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Research Article

Global Strong Solutions to Some Nonlinear Dirac Equations in Super-Critical Space

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We study the initial value problem of some nonlinear Dirac equations which are $L^m(\mathbb{R})$ critical. Corresponding to the structure of nonlinear terms, global strong solutions can be obtained in different Lebesgue spaces by using solution representation formula. The uniqueness of weak solutions is proved for the solution $U \in L^\infty([0,T]; Y^{m+2}(\mathbb{R}))$.

1. Introduction

In this study, we are interested in global strong solutions of nonlinear Dirac type equations:

$$\begin{aligned} -i\left(\partial_t U_1 + \partial_x U_1\right) &= \partial_{\overline{U}_1} F\left(U_1, U_2\right), \\ -i\left(\partial_t U_2 - \partial_x U_2\right) &= \partial_{\overline{U}_2} F\left(U_1, U_2\right), \end{aligned} \tag{1}$$

where $F(U_1, U_2) : \mathbb{C}^2 \to \mathbb{R}$ is a nonlinear potential function which will be specified later.

The following Thirring model with the potential $F(U_1, U_2) = |U_1|^2|U_2|^2$ has been studied in [1–4]:

$$\begin{aligned} -i\left(\partial_t U_1 + \partial_x U_1\right) &= \left|U_2\right|^2 U_1, \\ -i\left(\partial_t U_2 - \partial_x U_2\right) &= \left|U_1\right|^2 U_2, \\ U_1(0,x) = u_1(x), &\qquad U_2(0,x) = u_2(x). \end{aligned} \tag{2}$$

The initial value problem of (2) was studied in [1, 3] in terms of Sobolev space H^s $(s \geq 1)$. Low regularity well-posedness was discussed in [4] showing that there exists a time $T > 0$ and solution $U \in C([0,T], H^s(\mathbb{R}))$ $(s > 0)$ of the Cauchy problem (2). Their results were based on the observation of the null structure of Thirring equations and application of $X^{s,b}$ spaces which is a certain subspace of $C([0,T], H^s(\mathbb{R}))$. They also proved global well-posedness for $s > 1/2$ and unconditional uniqueness for $s > 1/4$. Nonlinear Dirac equations in $\mathbb{R}^{1+1}$ have been studied by several authors [5–9].

In the context of Bragg grating [10, 11], the nonlinear term F takes the form

$$F = \left|U_1\right|^4 + \left|U_2\right|^4 + 4\left|U_1\right|^2\left|U_2\right|^2, \tag{3}$$

which gives $\partial_{\overline{U}_1} F = (2|U_1|^2 + 4|U_2|^2)U_1$ and $\partial_{\overline{U}_2} F = (2|U_2|^2 + 4|U_1|^2)U_2$. We may consider the sixth and higher orders. The following potential term is introduced in the context of the Bose-Einstein condensates [12]:

$$F = \left(\left|U_1\right|^2 + \left|U_2\right|^2\right) \left|U_1\right|^2\left|U_2\right|^2, \tag{4}$$

which gives $\partial_{\overline{U}_1} F = (2|U_1|^2|U_2|^2 + |U_2|^4)U_1$ and $\partial_{\overline{U}_2} F = (2|U_2|^2|U_1|^2 + |U_1|^4)U_2$.

Several authors [4, 5, 7] have studied the initial value problem of the following Dirac equations with quadratic nonlinearities:

$$\begin{aligned} -i\left(\partial_t U_1 + \partial_x U_1\right) &= Q_1\left(U_1, U_2\right), \\ -i\left(\partial_t U_2 - \partial_x U_2\right) &= Q_2\left(U_1, U_2\right). \end{aligned} \tag{5}$$

If the nonlinear term is of $Q_1 = |U_2|^2$, $Q_2 = |U_1|^2$ type ($Q_1 = U_1\overline{U}_2$, $Q_2 = \overline{U}_1 U_2$, resp.), then one can obtain local well-posedness for the Sobolev space H^s $(s > 0)$ [4] (H^s $(s > -1/4)$ [7], resp.). Note that the scaling properties of quadratic Dirac equations give the critical Sobolev exponent $s_{\text{cr}} = -1/2$.

Now it seems natural to consider the following equations:

$$\begin{aligned} \partial_t U_1 + \partial_x U_1 &= i|U_1|^k |U_2|^{m-k} U_1, \\ \partial_t U_2 - \partial_x U_2 &= i|U_2|^k |U_1|^{m-k} U_2, \end{aligned} \tag{6}$$

which are generalization of the basic cases of the literature and model problem to investigate regularities of solutions according to the structure of nonlinearities. Here, m is positive integer and $k = 0, 1, \dots, m$.

The system (6) is invariant under the scaling

$$U^{\lambda}(t, x) = \lambda U(\lambda^m t, \lambda^m x), \tag{7}$$

from which we deduce a scale invariant Lebesgue space $L^m(\mathbb{R})$. We study the initial value problem of (6) in Lebesgue space. Let us denote

$$p(m, k) = \operatorname{Max}(m - k, 2k + 1). \tag{8}$$

Note that $p(m, k) \geq 2$ except for $m = 1$ and $k = 0$ where $p(1, 0) = 1$. We define

$$Y^p(\mathbb{R}) = L^2(\mathbb{R}) \cap L^p(\mathbb{R}), \tag{9}$$

where $p \geq 2$. We call Y^p as subcritical space for $p > m$, critical space for $p = m$, and supercritical space for $p < m$.

For the initial data $u_j \in L^2(\mathbb{R})(j = 1, 2)$, we have proved in [2] that there exists a global strong solution $U = (U_1, U_2)$ to the initial value problem (2) in the critical space

$$U \in C([0, \infty); L^2(\mathbb{R})). \tag{10}$$

Elementary and interesting approach was made. and special structure of nonlinearity was made use of in applying Fubini's theory in integration. The following is concerned with the existence of global strong solutions for the case of $m \geq 2$ or $k \geq 1$.

Theorem 1. *For the initial data $v_j \in Y^{p(m,k)}(\mathbb{R})$ $(j = 1, 2)$, there exists a global strong solution $V = (V_1, V_2)$ to the initial value problem* (6) *which satisfies*

$$V \in C([0, \infty); Y^{p(m,k)}(\mathbb{R})). \tag{11}$$

Remark 2. (1) Related with the title of this paper, we emphasize the case $p(m, k) < m$ where global strong solution can be constructed in the super-critical space.

(2) Taking the embedding $Y^q(\mathbb{R}) \hookrightarrow Y^r(\mathbb{R})$ $(q \geq r)$ into account, it is likely to say that the system (6) with small $p(m, k)$ has better smoothing property than equations with large $p(m, k)$. We can check, in the case of (4), the existence of solution $V \in C([0, \infty); Y^5(\mathbb{R}))$ because of $p(4, 2) = 5 > 4 = p(4, 0)$.

(3) For the case of $m = 1$ and $k = 0$, we can construct a global strong solution $V = (V_1, V_2)$ to the initial value problem (6) which satisfies

$$V \in C([0, \infty); L^1(\mathbb{R})) \tag{12}$$

for the initial data $v_j \in L^1(\mathbb{R})$ $(j = 1, 2)$.

Our second result is concerned with the uniqueness of weak solutions. The null structure of the nonlinearity of Thirring model which is L^2 critical problem was used in [2] to prove the uniqueness in Y^4. To treat the general nonlinearity in (6) of which we could not find the null structure, a different approach is considered.

Theorem 3. *Let U and V be solutions of* (6) *in the distribution sense with same initial data. Moreover, one assumes that*

$$U, V \in L^{\infty}([0, T]; Y^{m+2}(\mathbb{R})). \tag{13}$$

Then, one has $\|(U - V)(t, \cdot)\|_{L^2} = 0$ for $0 \leq t \leq T$.

Theorems 1 and 3 imply that if $p(m, k) \geq m + 2$ then a strong solution to the initial value problem (6) is unique and is in fact a well-posed solution.

Theorem 1 is proved in Section 2 and Theorem 3 in Section 2.2. We use C to denote various constants and $A \lesssim B$ to denote an estimate of the form $A \leq CB$.

2. Proof of Theorem 1

To construct global strong solution of (6), we basically follow the idea of [2] with modification. We will find a solution representation formula. Then, global strong solutions can be obtained by constructing an explicit approximation and using Fubini's theorem.

2.1. Representation Formula of Classical Solution. In this subsection, we consider $C^{\infty}((0, T) \times \mathbb{R})$ solutions which satisfy (6) in the classical sense. An explicit representation of its solution is given in terms of initial data [2, 13]. It is interesting to express the solution of nonlinear partial differential equations in terms of initial data.

Integrating (6) along the outgoing and ingoing characteristics, we obtain

$$\begin{aligned} U_1(t, x) &= u_1(x - t) \\ &\quad \times \exp\left(i \int_0^t |U_1(s, x - t + s)|^k \right. \\ &\qquad \left. \times |U_2(s, x - t + s)|^{m-k} ds\right), \\ U_2(t, x) &= u_2(x + t) \\ &\quad \times \exp\left(i \int_0^t |U_2(s, x + t - s)|^k \right. \\ &\qquad \left. \times |U_1(s, x + t - s)|^{m-k} ds\right). \end{aligned} \tag{14}$$

Taking into account

$$|U_1(t, x)| = |u_1(x - t)|, \qquad |U_2(t, x)| = |u_2(x + t)|, \tag{15}$$

(14) becomes

$$U_1(t,x) = u_1(x-t) \times \exp\left(\frac{i}{2}|u_1(x-t)|^k \int_{x-t}^{x+t} |u_2(y)|^{m-k} dy\right),$$
$$U_2(t,x) = u_2(x+t) \times \exp\left(\frac{i}{2}|u_2(x+t)|^k \int_{x-t}^{x+t} |u_1(y)|^{m-k} dy\right). \quad (16)$$

In the case of $k = m$, we have the following obvious representation formula of solution:

$$U_1(t,x) = u_1(x-t)\exp\left(it|u_1(x-t)|^m\right),$$
$$U_2(t,x) = u_2(x+t)\exp\left(it|u_1(x-t)|^m\right). \quad (17)$$

2.2. Global Strong Solutions in $Y^p(\mathbb{R})$. We now present the proof of Theorem 1. Let us introduce a strong solution of the initial value problem of the system (6).

Definition 4. Consider the Cauchy problem (6) with initial data $v = (v_1, v_2) \in Y^{p(m,k)}(\mathbb{R}))$. It is said that $V = (V_1, V_2)$ is a strong solution to the Cauchy problem on the time interval $[0,T]$ provided that

$$(V_1, V_2) \in C\left([0,T]; Y^{p(m,k)}(\mathbb{R})\right) \quad (18)$$

satisfy (6) in the sense of distributions. That is, for any $\phi \in C_0^\infty((-T,T)\times\mathbb{R})$, we have

$$\int_0^T \int_{\mathbb{R}} V_1\partial_t\phi + V_1\partial_x\phi + i|V_1|^k|V_2|^{m-k}V_1\phi\, dx dt + \int_{\mathbb{R}} v_1(x)\phi(0,x)\,dx = 0, \quad (19)$$

$$\int_0^T \int_{\mathbb{R}} V_2\partial_t\phi - V_2\partial_x\phi + i|V_2|^k|V_1|^{m-k}V_2\phi\, dx dt + \int_{\mathbb{R}} v_2(x)\phi(0,x)\,dx = 0. \quad (20)$$

Remark 5. We say that V is a weak solution if $V \in L^\infty([0,T]; Y^{p(m,k)}(\mathbb{R}))$ satisfies (6) in the sense of distribution.

For the initial data $v_j \in Y^{p(m,k)}(\mathbb{R})$ $(j = 1,2)$, we propose that the following functions V_j are the global strong solution of (6):

$$V_1(t,x) = v_1(x-t) \times \exp\left(\frac{i}{2}|v_1(x-t)|^k \int_{x-t}^{x+t} |v_2(y)|^{m-k} dy\right),$$
$$V_2(t,x) = v_2(x+t) \times \exp\left(\frac{i}{2}|v_2(x+t)|^k \int_{x-t}^{x+t} |v_1(y)|^{m-k} dy\right). \quad (21)$$

For the smooth function sequence $\{u_j^{(n)}\}_{n=1}^\infty$ $(j = 1,2)$ which converges to v_j in $Y^{p(m,k)}(\mathbb{R})$ as $n \to \infty$, we consider sequence of classical solutions $U_j^{(n)}$ of (6). First of all, we estimate the difference of $(V_j, U_j^{(n)})$. Using the representation (16) and (21), we have

$$\begin{aligned}
&\left|\left(V_1 - U_1^{(n)}\right)(t,x)\right| \\
&\le \left|\exp\left(\frac{i}{2}\left|u_1^{(n)}(x-t)\right|^k \int_{x-t}^{x+t}\left|u_2^{(n)}(y)\right|^{m-k} dy\right) \times \left(v_1 - u_1^{(n)}\right)(x-t)\right| \\
&\quad + \left|v_1(x-t)\left(e^{(i/2)|v_1(x-t)|^k \int_{x-t}^{x+t}|v_2(y)|^{m-k}dy} - e^{(i/2)|u_1^{(n)}(x-t)|^k \int_{x-t}^{x+t}|u_2^{(n)}(y)|^{m-k}dy}\right)\right| \\
&\le \left|\left(v_1 - u_1^{(n)}\right)(x-t)\right| \\
&\quad + \frac{1}{2}|v_1(x-t)|\left||v_1(x-t)|^k \int_{x-t}^{x+t}|v_2(y)|^{m-k}dy - \left|u_1^{(n)}(x-t)\right|^k \int_{x-t}^{x+t}\left|u_2^{(n)}(y)\right|^{m-k}dy\right| \\
&= (\mathrm{A}) + (\mathrm{B}),
\end{aligned} \quad (22)$$

where we used $|e^{ix} - e^{iy}| \le |x-y|$ for $x, y \in \mathbb{R}$. The L^q norm of the first term (A) can be treated as follows:

$$\left(\int_{\mathbb{R}}\left|\left(v_1 - u_1^{(n)}\right)(x-t)\right|^q dx\right)^{1/q} = \left\|v_1 - u_1^{(n)}\right\|_{L^q}. \quad (23)$$

To estimate the L^p norm of the second term (B), we consider the following three cases.

(i) For the case $k = 0$, we have with $2 \le p \le m$

$$\int_{\mathbb{R}}|v_1(x-t)|^p\left|\int_{x-t}^{x+t}|v_2(y)|^m dy - \int_{x-t}^{x+t}\left|u_2^{(n)}(y)\right|^m dy\right|^p dx \lesssim \left\|v_2 - u_2^{(n)}\right\|_{L^m}^p\left(\|v_2\|_{L^m}^{m-1} + \left\|u_2^{(n)}\right\|_{L^m}^{m-1}\right)^p\|v_1\|_{L^p}^p. \quad (24)$$

(ii) For the case $k = m$, we have with $1 \le p \le (2m+1)/(m+1)$

$$\begin{aligned}
&(2t)^p\int_{\mathbb{R}}|v_1(x-t)|^p\left||v_1(x-t)|^m - \left|u_1^{(n)}(x-t)\right|^m\right|^p dx \\
&\le (2t)^p\|v_1\|_{L^{(m+1)p}}^p\left\|v_1 - u_1^{(n)}\right\|_{L^{(m+1)p}}^p \times \left(\|v_1\|_{L^{(m+1)p}}^{(m-1)p} + \left\|u_1^{(n)}\right\|_{L^{(m+1)p}}^{(m-1)p}\right).
\end{aligned} \quad (25)$$

(iii) For other cases $1 \le k \le m-1$, we have with $1 \le p \le p(m,k)/(k+1)$

$$\begin{aligned}
&\int_{\mathbb{R}} \left|v_1(x-t)\right|^p \left| \left|v_1(x-t)\right|^k \int_{x-t}^{x+t} \left|v_2(y)\right|^{m-k} dy \right.\\
&\qquad\qquad \left. - \left|u_1^{(n)}(x-t)\right|^k \int_{x-t}^{x+t} \left|u_2^{(n)}(y)\right|^{m-k} dy \right|^p dx \\
&\quad\lesssim \sup_x \left(\int_{x-t}^{x+t} |v_2(y)|^{m-k}\, dy \right)^p \\
&\qquad \times \int_{\mathbb{R}} \left|v_1(x-t)\right|^p \left| \left|v_1(x-t)\right|^k - \left|u_1^{(n)}(x-t)\right|^k \right|^p dx \\
&\qquad + \sup_x \left(\int_{x-t}^{x+t} \left| \left|v_2(y)\right|^{m-k} - \left|u_2^{(n)}(y)\right|^{m-k} \right| dy \right)^p \\
&\qquad \times \int_{\mathbb{R}} \left|v_1(x-t)\right|^p \left|u_1^{(n)}(x-t)\right|^{kp} dx \\
&\quad\lesssim \sup_x \left(\int_{x-t}^{x+t} \left|v_2(y)\right|^{m-k} dy \right)^p \left\|v_1\right\|_{L^{(k+1)p}}^p \\
&\qquad \times \left\|v_1 - u_1^{(n)}\right\|_{L^{(k+1)p}}^p \left(\left\|v_1\right\|_{L^{(k+1)p}}^{(k-1)p} + \left\|u_1^{(n)}\right\|_{L^{(k+1)p}}^{(k-1)p} \right) \\
&\qquad + \sup_x \left(\int_{x-t}^{x+t} \left| \left|v_2(y)\right|^{m-k} - \left|u_2^{(n)}(y)\right|^{m-k} \right| dy \right)^p \\
&\qquad \times \left\|v_1\right\|_{L^{(k+1)p}}^p \left\|u_1^{(n)}\right\|_{L^{(k+1)p}}^{kp},
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
&\sup_x \left(\int_{x-t}^{x+t} \left|v_2(y)\right|^{m-k} dy \right)^p \\
&\qquad \lesssim \begin{cases} (2t)^{p/2} \left\|v_2\right\|_{L^2}^p & \text{for } m-k=1, \\ \left\|v_2\right\|_{L^{m-k}}^{(m-k)p} & \text{for } m-k \ge 2, \end{cases} \\
&\sup_x \left(\int_{x-t}^{x+t} \left| \left|v_2(y)\right|^{m-k} - \left|u_2^{(n)}(y)\right|^{m-k} \right| dy \right)^p \\
&\qquad \lesssim \begin{cases} (2t)^{p/2} \left\|v_2 - u_2^{(n)}\right\|_{L^2}^p & \text{for } m-k=1, \\ \left\|v_2 - u_2^{(n)}\right\|_{L^{m-k}}^p \left(\left\|v_2\right\|_{L^{m-k}}^{m-k-1} + \left\|u_2^{(n)}\right\|_{L^{m-k}}^{m-k-1} \right)^p & \\ \qquad\qquad \text{for } m-k \ge 2. & \end{cases}
\end{aligned} \tag{27}$$

Now, we are ready to prove that $V = (V_1, V_2)$ is a global strong solution to the Cauchy problem (6). It is easy to check (18) by considering the representation (21). Making use of the classical solution to (6), we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} V_1 \partial_t \phi + V_1 \partial_x \phi + i \left|V_1\right|^k \left|V_2\right|^{m-k} V_1 \phi \, dx dt \\
&\quad + \int_{\mathbb{R}} v_1(x) \phi(0,x)\, dx \\
&= \int_0^T \int_{\mathbb{R}} \left(V_1 - U_1^{(n)}\right) \left(\partial_t \phi + \partial_x \phi\right) dx dt \\
&\quad + \int_{\mathbb{R}} \left(v_1 - u_1^{(n)}\right) \phi(0,x)\, dx \\
&\quad + \int_0^T \int_{\mathbb{R}} i \left(\left|V_1\right|^k \left|V_2\right|^{m-k} V_1 \right. \\
&\qquad\qquad \left. - \left|U_1^{(n)}\right|^k \left|U_2^{(n)}\right|^{m-k} U_1^{(n)} \right) \phi \, dx dt \\
&= (1) + (2) + (3).
\end{aligned} \tag{28}$$

Considering (23)–(26) with $q = p = p(m,k)/(k+1)$, the first integral (1) can be estimated as follows:

$$\begin{aligned}
|(1)| &\lesssim T \sup_{0 \le t \le T} \left(\left\|\partial_t \phi(t,\cdot)\right\|_{L^{q'}} + \left\|\partial_x \phi(t,\cdot)\right\|_{L^{q'}} \right) \\
&\quad \times \sup_{0 \le t \le T} \left\| \left(V_1 - U_1^{(n)}\right)(t,\cdot) \right\|_{L^q},
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
&\sup_{0 \le t \le T} \left\| \left(V_1 - U_1^{(n)}\right)(t,\cdot) \right\|_{L^q} \\
&\lesssim \begin{cases}
\left\|v_1 - u_1^{(n)}\right\|_{L^q} \\
\quad + \Big[(2T)^{p/2} \left\|v_2\right\|_{L^2}^p \left\|v_1\right\|_{L^{p(m,k)}}^p \\
\qquad \times \left\|v_1 - u_1^{(n)}\right\|_{L^{p(m,k)}}^p \left(\left\|v_1\right\|_{L^{p(m,k)}}^{(k-1)p} + \left\|u_1^{(n)}\right\|_{L^{p(m,k)}}^{(k-1)p} \right) \\
\qquad + (2T)^{p/2} \left\|v_2 - u_2^{(n)}\right\|_{L^2}^p \left\|v_1\right\|_{L^{p(m,k)}}^p \left\|u_1^{(n)}\right\|_{L^{p(m,k)}}^{kp} \Big]^{1/p} \\
\qquad\qquad\qquad \text{for } m-k=1, \\
\left\|v_1 - u_1^{(n)}\right\|_{L^q} \\
\quad + \Big[\left\|v_2\right\|_{L^{m-k}}^{(m-k)p} \left\|v_1\right\|_{L^{p(m,k)}}^p \left\|v_1 - u_1^{(n)}\right\|_{L^{p(m,k)}}^p \\
\qquad \times \left(\left\|v_1\right\|_{L^{p(m,k)}}^{(k-1)p} + \left\|u_1^{(n)}\right\|_{L^{p(m,k)}}^{(k-1)p} \right) \\
\qquad + \left\|v_2 - u_2^{(n)}\right\|_{L^{m-k}}^p \left(\left\|v_2\right\|_{L^{m-k}}^{m-k-1} + \left\|u_2^{(n)}\right\|_{L^{m-k}}^{m-k-1} \right)^p \\
\qquad \times \left\|v_1\right\|_{L^{p(m,k)}}^p \left\|u_1^{(n)}\right\|_{L^{p(m,k)}}^{kp} \Big]^{1/p} \\
\qquad\qquad\qquad \text{for } m-k \ge 2.
\end{cases}
\end{aligned} \tag{30}$$

The integral (2) can be bounded easily:

$$\int_{\mathbb{R}} \left(v_1 - u_1^{(n)}\right) \phi(0,x)\, dx \le \left\|v_1 - u_1^{(n)}\right\|_{L^2} \left\|\phi(0,x)\right\|_{L^2}. \tag{31}$$

To take care of the integral (3), we decompose the integral by using representation (16) and (21) with the notation $E(v_1, v_2, k) = \exp((i/2)|v_1(x-t)|^k \int_{x-t}^{x+t} |v_2(y)|^{m-k} dy)$:

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \left(|V_1|^k |V_2|^{m-k} V_1 \right.\\
&\qquad \left. - \left|U_1^{(n)}\right|^k \left|U_2^{(n)}\right|^{m-k} U_1^{(n)}\right) \phi(t,x)\, dx\,dt \\
&\quad = \int_0^T \int_{\mathbb{R}} \left(|v_1(x-t)|^k - \left|u_1^{(n)}(x-t)\right|^k\right) |v_2(x+t)|^{m-k} \\
&\qquad \times v_1(x-t) E(v_1, v_2, k) \phi(t,x)\, dx\,dt \\
&\qquad + \int_0^T \int_{\mathbb{R}} \left|u_1^{(n)}(x-t)\right|^k \\
&\qquad \times \left(|v_2(x+t)|^{m-k} - \left|u_2^{(n)}(x+t)\right|^{m-k}\right) \\
&\qquad \times v_1(x-t) E(v_1, v_2, k) \phi(t,x)\, dx\,dt \\
&\qquad + \int_0^T \int_{\mathbb{R}} \left|u_1^{(n)}(x-t)\right|^k \left|u_2^{(n)}(x+t)\right|^{m-k} \\
&\qquad \times \left(v_1 - u_1^{(n)}\right)(x-t) E(v_1, v_2, k) \phi(t,x)\, dx\,dt \\
&\qquad + \int_0^T \int_{\mathbb{R}} \left|u_1^{(n)}(x-t)\right|^k \left|u_2^{(n)}(x+t)\right|^{m-k} u_1^{(n)}(x-t) \\
&\qquad \times \left(E(v_1, v_2, k) - E\left(u_1^{(n)}, u_2^{(n)}, k\right)\right) \phi(t,x)\, dx\,dt \\
&\quad = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}),
\end{aligned} \tag{32}$$

where we understand that the integral (I) does not exist for $k = 0$ and the integral (II) for $k = m$.

Changing variables $x + t = y$ and $x - t = s$, the first integral (I) can be bounded as follows:

$$\begin{aligned}
|(\mathrm{I})| &\leq \int_{\mathbb{R}} \int_s^{s+2T} \left||v_1(s)|^k - \left|u_1^{(n)}(s)\right|^k\right| |v_2(y)|^{m-k} \\
&\qquad \times |v_1(s)| |\phi(y,s)|\, dy\,ds \\
&\lesssim \sup_s \left(\int_s^{s+2T} |v_2(y)|^{m-k} |\phi(y,s)|\, dy\right) \\
&\qquad \times \int_{\mathbb{R}} \left|\left(v_1 - u_1^{(n)}\right)(s)\right| \\
&\qquad\qquad \times \left(|v_1(s)|^{k-1} + \left|u_1^{(n)}(s)\right|^{k-1}\right) |v_1(s)|\, ds \\
&\lesssim \sup_s \left(\int_s^{s+2T} |v_2(y)|^{m-k} |\phi(y,s)|\, dy\right) \left\|v_1 - u_1^{(n)}\right\|_{L^{k+1}} \\
&\qquad \times \left(\|v_1\|_{L^{k+1}}^{k-1} + \left\|u_1^{(n)}\right\|_{L^{k+1}}^{k-1}\right) \|v_1\|_{L^{k+1}},
\end{aligned} \tag{33}$$

where we note that $k \geq 1$. We also have

$$\begin{aligned}
&\sup_s \int_s^{s+2T} |v_2(y)|^{m-k} |\phi(y,s)|\, dy \\
&\quad \leq \begin{cases}
\sup_s \int_s^{s+2T} |\phi(y,s)|\, dy & \text{if } m-k = 0, \\
\|v_2\|_{L^2} \sup_s \left(\int_s^{s+2T} |\phi(y,s)|^2 dy\right)^{1/2} & \text{if } m-k = 1, \\
M \|v_2\|_{L^{m-k}}^{m-k} \quad \text{if } m-k \geq 2,
\end{cases}
\end{aligned} \tag{34}$$

where we denote $M = \sup_{\Sigma_T} |\phi(t,x)|$ where $\Sigma_T = \{(x,t) \mid x \in \mathbb{R}, 0 \leq t \leq T\}$.

For the integral (II), we consider two cases. Note that the integral (II) disappears if $k = m$. For the first case $m - k = 1$, we have

$$\begin{aligned}
|(\mathrm{II})| &\leq \int_{\mathbb{R}} \int_s^{s+2T} \left|u_1^{(n)}(s)\right|^k \left||v_2(y)| - \left|u_2^{(n)}(y)\right|\right| \\
&\qquad \times |v_1(s)| |\phi(y,s)|\, dy\,ds \\
&\lesssim \sup_s \left(\int_s^{s+2T} \left||v_2(y)| - \left|u_2^{(n)}(y)\right|\right| |\phi(y,s)|\, dy\right) \\
&\qquad \times \int_{\mathbb{R}} \left|u_1^{(n)}(s)\right|^k |v_1(s)|\, ds \\
&\lesssim \sup_s \left(\int_s^{s+2T} |\phi(y,s)|^2 dy\right)^{1/2} \\
&\qquad \times \left\|v_2 - u_2^{(n)}\right\|_{L^2} \left\|u_1^{(n)}\right\|_{L^{k+1}}^k \|v_1\|_{L^{k+1}},
\end{aligned} \tag{35}$$

where $k \geq 1$ because $m = 1$ and $k = 0$ case is excluded. For the second case $m - k \geq 2$, we have

$$\begin{aligned}
|(\mathrm{II})| &\leq \int_{\mathbb{R}} \int_{y-2T}^{y} \left|u_1^{(n)}(s)\right|^k \left||v_2(y)|^{m-k} - \left|u_2^{(n)}(y)\right|^{m-k}\right| \\
&\qquad \times |v_1(s)| |\phi(y,s)|\, ds\,dy \\
&\lesssim \sup_y \left(\int_{y-2T}^{y} \left|u_1^{(n)}(s)\right|^k |v_1(s)| |\phi(y,s)|\, ds\right) \\
&\qquad \times \left\|v_2 - u_2^{(n)}\right\|_{L^{m-k}} \left(\|v_2\|_{L^{m-k}}^{m-k-1} + \left\|u_2^{(n)}\right\|_{L^{m-k}}^{m-k-1}\right),
\end{aligned} \tag{36}$$

where we have

$$\begin{aligned}
&\sup_y \int_{y-2T}^{y} \left|u_1^{(n)}(s)\right|^k |v_1(s)| |\phi(y,s)|\, ds \\
&\quad \leq \begin{cases}
\|v_1\|_{L^2} \sup_y \left(\int_{y-2T}^{y} |\phi(y,s)|^2 ds\right)^{1/2} & \text{if } k = 0, \\
M \left\|u_1^{(n)}\right\|_{L^{k+1}}^k \|v_1\|_{L^{k+1}} & \text{if } k \geq 1.
\end{cases}
\end{aligned} \tag{37}$$

The integral (III) can be treated in a similar way to (II). For the case $k = 0$, we have

$$
\begin{aligned}
&|(\mathrm{III})| \\
&\lesssim \sup_y \left(\int_{y-2T}^{y} \left| \left(v_1 - u_1^{(n)}\right)(s) \right| |\phi(y,s)|\, ds \right) \left\| u_2^{(n)} \right\|_{L^m}^m \\
&\lesssim \sup_y \left(\int_{y-2T}^{y} |\phi(y,s)|^2 ds \right)^{1/2} \\
&\quad \times \left\| v_1 - u_1^{(n)} \right\|_{L^2} \left\| u_2^{(n)} \right\|_{L^m}^m .
\end{aligned} \tag{38}
$$

For the case $k \geq 1$, we have

$$
\begin{aligned}
|(\mathrm{III})| &\lesssim \sup_s \left(\int_s^{s+2T} \left| u_2^{(n)}(y) \right|^{m-k} |\phi(y,s)|\, dy \right) \\
&\quad \times \left\| v_1 - u_1^{(n)} \right\|_{L^{k+1}} \left\| u_1^{(n)} \right\|_{L^{k+1}}^k \\
&\lesssim M \left\| u_2^{(n)} \right\|_{L^{m-k}} \left\| v_1 - u_1^{(n)} \right\|_{L^{k+1}} \left\| u_1^{(n)} \right\|_{L^{k+1}}^k .
\end{aligned} \tag{39}
$$

Taking into account $|e^{ix} - e^{iy}| \leq |x - y|$ for $x, y \in \mathbb{R}$, the integral (IV) can be bounded for the case $k = 0$:

$$
\begin{aligned}
&\int_{\mathbb{R}} \int_{y-2T}^{y} \left| u_1^{(n)}(s) \right| \left| u_2^{(n)}(y) \right|^m \\
&\quad \times \left| \int_s^y |v_2(h)|^m dh - \int_s^y \left| u_2^{(n)}(h) \right|^m dh \right| |\phi(y,s)|\, ds\, dy \\
&\lesssim \sup_{y,s} \left(\int_s^y \left| |v_2(h)|^m - \left| u_2^{(n)}(h) \right|^m \right| dh \right) \\
&\quad \times \int_{\mathbb{R}} \left| u_2^{(n)}(y) \right|^m \int_{y-2T}^{y} \left| u_1^{(n)}(s) \right| |\phi(y,s)|\, ds\, dy,
\end{aligned} \tag{40}
$$

where supremum $\sup_{y,s}$ is taken over $s \in \mathbb{R}$ and $s \leq y \leq s + 2T$. Then, we obtain

$$
\begin{aligned}
(\mathrm{IV}) &\lesssim \left\| v_2 - u_2^{(n)} \right\|_{L^m} \left(\|v_2\|_{L^m}^{m-1} + \left\| u_2^{(n)} \right\|_{L^m}^{m-1} \right) \\
&\quad \times \left\| u_2^{(n)} \right\|_{L^m}^m \left\| u_1^{(n)} \right\|_{L^2} \sup_y \left(\int_{y-2T}^{y} |\phi(y,s)|^2 ds \right)^{1/2} .
\end{aligned} \tag{41}
$$

For the case $k \geq 1$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \int_s^{s+2T} \left| u_1^{(n)}(s) \right|^{k+1} \left| u_2^{(n)}(y) \right|^{m-k} \\
&\quad \times \Bigg| |v_1(s)|^k \int_s^y |v_2(h)|^{m-k} dh \\
&\quad - \left| u_1^{(n)}(s) \right|^k \int_s^y \left| u_2^{(n)}(h) \right|^{m-k} dh \Bigg| |\phi|\, dy\, ds \\
&\leq \int_{\mathbb{R}} \int_s^{s+2T} \left| u_1^{(n)}(s) \right|^{k+1} \left| u_2^{(n)}(y) \right|^{m-k} |v_1(s)|^k \\
&\quad \times \int_s^y \left| |v_2(h)|^{m-k} - \left| u_2^{(n)}(h) \right|^{m-k} \right| dh\, |\phi(y,s)|\, dy\, ds \\
&\quad + \int_{\mathbb{R}} \int_s^{s+2T} \left| u_1^{(n)}(s) \right|^{k+1} \left| u_2^{(n)}(y) \right|^{m-k} \\
&\quad \times \left| |v_1(s)|^k - \left| u_1^{(n)}(s) \right|^k \right| \\
&\quad \times \int_s^y \left| u_2^{(n)}(h) \right|^{m-k} dh\, |\phi(y,s)|\, dy\, ds \\
&\lesssim \sup_{y,s} \left(\int_s^y \left| |v_2(h)|^{m-k} - \left| u_2^{(n)}(h) \right|^{m-k} \right| dh \right) \\
&\quad \times \int_{\mathbb{R}} \left| u_1^{(n)}(s) \right|^{k+1} |v_1(s)|^k \int_s^{s+2T} \left| u_2^{(n)}(y) \right|^{m-k} |\phi|\, dy\, ds \\
&\quad + \sup_{y,s} \left(\int_s^y \left| u_2^{(n)}(h) \right|^{m-k} dh \right) \int_{\mathbb{R}} \left| u_1^{(n)}(s) \right|^{k+1} \\
&\quad \times \left| |v_1(s)|^k - \left| u_1^{(n)}(s) \right|^k \right| \int_s^{s+2T} \left| u_2^{(n)}(y) \right|^{m-k} |\phi|\, dy\, ds .
\end{aligned} \tag{42}
$$

Considering $s \leq y \leq s + 2T$, we can bound (IV) as follows. For the case $m - k = 0$, we have

$$
\begin{aligned}
|(\mathrm{IV})| &\lesssim 2T \left\| u_1^{(n)} \right\|_{L^{2m+1}}^{m+1} \left\| v_1 - u_1^{(n)} \right\|_{L^{2m+1}} \\
&\quad \times \left(\|v_1\|_{L^{2m+1}}^{m-1} + \left\| u_1^{(n)} \right\|_{L^{2m+1}}^{m-1} \right) \sup_s \left(\int_s^{s+2T} |\phi(y,s)|\, dy \right).
\end{aligned} \tag{43}
$$

For the case $m - k = 1$, we have

$$
\begin{aligned}
|(\mathrm{IV})| &\lesssim \sqrt{2T} \left\| v_2 - u_2^{(n)} \right\|_{L^2} \left\| u_1^{(n)} \right\|_{L^{2m-1}}^m \|v_1\|_{L^{2m-1}}^{m-1} \\
&\quad \times \left\| u_2^{(n)} \right\|_{L^2} \sup_s \left(\int_s^{s+2T} |\phi(y,s)|^2 dy \right)^{1/2} .
\end{aligned} \tag{44}
$$

For other cases ($k \geq 1$ and $m - k \geq 2$), we finally have

$$
\begin{aligned}
|(\mathrm{IV})| &\lesssim M \left\| v_2 - u_2^{(n)} \right\|_{L^{m-k}} \left(\|v_2\|_{L^{m-k}}^{m-k-1} + \left\| u_2^{(n)} \right\|_{L^{m-k}}^{m-k-1} \right) \\
&\quad \times \left\| u_2^{(n)} \right\|_{L^{m-k}}^{m-k} \left\| u_1^{(n)} \right\|_{L^{2k+1}}^{k+1} \|v_1\|_{L^{2k+1}}^{k+1} \\
&\quad + M \left\| u_2^{(n)} \right\|_{L^{m-k}}^{2m-2k} \left\| u_1^{(n)} \right\|_{L^{2k+1}}^{k+1} \left\| v_1 - u_1^{(n)} \right\|_{L^{2k+1}} \\
&\quad \times \left(\|v_1\|_{L^{2k+1}}^{k-1} + \left\| u_1^{(n)} \right\|_{L^{2k+1}}^{k-1} \right).
\end{aligned} \tag{45}
$$

Then, we have $|(1)| + |(2)| + |(3)| \to 0$ as $n \to \infty$ which implies that (19) holds.

Proof of Theorem 3

Here, we prove Theorem 3 which shows the uniqueness of weak solutions to (6). Before proving Theorem 3, we introduce a lemma given in [14] for easy reference. Consider the following equations on the time interval $[-L, L]$:

$$\begin{aligned}(\partial_t + \partial_x)\, v_+ &= f_+,\\ (\partial_t - \partial_x)\, v_- &= f_-.\end{aligned} \tag{46}$$

Lemma 6. *Suppose that $f_\pm \in L^1([-L, L] \times \mathbb{R})$ and $v_\pm \in L^\infty([-L, L]; L^1(\mathbb{R}))$ satisfy (46) in the sense of distribution. Then, one has*

$$\begin{aligned}&\int_{-L+T}^{L-T} \left|v_+(T, y)\right| dy\\ &\quad\le \int_{-L}^{L} \left|v_+(0, y)\right| dy + \int_0^T \int_{-L+t}^{L-t} \left|f_+(t, y)\right| dy\,dt,\\ &\int_{-L+T}^{L-T} \left|v_-(T, y)\right| dy\\ &\quad\le \int_{-L}^{L} \left|v_-(0, y)\right| dy + \int_0^T \int_{-L+t}^{L-t} \left|f_-(t, y)\right| dy\,dt.\end{aligned} \tag{47}$$

Now let U_j and V_j be two weak solutions of (6) with the same initial data. We define $\omega_j = U_j - V_j$. Then, we have equations for ω_j:

$$\begin{aligned}\partial_t\omega_1 + \partial_x\omega_1 &= i\left(|U_1| - |V_1|\right)|U_1|^{k-1}|U_2|^{m-k}U_1\\ &\quad + \cdots + i|V_1|^k\left(|U_2| - |V_2|\right)|U_2|^{m-k-1}U_1\\ &\quad + \cdots + i|V_1|^k|V_2|^{m-k-1}\left(|U_2| - |V_2|\right)U_1\\ &\quad + i|V_1|^k|V_2|^{m-k}\omega_1,\\ \partial_t\omega_2 - \partial_x\omega_2 &= i\left(|U_2| - |V_2|\right)|U_2|^{k-1}|U_1|^{m-k}U_2\\ &\quad + \cdots + i|V_2|^k\left(|U_1| - |V_1|\right)|U_1|^{m-k-1}U_2\\ &\quad + \cdots + i|V_2|^k|V_1|^{m-k-1}\left(|U_1| - |V_1|\right)U_2\\ &\quad + i|V_2|^k|V_1|^{m-k}\omega_2.\end{aligned} \tag{48}$$

Multiplying by complex conjugates, respectively, and taking the real parts, we obtain

$$\begin{aligned}&\partial_t|\omega_1|^2 + \partial_x|\omega_1|^2\\ &\quad = 2\left(|U_1| - |V_1|\right)|U_1|^{k-1}|U_2|^{m-k}\,\mathrm{Im}\left(\overline{U}_1\omega_1\right)\\ &\qquad + \cdots + 2|V_1|^k|V_2|^{m-k-1}\left(|U_2| - |V_2|\right)\mathrm{Im}\left(\overline{U}_1\omega_1\right),\\ &\partial_t|\omega_2|^2 - \partial_x|\omega_2|^2\\ &\quad = 2\left(|U_2| - |V_2|\right)|U_2|^{k-1}|U_1|^{m-k}\,\mathrm{Im}\left(\overline{U}_2\omega_2\right)\\ &\qquad + \cdots + 2|V_2|^k|V_1|^{m-k-1}\left(|U_1| - |V_1|\right)\mathrm{Im}\left(\overline{U}_2\omega_2\right),\end{aligned} \tag{49}$$

where we note that $\mathrm{Re}(i|V_1|^k|V_2|^{m-k}|\omega_1|^2) = 0 = \mathrm{Re}(i|V_2|^k|V_1|^{m-k}|\omega_2|^2)$. Applying Lemma 6 to (49) and considering $\omega_j(0, x) = 0$, we have

$$\begin{aligned}&\int_{-L+T}^{L-T} \left|\omega_1(t, y)\right|^2 dy\\ &\quad\lesssim \int_0^T \left\||U_1| - |V_1|\right\|_{L^{m+2}} \|U_1\|_{L^{m+2}}^k \|U_2\|_{L^{m+2}} \|\omega_1\|_{L^{m+2}}\\ &\qquad + \cdots + \|V_1\|_{L^{m+2}}^k \|V_2\|_{L^{m+2}}^{m-k-1} \left\||U_2| - |V_2|\right\|_{L^{m+2}}\\ &\qquad \times \|U_1\|_{L^{m+2}} \|\omega_1\|_{L^{m+2}} dt,\end{aligned} \tag{50}$$

where we understand $\|f\|_{L^{m+2}} = (\int_{-L+t}^{L-t} |f(t, x)|^{m+2} dx)^{1/(m+2)}$.

To estimate $\||U_j(t, \cdot)| - |V_j(t, \cdot)|\|_{L^{m+2}}$ $(j = 1, 2)$ in (50) we multiply (6) by $|U_j|^m \overline{U}_j$ to obtain

$$\begin{aligned}\partial_t|U_1|^{m+2} + \partial_x|U_1|^{m+2} &= 0,\\ \partial_t|U_2|^{m+2} - \partial_x|U_2|^{m+2} &= 0.\end{aligned} \tag{51}$$

With the same equations for V_j, we obtain

$$\begin{aligned}\partial_t\left(|U_1|^{m+2} - |V_1|^{m+2}\right) + \partial_x\left(|U_1|^{m+2} - |V_1|^{m+2}\right) &= 0,\\ \partial_t\left(|U_2|^{m+2} - |V_2|^{m+2}\right) - \partial_x\left(|U_2|^{m+2} - |V_2|^{m+2}\right) &= 0.\end{aligned} \tag{52}$$

Applying Lemma 6 and considering $|U_j(0, x)|^{m+2} - |V_j(0, x)|^{m+2} = 0$, we have

$$\begin{aligned}&\int_{-L+t}^{L-t} \left||U_j(t, x)| - |V_j(t, x)|\right|^{m+2} dx\\ &\quad\le \int_{-L+t}^{L-t} \left||U_j(t, x)|^{m+2} - |V_j(t, x)|^{m+2}\right| dx \le 0.\end{aligned} \tag{53}$$

Considering the inequality (50), we conclude that $\int_{-L+T}^{L-T} |\omega_1(t, y)|^2 dy = 0$.

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