amenable.*

(iii) G *is finite.*

Proof. (i) $\Rightarrow$ (ii) If $IL_J(G,\omega)^{**}$ is amenable, then so is $IL_J(G,\omega)$ by [39] and also [40]. Then Theorem 25 implies that G is discrete and by Proposition 15, $IL_J(G,\omega) = \ell^1(G,\omega)$ and $\|f\|_{IL_J} = \|f\|_{1,\omega}$, for each $f \in \ell^1(G,\omega)$. It follows that $\ell^1(G,\omega)^{**}$ is amenable.

(ii) $\Rightarrow$ (iii) It is obtained from [41, Theorem 4].

(iii) $\Rightarrow$ (i) If G is finite, then $IL_J(G,\omega) = L^1(G,\omega)$, as Banach algebras and the result is obtained by [41, Theorem 4]. □

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