Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 284865, 7 pages
http://dx.doi.org/10.1155/2013/284865

Research Article

Abundant Exact Solition-Like Solutions to the Generalized Bretherton Equation with Arbitrary Constants

Xiuqing Yu, Fengsheng Xu, and Lihua Zhang

Department of Mathematics Sciences, Dezhou University, Dezhou 253023, China

Correspondence should be addressed to Xiuqing Yu; sddzyxq@163.com

Received 20 January 2013; Accepted 26 February 2013

Academic Editor: Abdel-Maksoud A. Soliman

The Riccati equation is employed to construct exact travelling wave solutions to the generalized Bretherton equation. Taking full advantage of the Riccati equation which has more new solutions, abundant new multiple solition-like solutions are obtained for the generalized Bretherton equation.

1. Introduction

Nonlinear partial differential equations (PDE) are widely chosen to describe complex phenomena in physics sciences. Searching for exact solutions to nonlinear differential equations plays more and more important role in nonlinear science. Recently, various direct methods have been proposed, such as the tanh-function method [1, 2], the Jacobi elliptic function expansion method [3, 4], the F-expansion [5–8], sine-cosine method [9, 10], and the homogeneous balance method [11–13]. Among them, the tanh-function method is improved continuously [14–17] as one of the most effectively straightforward methods for constructing exact solutions to PDEs. In the paper an extended tanh-function method is used to solve the generalized Bretherton equation with arbitrary constants.

In [18], Bretherton introduced the partial differential equation

$$u_{tt} + u_{xx} + u_{xxxx} + u - u^2 = 0 \tag{1a}$$

in time and one spatial dimension as a model of a dispersive wave system to study the resonant nonlinear interaction between three liner models. The modified Bretherton equation

$$u_{tt} + u_{xx} + u_{xxxx} + u - u^3 = 0 \tag{1b}$$

was studied by Kudryashov [19], Kudryashov et al. [20], and Berloff and Howard [21], and its travelling wave solutions were obtained.

Our aim in this paper is to investigate multiple soliton-like solutions to the generalized Bretherton equation in [22] by using the solutions to the Riccati equation:

$$u_{tt} + \alpha u_{xx} + \beta u_{xxxx} + \delta u + \gamma u^3 = 0. \tag{1c}$$

2. Multiple Soliton-Like Solutions to the Generalized Bretherton Equation

We assume the travelling wave variable

$$u(x,t) = u(\xi), \quad \xi = x - Vt, \tag{2}$$

where V is the speed of the travelling wave.

Making use of the travelling wave transformation (2), (1c) is converted into an ordinary differential equation (ODE) for $u = u(\xi)$ as follows:

$$\left(V^2 + \alpha\right) u'' + \beta u^{(4)} + \delta u + \gamma u^3 = 0. \tag{3}$$

We assume that the solutions to (3) can be expressed in the form

$$u(\xi) = \sum_{i=-M}^{M} a_i \phi^i, \tag{4}$$

where ϕ is a solution of the Riccati equation,

$$\phi' = A + B\phi + C\phi^2, \tag{5}$$

where a_i $(i = 0, \pm 1, \pm 2, \ldots, \pm M)$ A, B, and C are constants to be determined later, and either a_{-M} or a_M can be zero, but they cannot be zero together.

Substituting (4) into (3) together with (5) and considering the homogeneous balance between the highest-order derivative $u^{(4)}$ and the nonlinear term u^3, we obtain $M = 2$. Thus the solution to (3) takes the following form:

$$u(\xi) = a_2\phi^2 + a_1\phi + a_0 + a_{-1}\phi^{-1} + a_{-2}\phi^{-2}. \tag{6}$$

Substituting (6) with (5) into (3) and collecting all the terms of the same power of ϕ, the left-hand side of (3) is converted into another polynomial of ϕ. Setting the coefficients of ϕ^i $(i = 0, \pm 1, \pm 2)$ to zero yields a set of algebraic equations

$$\begin{gathered}
\gamma a_{-2}^3 - 120\beta a_{-2}A^4 = 0, \\
3\gamma a_{-1}a_{-2}^2 + 24\beta a_{-1}A^4 + 96\beta a_{-2}BA^3 = 0, \\
330\beta a_{-2}B^2A^2 + 60\beta a_{-1}BA^3 + 3\gamma a_{-1}^2 a_{-2} + 6\alpha a_{-2}A^2 \\
+ 3\gamma a_0 a_{-2}^2 + 6V^2 a_{-2}A^2 = 0, \\
40\beta a_{-1}A^3C + 3\gamma a_1 a_{-2}^2 + 2V^2 a_{-1}A^2 + 10V^2 a_{-2}BA + 2\alpha a_{-1}A^2 \\
+ 50\beta a_{-1}B^2A^2 + 6\gamma a_{-1}a_0 a_{-2} + 130\beta B^3 A + \gamma a_{-1}^3 \\
+ 440\beta a_{-2}CBA^2 + 10\alpha a_{-2}BA = 0, \\
6\gamma a_1 a_{-1}a_{-2} + 60\beta a_{-1}A^2CB + 3\gamma a_2 a_{-2}^2 + \delta a_{-2} \\
+ 3\gamma a_{-1}^2 a_0 + 3\alpha a_{-1}BA + 8V^2 a_{-2}CA \\
+ 8\alpha a_{-2}CA + 136\beta a_{-2}C^2A^2 + 3\gamma a_0^2 a_{-2} \\
+ 4\alpha a_{-2}B^2 + 4V^2 a_{-2}B^2 + 16\beta a_{-2}B^4 \\
+ 3V^2 a_{-1}BA + 15\beta a_{-1}B^3A + 232\beta a_{-2}CB^2A = 0, \\
16\beta a_{-1}A^2C^2 + 6V^2 a_{-2}CB + 2\alpha a_{-1}AC + \beta a_{-1}B^4 \\
+ \delta a_{-1} + 6\gamma a_2 a_{-1}a_{-2} + 30\beta a_{-2}CB^3 + 3\gamma a_{-1}a_0^2 \\
+ 2V^2 a_{-1}AC + 6\gamma a_1 a_0 a_{-2} + \alpha a_{-1}B^2 \\
+ 120\beta a_{-2}C^2AB + 6\alpha a_{-2}CB + 22\beta a_{-1}ACB^2 \\
+ a_{-1}V^2B^2 + 3\gamma a_1 a_{-1}^2 = 0, \\
2\alpha a_{-2}C^2 + 2\alpha a_2 A^2 + 2a_2 V^2 A^2 + 2a_{-2}C^2 + 3\gamma a_1^2 a_{-2} \\
+ 3\gamma a_2 a_{-1}^2 + 8\beta a_{-1}AC^2B + \delta a_0 + a_1 V^2 BA \\
+ V^2 a_{-1}BC + \alpha a_1 BA + \alpha a_{-1}BC + 16\beta a_2 CA^3 \\
+ \beta a_1 B^3 A + 14\beta a_2 B^2A^2 + 16\beta a_{-2}C^3A + \beta a_{-1}B^3C \\
+ 14\beta a_{-2}C^2B^2 + 6\gamma a_2 a_0 a_{-2} + 6\gamma a_1 a_{-1}a_0 \\
+ 8\beta\alpha_1 CBA^2 + \gamma a_0^3 = 0, \\
6\gamma a_2 a_{-1}a_0 + \alpha a_{-1}B^2 + 6a_2 V^2 BA + a_1 V^2 B^2 \\
+ 3\gamma a_1 a_0^2 + 2a_1 V^2 CA + 2\alpha a_1 CA + 3\gamma a_1^2 a_{-1} + \delta a_1 \\
+ 6\alpha a_2 BA + 22\beta a_1 CB^2A + 30\beta a_2 B^3 A \\
+ 16\beta a_1 C^2A^2 + \beta a_1 B^4 \\
+ 6\gamma a_2 a_1 a_{-2} + 120\beta a_2 CBA^2 = 0, \\
4\alpha a_2 B^2 + 8\alpha a_2 CA + 136\beta a_2 C^2A^2 + 16\beta a_2 B^4 \\
+ 3\gamma a_2 a_0^2 + \delta a_2 + 3V^2 BC + 3\alpha a_1 CB \\
+ 15\beta a_1 CB^3 + 4a_2 V^2 B^2 + 60\beta a_1 ABC^2 \\
+ 8a_2 V^2 CA + 3\gamma a_1^2 a_0 + 23\beta a_2 CB^2 A \\
+ 3\gamma a_2^2 a_{-2} + 6\gamma a_2 a_1 a_{-1} = 0, \\
40\beta a_1 C^3 A + 2\alpha a_1 C^2 + 50\beta a_1 C^2B^2 + 10\alpha a_2 CB \\
+ 6\gamma a_2 a_1 a_0 + \gamma a_1^3 + 10a_2 V^2 BC \\
+ 3\gamma a_2^2 a_{-1} + 130\beta a_2 CB^3 + 440\beta a_2 C^2BA \\
+ 2a_1 V^2 C^2 = 0, \\
3\gamma a_2^2 a_0 + 3\gamma a_1^2 a_2 + 6a_2 V^2 C^2 + 60\beta a_1 C^3 B \\
+ 240\beta a_2 C^3 A + 330\beta a_2 C^2B^2 + 6\alpha a_2 C^2 = 0, \\
3\gamma a_2^2 a_1 + 24\beta a_1 C^4 + 336\beta a_2 C^3 B = 0, \\
\gamma a_2^3 + 120\beta a_2 C^4 = 0.
\end{gathered} \tag{7}$$

Solving (7) with the help of the symbolic computation software Maple, we obtain the following.

Case 1. One has

$$\begin{gathered}
a_{-1} = 0, \qquad a_{-2} = 0, \qquad a_2 = 2C^2\sqrt{\frac{-30\beta}{\gamma}}, \\
a_1 = 2BC\sqrt{\frac{-30\beta}{\gamma}}, \qquad a_0 = 2AC\sqrt{\frac{-30\beta}{\gamma}}, \\
V = \pm\sqrt{-\alpha - 5B^2\beta + 20CA\beta}, \\
\delta = 4\beta\left(-8CAB^2 + B^4 + 16A^2C^2\right).
\end{gathered} \tag{8a}$$

Case 2. One has

$$\begin{gathered}
a_{-1} = 0, \qquad a_{-2} = 0, \qquad a_2 = -2C^2\sqrt{\frac{-30\beta}{\gamma}}, \\
a_1 = -2BC\sqrt{\frac{-30\beta}{\gamma}}, \qquad a_0 = -2AC\sqrt{\frac{-30\beta}{\gamma}},
\end{gathered}$$

$$V = \pm\sqrt{-\alpha - 5B^2\beta + 20CA\beta},$$
$$\delta = 4\beta\left(-8CAB^2 + B^4 + 16A^2C^2\right), \tag{8b}$$

where A, B, and C are arbitrary constants, but C cannot be zero.

Case 3. One has

$$a_{-1} = 0, \quad a_2 = 0, \quad a_1 = 0, \quad a_{-2} = 2A^2\sqrt{\frac{30\beta}{\gamma}},$$
$$a_0 = 2AC\sqrt{\frac{30\beta}{\gamma}}, \quad B = 0,$$
$$V = \pm\sqrt{-\alpha - 60CA\beta}, \quad \delta = -16\beta A^2C^2. \tag{8c}$$

Case 4. One has

$$a_{-1} = 0, \quad a_2 = 0, \quad a_1 = 0, \quad a_{-2} = 2A^2\sqrt{\frac{30\beta}{\gamma}},$$
$$a_0 = -\frac{-15 \pm \sqrt{165}}{15}CA\sqrt{\frac{30\beta}{\gamma}}, \quad B = 0,$$
$$V = \varepsilon\sqrt{-\alpha + \left(-30 \pm 2\sqrt{165}\right)CA\beta},$$
$$\delta = -4A^2C^2\beta\left(13 \pm \sqrt{165}\right), \tag{8d}$$

where A, B, and C are arbitrary constants, but A cannot be zero.

Case 5. One has

$$a_{-1} = 0, \quad a_2 = 0, \quad a_1 = 0, \quad a_{-2} = -2A^2\sqrt{\frac{30\beta}{\gamma}},$$
$$a_0 = -2AC\sqrt{\frac{30\beta}{\gamma}}, \quad B = 0,$$
$$V = \pm\sqrt{-\alpha - 60CA\beta}, \quad \delta = -16BA^2C^2. \tag{8e}$$

Case 6. One has

$$a_{-1} = 0, \quad a_2 = 0, \quad a_1 = 0, \quad a_{-2} = -2A^2\sqrt{\frac{30\beta}{\gamma}},$$
$$a_0 = \frac{\left(-15 \pm \sqrt{165}\right)CA}{15}\sqrt{\frac{30\beta}{\gamma}}, \quad B = 0,$$
$$V = \varepsilon\sqrt{-\alpha + \left(-30 \pm 2\sqrt{165}\right)CA\beta},$$
$$\delta = -4A^2C^2\beta\left(13 \pm \sqrt{165}\right), \tag{8f}$$

where A, B, and C are arbitrary constants, while C cannot be zero in Cases 1 and 2 and A cannot be zero in Cases 3–6. ε is an arbitrary element of $\{-1, 1\}$.

Substituting ((8a), (8b), (8c), (8d), (8e), (8f)) into (6) respectively and taking advantage of solitions to (5), we can find the following solutions which contain multiple solition-like and triangular periodic solutions for the generalized Bretherton equation.

When $\delta = 4\beta$,

$$u_{1a} = \theta\sqrt{\frac{-15\beta}{2\gamma}}\left(\coth\left(x + \varepsilon\sqrt{-\alpha - 5\beta t}\right) \pm \operatorname{csch}\left(x + \varepsilon\sqrt{-\alpha - 5\beta t}\right)\right)^2 - \theta\sqrt{\frac{-15\beta}{2\gamma}}, \tag{9a}$$

$$u_{1b} = \theta\sqrt{\frac{-15\beta}{2\gamma}}\left(\frac{\tanh\left(x + \varepsilon\sqrt{-\alpha - 5\beta t}\right)}{\left(1 \pm \operatorname{sech}\left(x + \varepsilon\sqrt{-\alpha - 5\beta t}\right)\right)}\right)^2 - \theta\sqrt{\frac{-15\beta}{2\gamma}}, \tag{9b}$$

$$u_{1c} = \theta\sqrt{\frac{-15\beta}{2\gamma}}\left(\tan\left(x + \varepsilon\sqrt{-\alpha + 5\beta t}\right) \pm \sec\left(x + \varepsilon\sqrt{-\alpha + 5\beta t}\right)\right)^2 + \theta\sqrt{\frac{-15\beta}{2\gamma}}, \tag{9c}$$

$$u_{1d} = \theta\sqrt{\frac{-15\beta}{2\gamma}}\left(\csc\left(x + \varepsilon\sqrt{-\alpha + 5\beta t}\right) - \cot\left(x + \varepsilon\sqrt{-\alpha + 5\beta t}\right)\right)^2 + \theta\sqrt{\frac{-15\beta}{2\gamma}}, \tag{9d}$$

$$u_{1e} = \theta\sqrt{\frac{-15\beta}{2\gamma}}\left(\frac{\tan\left(x + \varepsilon\sqrt{-\alpha + 5\beta t}\right)}{\left(1 \pm \sec\left(x + \varepsilon\sqrt{-\alpha + 5\beta t}\right)\right)}\right)^2 + \theta\sqrt{\frac{-15\beta}{2\gamma}}, \tag{9e}$$

$$u_{1f} = \theta\sqrt{\frac{-15\beta}{2\gamma}}\left(\cot\left(x + \varepsilon\sqrt{-\alpha + 5\beta t}\right) \pm \csc\left(x + \varepsilon\sqrt{-\alpha + 5\beta t}\right)\right)^2 + \theta\sqrt{\frac{-15\beta}{2\gamma}}, \tag{9f}$$

$$u_{1g} = \theta\sqrt{\frac{-15\beta}{2\gamma}}\left(\sec\left(x+\varepsilon\sqrt{-\alpha+5\beta t}\right) - \tan\left(x+\varepsilon\sqrt{-\alpha+5\beta t}\right)\right)^2 + \theta\sqrt{\frac{-15\beta}{2\gamma}}, \tag{9g}$$

$$u_{1h} = \theta\sqrt{\frac{-15\beta}{2\gamma}}\left(\frac{\cot\left(x+\varepsilon\sqrt{-\alpha+5\beta t}\right)}{\left(1\pm\csc\left(x+\varepsilon\sqrt{-\alpha+5\beta t}\right)\right)}\right)^2 + \theta\sqrt{\frac{-15\beta}{2\gamma}}, \tag{9h}$$

when $\delta = 64\beta$,

$$u_{2a} = 2\theta\sqrt{\frac{-30\beta}{\gamma}}\tanh^2\left(x\pm\sqrt{-\alpha-20\beta t}\right) - 2\theta\sqrt{\frac{-30\beta}{\gamma}}, \tag{10a}$$

$$u_{2b} = 2\theta\sqrt{\frac{-30\beta}{\gamma}}\coth^2\left(x\pm\sqrt{-\alpha-20\beta t}\right) - 2\theta\sqrt{\frac{-30\beta}{\gamma}}, \tag{10b}$$

$$u_{2c} = 2\theta\sqrt{\frac{-30\beta}{\gamma}}\tan^2\left(x\pm\sqrt{-\alpha+20\beta t}\right) + 2\theta\sqrt{\frac{-30\beta}{\gamma}}, \tag{10c}$$

$$u_{2d} = 2\theta\sqrt{\frac{-30\beta}{\gamma}}\cot^2\left(x\pm\sqrt{-\alpha+20\beta t}\right) + 2\theta\sqrt{\frac{-30\beta}{\gamma}}, \tag{10d}$$

$$u_{2e} = 8\theta\sqrt{\frac{-30\beta}{\gamma}}\left(\frac{\tan\left(x+\varepsilon\sqrt{-\alpha+20\beta t}\right)}{\left(1\pm\tan\left(x+\varepsilon\sqrt{-\alpha+20\beta t}\right)\right)}\right)^2 \mp 8\theta\sqrt{\frac{-30\beta}{\gamma}}\frac{\tan\left(x+\varepsilon\sqrt{-\alpha+20\beta t}\right)}{\left(1\pm\tan\left(x+\varepsilon\sqrt{-\alpha+20\beta t}\right)\right)} + 4\theta\sqrt{\frac{-30\beta}{\gamma}}, \tag{10e}$$

$$u_{2f} = 8\theta\sqrt{\frac{-30\beta}{\gamma}}\left(\frac{\cot\left(x+\varepsilon\sqrt{-\alpha+20\beta t}\right)}{\left(1\pm\cot\left(x+\varepsilon\sqrt{-\alpha+20\beta t}\right)\right)}\right)^2 \mp 8\theta\sqrt{\frac{-30\beta}{\gamma}}\frac{\cot\left(x+\varepsilon\sqrt{-\alpha+20\beta t}\right)}{\left(1\pm\cot\left(x+\varepsilon\sqrt{-\alpha+20\beta t}\right)\right)} + 4\theta\sqrt{\frac{-30\beta}{\gamma}}, \tag{10f}$$

when $\delta = 1024\beta$,

$$u_{3a} = 32\theta\sqrt{\frac{-30\beta}{\gamma}}\left(\frac{\tanh\left(x\pm\sqrt{-\alpha-80\beta t}\right)}{\left(1+\tanh^2\left(x\pm\sqrt{-\alpha-80\beta t}\right)\right)}\right)^2 - 8\theta\sqrt{\frac{-30\beta}{\gamma}}, \tag{11a}$$

$$u_{3b} = 32\theta\sqrt{\frac{-30\beta}{\gamma}}\left(\frac{\tan\left(x\pm\sqrt{-\alpha+80\beta t}\right)}{\left(1-\tan^2\left(x\pm\sqrt{-\alpha+80\beta t}\right)\right)}\right)^2 + 8\theta\sqrt{\frac{-30\beta}{\gamma}}, \tag{11b}$$

$$u_{3c} = 32\theta\sqrt{\frac{-30\beta}{\gamma}}\left(\frac{\cot\left(x\pm\sqrt{-\alpha+80\beta t}\right)}{\left(1-\cot^2\left(x\pm\sqrt{-\alpha+80\beta t}\right)\right)}\right)^2 + 8\theta\sqrt{\frac{-30\beta}{\gamma}}, \tag{11c}$$

when $\delta = 0$,

$$u_4 = \frac{\pm 2\sqrt{-30\beta/\gamma}}{\left(\left(x+\varepsilon\sqrt{-\alpha t}\right)^2 + c_0\right)^2}. \tag{12}$$

When $\delta = -\beta$,

$$u_{5a} = \theta\sqrt{\frac{15\beta}{2\gamma}}\left(\coth\left(x+\varepsilon\sqrt{-\alpha+15\beta t}\right) \pm \operatorname{csch}\left(x+\varepsilon\sqrt{-\alpha+15\beta t}\right)\right)^{-2} - \theta\sqrt{\frac{15\beta}{2\gamma}}, \tag{13a}$$

$$u_{5b} = \theta\sqrt{\frac{15\beta}{2\gamma}}\left(\frac{\tanh\left(x+\varepsilon\sqrt{-\alpha+15\beta t}\right)}{\left(1\pm\operatorname{sech}\left(x+\varepsilon\sqrt{-\alpha+15\beta t}\right)\right)}\right)^{-2} - \theta\sqrt{\frac{15\beta}{2\gamma}}, \tag{13b}$$

$$u_{5c} = \theta\sqrt{\frac{15\beta}{2\gamma}}\left(\tan\left(x+\varepsilon\sqrt{-\alpha-15\beta t}\right) \pm \sec\left(x+\varepsilon\sqrt{-\alpha-15\beta t}\right)\right)^{-2} + \theta\sqrt{\frac{15\beta}{2\gamma}}, \tag{13c}$$

$$u_{5d} = \theta\sqrt{\frac{15\beta}{2\gamma}}\left(\frac{\tan\left(x+\varepsilon\sqrt{-\alpha-15\beta t}\right)}{\left(1\pm\sec\left(x+\varepsilon\sqrt{-\alpha-15\beta t}\right)\right)}\right)^{-2} + \theta\sqrt{\frac{15\beta}{2\gamma}}, \tag{13d}$$

$$u_{5e} = \theta\sqrt{\frac{15\beta}{2\gamma}}\left(\cot\left(x+\varepsilon\sqrt{-\alpha-15\beta t}\right) \pm \csc\left(x+\varepsilon\sqrt{-\alpha-15\beta t}\right)\right)^{-2} + \theta\sqrt{\frac{15\beta}{2\gamma}}, \tag{13e}$$

$$u_{5f} = \theta\sqrt{\frac{15\beta}{2\gamma}}\left(\sec\left(x+\varepsilon\sqrt{-\alpha-15\beta t}\right) - \tan\left(x+\varepsilon\sqrt{-\alpha-15\beta t}\right)\right)^{-2} + \theta\sqrt{\frac{15\beta}{2\gamma}}, \tag{13f}$$

$$u_{5g} = \theta\sqrt{\frac{15\beta}{2\gamma}}\left(\frac{\cot\left(x+\varepsilon\sqrt{-\alpha-15\beta t}\right)}{\left(1\pm\csc\left(x+\varepsilon\sqrt{-\alpha-15\beta t}\right)\right)}\right)^{-2} + \theta\sqrt{\frac{-15\beta}{2\gamma}}, \tag{13g}$$

when $\delta = -16\beta$,

$$u_{6a} = 2\theta\sqrt{\frac{30\beta}{\gamma}}\tanh^{-2}\left(x\pm\sqrt{-\alpha+60\beta t}\right) - 2\theta\sqrt{\frac{30\beta}{\gamma}}, \tag{14a}$$

$$u_{6b} = 2\theta\sqrt{\frac{30\beta}{\gamma}}\coth^{-2}\left(x\pm\sqrt{-\alpha+60\beta t}\right) - 2\theta\sqrt{\frac{30\beta}{\gamma}}, \tag{14b}$$

$$u_{6c} = 2\theta\sqrt{\frac{30\beta}{\gamma}}\tan^{-2}\left(x\pm\sqrt{-\alpha-60\beta t}\right) + 2\theta\sqrt{\frac{30\beta}{\gamma}}, \tag{14c}$$

$$u_{6d} = 2\theta\sqrt{\frac{30\beta}{\gamma}}\cot^{-2}\left(x\pm\sqrt{-\alpha-60\beta t}\right) + 2\theta\sqrt{\frac{30\beta}{\gamma}}, \tag{14d}$$

when $\delta = -64\beta$,

$$u_{7a} = 2\theta\sqrt{\frac{30\beta}{\gamma}}\left(\frac{\tanh\left(x\pm\sqrt{-\alpha-240\beta t}\right)}{\left(1-\tanh^2\left(x\pm\sqrt{-\alpha-240\beta t}\right)\right)}\right)^{-2} + 8\theta\sqrt{\frac{30\beta}{\gamma}}, \tag{15a}$$

$$u_{7b} = 2\theta\sqrt{\frac{30\beta}{\gamma}}\left(\frac{\tanh\left(x\pm\sqrt{-\alpha+240\beta t}\right)}{\left(1+\tanh^2\left(x\pm\sqrt{-\alpha+240\beta t}\right)\right)}\right)^{-2} - 8\theta\sqrt{\frac{30\beta}{\gamma}}, \tag{15b}$$

$$u_{7c} = 2\theta\sqrt{\frac{30\beta}{\gamma}}\left(\frac{\cot\left(x\pm\sqrt{-\alpha-240\beta t}\right)}{\left(1-\cot^2\left(x\pm\sqrt{-\alpha-240\beta t}\right)\right)}\right)^{-2} + 8\theta\sqrt{\frac{30\beta}{\gamma}}, \tag{15c}$$

when $\delta = -(13\pm\sqrt{165})\beta$,

$$u_{8a} = \theta\sqrt{\frac{15\beta}{2\gamma}} \times\left(\coth\left(x+\varepsilon\sqrt{-\alpha-\frac{1}{4}\left(-30+2\sigma\sqrt{165}\right)\beta t}\right) \pm \operatorname{csch}\left(x+\varepsilon\sqrt{-\alpha-\frac{1}{4}\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)\right)^{-2} + \frac{1}{4}\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1+\sigma\frac{\sqrt{165}}{15}\right), \tag{16a}$$

$$u_{8b} = \theta\sqrt{\frac{15\beta}{2\gamma}} \times\left(\frac{\tanh\left(x+\varepsilon\sqrt{-\alpha-(1/4)\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)}{\left(1\pm\operatorname{sech}\left(x+\varepsilon\sqrt{-\alpha-(1/4)\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)\right)}\right)^{-2} + \frac{1}{4}\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1+\sigma\frac{\sqrt{165}}{15}\right), \tag{16b}$$

$$u_{8c} = \theta\sqrt{\frac{15\beta}{2\gamma}}\left(\tan\left(x+\varepsilon\sqrt{-\alpha+\frac{1}{4}\left(-30+2\sigma\sqrt{165}\right)\beta t}\right) \pm \sec\left(x+\varepsilon\sqrt{-\alpha+\frac{1}{4}\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)\right)^{-2} - \frac{1}{4}\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1+\sigma\frac{\sqrt{165}}{15}\right), \tag{16c}$$

$$u_{8d} = \theta\sqrt{\frac{15\beta}{2\gamma}} \times\left(\csc\left(x+\varepsilon\sqrt{-\alpha+\frac{1}{4}\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)\right.$$

$$-\cot\left(x+\varepsilon\sqrt{-\alpha+\frac{1}{4}\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)\Bigg)^{-2}$$
$$-\frac{1}{4}\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1+\sigma\frac{\sqrt{165}}{15}\right), \tag{16d}$$

$$u_{8e}=\theta\sqrt{\frac{15\beta}{2\gamma}}\times\left(\frac{\tan\left(x+\varepsilon\sqrt{-\alpha+(1/4)\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)}{\left(1\pm\sec\left(x+\varepsilon\sqrt{-\alpha+(1/4)\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)\right)}\right)^{-2}-\frac{1}{4}\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1+\sigma\frac{\sqrt{165}}{15}\right), \tag{16e}$$

$$u_{8f}=\theta\sqrt{\frac{15\beta}{2\gamma}}\times\left(\cot\left(x+\varepsilon\sqrt{-\alpha+\frac{1}{4}\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)\pm\csc\left(x+\varepsilon\sqrt{-\alpha+\frac{1}{4}\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)\right)^{-2}-\frac{1}{4}\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1+\sigma\frac{\sqrt{165}}{15}\right), \tag{16f}$$

$$u_{8g}=\theta\sqrt{\frac{15\beta}{2\gamma}}\times\left(\sec\left(x+\varepsilon\sqrt{-\alpha+\frac{1}{4}\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)-\tan\left(x+\varepsilon\sqrt{-\alpha+\frac{1}{4}\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)\right)^{-2}-\frac{1}{4}\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1+\sigma\frac{\sqrt{165}}{15}\right), \tag{16g}$$

$$u_{8h}=\theta\sqrt{\frac{15\beta}{2\gamma}}\times\left(\frac{\cot\left(x+\varepsilon\sqrt{-\alpha+(1/4)\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)}{\left(1\pm\csc\left(x+\varepsilon\sqrt{-\alpha+(1/4)\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)\right)}\right)^{-2}-\frac{1}{4}\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1+\sigma\frac{\sqrt{165}}{15}\right), \tag{16h}$$

when $\delta=-4(13\pm\sqrt{165})\beta$,

$$u_{9a}=2\theta\sqrt{\frac{30\beta}{\gamma}}\tanh^{-2}\left(x+\varepsilon\sqrt{-\alpha-\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)+\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1+\sigma\frac{\sqrt{165}}{15}\right), \tag{17a}$$

$$u_{9b}=2\theta\sqrt{\frac{30\beta}{\gamma}}\coth^{-2}\left(x+\varepsilon\sqrt{-\alpha-\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)+\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1+\sigma\frac{\sqrt{165}}{15}\right), \tag{17b}$$

$$u_{9c}=2\theta\sqrt{\frac{30\beta}{\gamma}}\tan^{-2}\left(x+\varepsilon\sqrt{-\alpha+\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)-\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1+\sigma\frac{\sqrt{165}}{15}\right), \tag{17c}$$

$$u_{9d}=2\theta\sqrt{\frac{30\beta}{\gamma}}\cot^{-2}\left(x+\varepsilon\sqrt{-\alpha+\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)-\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1+\sigma\frac{\sqrt{165}}{15}\right), \tag{17d}$$

when $\delta=-64(13\pm\sqrt{165})\beta$,

$$u_{10a}=2\theta\sqrt{\frac{30\beta}{\gamma}}\times\left(\frac{\tanh\left(x+\varepsilon\sqrt{-\alpha-4\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)}{\left(1+\tanh^{2}\left(x+\varepsilon\sqrt{-\alpha-4\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)\right)}\right)^{-2}+4\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1+\sigma\frac{\sqrt{165}}{15}\right), \tag{18a}$$

$$u_{10b}=2\theta\sqrt{\frac{30\beta}{\gamma}}\times\left(\frac{\tanh\left(x+\varepsilon\sqrt{-\alpha+4\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)}{\left(1-\tanh^{2}\left(x+\varepsilon\sqrt{-\alpha+4\left(-30+2\sigma\sqrt{165}\right)\beta t}\right)\right)}\right)^{-2}-4\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1+\sigma\frac{\sqrt{165}}{15}\right), \tag{18b}$$

$$u_{10c} = 2\theta\sqrt{\frac{30\beta}{\gamma}} \times \left(\frac{\cot\left(x + \varepsilon\sqrt{-\alpha + 4\left(-30 + 2\sigma\sqrt{165}\right)\beta t}\right)}{\left(1 - \cot^2\left(x + \varepsilon\sqrt{-\alpha + 4\left(-30 + 2\sigma\sqrt{165}\right)\beta t}\right)\right)} \right)^{-2} - 4\theta\sqrt{\frac{30\beta}{\gamma}}\left(-1 + \sigma\frac{\sqrt{165}}{15}\right), \tag{18c}$$

where ε, θ, and σ are arbitrary elements of $\{-1, 1\}$.

3. Conclusion

In this paper, we have used solutions to the Riccati equation to solve the generalized Bretherton equation with arbitrary constants and obtained abundant new multiple solition-like and triangular periodic solutions. It is significant to observe practical denotation of the obtained solutions, so the obtained solutions involving arbitrary constants in this paper have potential applications in dispersive wave systems to research for resonant nonlinear interactions.

Acknowledgment

This work is supported by the Natural Science Foundation of Shandong Province of China (ZR2010AL019).

References

[1] E. J. Parkes and B. R. Duffy, "An automated tanh-function method for finding solitary wave solutions to nonlinear evolution equations," *Computer Physics Communications*, vol. 98, no. 3, pp. 288–300, 1996.

[2] E. Fan, "Extended tanh-function method and its applications to nonlinear equations," *Physics Letters A*, vol. 277, no. 4-5, pp. 212–218, 2000.

[3] S. Liu, Z. Fu, S. Liu, and Q. Zhao, "Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations," *Physics Letters A*, vol. 289, no. 1-2, pp. 69–74, 2001.

[4] Z. Fu, S. Liu, S. Liu, and Q. Zhao, "New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations," *Physics Letters A*, vol. 290, no. 1-2, pp. 72–76, 2001.

[5] M. Wang and Y. Zhou, "The periodic wave solutions for the Klein-Gordon-Schrödinger equations," *Physics Letters A*, vol. 318, no. 1-2, pp. 84–92, 2003.

[6] Y. Zhou, M. Wang, and T. Miao, "The periodic wave solutions and solitary wave solutions for a class of nonlinear partial differential equations," *Physics Letters A*, vol. 323, no. 1-2, pp. 77–88, 2004.

[7] M. Wang and X. Li, "Extended F-expansion method and periodic wave solutions for the generalized Zakharov equations," *Physics Letters A*, vol. 343, no. 1-3, pp. 48–54, 2005.

[8] M. Wang and X. Li, "Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation," *Chaos, Solitons and Fractals*, vol. 24, no. 5, pp. 1257–1268, 2005.

[9] A.-M. Wazwaz, "A study on nonlinear dispersive partial differential equations of compact and noncompact solutions," *Applied Mathematics and Computation*, vol. 135, no. 2-3, pp. 399–409, 2003.

[10] A.-M. Wazwaz, "A construction of compact and noncompact solutions for nonlinear dispersive equations of even order," *Applied Mathematics and Computation*, vol. 135, no. 2-3, pp. 411–424, 2003.

[11] M. Wang, "Solitary wave solutions for variant Boussinesq equations," *Physics Letters A*, vol. 199, no. 3-4, pp. 169–172, 1995.

[12] M. Wang, "Exact solutions for a compound KdV-Burgers equation," *Physics Letters A*, vol. 213, no. 5-6, pp. 279–287, 1996.

[13] M. Wang, Y. Zhou, and Z. B. Li, "Application of a homogeneous balance method to exact solutions of nonlinear evolution equations in mathematical physics," *Physics Letters A*, vol. 216, pp. 67–75, 1996.

[14] C. Bai and H. Zhao, "New explicit exact solutions for the (2 + 1)-dimensional higher-order Broer-Kaup system," *Communications in Theoretical Physics*, vol. 41, no. 4, pp. 521–526, 2004.

[15] L. Zhang, X. Liu, and C. Bai, "New multiple soliton-like and periodic solutions for (2+1)-dimensional canonical generalized KP equation with variable coefficients," *Communications in Theoretical Physics*, vol. 46, pp. 793–798, 2006.

[16] Sirendaoreji, "Auxiliary equation method and new solutions of Klein-Gordon equations," *Chaos, Solitons and Fractals*, vol. 31, no. 4, pp. 943–950, 2007.

[17] Sirendaoreji, "A new auxiliary equation and exact travelling wave solutions of nonlinear equations," *Physics Letters A*, vol. 356, no. 2, pp. 124–130, 2006.

[18] F. P. Bretherton, "Resonant interactions between waves. The case of discrete oscillations," *Journal of Fluid Mechanics*, vol. 20, pp. 457–479, 1964.

[19] N. A. Kudryashov, "On types of nonlinear nonintegrable equations with exact solutions," *Physics Letters A*, vol. 155, no. 4-5, pp. 269–275, 1991.

[20] N. A. Kudryashov, D. I. Sinelshchikov, and M. V. Demina, "Exact solutions of the generalized Bretherton equation," *Physics Letters A*, vol. 375, no. 7, pp. 1074–1079, 2011.

[21] N. G. Berloff and L. N. Howard, "Nonlinear wave interactions in nonlinear nonintegrable systems," *Studies in Applied Mathematics*, vol. 100, no. 3, pp. 195–213, 1998.

[22] M. A. Akbar, H. Norhashidah, A. Mohd, and E. M. E. Zayed, "Abundant exact traveling wave solutions of generalized Bretherton equation via improved (G'/G)-expansion method," *Communications in Theoretical Physics*, vol. 57, no. 2, pp. 173–178, 2012.