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Letter to the Editor

Approximately Ternary Homomorphisms on C^*-Ternary Algebras

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Gordji et al. established the Hyers-Ulam stability and the superstability of C^*-ternary homomorphisms and C^*-ternary derivations on C^*-ternary algebras, associated with the following functional equation: $f((x_2-x_1)/3)+f((x_1-3x_3)/3)+f((3x_1+3x_3-x_2)/3) = f(x_1)$, by the direct method. Under the conditions in the main theorems, we can show that the related mappings must be zero. In this paper, we correct the conditions and prove the corrected theorems. Furthermore, we prove the Hyers-Ulam stability and the superstability of C^*-ternary homomorphisms and C^*-ternary derivations on C^*-ternary algebras by using a fixed point approach.

1. Introduction

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \rightarrowtail [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x[w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x, x, x]\| = \|x\|^3$. If a C^*-ternary algebra $(A[\cdot, \cdot, \cdot])$ has an identity, that is, an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [x, e, y]$ and $x^* := [e, x, e]$, is a unital C^*-algebra. Conversely, if $(A, \circ)$ is a unital C^*-algebra, then $[x, y, z] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra. A $\mathbb{C}$-linear mapping $H : A \rightarrow B$ between C^*-ternary algebras is called a C^*-ternary homomorphism if

$$H([x, y, z]) = [H(x), H(y), H(z)] \tag{1}$$

for all $x, y, z \in A$. A $\mathbb{C}$-linear mapping $\delta : A \rightarrow A$ is called a C^*-ternary derivation if

$$\begin{aligned}\delta([x, y, z]) = {}& [\delta(x), y, z] + [x, \delta(y), z] \\ & + [x, y, \delta(z)] \quad (x, y, z \in A).\end{aligned} \tag{2}$$

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their applications in physics (see [1–4]).

The stability problem of functional equations is originated from the following question of Ulam [5]: *under what condition does there exist an additive mapping near an approximately additive mapping?* In 1941, Hyers [6] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Rassias [7] extended the theorem of Hyers by considering the unbounded Cauchy difference $\|f(x+y)-f(x)-f(y)\| \leq \varepsilon(\|x\|^p+\|y\|^p)$, $(\varepsilon > 0,\ p \in [0, 1))$. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [8–12]).

Gordji et al. [13] proved the Hyers-Ulam stability and the superstability of C^*-ternary homomorphisms and C^*-ternary derivations on C^*-ternary algebras, associated with the functional equation

$$\begin{aligned}& f\left(\frac{x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3x_3}{3}\right) + f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) \\ & \quad = f(x_1)\end{aligned} \tag{3}$$

by applying the direct method. Under the conditions in the main theorems of [13], we can show that the related mappings must be zero.

In this paper, we change the conditions of [13] and establish the corrected theorems. Moreover, we prove the Hyers-Ulam stability and the superstability of C^*-ternary homomorphisms and C^*-ternary derivations on C^*-ternary algebras by employing a fixed point method. In fact, we show that some results of [13] are the special cases of our results.

2. Superstability: Direct Method

Throughout this paper, we assume that A is a C^*-ternary algebra with norm $\|\cdot\|$ and that B is a C^*-ternary algebra with norm $\|\cdot\|$. Moreover, we assume that $n_0 \in \mathbb{N}$ is a positive integer and suppose that $\mathbb{T}^1_{1/n_0} := \{e^{i\theta};\ 0 \le \theta \le 2\pi/n_0\}$.

In this section, we modify some results of [13]. Recall that a functional equation is called *superstable* if every approximate solution is an exact solution of it.

Lemma 1 (see [13]). *Let $f : A \to B$ be a mapping such that*

$$\left\| f\left(\frac{x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3x_3}{3}\right) + f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) \right\| \le \|f(x_1)\| \tag{4}$$

for all $x_1, x_2, x_3 \in A$. Then, f is additive.

We correct the statements of [13, Theorem 2.2] as follows.

Theorem 2. *Let $p \ne 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$\left\| f\left(\frac{\mu x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3\mu x_3}{3}\right) + \mu f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) \right\| \le \|f(x_1)\|, \tag{5}$$

$$\begin{aligned} &\|f([x_1, x_2, x_3]) - [f(x_1), f(x_2), f(x_3)]\| \\ &\quad \le \theta\left(\|x_1\|^{3p} + \|x_2\|^{3p} + \|x_3\|^{3p}\right) \end{aligned} \tag{6}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $x_1, x_2, x_3 \in A$. Then, the mapping $f : A \to B$ is a C^*-ternary homomorphism.

Proof. The proof is the same as in the proof of [13, Theorem 2.2]. □

In the following result, we correct Theorem 3 from [13]. Since the proof is similar, it is omitted.

Theorem 3. *Let $p \ne 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (5) *and*

$$\begin{aligned} &\|f([x_1, x_2, x_3]) - [f(x_1), x_2, x_3] \\ &\quad - [x_1, f(x_2), x_3] - [x_1, x_2, f(x_3)]\| \\ &\quad \le \theta\left(\|x_1\|^{3p} + \|x_2\|^{3p} + \|x_3\|^{3p}\right) \end{aligned} \tag{7}$$

for all $x_1, x_2, x_3 \in A$. Then, the mapping $f : A \to A$ is a C^*-ternary derivation.

3. Hyers-Ulam Stability: Direct Method

In this section, we prove the Hyers-Ulam stability of C^*-ternary homomorphisms and C^*-ternary derivations on C^*-ternary algebras by the direct method.

Theorem 4. *Let $p > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (6) *and*

$$\begin{aligned} &\left\| f\left(\frac{\mu x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3\mu x_3}{3}\right) \right. \\ &\quad \left. + \mu f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) - \mu f(x_1) \right\| \\ &\quad \le \theta\left(\|x_1\|^p + \|x_2\|^p + \|x_3\|^p\right) \end{aligned} \tag{8}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $x_1, x_2, x_3 \in A$. Then, there exists a unique C^*-ternary homomorphism $H : A \to B$ such that

$$\|H(x_1) - f(x_1)\| \le \frac{3^p(1 + 2^p)\theta\|x_1\|^p}{3^p - 3} \tag{9}$$

for all $x_1 \in A$.

Proof. Letting $\mu = 1$, $x_2 = 2x_1$, and $x_3 = 0$ in (8), we get

$$\left\| 3f\left(\frac{x_1}{3}\right) - f(x_1) \right\| \le (1 + 2^p)\theta\|x_1\|^p \tag{10}$$

for all $x_1 \in A$. By induction, we have

$$\left\| 3^n f\left(\frac{x_1}{3^n}\right) - f(x_1) \right\| \le (1 + 2^p)\theta\|x_1\|^p \sum_{i=0}^{n-1} 3^{i(1-p)} \tag{11}$$

for all $x_1 \in A$. Hence,

$$\begin{aligned} &\left\| 3^{n+m} f\left(\frac{x_1}{3^{n+m}}\right) - 3^m f\left(\frac{x_1}{3^m}\right) \right\| \\ &\quad \le (1 + 2^p)\theta\|x_1\|^p \sum_{i=0}^{n-1} 3^{(i+m)(1-p)} \\ &\quad \le (1 + 2^p)\theta\|x_1\|^p \sum_{i=m}^{n+m-1} 3^{i(1-p)} \end{aligned} \tag{12}$$

for all nonnegative integers m and n with $n \ge m$ and all $x_1 \in A$. It follows that the sequence $\{3^n f(x_1/3^n)\}$ is a Cauchy sequence for all $x_1 \in A$. Since B is complete, the sequence $\{3^n f(x_1/3^n)\}$ converges. Thus, one can define the mapping $H : A \to B$ by

$$H(x_1) := \lim_{n\to\infty} 3^n f\left(\frac{x_1}{3^n}\right) \tag{13}$$

for all $x_1 \in A$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (12), we get (9). It follows from (8) that

$$\begin{aligned} &\left\| H\left(\frac{\mu x_2 - x_1}{3}\right) + H\left(\frac{x_1 - 3\mu x_3}{3}\right) \right. \\ &\quad \left. + \mu H\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) - \mu H(x_1) \right\| \\ &\quad = \lim_{n\to\infty} 3^n \left[\left\| f\left(\frac{\mu x_2 - x_1}{3^{n+1}}\right) + f\left(\frac{x_1 - 3\mu x_3}{3^{n+1}}\right) \right.\right. \\ &\qquad\qquad \left.\left. + \mu f\left(\frac{3x_1 + 3x_3 - x_2}{3^{n+1}}\right) - \mu f\left(\frac{x_1}{3^n}\right) \right\| \right] \\ &\quad \le \lim_{n\to\infty} \frac{3^n \theta}{3^{np}} \left(\|x_1\|^p + \|x_2\|^p + \|x_3\|^p \right) = 0 \end{aligned} \tag{14}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $x_1, x_2, x_3 \in A$. So

$$\begin{aligned} &H\left(\frac{\mu x_2 - x_1}{3}\right) + H\left(\frac{x_1 - 3\mu x_3}{3}\right) \\ &\quad + \mu H\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) = \mu H(x_1) \end{aligned} \tag{15}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $x_1, x_2, x_3 \in A$. Put $\mu = 1$ in (15). Then the mapping $H : A \to B$ satisfies the inequality (4), and thus, the mapping $H : A \to B$ is additive. Letting $x_1 = x_2 = 0$ in (15), we get $H(-3\mu x_3/3) + \mu H(3x_3/3) = 0$ and so $H(\mu x_3) = \mu H(x_3)$ for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $x_3 \in A$. By the same reasoning as in the proof of [14, Theorem 2.2], the mapping H is $\mathbb{C}$-linear. Now, let $H' : A \to B$ be another additive mapping satisfying (9). Then, we have

$$\begin{aligned} &\left\| H(x_1) - H'(x_1) \right\| \\ &\quad = 3^n \left\| H\left(\frac{x_1}{3^n}\right) - H'\left(\frac{x_1}{3^n}\right) \right\| \\ &\quad \le 3^n \left(\left\| H\left(\frac{x_1}{3^n}\right) - f\left(\frac{x_1}{3^n}\right) \right\| + \left\| H'\left(\frac{x_1}{3^n}\right) - f\left(\frac{x_1}{3^n}\right) \right\| \right) \\ &\quad \le \frac{2 \cdot 3^n (1 + 2^p) 3^p}{3^{np} (3^p - 3)} \theta \|x\|^p, \end{aligned} \tag{16}$$

which tends to zero as $n \to \infty$ for all $x_1 \in A$. Thus, we can conclude that $H(x_1) = H'(x_1)$ for all $x_1 \in A$. This shows the uniqueness of H. It follows from (6) that

$$\begin{aligned} &\left\| H([x_1, x_2, x_3]) - [H(x_1), H(x_2), H(x_3)] \right\| \\ &\quad = \lim_{n\to\infty} 27^n \left\| f\left(\frac{[x_1, x_2, x_3]}{27^n}\right) \right. \\ &\qquad\qquad \left. - \left[f\left(\frac{x_1}{3^n}\right), f\left(\frac{x_2}{3^n}\right), f\left(\frac{x_3}{3^n}\right) \right] \right\| \\ &\quad \le \lim_{n\to\infty} \frac{27^n \theta}{27^{np}} \left(\|x_1\|^{3p} + \|x_2\|^{3p} + \|x_3\|^{3p} \right) = 0 \end{aligned} \tag{17}$$

for all $x_1, x_2, x_3 \in A$. Therefore, the mapping H is a unique C^*-ternary homomorphism satisfying (9). □

Theorem 5. *Let $p < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying (6) and (8). Then, there exists a unique C^*-ternary homomorphism $H : A \to B$ such that*

$$\left\| H(x_1) - f(x_1) \right\| \le \frac{3^p (1 + 2^p) \theta \|x_1\|^p}{3 - 3^p} \tag{18}$$

for all $x_1 \in A$.

Proof. The proof is similar to the proof of Theorem 4. □

In the following theorem, we prove the Hyers-Ulam stability of derivations on C^*-ternary algebras via the direct method.

Theorem 6. *Let $p > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (7) and*

$$\begin{aligned} &\left\| f\left(\frac{\mu x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3\mu x_3}{3}\right) \right. \\ &\quad \left. + \mu f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) - \mu f(x_1) \right\| \\ &\quad \le \theta \left(\|x_1\|^p + \|x_2\|^p + \|x_3\|^p \right) \end{aligned} \tag{19}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $x_1, x_2, x_3 \in A$. Then, there exists a unique C^*-ternary derivation $D : A \to A$ such that

$$\left\| D(x_1) - f(x_1) \right\| \le \frac{3^p (1 + 2^p) \theta \|x_1\|^p}{3^p - 3} \tag{20}$$

for all $x_1 \in A$.

Proof. By the same reasoning as in the proof of Theorem 4, there exists a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (20) which is defined by

$$D(x_1) := \lim_{n\to\infty} 3^n f\left(\frac{x_1}{3^n}\right) \tag{21}$$

for all $x_1 \in A$. The inequality (7) implies that

$$\begin{aligned} &\left\| D([x_1, x_2, x_3]) - [D(x_1), x_2, x_3] \right. \\ &\quad \left. - [x_1, D(x_2), x_3] - [x_1, x_2, D(x_3)] \right\| \\ &\quad = \lim_{n\to\infty} 27^n \left(\left\| f\left(\frac{[x_1, x_2, x_3]}{27^n}\right) - \left[f\left(\frac{x_1}{3^n}\right), \frac{x_2}{3^n}, \frac{x_3}{3^n} \right] \right.\right. \\ &\qquad\qquad \left.\left. - \left[\frac{x_1}{3^n}, f\left(\frac{x_2}{3^n}\right), \frac{x_3}{3^n} \right] - \left[\frac{x_1}{3^n}, \frac{x_2}{3^n}, f\left(\frac{x_3}{3^n}\right) \right] \right\| \right) \\ &\quad \le \lim_{n\to\infty} \frac{27^n \theta}{27^{np}} \left(\|x_1\|^{3p} + \|x_2\|^{3p} + \|x_3\|^{3p} \right) = 0 \end{aligned} \tag{22}$$

for all $x_1, x_2, x_3 \in A$. So

$$\begin{aligned} D([x_1, x_2, x_3]) &= [D(x_1), x_2, x_3] \\ &\quad + [x_1, D(x_2), x_3] + [x_1, x_2, D(x_3)] \end{aligned} \tag{23}$$

for all $x_1, x_2, x_3 \in A$. Consequently, the mapping D is a unique C^*-ternary derivation satisfying (20). □

The following consequence is analogous to Theorem 4 for C^*-ternary derivations and its proof is similar to the proof of Theorems 4 and 6.

Theorem 7. *Let $p < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (7) *and* (19)*. Then, there exists a unique C^*-ternary derivation $D : A \to A$ such that*

$$\|D(x_1) - f(x_1)\| \le \frac{3^p(1+2^p)\theta\|x_1\|^p}{3-3^p} \tag{24}$$

for all $x_1 \in A$.

4. Superstability: A Fixed Point Approach

In this section, we prove the superstability of C^*-ternary homomorphisms and of C^*-ternary derivations on C^*-ternary algebras by using the fixed point method (Theorem 8).

Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(3) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$.

We recall a fundamental result in the fixed point theory from [15] which is a useful tool to achieve our purposes in the sequel.

Theorem 8. *Let (X, d) be a complete generalized metric space, and let $J : X \to X$ be a strictly contractive mapping with the Lipschitz constant $\alpha < 1$. Then, for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{25}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(i) *$d(J^n x, J^{n+1} x) < \infty$, for all $n \ge n_0$;*

(ii) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(iii) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(iv) *$d(y, y^*) \le (1/(1-\alpha)) d(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Rassias [16] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. In 2003, Cădariu and Radu applied a fixed point method to the investigation of the Jensen functional equation [17]. They presented a short and a simple proof for the Cauchy functional equation and the quadratic functional equation in [18, 19], respectively. By using the fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors. For instance, the Hyers-Ulam stability and the superstability of a ternary quadratic derivation on ternary Banach algebras and C^*-ternary rings by using Theorem 8 are investigated in [20]. Recently, in [21], Park and Bodaghi proved the stability and the superstability of $*$-derivations associated with the Cauchy functional equation and the Jensen functional equation by the mentioned theorem (for more applications, see [22–28]).

From now on, we denote $\overbrace{A \times A \times \cdots \times A}^{n\text{-times}}$ by A^n. We prove the superstability of C^*-ternary homomorphism on C^*-ternary algebras by employing Theorem 8 as follows.

Theorem 9. *Let $\varphi : A^3 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x_1, x_2, x_3) \le 27\alpha\varphi\left(\frac{x_1}{3}, \frac{x_2}{3}, \frac{x_2}{3}\right) \tag{26}$$

for all $x_1, x_2, x_3 \in A$. Let $f : A \to B$ be a mapping satisfying (5) *and*

$$\begin{aligned} &\|f([x_1, x_2, x_3]) - [f(x_1), f(x_2), f(x_3)]\| \\ &\quad \le \varphi(x_1, x_2, x_3) \end{aligned} \tag{27}$$

for all $x_1, x_2, x_3 \in A$. Then, the mapping $f : A \to B$ is a C^*-ternary homomorphism.

Proof. Since the proof is similar to the proof of [13, Theorem 2.2], we only show some parts of it. From the proof of [13, Theorem 2.2], one can show that the mapping $f : A \to B$ is $\mathbb{C}$-linear. The inequality (26) implies that

$$\lim_{n\to\infty} \frac{1}{27^n} \varphi(3^n x_1, 3^n x_2, 3^n x_3) = 0 \tag{28}$$

for all $x_1, x_2, x_3 \in A$. Since f is additive, it follows from (27) and (28) that

$$f([x_1, x_2, x_3]) = [f(x_1), f(x_2), f(x_3)] \tag{29}$$

for all $x_1, x_2, x_3 \in A$. Thus, the mapping $f : A \to B$ is a C^*-ternary homomorphism. □

Theorem 10. *Let $\varphi : A^3 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x_1, x_2, x_3) \le \frac{\alpha}{3}\varphi(3x_1, 3x_2, 3x_3) \tag{30}$$

for all $x_1, x_2, x_3 \in A$. Let $f : A \to B$ be a mapping satisfying (5) *and* (27)*. Then, the mapping $f : A \to B$ is a C^*-ternary homomorphism.*

Proof. Similar to the proof of Theorem 9, the mapping $f : A \to B$ is $\mathbb{C}$-linear. It also follows from (30) that

$$\lim_{n\to\infty} 3^n \varphi\left(\frac{x_1}{3^n}, \frac{x_2}{3^n}, \frac{x_3}{3^n}\right) = 0 \tag{31}$$

for all $x_1, x_2, x_3 \in A$. Since f is additive, we can deduce from (27) and (31) that

$$f([x_1, x_2, x_3]) = [f(x_1), f(x_2), f(x_3)] \tag{32}$$

for all $x_1, x_2, x_3 \in A$. Therefore, the mapping f is a C^*-ternary homomorphism. □

Remark 11. Theorem 2 follows from Theorems 9 and 10 by taking $\varphi(x_1, x_2, x_3) = \theta(\|x_1\|^{3p} + \|x_2\|^{3p} + \|x_3\|^{3p})$ for all $x_1, x_2, x_3 \in A$.

In analogy with Theorems 9 and 10, we have the following theorems for the superstability of C^*-ternary derivations on C^*-ternary algebras.

Theorem 12. *Let $\varphi : A^3 \to [0, \infty)$ be a function satisfying* (26). *Let $f : A \to A$ be a mapping satisfying* (5) *and*

$$\begin{aligned} &\| f([x_1, x_2, x_3]) - [f(x_1), x_2, x_3] \\ &\quad - [x_1, f(x_2), x_3] - [x_1, x_2, f(x_3)] \| \\ &\quad \le \varphi(x_1, x_2, x_3) \end{aligned} \tag{33}$$

for all $x_1, x_2, x_3 \in A$. Then, the mapping $f : A \to A$ is a C^*-ternary derivation.

Proof. The proof is similar to the proof of Theorem 9. □

Theorem 13. *Let $\varphi : A^3 \to [0, \infty)$ be a function satisfying* (30). *Let $f : A \to A$ be a mapping satisfying* (5) *and* (33). *Then, the mapping $f : A \to A$ is a C^*-ternary derivation.*

Proof. Refer to the proof of Theorem 10. □

Note that Theorem 3 follows immediately from Theorems 12 and 13 by putting $\varphi(x_1, x_2, x_3) = \theta(\|x_1\|^{3p} + \|x_2\|^{3p} + \|x_3\|^{3p})$ for all $x_1, x_2, x_3 \in A$.

5. Hyers-Ulam Stability: Fixed Point Method

In this section, we apply Theorem 8 to prove the Hyers-Ulam stability of C^*-ternary homomorphisms and C^*-ternary derivations on C^*-ternary algebras.

Theorem 14. *Let $\varphi : A^3 \to [0, \infty)$ be a function satisfying* (30). *Let $f : A \to B$ be a mapping satisfying* (27) *and*

$$\begin{aligned} &\left\| f\left(\frac{\mu x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3\mu x_3}{3}\right) \right. \\ &\quad \left. + \mu f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) - \mu f(x_1) \right\| \\ &\quad \le \varphi(x_1, x_2, x_3) \end{aligned} \tag{34}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $x_1, x_2, x_3 \in A$. Then, there exists a unique C^*-ternary homomorphism $H : A \to B$ such that

$$\|H(x_1) - f(x_1)\| \le \frac{\alpha}{1 - \alpha} \varphi(x_1, 2x_1, 0) \tag{35}$$

for all $x_1 \in A$.

Proof. Letting $\mu = 1$, $x_2 = 2x_1$, and $x_3 = 0$ in (34), we get

$$\left\| 3f\left(\frac{x_1}{3}\right) - f(x_1) \right\| \le (x_1, 2x_1, 0) \tag{36}$$

for all $x_1 \in A$. Consider the set

$$S := \{h : A \longrightarrow B\} \tag{37}$$

and introduce the generalized metric on S as follows:

$$\begin{aligned} d(g, h) = \inf \{ \mu \in \mathbb{R}^+ : &\|g(x) - h(x)\| \\ &\le \mu \varphi(x, 2x, 0), \ \forall x \in A \}, \end{aligned} \tag{38}$$

where, as usual, $\inf \phi = +\infty$. Similar to the proof of [29, Theorem 2.2], we can show that d is a generalized metric on S and the metric space (S, d) is complete. We now define the linear mapping $J : S \to S$ via $Jg(x) := (1/3)g(3x)$ for all $x \in A$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varphi(x, 2x, 0) \tag{39}$$

for all $x \in A$. Hence

$$\|Jg(x) - Jh(x)\| = \left\| \frac{1}{3} g(3x) - \frac{1}{3} h(3x) \right\| \le \alpha \varphi(x, 2x, 0) \tag{40}$$

for all $x \in A$. Thus, $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le \alpha \varepsilon$. This means that

$$d(Jg, Jh) \le \alpha d(g, h) \tag{41}$$

for all $g, h \in S$. It follows from (36) that

$$\left\| f(x_1) - \frac{1}{3} f(3x_1) \right\| \le \frac{1}{3} \varphi(3x_1, 6x_1, 0) \le \alpha \varphi(x_1, 2x_1, 0) \tag{42}$$

for all $x_1 \in A$. So $d(f, Jf) \le \alpha$. By Theorem 8, there exists a mapping $H : A \to B$ satisfies the following:

(1) H is a fixed point of J, that is,

$$H(3x) = 3H(x) \tag{43}$$

for all $x \in A$. Indeed, the mapping H is a unique fixed point of J in the set $M = \{g \in S : d(h, g) < \infty\}$. This implies that H satisfying (43) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - H(x)\| \le \mu \varphi(x, 2x, 0) \tag{44}$$

for all $x \in A$;

(2) $d(J^n f, H) \to 0$ as $n \to \infty$, and thus, we have the following equality:

$$\lim_{n \to \infty} \frac{1}{3^n} f(3^n x) = H(x) \quad (x \in A); \tag{45}$$

(3) $d(f, H) \le (1/(1 - \alpha)) d(f, Jf)$, which implies the followin inequality:

$$d(f, H) \le \frac{\alpha}{1 - \alpha}. \tag{46}$$

This shows that the inequality (35) holds. The rest of the proof is similar to the proof of Theorem 4. □

Theorem 15. *Let $\varphi : A^3 \to [0,\infty)$ be a function satisfying* (26). *Let $f : A \to B$ be a mapping satisfying* (27) *and* (34). *Then, there exists a unique C^*-ternary homomorphism $H : A \to B$ such that*

$$\|H(x_1) - f(x_1)\| \leq \frac{1}{1-\alpha}\varphi(x_1, 2x_1, 0) \tag{47}$$

for all $x_1 \in A$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 14. Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 3g\left(\frac{x}{3}\right) \tag{48}$$

for all $x \in X$. The inequality (36) implies that $d(f, Jf) \leq 1$. So $d(f, H) \leq 1/(1-\alpha)$. Thus, we obtain the inequality (47). The rest of the proof is similar to the proofs of Theorems 4 and 14. □

The following parallel results for the Hyers-Ulam stability of derivations on C^*-ternary algebras can be proved in similar ways to the proofs of Theorems 6 and 14, and so we omit their proofs.

Theorem 16. *Let $\varphi : A^3 \to [0,\infty)$ be a function satisfying* (30). *Let $f : A \to A$ be a mapping satisfying* (33) *and*

$$\begin{aligned} &\left\| f\left(\frac{\mu x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3\mu x_3}{3}\right) \right. \\ &\quad \left. + \mu f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) - \mu f(x_1) \right\| \\ &\quad \leq \varphi(x_1, x_2, x_3) \end{aligned} \tag{49}$$

for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $x_1, x_2, x_3 \in A$. Then, there exists a unique C^*-ternary derivation $D : A \to A$ such that

$$\|D(x_1) - f(x_1)\| \leq \frac{\alpha}{1-\alpha}\varphi(x_1, 2x_1, 0) \tag{50}$$

for all $x_1 \in A$.

Theorem 17. *Let $\varphi : A^3 \to [0,\infty)$ be a function satisfying* (26). *Let $f : A \to A$ be a mapping satisfying* (33) *and* (49). *Then, there exists a unique C^*-ternary derivation $D : A \to A$ such that*

$$\|D(x_1) - f(x_1)\| \leq \frac{1}{1-\alpha}\varphi(x_1, 2x_1, 0) \tag{51}$$

for all $x_1 \in A$.

Remark 18. All results of Section 3 are the direct consequences of the results of this section as follows:

(i) Theorem 4 follows from Theorem 15 by taking $\varphi(x_1, x_2, x_3) = \theta(\|x_1\|^{3p} + \|x_2\|^{3p} + \|x_3\|^{3p})$ for all $x_1, x_2, x_3 \in A$, and $\alpha = 3^{1-p}$;

(ii) we can obtain Theorem 5 from Theorem 14 by letting $\varphi(x_1, x_2, x_3) = \theta(\|x_1\|^{3p} + \|x_2\|^{3p} + \|x_3\|^{3p})$ for all $x_1, x_2, x_3 \in A$, and $\alpha = 3^{p-1}$;

(iii) if we put $\varphi(x_1, x_2, x_3) = \theta(\|x_1\|^{3p} + \|x_2\|^{3p} + \|x_3\|^{3p})$ for all $x_1, x_2, x_3 \in A$, and $\alpha = 3^{1-p}$ in Theorem 17, then we conclude Theorem 6;

(iv) putting $\varphi(x_1, x_2, x_3) = \theta(\|x_1\|^{3p} + \|x_2\|^{3p} + \|x_3\|^{3p})$ for all $x_1, x_2, x_3 \in A$, and $\alpha = 3^{p-1}$ in Theorem 16, we get Theorem 7.

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