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Research Article

Refinements of Hardy-Type Inequalities

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Using Hu Ke's inequality, which is a sharped Hölder's inequality, we present some new refinements of Hardy-type inequalities proposed by Imoru.

1. Introduction

Let $f(x) \geq 0$, $f(x) \in L^p(0,\infty)$, $p > 1$. Then the famous Hardy's inequality [1, Theorem 319] reads as

$$\int_0^\infty \left[\int_0^\infty K(x,y) f(x)\, dx\right]^p dy \leq \left[\int_0^\infty K(x,1) x^{-1/p} dx\right]^p \int_0^\infty f^p(x)\, dx, \quad (1)$$

where $K(x,y)$ is nonnegative and homogeneous of degree -1. The sign of the inequality in (1) is reversed if $0 < p < 1$. The special cases of inequality (1) are the subject of the following theorem, which is also due to Hardy et al. [1, Theorem 330].

Theorem A. *Let $p > 1$, $f(x)$ be nonnegative and Lebesgue integrable on $[0,a]$ or $[a,\infty)$ for every $a > 0$, according to $r > 1$ or $r < 1$. Then*

$$\int_0^\infty y^{-r} F^p(y)\, dy \leq \left(\frac{p}{r-1}\right)^p \int_0^\infty x^{-r} [x f(x)]^p dx \quad (r > 1),$$
$$\int_0^\infty y^{-r} F^p(y)\, dy \leq \left(\frac{p}{1-r}\right)^p \int_0^\infty x^{-r} [x f(x)]^p dx \quad (r < 1), \quad (2)$$

where

$$F(y) = \begin{cases} \int_0^y f(x)\, dx & (r > 1) \\ \int_y^\infty f(x)\, dx & (r < 1). \end{cases} \quad (3)$$

The signs of the inequalities are reversed if $0 < p < 1$.

As is well known, inequalities (2) play a very important role in both theory and applications. Ever since Hardy discovered inequalities (2), they have been studied by many authors, who either reproved them using various techniques or improved, generalized, and applied them in many different ways (see e.g. [2–22] and references therein). For further remarks concerning the improvements and properties of inequalities (2) and their generalizations, see for example, [23] or [24].

In the year 1977, Imoru [6] obtained the following integral inequalities which are related to Hardy's (see Theorem A).

Theorem B. *Let g be continuous and nondecreasing on $[0,\infty]$ with $g(0) = 0$, $g(x) > 0$ for $x > 0$ and $g(\infty) = \infty$. Let $f(x)$ be nonnegative and Lebesgue integrable with respect to $g(x)$ on $[0,b]$ or on $[a,\infty)$ according to $r > 1$ or $r < 1$, where $a, b > 0$. Suppose*

$$F(x) = \begin{cases} \int_0^x f(t)\, dg(t) & (r > 1) \\ \int_x^\infty f(t)\, dg(t) & (r < 1). \end{cases} \quad (4)$$

If $p \geq 1$, then

$$\int_0^b g^{-r}(x) F^p(x)\, dg(x) + \frac{p}{r-1} g^{1-r}(b) F^p(b) \leq \left(\frac{p}{r-1}\right)^p \int_0^b g^{-r}(x) [f(x) g(x)]^p dg(x) \quad (r > 1),$$

$$\int_a^\infty g^{-r}(x)\,F^p(x)\,dg(x)+\frac{p}{1-r}g^{1-r}(a)\,F^p(a) \le \left(\frac{p}{1-r}\right)^p\int_a^\infty g^{-r}(x)\,[f(x)\,g(x)]^p dg(x)\quad (r<1), \tag{5}$$

with both signs of inequalities reversed if $0<p\le 1$.

Later, in 1981, Chan in [2] derived several exponential generalizations of the Imoru's inequalities (5). In 1985, Imoru in [7] presented further extensions of (5). Moreover, in 1988, Yang et al. [22] gave some new generalizations of (5). Recently, Oguntuase and Imoru in [10] obtained other generalizations of the Yang et al.'s results.

The main purpose of this work is to give some improvements of inequalities (5) by using Hu Ke's inequality which is a sharp Hölder's inequality.

2. A Set of Lemmas

In this section, we will prove lemmas, which play crucial roles in proving our main results.

Lemma 1 (see [25] Hu Ke's inequality). *Let* $f(x), g(x)$, *and* $e(x)$ *be integrable functions defined on* $[0,+\infty)$ *and* $f(x), g(x)\ge 0, 1-e(x)+e(y)\ge 0$ *for all* $x, y\in[0,+\infty)$, *and let* $p>1, 1/p+1/q=1$. *Then*

$$\int_0^\infty f(x)\,g(x)\,dx \le \left(\int_0^\infty f^p(x)\,dx\right)^{1/p}\left(\int_0^\infty g^q(x)\,dx\right)^{1/q} \times\left[1-\left(\frac{\int_0^\infty f^p(x)\,e(x)\,dx}{\int_0^\infty f^p(x)\,dx}-\frac{\int_0^\infty g^q(x)\,e(x)\,dx}{\int_0^\infty g^q(x)\,dx}\right)^2\right]^{\tau/2}, \tag{6}$$

where $\tau=\min\{1/p,1/q\}$.

Lemma 2 (see [26]). *Let* $f(x), g(x)$, *and* $e(x)$ *be integrable functions defined on* $[0,+\infty)$ *and* $f(x)\ge 0, g(x)>0, 1-e(x)+e(y)\ge 0$ *for all* $x, y\in[0,+\infty)$, *let* $q<0, 1/p+1/q=1$, *and let* $f(x), g(x)\in L^q(0,\infty)$. *Then*

$$\int_0^\infty f(x)\,g(x)\,dx \ge \left(\int_0^\infty f^p(x)\,dx\right)^{1/p}\left(\int_0^\infty g^q(x)\,dx\right)^{1/q} \times\left[1-\left(\frac{\int_0^\infty f(x)\,g(x)\,e(x)\,dx}{\int_0^\infty f(x)\,g(x)\,dx}-\frac{\int_0^\infty g^q(x)\,e(x)\,dx}{\int_0^\infty g^q(x)\,dx}\right)^2\right]^{\rho/2}, \tag{7}$$

where $\rho=\max\{-1,1/q\}$.

Lemma 3. *Let* g *be continuous and nondecreasing on* $[a,b]$. *Let* $\varphi(x,t)$ *and* $e(x)$ *be integrable functions and* $\varphi(x,t)\ge 0$, $1-e(x)+e(y)\ge 0$ *for all* $x, y, t\in[0,+\infty)$, *and let* ϕ *be nondecreasing. If* $p\ge 1$, *then*

$$\int_0^b g^{-1}(x)\left[\int_0^x \varphi(x,t)\,d\phi(t)\right]dg(x) \ge \int_0^b\left\{g^{-1}(x)\left[\int_0^x \varphi^{1/p}(x,t)\,d\phi(t)\right]^p\times\left[\int_0^x d\phi(t)\right]^{1-p} \times\left[1-\left(\frac{\int_0^x \varphi(x,t)\,e(t)\,d\phi(t)}{\int_0^x \varphi(x,t)\,d\phi(t)}-\frac{\int_0^x e(t)\,d\phi(t)}{\int_0^x d\phi(t)}\right)^2\right]^{\beta/2}\right\}dg(x), \tag{8}$$

$$\int_a^\infty g^{-1}(x)\left[\int_x^\infty \varphi(x,t)\,d\phi(t)\right]dg(x) \ge \int_a^\infty\left\{g^{-1}(x)\left[\int_x^\infty \varphi^{1/p}(x,t)\,d\phi(t)\right]^p\times\left[\int_x^\infty d\phi(t)\right]^{1-p} \times\left[1-\left(\frac{\int_x^\infty \varphi(x,t)\,e(t)\,d\phi(t)}{\int_x^\infty \varphi(x,t)\,d\phi(t)}-\frac{\int_x^\infty e(t)\,d\phi(t)}{\int_x^\infty d\phi(t)}\right)^2\right]^{\beta/2}\right\}dg(x), \tag{9}$$

where $\beta=\max\{-1,1-p\}$. *If* $0<p<1$, *then*

$$\int_a^\infty g^{-1}(x)\left[\int_x^\infty \varphi(x,t)\,d\phi(t)\right]dg(x) \le \int_a^\infty\left\{g^{-1}(x)\left[\int_x^\infty \varphi^{1/p}(x,t)\,d\phi(t)\right]^p\times\left[\int_x^\infty d\phi(t)\right]^{1-p}\right.$$

$$\times \left[1-\left(\frac{\int_x^\infty \varphi^{1/p}(x,t)\,e(t)\,d\phi(t)}{\int_x^\infty \varphi^{1/p}(x,t)\,d\phi(t)} - \frac{\int_x^\infty e(t)\,d\phi(t)}{\int_x^\infty d\phi(t)}\right)^2\right]^{\gamma/2}\Bigg\}\,dg(x),$$

$$\int_a^\infty g^{-1}(x)\left[\int_x^\infty \varphi(x,t)\,d\phi(t)\right]dg(x)$$
$$\leq \int_a^\infty \Bigg\{g^{-1}(x)\left[\int_x^\infty \varphi^{1/p}(x,t)\,d\phi(t)\right]^p \times \left[\int_x^\infty d\phi(t)\right]^{1-p}$$
$$\times \left[1-\left(\frac{\int_x^\infty \varphi^{1/p}(x,t)\,e(t)\,d\phi(t)}{\int_x^\infty \varphi^{1/p}(x,t)\,d\phi(t)} - \frac{\int_x^\infty e(t)\,d\phi(t)}{\int_x^\infty d\phi(t)}\right)^2\right]^{\gamma/2}\Bigg\}\,dg(x), \tag{10}$$

where $\gamma = \min\{p, 1-p\}$.

Proof. From Lemmas 1 and 2, the conclusion is easy to obtain. □

Lemma 4. *Let g be continuous and nondecreasing on $[0,\infty]$ with $g(0) = 0$, $g(x) > 0$ for $x > 0$ and $g(\infty) = \infty$. Let $\delta = (1-r)/p$, $r \neq 1$, and $f(x)$, $e(x)$ be nonnegative and Lebesgue integrable with respect to $g(x)$ on $[0,b]$ or on $[a,\infty)$ according to $r > 1$ or $r < 1$, where $a, b > 0$, and let $1 - e(x) + e(y) \geq 0$ for all $x, y \in [0,+\infty)$. Suppose*

$$\lambda(x) = \begin{cases} \int_0^x [g(t)]^{(p-1)(1+\delta)} f^p(t)\,dg(t) & (r > 1) \\ \int_x^\infty [g(t)]^{(p-1)(1+\delta)} f^p(t)\,dg(t) & (r < 1). \end{cases} \tag{11}$$

If $p \geq 1$, then

$$\int_0^b g^{\delta-1}(x)\,\lambda(x)\,dg(x)$$
$$\geq \left(-\delta^{-1}\right)^{1-p}$$
$$\times \int_0^b \Bigg\{[g(x)]^{\delta p-1} F^p(x)$$
$$\times \left[1-\left(\frac{\int_0^x g^{-(1+\delta)}(t)\,e(t)\,dg(t)}{\int_0^x g^{-(1+\delta)}(t)\,dg(t)} - \frac{\int_0^x g^{(1+\delta)(p-1)}(t)\,f^p(t)\,e(t)\,dg(t)}{\int_0^x g^{(1+\delta)(p-1)}(t)\,f^p(t)\,dg(t)}\right)^2\right]^{\beta/2}\Bigg\}\,dg(x)$$
$$(r > 1), \tag{12}$$

$$\int_a^\infty g^{\delta-1}(x)\,\lambda(x)\,dg(x)$$
$$\geq \delta^{p-1}$$
$$\times \int_a^\infty \Bigg\{[g(x)]^{\delta p-1} F^p(x)$$
$$\times \left[-\left(\frac{\int_x^\infty g^{-(1+\delta)}(t)\,e(t)\,dg(t)}{\int_x^\infty g^{-(1+\delta)}(t)\,dg(t)}\right.\right.^{\beta/2}$$
$$\left.\left. - \frac{\int_x^\infty g^{(1+\delta)(p-1)}(t)\,f^p(t)\,e(t)\,dg(t)}{\int_x^\infty g^{(1+\delta)(p-1)}(t)\,f^p(t)\,dg(t)}\right)^2\right]^{\beta/2}\Bigg\}\,dg(x)$$
$$(r < 1), \tag{13}$$

where $F(x)$ is as in Theorem B, $\beta = \max\{-1, 1-p\}$. If $0 < p < 1$, then

$$\int_0^b g^{\delta-1}(x)\lambda(x)\,dg(x) \le \left(-\delta^{-1}\right)^{1-p} \times \int_0^b \left\{ [g(x)]^{\delta p-1} F^p(x) \times \left[1 - \left(\frac{\int_0^x f(t)e(t)\,dg(t)}{\int_0^x f(t)\,dg(t)} - \frac{\int_0^x g^{-(1+\delta)}(t)e(t)\,dg(t)}{\int_0^x g^{-(1+\delta)}(t)\,dg(t)}\right)^2\right]^{\gamma/2}\right\} dg(x) \quad (r > 1),$$

$$\int_a^\infty g^{\delta-1}(x)\lambda(x)\,dg(x) \le \delta^{p-1} \times \int_a^\infty \left\{ [g(x)]^{\delta p-1} F^p(x) \times \left[1 - \left(\frac{\int_x^\infty f(t)e(t)\,dg(t)}{\int_x^\infty f(t)\,dg(t)} - \frac{\int_x^\infty g^{-(1+\delta)}(t)e(t)\,dg(t)}{\int_x^\infty g^{-(1+\delta)}(t)\,dg(t)}\right)^2\right]^{\gamma/2}\right\} dg(x) \quad (r < 1), \tag{14}$$

where $F(x)$ is as in Theorem B, $\gamma = \min\{p, 1-p\}$.

Proof. We only prove inequality (12); the proofs of (13) and (14) are similar. Let $\varphi(x,t) = g^\delta(x)[g(t)]^{p(1+\delta)} f^p(t)$, $d\phi(t) = [g(t)]^{-(1+\delta)} dg(t)$ in inequality (8). Then, if $r, p > 1$, we have

$$\begin{aligned}
\int_0^b g^{\delta-1}(x)\lambda(x)\,dg(x) &= \int_0^b g^{-1}(x)\left[\int_0^x \varphi(x,t)\,d\phi(t)\right] dg(x) \\
&\ge \int_0^b \left\{ g^{-1}(x)\left[\int_0^x \varphi^{1/p}(x,t)\,d\phi(t)\right]^p \times \left[\int_0^x d\phi(t)\right]^{1-p} \times \left[1 - \left(\frac{\int_0^x \varphi(x,t)e(t)\,d\phi(t)}{\int_0^x \varphi(x,t)\,d\phi(t)} - \frac{\int_0^x e(t)\,d\phi(t)}{\int_0^x d\phi(t)}\right)^2\right]^{\beta/2}\right\} dg(x) \\
&= \int_0^b \left\{ g^{-1+\delta}(x)\left[\int_0^x f(t)\,dg(t)\right]^p \left[\int_0^x g^{-(1+\delta)}(t)\,dg(t)\right]^{1-p} \right. \\
&\quad \left. \times \left[1 - \left(\frac{\int_0^x g^{(1+\delta)(p-1)}(t) f^p(t)e(t)\,dg(t)}{\int_0^x g^{(1+\delta)(p-1)}(t) f^p(t)\,dg(t)} - \frac{\int_0^x g^{-(1+\delta)}(t)e(t)\,dg(t)}{\int_0^x g^{-(1+\delta)}(t)\,dg(t)}\right)^2\right]^{\beta/2}\right\} dg(x) \\
&= \left(-\delta^{-1}\right)^{1-p} \times \int_0^b \left\{ [g(x)]^{\delta p-1} F^p(x) \times \left[1 - \left(\frac{\int_0^x g^{(1+\delta)(p-1)}(t) f^p(t)e(t)\,dg(t)}{\int_0^x g^{(1+\delta)(p-1)}(t) f^p(t)\,dg(t)} - \frac{\int_0^x g^{-(1+\delta)}(t)e(t)\,dg(t)}{\int_0^x g^{-(1+\delta)}(t)\,dg(t)}\right)^2\right]^{\beta/2}\right\} dg(x).
\end{aligned} \tag{15}$$

This proves inequality (12). Lemma 4 is proved. □

Lemma 5. *With notation as in Lemma 4, one has the results as follows. If $p > 1$, then*

$$g^\delta(b)\lambda(b) \ge \left(-\delta^{-1}\right)^{1-p} g^{\delta p}(b) F^p(b) \times \left[1 - \left(\frac{\int_0^b g^{-(1+\delta)}(t)e(t)\,dg(t)}{\int_0^b g^{-(1+\delta)}(t)\,dg(t)} - \frac{\int_0^b g^{(1+\delta)(p-1)}(t) f^p(t)e(t)\,dg(t)}{\int_0^b g^{(1+\delta)(p-1)}(t) f^p(t)\,dg(t)}\right)^2\right]^{\beta/2} \quad (r > 1), \tag{16}$$

$$g^\delta(a)\lambda(a) \ge \delta^{p-1} g^{\delta p}(a) F^p(a) \times \left[1 - \left(\frac{\int_a^\infty g^{-(1+\delta)}(t)e(t)\,dg(t)}{\int_a^\infty g^{-(1+\delta)}(t)\,dg(t)} - \frac{\int_a^\infty g^{(1+\delta)(p-1)}(t) f^p(t)e(t)\,dg(t)}{\int_a^\infty g^{(1+\delta)(p-1)}(t) f^p(t)\,dg(t)}\right)^2\right]^{\beta/2} \quad (r < 1). \tag{17}$$

If $0 < p < 1$, *then*

$$\begin{aligned}
&g^{\delta}(b)\lambda(b) \\
&\quad\le \left(-\delta^{-1}\right)^{1-p} g^{\delta p}(b) F^{p}(b) \\
&\quad\times\left[1-\left(\frac{\int_0^b f(t)e(t)\,dg(t)}{\int_0^b f(t)\,dg(t)}\right.\right. \\
&\qquad\left.\left.-\frac{\int_0^b g^{-(1+\delta)}(t)e(t)\,dg(t)}{\int_0^b g^{-(1+\delta)}(t)\,dg(t)}\right)^2\right]^{\gamma/2} \quad (r>1), \\
&g^{\delta}(a)\lambda(a) \\
&\quad\le \delta^{p-1} g^{\delta p}(a) F^{p}(a) \\
&\quad\times\left[1-\left(\frac{\int_a^\infty f(t)e(t)\,dg(t)}{\int_a^\infty f(t)\,dg(t)}\right.\right. \\
&\qquad\left.\left.-\frac{\int_a^\infty g^{-(1+\delta)}(t)e(t)\,dg(t)}{\int_a^\infty g^{-(1+\delta)}(t)\,dg(t)}\right)^2\right]^{\gamma/2} \quad (r<1).
\end{aligned} \tag{18}$$

Proof. We only prove inequality (16); the proofs of (17) and (18) are similar. If r, $p > 1$, by using inequality (8), we have

$$\begin{aligned}
&g^{\delta}(b)\lambda(b) \\
&\quad= g^{\delta}(b)\int_0^b g^{(p-1)(1+\delta)}(t) f^{p}(t)\,dg(t) \\
&\quad= \int_0^b \varphi(b,t)\,d\phi(t) \\
&\quad\ge \left[\int_0^b \varphi^{1/p}(b,t)\,d\phi(t)\right]^{p}\times\left[\int_0^b d\phi(t)\right]^{1-p} \\
&\quad\times\left[1-\left(\frac{\int_0^b \varphi(b,t)e(t)\,d\phi(t)}{\int_0^b \varphi(b,t)\,d\phi(t)}-\frac{\int_0^b e(t)\,d\phi(t)}{\int_0^b d\phi(t)}\right)^2\right]^{\beta/2} \\
&\quad= \left(-\delta^{-1}\right)^{1-p} g^{\delta p}(b) F^{p}(b) \\
&\quad\times\left[1-\left(\frac{\int_0^b g^{-(1+\delta)}(t)e(t)\,dg(t)}{\int_0^b g^{-(1+\delta)}(t)\,dg(t)}\right.\right. \\
&\qquad\left.\left.-\frac{\int_0^b g^{(1+\delta)(p-1)}(t) f^{p}(t)e(t)\,dg(t)}{\int_0^b g^{(1+\delta)(p-1)}(t) f^{p}(t)\,dg(t)}\right)^2\right]^{\beta/2},
\end{aligned} \tag{19}$$

where $\varphi(b,t) = g^{\delta}(b)(g(t))^{p(1+\delta)} f^{p}(t)$, $d\phi(t) = (g(t))^{-(1+\delta)}dg(t)$. This proves inequality (16). Lemma 5 is proved. □

Lemma 6 (see [23]). *If* $x > -1$, $\alpha > 1$, *or* $\alpha < 0$, *then*

$$(1+x)^{\alpha} \ge 1+\alpha x. \tag{20}$$

The inequality is reversed for $0 < \alpha < 1$.

3. Refinements of Hardy-Type Inequalities

Theorem 7. *Let* g *be continuous and nondecreasing on* $[0,\infty]$ *with* $g(0) = 0$, $g(x) > 0$ *for* $x > 0$ *and* $g(\infty) = \infty$. *Let* $f(x)$, *and* $e(x)$ *be nonnegative and Lebesgue integrable with respect to* $g(x)$ *on* $[0,b]$ *or on* $[a,\infty)$ *according to* $r > 1$ *or* $r < 1$, *where* a, $b > 0$, *and let* $1 - e(x) + e(y) \ge 0$ *for all* x, $y \in [0,+\infty)$. *Suppose* $F(x)$ *is as in Theorem B. If* $p \ge 1$, *then*

$$\begin{aligned}
&\int_0^b g^{-r}(x)F^{p}(x)\left[1-\frac{\beta}{2}\omega^2(f,g,e;x)\right]dg(x) \\
&\quad+\frac{p}{r-1}g^{1-r}(b)F^{p}(b)\left[1-\frac{\beta}{2}\omega^2(f,g,e;b)\right] \\
&\quad\le\left(\frac{p}{r-1}\right)^{p}\int_0^b g^{-r}(x)\left[f(x)g(x)\right]^{p}dg(x) \quad (r>1),
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\int_a^\infty g^{-r}(x)F^{p}(x)\left[1-\frac{\beta}{2}\varpi^2(f,g,e;x)\right]dg(x) \\
&\quad+\frac{p}{1-r}g^{1-r}(b)F^{p}(b)\left[1-\frac{\beta}{2}\varpi^2(f,g,e;a)\right] \\
&\quad\le\left(\frac{p}{1-r}\right)^{p}\int_0^b g^{-r}(x)\left[f(x)g(x)\right]^{p}dg(x) \quad (r<1),
\end{aligned} \tag{22}$$

where $\beta = \max\{-1, 1-p\}$, $\delta = (1-r)/p$,

$$\begin{aligned}
\omega(f,g,e;x) &= \frac{\int_0^x g^{-(1+\delta)}(t)e(t)\,dg(t)}{\int_0^x g^{-(1+\delta)}(t)\,dg(t)} \\
&\quad-\frac{\int_0^x g^{(1+\delta)(p-1)}(t)f^{p}(t)e(t)\,dg(t)}{\int_0^x g^{(1+\delta)(p-1)}(t)f^{p}(t)\,dg(t)}, \\
\varpi(f,g,e;x) &= \frac{\int_x^\infty g^{-(1+\delta)}(t)e(t)\,dg(t)}{\int_x^\infty g^{-(1+\delta)}(t)\,dg(t)} \\
&\quad-\frac{\int_x^\infty g^{(1+\delta)(p-1)}(t)f^{p}(t)e(t)\,dg(t)}{\int_x^\infty g^{(1+\delta)(p-1)}(t)f^{p}(t)\,dg(t)}.
\end{aligned} \tag{23}$$

If $0 < p < 1$*, then*

$$\int_0^b g^{-r}(x) F^p(x)\left[1-\frac{\gamma}{2}\mu^2(f,g,e;x)\right]dg(x) + \frac{p}{r-1}g^{1-r}(b)F^p(b)\left[1-\frac{\gamma}{2}\mu^2(f,g,e;b)\right] \geq \left(\frac{p}{r-1}\right)^p\int_0^b g^{-r}(x)[f(x)g(x)]^p dg(x) \quad (r>1),$$
$$\int_a^\infty g^{-r}(x) F^p(x)\left[1-\frac{\gamma}{2}\nu^2(f,g,e;x)\right]dg(x) + \frac{p}{1-r}g^{1-r}(b)F^p(b)\left[1-\frac{\gamma}{2}\nu^2(f,g,e;a)\right] \geq \left(\frac{p}{1-r}\right)^p\int_0^b g^{-r}(x)[f(x)g(x)]^p dg(x) \quad (r<1), \tag{24}$$

where $\gamma = \min\{p, 1-p\}$, $\delta = (1-r)/p$,

$$\mu(f,g,e;x) = \frac{\int_0^x f(t)e(t)dg(t)}{\int_0^x f(t)dg(t)} - \frac{\int_0^x g^{-(1+\delta)}(t)e(t)dg(t)}{\int_0^x g^{-(1+\delta)}(t)dg(t)},$$
$$\nu(f,g,e;x) = \frac{\int_x^\infty f(t)e(t)dg(t)}{\int_x^\infty f(t)dg(t)} - \frac{\int_x^\infty g^{-(1+\delta)}(t)e(t)dg(t)}{\int_x^\infty g^{-(1+\delta)}(t)dg(t)}. \tag{25}$$

Proof. We only prove the case $p \geq 1$; the proof of case $0 < p < 1$ is similar.

(i) When $r > 1$, by using the nondecreasing property of g, we have

$$0 < \lambda(x) = \int_0^x g^{-(1-p)(1+\delta)}(t) f^p(t) dg(t) = \int_0^x g^{(r-1)/p}(t)\left[g^{p-r}(t) f^p(t)\right]dg(t) \leq g^{(r-1)/p}(x)\int_0^x g^{p-r}(t) f^p(t) dg(t), \tag{26}$$

and hence

$$\lim_{x\to 0^+} g^\delta(x)\lambda(x) = 0, \tag{27}$$

from which and from inequality (12) we have, on using integration by parts, □

$$\int_0^b g^{\delta-1}(x)\lambda(x)dg(x) = \delta^{-1}g^\delta(b)\lambda(b) - \delta^{-1}\int_0^b g^{\delta p-1}(x)[g(x)f(x)]^p dg(x) \geq \left(-\delta^{-1}\right)^{1-p} \times \int_0^b \left\{[g(x)]^{\delta p-1}F^p(x) \times \left[-\left(\frac{\int_0^x g^{-(1+\delta)}(t)e(t)dg(t)}{\int_0^x g^{-(1+\delta)}(t)dg(t)} - \left(\left(\int_0^x g^{(1+\delta)(p-1)}(t)f^p(t)e(t)dg(t)\right) \times \left(\int_0^x g^{(1+\delta)(p-1)}(t)f^p(t)dg(t)\right)^{-1}\right)\right)^2\right]^{\beta/2}\right\}dg(x). \tag{28}$$

That is,

$$\int_0^b\left\{[g(x)]^{\delta p-1}F^p(x) \times \left[1-\left(\frac{\int_0^x g^{-(1+\delta)}(t)e(t)dg(t)}{\int_0^x g^{-(1+\delta)}(t)dg(t)} - \frac{\int_0^x g^{(1+\delta)(p-1)}(t)f^p(t)e(t)dg(t)}{\int_0^x g^{(1+\delta)(p-1)}(t)f^p(t)dg(t)}\right)^2\right]^{\beta/2}\right\}dg(x) \leq -\left(\frac{p}{r-1}\right)^p g^\delta(b)\lambda(b) + \left(\frac{p}{r-1}\right)^p\int_0^b g^{-r}(x)[g(x)f(x)]^p dg(x). \tag{29}$$

Combining inequalities (16), (20), and (29) yields inequality (21).

(ii) When $r < 1$, by the same method as in case (i), we obtain

$$0 < \lambda(x) = \int_x^\infty g^{-(1-p)(1+\delta)}(t) f^p(t) dg(t) \leq g^{(r-1)/p}(x)\int_x^\infty g^{p-r}(t) f^p(t) dg(t), \tag{30}$$

and hence

$$\lim_{x\to\infty} g^\delta(x)\lambda(x) = 0, \tag{31}$$

from which and from inequality (13) we have, on using integration by parts,

$$\begin{aligned}
&\int_a^{\infty} g^{\delta-1}(x)\,\lambda(x)\,dg(x)\\
&= (-\delta)^{-1} g^{\delta}(a)\,\lambda(a) + \delta^{-1}\int_a^{\infty} g^{\delta p-1}(x)\,[g(x)\,f(x)]^{p}\,dg(x)\\
&\geq \delta^{p-1}\\
&\quad\times\int_a^{\infty}\Bigg\{[g(x)]^{\delta p-1}F^{p}(x)\\
&\qquad\times\Bigg[1-\Bigg(\frac{\int_x^{\infty} g^{-(1+\delta)}(t)\,e(t)\,dg(t)}{\int_x^{\infty} g^{-(1+\delta)}(t)\,dg(t)}\\
&\qquad\quad-\frac{\int_x^{\infty} g^{(1+\delta)(p-1)}(t)\,f^{p}(t)\,e(t)\,dg(t)}{\int_x^{\infty} g^{[1+\delta](p-1)}(t)\,f^{p}(t)\,dg(t)}\Bigg)^{2}\Bigg]^{\beta/2}\Bigg\}\,dg(x).
\end{aligned}\tag{32}$$

That is,

$$\begin{aligned}
&\int_a^{\infty}\Bigg\{(g(x))^{\delta p-1}F^{p}(x)\\
&\quad\times\Bigg[1-\Bigg(\frac{\int_x^{\infty} g^{-(1+\delta)}(t)\,e(t)\,dg(t)}{\int_x^{\infty} g^{-(1+\delta)}(t)\,dg(t)}\\
&\qquad-\frac{\int_x^{\infty} g^{(1+\delta)(p-1)}(t)\,f^{p}(t)\,e(t)\,dg(t)}{\int_x^{\infty} g^{(1+\delta)(p-1)}(t)\,f^{p}(t)\,dg(t)}\Bigg)^{2}\Bigg]^{\beta/2}\Bigg\}\,dg(x)\\
&\leq -\left(\frac{p}{1-r}\right)^{p} g^{\delta}(b)\,\lambda(b)\\
&\quad+\left(\frac{p}{1-r}\right)^{p}\int_0^{b} g^{-r}(x)\,[g(x)\,f(x)]^{p}\,dg(x).
\end{aligned}\tag{33}$$

Combining inequalities (17), (20), and (33) yields inequality (22). The proof of Theorem 7 is complete.

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