Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 374938, 7 pages
http://dx.doi.org/10.1155/2013/374938

Research Article

Complete Controllability of Impulsive Fractional Linear Time-Invariant Systems with Delay

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Received 27 February 2013; Accepted 23 June 2013

Academic Editor: G. M. N'Guérékata

Some flaws on impulsive fractional differential equations (systems) have been found. This paper is concerned with the complete controllability of impulsive fractional linear time-invariant dynamical systems with delay. The criteria on the controllability of the system, which is sufficient and necessary, are established by constructing suitable control inputs. Two examples are provided to illustrate the obtained results.

1. Introduction

Recently, a variety of problems such as the existence, uniqueness of mild solution for the initial value problem, periodic boundary value problems, antiperiodic boundary value problems, and Ulam stability for impulsive fractional differential equations have been considered due to their important role in modeling natural phenomena such as medicine, biology, and optimal control; see the paper [1–16].

The concept of controllability plays an important role in the analysis and design of control systems. With the developments of theories of impulsive fractional differential equations, there have been a few papers devoted to the controllability of impulsive fractional differential systems; see [17–20]. In [17], the author discussed the controllability of impulsive fractional linear time-invariant systems through constructing a suitable control input in time domain. By fixed point theorem, the controllability of integrodifferential systems was investigated in [18–20]. It should be mentioned that the controllability for linear fractional dynamical systems has been investigated by several scholars [21–26] while the theory of controllability for impulsive fractional linear time-invariant systems is still in the initial stage [17].

The impulsive fractional differential equations (systems) which had been investigated earlier often have the form

$$ {}^{c}D_{0,t}^{\alpha}x(t) = f(t, x(t)), \quad t \in J' := J \setminus \{t_1, t_2, \dots, t_m\}, \quad J := [0, T], \tag{1}$$

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad k = 1, 2, \dots, m, \tag{2}$$

$$x(0) = x_0 \tag{3}$$

or

$$ {}^{c}D_{0,t}^{\alpha}x(t) = Ax(t) + Bu(t), \quad t \in J' := J \setminus \{t_1, t_2, \dots, t_m\}, \quad J := [0, T], \tag{4}$$

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad k = 1, 2, \dots, m, \tag{5}$$

$$y(t) = Cx(t) + Du(t), \tag{6}$$

$$x(0) = x_0 \tag{7}$$

and so forth, where ${}^{c}D_{0,t}^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$ with lower limit zero, $x_0 \in \mathbb{R}$, f is jointly continuous, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, t_k satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $x(t_k^+) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)$, and $x(t_k^-) = \lim_{\epsilon \to 0^-} x(t_k + \epsilon)$ represent, respectively, the right and the left limits of $x(t)$ at $t = t_k$, A, B, C, D are the known constant matrices, $x(t)$, $y(t)$, $u(t)$ are vectors with appropriate dimensions.

However, the function $x(t)$ defined on $[0, T]$ is continuous everywhere except for finite number of points t_k, $k = 1, 2, \dots, m$, at which the limits $x(t_k^+)$ and $x(t_k^-)$ exist with

$x(t_k) = x(t_k^-)$. If there exists some $k \in \{1, 2, \ldots, m\}$ such that $t_k \in (0, t)$, $0 < \alpha < 1$, and $x(t_k^+) - x(t_k^-) \neq 0$, then ${}^cD_{0,t}^{\alpha}x(t)$ does not exist since $\dot{x}(t)$ is meaningless at the impulsive moment t_k. That is to say $\dot{x}(t_k)$ is meaningless. As a result, investigating (1)–(6) is meaningless.

Motivated by this fact, this paper is concerned with the complete controllability of the impulsive fractional linear time-invariant system with delay

$$\begin{gathered} {}^cD_{t_i,t}^{\alpha}x(t) = Ax(t) + Bx(t-\tau) + Gu(t), \\ t \in (t_i, t_{i+1}), \quad i = 0, 1, \ldots, k, \end{gathered} \tag{8}$$

$$\Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i^-)), \quad i = 1, 2, \ldots, k, \tag{9}$$

$$x(0^+) = \omega, \tag{10}$$

$$y(t) = Ex(t) + Fu(t), \tag{11}$$

$$x(t) = \phi(t), \quad -\tau \leq t \leq 0 \tag{12}$$

in n-dimensional Euclidean space, where $0 = t_0 < t_1 < t_2 < t_3 < \cdots < t_k < t_{k+1} = T < \infty$, $J = [0, T]$, ${}^cD_{t_i,t}^{\alpha}$ denotes the Caputo's derivative of order α with the lower limit t_i, $i = 0, 1, \ldots, k$, $0 < \alpha < 1$, $\omega \in \mathbb{R}^n$, A, B, G, E, F are known constant matrices with appropriate dimensions, the state variable $x(t) \in \mathbb{R}^n$, the initial function $\phi(t) \in \mathbb{D} = \{\psi : [-r, 0] \to \mathbb{R}^n \mid \psi \text{ is continuous on } [-\tau, 0]\}$, the delay $0 < \tau < \infty$, $\|\phi\|_{\mathbb{D}} = \sup\{\|\phi(t)\|_{\mathbb{R}^n}, t \in [-\tau, 0]\}$, the control input $u(t) \in \mathbb{R}^p$, the output $y(t) \in \mathbb{R}^m$, $I_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, \ldots, k$.

In this paper, the methods used is to construct a suitable control input function in time domain. The results obtained is sufficient and necessary, which are convenient for computation.

2. Preliminaries

In this section, we begin with some notations, definitions, and lemmas. Throughout this paper, ${}^cD_{a,t}^{\alpha}f(t)$ or ${}^cD_a^{\alpha}f(t)$ denotes the Caputo's derivative of order α with the lower limit a for the function f, $I_a^{\alpha}f(t)$ or $I_{a,t}^{\alpha}f(t)$ denotes integral of order α with lower limit a for the function f, $\overline{f}(s) = L[f(t); s] = \int_0^{\infty} e^{-st}f(t)dt$ denotes the Laplace transform of the function $f(t)$, and "$|M|$" denotes the norm of the matrix "M," "M^*" denotes the transpose of the matrix "M". Let $C(J, \mathbb{R}^n)$ be the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm $\|u\|_{C(J,\mathbb{R}^n)} = \sup\{\|u(t)\|, t \in J\}$. Let the Banach space $\mathrm{PC}(J, \mathbb{R}^n)$ be

$$\begin{aligned} &\mathrm{PC}(J, \mathbb{R}^n) \\ &= \{u : J \longrightarrow \mathbb{R}^n \mid u \in C((t_i, t_{i+1}], \mathbb{R}^n),\ i = 0, 1, \ldots, k, \\ &\quad u(t_i^-), u(t_i^+) \text{ exist, } u(t_i^-) = u(t_i),\ i = 1, 2, \ldots, k\} \end{aligned} \tag{13}$$

and the norm $\|u\|_{\mathrm{PC}(J,\mathbb{R}^n)} = \sup\{\|u(t)\| : t \in J\}$.

Definition 1 (see [27]). The fractional integral of order α with the lower limit $a \in \mathbb{R}$ for a function f is defined as

$$I_{a,t}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\,ds, \quad t > a,\ \alpha > 0 \tag{14}$$

Provided that the right-hand side is pointwise defined on $[a, +\infty)$, where Γ is the Gamma function.

Definition 2 (see [27]). The Caputo's derivative of order α with the lower limit $a \in \mathbb{R}$ for a function $f : [a, \infty) \to \mathbb{R}$ can be written as

$$\begin{aligned} &{}^cD_{a,t}^{\alpha}f(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}f^{(n)}(s)\,ds = I_{a,t}^{n-\alpha}\left(f^{(n)}(t)\right), \\ &\qquad t > a, \quad 0 < n-1 < \alpha \leq n. \end{aligned} \tag{15}$$

Particularly, when $0 < \alpha < 1$, it holds

$$\begin{aligned} &{}^cD_{a,t}^{\alpha}f(t) \\ &= \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}f'(s)\,ds = I_{a,t}^{1-\alpha}f'(t), \quad t > a. \end{aligned} \tag{16}$$

The Laplace transform of ${}^cD_{0,t}^{\alpha}f(t)$ is

$$\begin{aligned} &L\left[{}^cD_{0,t}^{\alpha}f(t); s\right] \\ &= \int_0^{+\infty} e^{-st}\left({}^cD_{0,t}^{\alpha}f(t)\right)dt \\ &= s^{\alpha}\overline{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}f^{(k)}(0), \quad n-1 < \alpha \leq n, \end{aligned} \tag{17}$$

where $\overline{f}(s)$ is the Laplace transform of $f(t)$.

In particular, for $0 < \alpha < 1$, it holds

$$\int_0^{+\infty} e^{-st}\left({}^cD_{0,t}^{\alpha}f(t)\right)dt = s^{\alpha}\overline{f}(s) - s^{\alpha-1}f(0). \tag{18}$$

Definition 3 (see [27]). The two-parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0,\ \beta > 0,\ z \in \mathbb{C}. \tag{19}$$

The Laplace transform of Mittag-Leffler function is

$$\begin{aligned} &L\left[t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm at^{\alpha}); s\right] \\ &= \int_0^{\infty} e^{-st}t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm at^{\alpha})\,dt \\ &= \frac{k!s^{\alpha-\beta}}{(s^{\alpha} \mp a)^{k+1}}, \quad \mathrm{Re}(s) > |a|^{1/\alpha}, \end{aligned} \tag{20}$$

where $\mathrm{Re}(s)$ denotes the real parts of s.

In addition, the Laplace transform of $t^{\alpha-1}$ is

$$L\left[t^{\alpha-1};s\right]=\Gamma(\alpha)s^{-\alpha},\quad \operatorname{Re}(s)>0. \tag{21}$$

Lemma 4 (see [28]). *Let $0<\operatorname{Re}(\alpha)\le 1$. If $x(t)\in C[a,b]$, then*

$$I_{a,t}^{\alpha}\left({}^{c}D_{a,t}^{\alpha}x(t)\right)=x(t)-x(a), \tag{22}$$

where $C[a,b]$ denotes the set of continuous functions on $[a,b]$.

3. Main Results

Definition 5 (complete controllability). The system (8)–(12) is said to be completely controllable on the interval $J=[0,T]$ if, for any $t_*>0$ ($t_*\in(0,T]$), $\phi(t)\in\mathbb{D}$, and $Z\in\mathbb{R}^n$, there exists an admissible control input $u(t)$ such that the state variable $x(t)$ of the system (8)–(12) satisfies $x(t_*)=Z$.

Using the Laplace transform method, we can easily obtain the following lemma.

Lemma 6. *The movement orbit of the state variable $x(t)$ of the system* (8)–(12) *can be written as*

$$x(t)=\begin{cases}\omega E_{\alpha,1}(At^{\alpha})\\ \quad+\displaystyle\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha})\\ \qquad\times(Bx(s-\tau)+Gu(s))\,ds, & t\in(0,t_1];\\ (x(t_1)+I_1(x(t_1)))E_{\alpha,1}(At^{\alpha})\\ \quad+\displaystyle\int_{t_1}^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha})\\ \qquad\times(Bx(s-\tau)+Gu(s))\,ds, & t\in(t_1,t_2];\\ \quad\vdots\\ (x(t_i)+I_i(x(t_i)))E_{\alpha,1}(At^{\alpha})\\ \quad+\displaystyle\int_{t_i}^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha})\\ \qquad\times(Bx(s-\tau)+Gu(s))\,ds, & t\in(t_i,t_{i+1}],\\ & i=2,3,\ldots,k.\end{cases} \tag{23}$$

Theorem 7. *The system* (8)–(12) *is completely controllable on $[0,T]$ if and only if the controllability matrices*

$$W_c[t_i,t_{i+1}]=\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_{i+1}-s)^{\alpha}\right)\times GG^*E_{\alpha,\alpha}\left(A^*(t_{i+1}-s)^{\alpha}\right)ds \tag{24}$$

are nonsingular, $i=0,1,2,\ldots,k$.

Proof. Sufficiency. Suppose that $W_c[t_i,t_{i+1}]$ is nonsingular; then $W_c^{-1}[t_i,t_{i+1}]$ is well defined, $i=0,1,2,\ldots,k$.

For $t\in(0,t_1]$, it follows from the formula (23) that

$$x(t)=\omega E_{\alpha,1}(at^{\alpha})+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha})\times(Bx(s-\tau)+Gu(s))\,ds,\quad t\in(0,t_1]. \tag{25}$$

For all $Z_0\in\mathbb{R}^n$, choosing

$$u(t)=G^*E_{\alpha,\alpha}\left(A^*(t_1-t)^{\alpha}\right)W_c^{-1}[0,t_1]\cdot\left[Z_0-\omega E_{\alpha,1}(At_1^{\alpha})-\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_1-s)^{\alpha}\right)Bx(s-\tau)\,ds\right] \tag{26}$$

and inserting (26) into (25) yields $x(t_1)=Z_0$. Thus, the system (8)–(12) is completely controllable on $[0,t_1]$.

Similarly, for $t\in(t_1,t_2]$, it follows from the formula (23) that

$$x(t)=(x(t_1)+I_1(x(t_1)))E_{\alpha,1}(At^{\alpha})+\int_{t_1}^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha})\times(Bx(s-\tau)+Gu(s))\,ds,\quad t\in(t_1,t_2]. \tag{27}$$

Since the system (8)–(12) is completely controllable on $[0,t_1]$, there exists a control input $u_1(t)$ such that $x(t_1)=0$. By (27), it follows that

$$x(t)=I_1(0)E_{\alpha,1}(At^{\alpha})+\int_{t_1}^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha})\times(Bx(s-\tau)+Gu(s))\,ds,\quad t\in(t_1,t_2]. \tag{28}$$

For all $Z_1\in\mathbb{R}^n$, choosing

$$u(t)=G^*E_{\alpha,\alpha}\left(A^*(t_2-t)^{\alpha}\right)W_c^{-1}[t_1,t_2]\cdot\left[Z_1-I_1(0)E_{\alpha,1}(At_2^{\alpha})-\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_2-t)^{\alpha}\right)Bx(s-\tau)\,ds\right], \tag{29}$$

together with (28) yields $x(t_2)=Z_1$. Thus, the system (8)–(12) is completely controllable on $[t_1,t_2]$.

By similar arguments, we can prove that the system (8)–(12) is completely controllable on $[t_i,t_{i+1}]$, $i=2,\ldots,k$.

Consequently, the system (8)–(12) is completely controllable on $J=[0,T]$.

Necessity. Suppose that the system (8)–(12) is completely controllable on $J=[0,T]$.

If $W_c[t_0,t_1]$ is singular, then there exists a nonzero vector Z_0 such that

$$Z_0^*W_c[0,t_1]Z_0=0. \tag{30}$$

That is

$$\int_{t_0}^{t_1}Z_0^*(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_1-s)^{\alpha}\right)GG^*E_{\alpha,\alpha}\times\left(A^*(t_1-s)^{\alpha}\right)Z_0ds=0. \tag{31}$$

Then we have

$$Z_0^* E_{\alpha,\alpha}\left(A(t_1-s)^\alpha\right)G=0 \tag{32}$$

on $s \in [0,t_1]$. By the assumption that the system (8)–(12) is completely controllable on J, the system (8)–(12) is completely controllable on $[t_i,t_{i+1}]$, $i=0,1,2,\ldots,k$. There exist control inputs $u_0(t)$ and $\widehat{u}_0(t)$ such that

$$\begin{aligned}x(t_1)&=\omega E_{\alpha,1}(At_1^\alpha)\\&+\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_1-s)^\alpha\right)Bx(s-\tau)\,ds\\&+\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_1-s)^\alpha\right)Gu_0(s)\,ds=0,\end{aligned} \tag{33}$$

$$\begin{aligned}x(t_1)&=\omega E_{\alpha,1}(At_1^\alpha)\\&+\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_1-s)^\alpha\right)Bx(s-\tau)\,ds\\&+\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_1-s)^\alpha\right)G\widehat{u}_0(s)\,ds=Z_0.\end{aligned} \tag{34}$$

By (34), we have

$$\begin{aligned}&\omega E_{\alpha,1}(At_1^\alpha)+\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_1-s)^\alpha\right)Bx(s-\tau)\,ds\\&=Z_0-\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_1-s)^\alpha\right)G\widehat{u}_0(s)\,ds.\end{aligned} \tag{35}$$

Inserting (35) into (33) yields

$$\begin{aligned}Z_0&+\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_1-s)^\alpha\right)\\&\times G\left(u_0(s)-\widehat{u}_0(s)\right)ds=0.\end{aligned} \tag{36}$$

Multiplying Z_0^* on both side of (36) yields

$$\begin{aligned}Z_0^*Z_0&+\int_0^{t_1}(t_1-s)^{\alpha-1}Z_0^*E_{\alpha,\alpha}\left(A(t_1-s)^\alpha\right)\\&\times G\left(u_0(s)-\widehat{u}_0(s)\right)ds=0.\end{aligned} \tag{37}$$

By (32) and (37), we have $Z_0^*Z_0=0$. Thus, $Z_0=0$. This is a contradiction.

If $W_c[t_i,t_{i+1}]$ is singular for some $i\in\{1,\ldots,k\}$, then there exists a nonzero vector Z_i such that

$$Z_i^* W_c[t_i,t_{i+1}]Z_i=0. \tag{38}$$

That is

$$\begin{aligned}\int_{t_i}^{t_{i+1}}&Z_i^*(t_{i+1}-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_{i+1}-s)^\alpha\right)\\&\times GG^*E_{\alpha,\alpha}\left(A^*(t_{i+1}-s)^\alpha\right)Z_i\,ds=0.\end{aligned} \tag{39}$$

Then, it follows that

$$Z_i^* E_{\alpha,\alpha}\left(A(t_{i+1}-s)^\alpha\right)G=0 \tag{40}$$

on $s\in[t_i,t_{i+1}]$. By formula (23) and the assumption that the system (8)–(12) is completely controllable, there exist control inputs $u_{i-1}(t)$ and $u_i(t)$ such that $x(t_i)=0$ and

$$\begin{aligned}x(t_{i+1})&=I_i(0)E_{\alpha,1}(At_{i+1}^\alpha)\\&+\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_{i+1}-s)^\alpha\right)\\&\cdot\left(Bx(s-\tau)+Gu_i(s)\right)ds=0.\end{aligned} \tag{41}$$

Similarly, there exists a control input $\widehat{u}_i(t)$ such that

$$\begin{aligned}x(t_{i+1})&=I_i(0)E_{\alpha,1}(At_{i+1}^\alpha)\\&+\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_{i+1}-s)^\alpha\right)\\&\cdot\left(Bx(s-\tau)+G\widehat{u}_i(s)\right)ds=Z_i.\end{aligned} \tag{42}$$

By (42), we have

$$\begin{aligned}&I_i(0)E_{\alpha,1}(At_{i+1}^\alpha)\\&+\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_{i+1}-s)^\alpha\right)Bx(s-\tau)\,ds\\&=Z_i-\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t_{i+1}-s)^\alpha\right)G\widehat{u}_i(s)\,ds.\end{aligned} \tag{43}$$

Inserting (43) into (41) yields

$$\begin{aligned}Z_i&+\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}\\&\times E_{\alpha,\alpha}\left(A(t_{i+1}-s)^\alpha\right)G\left(u_i(s)-\widehat{u}_i(s)\right)ds=0.\end{aligned} \tag{44}$$

Multiplying Z_i^* on both side of (44) yields

$$\begin{aligned}Z_i^*Z_i&+\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}Z_i^*\\&\times E_{\alpha,\alpha}\left(A(t_{i+1}-s)^\alpha\right)G\left(u_i(s)-\widehat{u}_i(s)\right)ds=0.\end{aligned} \tag{45}$$

Combining (45) with (40) yields $Z_i^*Z_i=0$. Thus, $Z_i=0$. This is a contradiction.

Thus, $W_c[t_i,t_{i+1}]$ is nonsingular for $i=0,1,\ldots,k$. This completes the proof. □

Theorem 8. *The system* (8)–(12) *is completely controllable on* $[0,T]$ *if and only if*

$$\operatorname{rank}\left(G \mid AG \mid \cdots \mid A^{n-1}G\right)=n. \tag{46}$$

Proof. Necessity. Suppose that system (8)–(12) is completely controllable on $[0,T]$. Then, the system (8)–(12) is completely controllable on $[0,t_1]$. Then, for any $Z_0 \in \mathbb{R}^n$, there exists a control input $u_0(t)$ such that $x(t_1) = Z_0$. By the formula (23), it follows that

$$Z_0 = x(t_1) = \omega E_{\alpha,1}(At_1^{\alpha}) + \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}\left(A(t_1 - s)^{\alpha}\right) \times (Bx(s-\tau) + Gu(s))\,ds. \tag{47}$$

By Cayley-Hamilton theorem, we have

$$t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha}) = \sum_{j=0}^{n-1} c_j(t)A^j, \tag{48}$$

where $c_j(t)$ are functions in t, $j = 1,2,\ldots,n-1$. Combining the formula (48) and the equality (47), we have

$$\begin{aligned} x(t_1) &- \omega E_{\alpha,1}(At_1^{\alpha}) - \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}\left(A(t_1 - s)^{\alpha}\right) Bx(s-\tau)\,ds \\ &= \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}\left(A(t_1 - s)^{\alpha}\right) Gu(s)\,ds \\ &= \sum_{j=0}^{n-1} A^j G \int_0^{t_1} c_j(t_1 - s)\,u(s)\,ds \\ &= \left(G \mid AG \mid \cdots \mid A^{n-1}G\right) \cdot \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \end{pmatrix}, \end{aligned} \tag{49}$$

where $d_j = \int_0^{t_1} c_j(t_1 - s)u(s)ds$, $j = 0,1,2,\ldots,n-1$. For arbitrary state Z_0 and initial function $\phi(t)$, the system (8)–(12) is completely controllable on $[0,t_1]$ if and only if there exists a control input $u(t)$ such that (47) or (49) holds. Obviously, for arbitrary initial function $\phi(t)$ and Z_0, the sufficient and necessary condition to have a control input $u(t)$ satisfying (49) is that

$$\operatorname{rank}\left(G \mid AG \mid \cdots \mid A^{n-1}G\right) = n. \tag{50}$$

Sufficiency. Suppose that $\operatorname{rank}(G \mid AG \mid \cdots \mid A^{n-1}G) = n$. In order to prove that the system (8)–(12) is completely controllable on $[0,T]$, it is sufficient to prove that the system (8)–(12) is completely controllable on $[t_i, t_{i+1}]$, $i = 0,1,\ldots,k$, respectively.

The formula (23) together with (48) yields (49). By the assumption that $\operatorname{rank}(G \mid AG \mid \cdots \mid A^{n-1}G) = n$, the system (8)–(12) is completely controllable on $[0,t_1]$.

Now we prove that the system (8)–(12) is completely controllable on $[t_1,t_2]$. The complete controllability of the system (8)–(12) on $[0,t_1]$ implies that there exists a control input $u_0(t)$ such that $x(t_1) = 0$. Inserting $x(t_1) = 0$ into the formula (23), we have, for $t \in (t_1,t_2]$,

$$x(t) = I_1(0)E_{\alpha,1}(At^{\alpha}) + \int_{t_1}^{t} (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-s)^{\alpha}\right) \times (Bx(s-\tau) + Gu(s))\,ds, \quad t \in (t_1,t_2]. \tag{51}$$

Thus, it follows

$$x(t_2) = I_1(0)E_{\alpha,1}(At_2^{\alpha}) + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}\left(A(t_2 - s)^{\alpha}\right) \times (Bx(s-\tau) + Gu(s))\,ds. \tag{52}$$

By (48) it follows taht

$$\begin{aligned} x(t_2) &- I_1(0)E_{\alpha,1}(At_2^{\alpha}) - \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}\left(A(t_2 - s)^{\alpha}\right) Bx(s-\tau)\,ds \\ &= \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}\left(A(t_2 - s)^{\alpha}\right) Gu(s)\,ds \\ &= \sum_{j=0}^{n-1} A^j G \int_{t_1}^{t_2} c_j(t_2 - s)\,u(s)\,ds \\ &= \left(G \mid AG \mid \cdots \mid A^{n-1}G\right) \cdot \begin{pmatrix} d'_0 \\ d'_1 \\ \vdots \\ d'_{n-1} \end{pmatrix}, \end{aligned} \tag{53}$$

where $d'_j = \int_{t_1}^{t_2} c_j(t_2 - s)u(s)ds$, $j = 0,1,2,\ldots,n-1$. Similar to the previous arguments, we can conclude that system (8)–(12) is completely controllable on $(t_1,t_2]$.

Repeating the process on $(t_i,t_{i+1}]$, respectively, we can prove that the system (8)–(12) is completely controllable on $(t_i,t_{i+1}]$, $i = 2,\ldots,k$. In conclusion, the system (8)–(12) is completely controllable on $J = [0,T]$. This completes the proof. □

Remark 9. From Theorem 8, we can conclude that the complete controllability of the system (8)–(12) is unrelated to the matrix B and initial function $\phi(t)$. The matrices A, G determine if the the system (8)–(12) possesses complete controllability.

4. Examples

Example 1. Consider the system (8)–(12). Choose $\alpha = 1/2$, $J = [0,2]$, $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, $\Delta x(t_1) = x(t_1^+) - x(t_1^-) = 3$, $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $G = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$. Now, we employ Theorems 7 and 8 to prove if that the system (8)–(10) is completely controllable, respectively.

By computation, we have

$$\begin{aligned} E_{1/2,1/2}\left(A(1-s)^{1/2}\right) &= \sum_{k=0}^{\infty} \frac{A^k(1-s)^{k/2}}{\Gamma(k/2+1/2)} \\ &= \frac{1}{\Gamma(1/2)} I + \frac{1}{\Gamma(1)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (1-s)^{1/2}, \end{aligned} \quad (54)$$

$$GG^* = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, \quad (55)$$

$$\begin{aligned} E_{1/2,1/2}\left(A^*(1-s)^{1/2}\right) &= \sum_{k=0}^{\infty} \frac{(A^*)^k(1-s)^{k(1/2)}}{\Gamma(k(1/2)+1/2)} \\ &= \frac{1}{\Gamma(1/2)} I + \frac{1}{\Gamma(1)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (1-s)^{1/2}, \end{aligned} \quad (56)$$

$$\begin{aligned} E_{1/2,1/2}\left(A(2-s)^{1/2}\right) &= \sum_{k=0}^{\infty} \frac{A^k(2-s)^{k/2}}{\Gamma(k/2+1/2)} \\ &= \frac{1}{\Gamma(1/2)} I + \frac{1}{\Gamma(1)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (2-s)^{1/2}, \end{aligned} \quad (57)$$

$$\begin{aligned} E_{1/2,1/2}\left(A^*(2-s)^{1/2}\right) &= \sum_{k=0}^{\infty} \frac{(A^*)^k(2-s)^{k/2}}{\Gamma(k/2+1/2)} \\ &= \frac{1}{\Gamma(1/2)} I + \frac{1}{\Gamma(1)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (2-s)^{1/2}. \end{aligned} \quad (58)$$

By the formula (24)

$$\begin{aligned} W_c\left[t_i, t_{i+1}\right] &= \int_{t_i}^{t_{i+1}} (t_{i+1}-s)^{-1/2} \left[E_{1/2,1/2}\left(A(t_{i+1}-s)^{1/2}\right)G\right] \\ &\quad \times \left[G^* E_{1/2,1/2}\left(A^*(t_{i+1}-s)^{1/2}\right)\right] ds, \end{aligned} \quad (59)$$

we have

$$\begin{aligned} W_c[0,1] &= \begin{pmatrix} \frac{8}{\pi} + \frac{4}{\pi^{0.5}} + \frac{2}{3} & \frac{4}{\pi} + \frac{1}{\pi^{0.5}} \\ \frac{4}{\pi} + \frac{1}{\pi^{0.5}} & \frac{2}{\pi} \end{pmatrix}, \\ W_c[1,2] &= \begin{pmatrix} \frac{4}{\pi} + \frac{6}{\pi^{0.5}} + \frac{2}{3} & \frac{2}{\pi} + \frac{1}{\pi^{0.5}} \\ \frac{2}{\pi} + \frac{1}{\pi^{0.5}} & \frac{1}{\pi} \end{pmatrix}. \end{aligned} \quad (60)$$

It is obvious that $W_c[0,1]$ and $W_c[1,2]$ are nonsingular. By Theorem 7, the system is completely controllable.

On the other hand,

$$\operatorname{rank}(G \mid AG) = \operatorname{rank}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = 2. \quad (61)$$

By Theorem 8, the system is completely controllable.

Example 2. Consider the time-invariant system (8)–(12). Choose

$$A = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 6 & -1 \\ 1 & 7 & 1 \end{pmatrix}, \qquad \alpha = \frac{1}{3}, \qquad G = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}. \quad (62)$$

By computation, we have

$$\operatorname{rank}\left(G \mid AG \mid A^2G\right) = \operatorname{rank}\begin{pmatrix} 2 & -4 & 6 \\ 0 & -1 & 7 \\ 1 & 1 & -12 \end{pmatrix} = 3. \quad (63)$$

By Theorem 8, the system is completely controllable.

Acknowledgments

The authors would like to thank the referee for his or her valuable comments, which help us to improve the quality of the paper. This paper is supported by National Natural Science Foundation of China (11071001), Research Fund for Doctoral Program of Educational Ministry of China (20103401120002 and 20123401120001), Program of Natural Science Research in Anhui Universities (KJ2011A020 and KJ2013A032), Scientific Research Starting Fund for Dr. of Anhui University (023033190001, 023033190181), and the 211 Project of Anhui University (KJQN1001, 023033050055).

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