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Research Article

Positive Solutions of Fractional Differential Equation with p-Laplacian Operator

Teng Ren and Xiaochun Chen

College of Business Administration, Hunan University, Changsha, Hunan 410082, China

Correspondence should be addressed to Xiaochun Chen; chenxiaochun1601@vip.163.com

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The basic assumption of ecological economics is that resource allocation exists social optimal solution, and the social optimal solution and the optimal solution of enterprises can be complementary. The mathematical methods and the ecological model are one of the important means in the study of ecological economics. In this paper, we study an ecological model arising from ecological economics by mathematical method, that is, study the existence of positive solutions for the fractional differential equation with p-Laplacian operator $\mathscr{D}_{\mathbf{t}}{}^{\beta}(\varphi_p(\mathscr{D}_{\mathbf{t}}{}^{\alpha}x))(t) = f(t, x(t))$, $t \in (0, 1)$, $x(0) = 0$, $x(1) = ax(\xi)$, $\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(0) = 0$, and $\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(1) = b\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(\eta)$, where $\mathscr{D}_{\mathbf{t}}{}^{\alpha}$, $\mathscr{D}_{\mathbf{t}}{}^{\beta}$ are the standard Riemann-Liouville derivatives, p-Laplacian operator is defined as $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, and the nonlinearity f may be singular at both $t = 0, 1$ and $x = 0$. By finding more suitable upper and lower solutions, we omit some key conditions of some existing works, and the existence of positive solution is established.

1. Introduction

It is well known that differential equation models can describe many nonlinear phenomena such as applied mathematics, economic mathematics, and physical and biological processes. Undoubtedly, the application of differential equation in the economics, management science, and engineering that is most successful especially plays an important role in the construction of the model for the corresponding phenomenon. In fact, many economic processes such as ecological economics model, risk model, the CIR model, and the Gaussian model in [1] can be described by differential equations. Recently, fractional-order models have proved to be more accurate than integer-order models; that is, there are more degrees of freedom in the fractional-order models. So complicated dynamic phenomenon of fractional-order calculus system has received more and more attention; see [2–15].

In this paper, we study an ecological model arising from ecological economics by mathematical method, that is, study the existence of positive solutions for the following p-Laplacian fractional boundary value problem:

$$\begin{gathered}\mathscr{D}_{\mathbf{t}}{}^{\beta}\left(\varphi_p\left(\mathscr{D}_{\mathbf{t}}{}^{\alpha}x\right)\right)(t) = f(t, x(t)), \quad t \in (0,1), \\ x(0) = 0, \qquad x(1) = ax(\xi), \\ \mathscr{D}_{\mathbf{t}}{}^{\alpha}x(0) = 0, \\ \mathscr{D}_{\mathbf{t}}{}^{\alpha}x(1) = b\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(\eta),\end{gathered} \tag{1}$$

where $\mathscr{D}_{\mathbf{t}}{}^{\alpha}$, $\mathscr{D}_{\mathbf{t}}{}^{\beta}$ are the standard Riemann-Liouville derivatives with $1 < \alpha, \beta \le 2$, $0 \le a, b \le 1$, $0 < \xi$, and $\eta \le 1$, and p-Laplacian operator is defined as $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $(\varphi_p)^{-1} = \varphi_q$, and $(1/p) + (1/q) = 1$.

The upper and lower solutions method is a powerful tool to achieve the existence results for boundary value problem; see [2–6]. Recently, Zhang and Liu [2] considered the existence of positive solutions for the singular fourth-order p-Laplacian equation

$$\left[\varphi_p\left(u''\right)\right]'' = f(t, u(t)), \quad 0 < t < 1, \tag{2}$$

with the four-point boundary conditions

$$\begin{gathered} u(0) = 0, \qquad u(1) = au(\xi), \\ u''(0) = 0, \qquad u''(1) = bu''(\eta), \end{gathered} \tag{3}$$

where $\varphi_p(t) = |t|^{p-2}t$, $p > 1$, $0 < \xi, \eta < 1$, and $f \in C((0,1)\times(0,+\infty),\ [0,+\infty))$ may be singular at $t = 0$ and/or 1 and $u = 0$. By using the upper and lower solutions method and fixed-point theorems, the existence of positive solutions to the boundary value problem is obtained. In [2], a upper and lower solution condition (H3) is used.

There exist a continuous function $a(t)$ and some fixed positive number k, such that $a(t) \geq kt(1-t)$, $t \in [0,1]$, and

$$\int_0^1 G(t,r)\varphi_p^{-1}\left(\int_0^1 H(r,s) f(s,a(s))\,ds\right)dr = b(t) \geq a(t),$$
$$\int_0^1 G(t,r)\varphi_p^{-1}\left(\int_0^1 H(r,s) f(s,b(s))\,ds\right)dr \geq a(t), \tag{4}$$

where $G(t,s)$, $H(t,s)$ are the associated Green's functions for the relevant problems. And then, the condition (H3) was also adopted by Wang et al. [3] to deal with the p-Laplacian fractional boundary value problem (1). By using similar method as [2], the existence results of at least one positive solution for the above fractional boundary value problem are established. Recently, replaced (H3) with a simple integral condition, Jia et al. [8] studied the existence, uniqueness, and asymptotic behavior of positive solutions for the higher nonlocal fractional differential equation by using upper and lower solutions method.

In this paper, we restart to establish the existence of positive solutions for the BVP (1) when the nonlinearity f may be singular at both $t = 0, 1$ and $x = 0$. By finding more suitable upper and lower solutions of (1), we completely omit the condition (H3) in [2, 3] and integral condition in [8], thus our work improves essentially the results of [2, 3, 8].

2. Basic Definitions and Preliminaries

In this section, we present some necessary definitions and lemmas from fractional calculus theory, which can be found in the recent literatures [7, 16, 17].

Definition 1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$I^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} x(s)\,ds \tag{5}$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$\mathcal{D}_{\mathbf{t}}{}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} x(s)\,ds, \tag{6}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 3. *(1) If $x \in L^1(0,1)$, $\alpha > \beta > 0$, then*

$$\begin{gathered} I^{\alpha} I^{\beta} x(t) = I^{\alpha+\beta} x(t), \\ \mathcal{D}_{\mathbf{t}}{}^{\beta} I^{\alpha} x(t) = I^{\alpha-\beta} x(t), \\ \mathcal{D}_{\mathbf{t}}{}^{\alpha} I^{\alpha} x(t) = x(t). \end{gathered} \tag{7}$$

(2) If $\alpha > 0$, $\beta > 0$, then

$$\mathcal{D}_{\mathbf{t}}{}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}. \tag{8}$$

Lemma 4. *Let $\alpha > 0$, and let $f(x)$ be integrable, then*

$$I^{\alpha}\mathcal{D}_{\mathbf{t}}{}^{\alpha} f(x) = f(x) + c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + \cdots + c_n x^{\alpha-n}, \tag{9}$$

where $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$, and n is the smallest integer greater than or equal to α.

Definition 5. A continuous function $\phi(t)$ is called a lower solution of the BVP (1), if it satisfies

$$\begin{gathered} \mathcal{D}_{\mathbf{t}}{}^{\beta}\left(\varphi_p\left(\mathcal{D}_{\mathbf{t}}{}^{\alpha}\phi\right)\right)(t) \leq f(t, \phi(t)), \quad 0 < t < 1, \\ \phi(0) \leq 0, \qquad \phi(1) \leq a\phi(\xi), \\ \mathcal{D}_{\mathbf{t}}{}^{\alpha}\phi(0) \geq 0, \qquad \mathcal{D}_{\mathbf{t}}{}^{\alpha}\phi(1) \geq b\mathcal{D}_{\mathbf{t}}{}^{\alpha}\phi(\eta). \end{gathered} \tag{10}$$

Definition 6. A continuous function $\psi(t)$ is called an upper solution of the BVP (1), if it satisfies

$$\begin{gathered} \mathcal{D}_{\mathbf{t}}{}^{\beta}\left(\varphi_p\left(\mathcal{D}_{\mathbf{t}}{}^{\alpha}\psi\right)\right)(t) \geq f(t, \psi(t)), \quad 0 < t < 1 \\ \psi(0) \geq 0, \qquad \psi(1) \geq a\psi(\xi), \\ \mathcal{D}_{\mathbf{t}}{}^{\alpha}\psi(0) \leq 0, \qquad \mathcal{D}_{\mathbf{t}}{}^{\alpha}\psi(1) \leq b\mathcal{D}_{\mathbf{t}}{}^{\alpha}\psi(\eta). \end{gathered} \tag{11}$$

For forthcoming analysis, we first consider the following linear fractional differential equation:

$$\begin{gathered} \mathcal{D}_{\mathbf{t}}{}^{\alpha} x(t) + y(t) = 0, \quad t \in (0,1), \\ x(0) = 0, \qquad x(1) = ax(\xi). \end{gathered} \tag{12}$$

Lemma 7. *If $1 < \alpha \leq 2$ and $y \in L^1[0,1]$, then the boundary value problem (12) has the unique solution*

$$x(t) = \int_0^1 G(t,s)\, y(s)\,ds, \tag{13}$$

where

$$G(t,s) = k(t,s) + \frac{ak(\xi,s)\,t^{\alpha-1}}{1-a\xi^{\alpha-1}}, \tag{14}$$

$$k(t,s) = \begin{cases} \dfrac{(t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{15}$$

Proof. By applying Lemma 4, we may reduce (12) to an equivalent integral equation

$$x(t) = -I^{\alpha} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \quad c_1, c_2 \in \mathbb{R}. \tag{16}$$

From $x(0) = 0$ and (16), we have $c_2 = 0$. Consequently the general solution of (12) is

$$x(t) = -I^{\alpha} y(t) + c_1 t^{\alpha-1} = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + c_1 t^{\alpha-1}. \tag{17}$$

By (17), one has

$$\begin{gathered} x(1) = -\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + c_1, \\ x(\xi) = -\int_0^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + c_1 \xi^{\alpha-1}. \end{gathered} \tag{18}$$

And then, we have

$$c_1 = \frac{\int_0^1 (1-s)^{\alpha-1} y(s)\, ds - a\int_0^{\xi} (\xi-s)^{\alpha-1} y(s)\, ds}{\Gamma(\alpha)(1-a\xi^{\alpha-1})}. \tag{19}$$

So, the unique solution of problem (12) is

$$\begin{aligned} x(t) = & -\int_0^t \frac{(t-s)^{\alpha-1} y(s)}{\Gamma(\alpha)} ds + \frac{t^{\alpha-1}}{1-a\xi^{\alpha-1}} \\ & \times \left\{ \int_0^1 \frac{(1-s)^{\alpha-1} y(s)}{\Gamma(\alpha)} ds - a \int_0^{\xi} \frac{(\xi-s)^{\alpha-1} y(s)}{\Gamma(\alpha)} ds \right\} \\ = & -\int_0^t \frac{(t-s)^{\alpha-1} y(s)}{\Gamma(\alpha)} ds + \int_0^1 \frac{(1-s)^{\alpha-1} t^{\alpha-1} y(s)}{\Gamma(\alpha)} ds \\ & + \frac{a t^{\alpha-1}}{1-a\xi^{\alpha-1}} \int_0^1 \frac{(1-s)^{\alpha-1} \xi^{\alpha-1} y(s)}{\Gamma(\alpha)} ds \\ & - \frac{a t^{\alpha-1}}{1-a\xi^{\alpha-1}} \int_0^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds \\ = & \int_0^1 \left(k(t,s) + \frac{a t^{\alpha-1} y(s)}{1-a\xi^{\alpha-1}} k(\xi,s) \right) ds \\ = & \int_0^1 G(t,s) y(s)\, ds. \end{aligned} \tag{20}$$

The proof is completed. □

Lemma 8. *Let $y \in L^1[0,1]$, $1 < \alpha, \beta \le 2$, $0 < \xi, \eta < 1$, and $0 \le a, b \le 1$. The fractional boundary value problem*

$$\begin{gathered} \mathscr{D}_{\mathbf{t}}{}^{\beta} \left(\varphi_p \left(\mathscr{D}_{\mathbf{t}}{}^{\alpha} x \right) \right)(t) = y(t), \quad t \in (0,1), \\ x(0) = 0, \qquad x(1) = a x(\xi), \\ \mathscr{D}_{\mathbf{t}}{}^{\alpha} x(0) = 0, \qquad \mathscr{D}_{\mathbf{t}}{}^{\alpha} x(1) = b \mathscr{D}_{\mathbf{t}}{}^{\alpha} x(\eta), \end{gathered} \tag{21}$$

has unique solution

$$x(t) = \int_0^1 G(t,s) \varphi_q \left(\int_0^1 H(s,\tau) y(\tau)\, d\tau \right) ds, \tag{22}$$

where

$$\begin{gathered} H(t,s) = g(t,s) + \frac{a g(\eta,s) t^{\beta-1}}{1-b_1\eta^{\beta-1}}, \\ g(t,s) = \begin{cases} \dfrac{(t(1-s))^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le s \le t \le 1, \\ \dfrac{(t(1-s))^{\beta-1}}{\Gamma(\beta)}, & 0 \le t \le s \le 1, \end{cases} \end{gathered} \tag{23}$$

and $b_1 = b^{p-1}$, and $G(t,s)$ is defined by (14).

Proof. At first, by Lemma 4, (21) is equivalent to the integral equation

$$\varphi_p \left(\mathscr{D}_{\mathbf{t}}{}^{\alpha} x(t) \right) = I^{\beta} y(t) + c_3 t^{\beta-1} + c_4 t^{\beta-2}, \quad c_3, c_4 \in \mathbb{R}. \tag{24}$$

From $\mathscr{D}_{\mathbf{t}}{}^{\alpha} x(0) = 0$, $\mathscr{D}_{\mathbf{t}}{}^{\alpha} x(1) = b\mathscr{D}_{\mathbf{t}}{}^{\alpha} x(\eta)$, and (24), we have $c_4 = 0$. Consequently the general solution of (21) is

$$\begin{aligned} \varphi_p \left(\mathscr{D}_{\mathbf{t}}{}^{\alpha} x(t) \right) &= I^{\beta} y(t) + c_3 t^{\beta-1} \\ &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s)\, ds + c_3 t^{\beta-1}. \end{aligned} \tag{25}$$

It follows from (25) that

$$\varphi_p \left(\mathscr{D}_{\mathbf{t}}{}^{\alpha} x(1) \right) = \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} y(s)\, ds + c_3, \tag{26}$$

$$\varphi_p \left(\mathscr{D}_{\mathbf{t}}{}^{\alpha} x(\eta) \right) = \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} y(s)\, ds + c_3 \eta^{\alpha-1}. \tag{27}$$

Thus (25) and (26) imply

$$c_3 = -\frac{\int_0^1 (1-s)^{\beta-1} y(s)\, ds - b_1 \int_0^{\eta} (\eta-s)^{\beta-1} y(s)\, ds}{\Gamma(\beta)(1-b_1\eta^{\beta-1})}, \tag{28}$$

where $b_1 = b^{p-1}$. Similar to Lemma 7, we have

$$\varphi_p \left(\mathscr{D}_{\mathbf{t}}{}^{\alpha} x(t) \right) = -\int_0^1 H(t,s) y(s)\, ds. \tag{29}$$

Consequently, fractional boundary value problem (21) is equivalent to the following problem:

$$\begin{gathered} \mathscr{D}_{\mathbf{t}}{}^{\alpha} x(t) + \varphi_q \left(\int_0^1 H(t,s) y(s)\, ds \right), \quad t \in (0,1), \\ x(0) = 0, \qquad x(1) = a x(\xi). \end{gathered} \tag{30}$$

Lemma 7 implies that fractional boundary value problem (21) has a unique solution

$$x(t) = \int_0^1 G(t,s) \varphi_q \left(\int_0^1 H(s,\tau) y(\tau)\, d\tau \right) ds. \tag{31}$$

The proof is completed. □

Lemma 9. *Let $1 < \alpha, \beta \leq 2$, $0 < \xi, \eta < 1$, and $0 \leq a, b \leq 1$. The functions $G(t,s)$ and $H(t,s)$ defined by (14) and (23), respectively, are continuous on $[0,1]\times[0,1]$ and satisfy*

(i) $G(t,s) \geq 0$, $H(t,s) \geq 0$, *for* $t, s \in [0,1]$.

(ii) *For* $t, s \in [0,1]$,

$$\sigma_1(s)\, t^{\alpha-1} \leq G(t,s) \leq \sigma_2(s)\, t^{\alpha-1}, \tag{32}$$

where

$$\sigma_1(s) = \frac{ak(\xi,s)}{1-a\xi^{\alpha-1}},$$
$$\sigma_2(s) = \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{ak(\xi,s)}{1-a\xi^{\alpha-1}}. \tag{33}$$

Proof. The proof is obvious, so we omit the proof. □

Set

$$e(t) = t^{\alpha-1}. \tag{34}$$

We present the following assumptions:

(S1) $f : (0,1)\times(0,\infty) \to [0,+\infty)$ is is continuous and decreasing in x.

(S2) For any $\kappa > 0$,

$$0 < \int_0^1 \sigma_2(s)\,\varphi_q\left(\int_0^1 H(s,\tau)\, f(\tau,\kappa e(\tau))\,d\tau\right)ds < +\infty. \tag{35}$$

From Lemmas 7 and 9, it is easy to obtain the following conclusion.

Lemma 10. *If $x \in C([0,1], R)$ satisfies*

$$x(0) = 0, \qquad x(1) = \varphi_p(b)\, x(\eta), \tag{36}$$

and $\mathcal{D}_{\mathbf{t}}{}^{\beta} x(t) \geq 0$ for any $t \in (0,1)$, then $x(t) \leq 0$, for $t \in [0,1]$.

3. Main Results

Let $E = C[0,1]$, and

$$P = \{x \in E :\ \text{there exists positive number } 0 < l_x < 1,\ L_x > 1 \ \text{ such that } l_x e(t) \leq x(t) \leq L_x e(t),\ t \in [0,1]\}. \tag{37}$$

Clearly, $e(t) \in P$, so P is nonempty. For any $x \in P$, define an operator T by

$$(Tx)(t) = \int_0^1 G(t,s)\,\varphi_q\left(\int_0^1 H(s,\tau)\, f(\tau,x(\tau))\,d\tau\right)ds, \quad t \in [0,1]. \tag{38}$$

Theorem 11. *Suppose* (S1)-(S2) *hold. Then the BVP* (1) *has at least one positive solution x, and there exist two positive constants $0 < \mu_1 < 1$, $\mu_2 > 1$, such that*

$$\mu_1 e(t) \leq x(t) \leq \mu_2 e(t), \quad t \in [0,1]. \tag{39}$$

Proof. We firstly assert that T is well defined on P and $T(P) \subset P$, and T is decreasing in x.

In fact, for any $x \in P$, by the definition of P, there exist two positive numbers $0 < l_x < 1$, $L_x > 1$, such that $l_x e(t) \leq x(t) \leq L_x e(t)$ for any $t \in [0,1]$. It follows from Lemma 9 and (S1)–(S3) that

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)\,\varphi_q\left(\int_0^1 H(s,\tau)\, f(\tau,x(\tau))\,d\tau\right)ds \\ &\leq e(t)\int_0^1 \sigma_2(s)\,\varphi_q\left(\int_0^1 H(s,\tau)\, f(\tau,l_x e(\tau))\,d\tau\right)ds \\ &< +\infty.\end{aligned} \tag{40}$$

On the other hand, by Lemma 9, we also have

$$\begin{aligned}(Tx)(t) &\geq e(t)\int_0^1 \sigma_1(s)\,\varphi_q\left(\int_0^1 H(s,\tau)\, f(\tau,x(\tau))\,d\tau\right)ds \\ &\geq e(t)\int_0^1 \sigma_1(s)\,\varphi_q\left(\int_0^1 H(s,\tau)\, f(\tau,L_x e(\tau))\,d\tau\right)ds.\end{aligned} \tag{41}$$

Take

$$\begin{aligned}l'_x &= \min\left\{1, \int_0^1 \sigma_1(s)\,\varphi_q\left(\int_0^1 H(s,\tau)\, f(\tau,L_x e(\tau))\,d\tau\right)ds\right\} \\ L'_x &= \max\left\{1, \int_0^1 \sigma_2(s)\,\varphi_q\left(\int_0^1 H(s,\tau)\, f(\tau,l_x e(\tau))\,d\tau\right)ds\right\},\end{aligned} \tag{42}$$

then by (40) and (41),

$$l'_x e(t) \leq (Tx)(t) \leq L'_x e(t), \tag{43}$$

which implies that T is well defined, and $T(P) \subset P$. It follows from (S1) that the operator T is decreasing in x. And by direct computations, we have

$$\begin{gathered}\mathcal{D}_{\mathbf{t}}{}^{\beta}\left(\varphi_p\left(\mathcal{D}_{\mathbf{t}}{}^{\alpha}(Tx)\right)\right)(t) = f(t,(Tx)(t)), \quad t \in (0,1), \\ (Tx)(0) = 0, \qquad (Tx)(1) = a\,(Tx)(\xi), \\ \mathcal{D}_{\mathbf{t}}{}^{\alpha}(Tx)(0) = 0, \qquad \mathcal{D}_{\mathbf{t}}{}^{\alpha}(Tx)(1) = b\mathcal{D}_{\mathbf{t}}{}^{\alpha}(Tx)(\eta).\end{gathered} \tag{44}$$

Next we focus on lower and upper solutions of the fractional boundary value problem (1). Let

$$\begin{aligned}m(t) &= \min\{e(t), (Te)(t)\}, \\ n(t) &= \max\{e(t), (Te)(t)\},\end{aligned} \tag{45}$$

then, if $e(t) = (Te)(t)$, the conclusion of Theorem 11 holds. If $e(t) \neq (Te)(t)$, clearly, $m(t), n(t) \in P$ and

$$m(t) \leq e(t) \leq n(t). \tag{46}$$

We will prove that the functions $\phi(t) = Tn(t)$, $\psi(t) = Tm(t)$ are a couple of lower and upper solutions of the fractional boundary value problem (1), respectively.

From (S1), T is nonincreasing relative to x. Thus it follows from (45)-(46) that

$$\begin{gathered} \phi(t) = Tn(t) \leq Tm(t) = \psi(t), \\ Tn(t) \leq Te(t) \leq n(t), \\ Tm(t) \geq Te(t) \geq m(t), \end{gathered} \tag{47}$$

and $\phi(t), \psi(t) \in P$. And it follows from (44)–(47) that

$$\begin{aligned} &\mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}\phi\right)\right)(t) - f(t, \phi(t)) \\ &\quad = \mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}(Tn)\right)\right)(t) - f(t, (Tn)(t)) \\ &\quad \leq \mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}(Tn)\right)\right)(t) \\ &\qquad - f(t, n(t)) = 0, \quad t \in (0,1), \\ &\phi(0) = 0, \qquad \phi(1) = a\phi(\xi), \qquad \mathscr{D}_t^{\alpha}\phi(0) = 0, \\ &\qquad \mathscr{D}_t^{\alpha}\phi(1) = b\mathscr{D}_t^{\alpha}\phi(\eta), \\ &\mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}\psi\right)\right)(t) - f(t, \psi(t)) \\ &\quad = \mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}(Tm)\right)\right)(t) - f(t, (Tm)(t)) \\ &\quad \geq \mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}(Tm)\right)\right)(t) \\ &\qquad - f(t, m(t)) = 0, \quad t \in (0,1), \\ &\qquad \psi(0) = 0, \qquad \psi(1) = a\psi(\xi), \\ &\mathscr{D}_t^{\alpha}\psi(0) = 0, \qquad \mathscr{D}_t^{\alpha}\psi(1) = b\mathscr{D}_t^{\alpha}\psi(\eta). \end{aligned} \tag{48}$$

That is, $\phi(t)$ and $\psi(t)$ are a couple of lower and upper solutions of fractional boundary value problem (1), respectively.

Now let us define a function

$$F(t, x) = \begin{cases} f(t, \phi(t)), & x < \phi(t), \\ f(t, x(t)), & \phi(t) \leq x \leq \psi(t), \\ f(t, \psi(t)), & x > \psi(t). \end{cases} \tag{49}$$

It follows from (S1) and (49) that $F : (0,1) \times [0, +\infty) \to [0, +\infty)$ is continuous.

We will show that the fractional boundary value problem

$$\begin{gathered} \mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}x\right)\right)(t) = F(t, x(t)), \quad t \in (0,1), \\ x(0) = 0, \qquad x(1) = ax(\xi), \\ \mathscr{D}_t^{\alpha}x(0) = 0, \qquad \mathscr{D}_t^{\alpha}x(1) = b\mathscr{D}_t^{\alpha}x(\eta), \end{gathered} \tag{50}$$

has a positive solution. Let us consider the operator

$$\mathfrak{B}x(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) F(\tau, x(\tau))\, d\tau\right) ds. \tag{51}$$

Thus $\mathfrak{B} : C[0,1] \to C[0,1]$, and the fixed point of the operator $\mathfrak{B}$ is a solution of the BVP (50). Noting that $\Phi \in P$, then there exists a constant $0 < l_\Phi < 1$, such that $\Phi(t) \geq l_\Phi e(t)$, $t \in [0,1]$. Thus for all $x \in E$, it follows from Lemma 9, (49), and (S2) that

$$\begin{aligned} \mathfrak{B}x(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) F(\tau, x(\tau))\, d\tau\right) ds \\ &\leq e(t)\int_0^1 \sigma_2(s)\varphi_q\left(\int_0^1 H(s,\tau) F(\tau, x(\tau))\, d\tau\right) ds \\ &\leq e(t)\int_0^1 \sigma_2(s)\varphi_q\left(\int_0^1 H(s,\tau) f(\tau, l_\Phi e(\tau))\, d\tau\right) ds \\ &< +\infty. \end{aligned} \tag{52}$$

That is, the operator $\mathfrak{B}$ is uniformly bounded.

From the uniform continuity of $G(t,s)$ and Lebesgue dominated convergence theorem, we easily obtain that $\mathfrak{B}$ is equicontinuous. Thus by the means of the Arzela-Ascoli theorem, we have $\mathfrak{B} : E \to E$ is completely continuous. The Schauder fixed point theorem implies that $\mathfrak{B}$ has at least a fixed point x, such that $x = \mathfrak{B}x$.

At the end, we claim that

$$\phi(t) \leq x(t) \leq \psi(t), \quad t \in [0,1]. \tag{53}$$

In fact, since x is fixed point of $\mathfrak{B}$ and (44), we get

$$\begin{gathered} x(0) = 0, \qquad x(1) = ax(\xi), \\ \mathscr{D}_t^{\alpha}x(0) = 0, \qquad \mathscr{D}_t^{\alpha}x(1) = b\mathscr{D}_t^{\alpha}x(\eta), \\ \psi(0) = 0, \qquad \psi(1) = a\psi(\xi), \\ \mathscr{D}_t^{\alpha}\psi(0) = 0, \qquad \mathscr{D}_t^{\alpha}\psi(1) = b\mathscr{D}_t^{\alpha}\psi(\eta). \end{gathered} \tag{54}$$

Otherwise, suppose that $x(t) > \psi(t)$. According to the definition of F, we have

$$\mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}x\right)\right)(t) = F(t, x(t)) = f(t, \psi(t)). \tag{55}$$

On the other hand, it follows from ψ is an upper solution to (1) that

$$\mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}\psi\right)\right)(t) \geq f(t, \psi(t)). \tag{56}$$

Let $z(t) = \varphi_p(\mathscr{D}_t^{\alpha}\psi(t)) - \varphi_p(\mathscr{D}_t^{\alpha}x(t))$; it follows from (55) and (56) that

$$\begin{aligned} \mathscr{D}_t^{\beta}z(t) &= \mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}\psi\right)\right)(t) \\ &\quad - \mathscr{D}_t^{\beta}\left(\varphi_p\left(\mathscr{D}_t^{\alpha}x\right)\right)(t) \geq 0, \\ z(0) &= 0, \qquad z(1) = \varphi_p(b) z(\eta). \end{aligned} \tag{57}$$

It follows from Lemma 10 that

$$z(t) \leq 0, \tag{58}$$

and then

$$\varphi_p\left(\mathscr{D}_{\mathbf{t}}{}^{\alpha}\psi(t)\right) - \varphi_p\left(\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(t)\right) \leq 0. \tag{59}$$

Notice that φ_p is monotone increasing; we have

$$\mathscr{D}_{\mathbf{t}}{}^{\alpha}\psi(t) \leq \mathscr{D}_{\mathbf{t}}{}^{\alpha}x(t), \text{ that is, } \quad \mathscr{D}_{\mathbf{t}}{}^{\alpha}\left(\psi - x\right)(t) \leq 0. \tag{60}$$

It follows from Lemma 10 and (54) that

$$\psi(t) - x(t) \geq 0. \tag{61}$$

Thus we have $x(t) \leq \psi(t)$ on $[0,1]$, which contradicts $x(t) > \psi(t)$. Hence, $x(t) > \psi(t)$ is impossible.

By the same way, we also have $x(t) \geq \phi(t)$ on $[0,1]$. So

$$\phi(t) \leq x(t) \leq \psi(t), \quad t \in [0,1]. \tag{62}$$

Consequently, $F(t,x(t)) = f(t,x(t)), t \in [0,1]$. Then $x(t)$ is a positive solution of the problem (1).

Finally, by $\psi, \phi \in P$, we have

$$\mu_1 e(t) = l_{\phi}e(t) \leq \phi(t) \leq x(t) \leq \psi(t) \leq L_{\psi}e(t) = \mu_2 e(t), \tag{63}$$

where

$$0 < \mu_1 < 1, \qquad \mu_2 > 1. \tag{64}$$

□

Remark 12. In Theorem 11, we find more suitable lower and upper solutions, then we refine the proved process, and the key condition (H3) in [2, 3] is removed, but the existence of positive solution is still obtained, thus our result is essential improvement of [2, 3].

Theorem 13. *If* $f(t,x) : [0,1] \times [0,+\infty) \rightarrow [0,+\infty)$ *is continuous, decreasing in* x *and* $f(t,\kappa) \not\equiv 0$*, for any* $\kappa > 0$*, then the boundary value problem* (1) *has at least one positive solution* $x(t)$*, and there exist two positive constants* $0 < \mu_1 < 1$, $\mu_2 > 1$*, such that*

$$\mu_1 e(t) \leq x(t) \leq \mu_2 e(t), \quad t \in [0,1]. \tag{65}$$

Proof. The proof is similar to Theorem 11, so we omit it here. □

Example 14. Consider the following boundary value problem:

$$\begin{gathered}
\mathscr{D}_{\mathbf{t}}{}^{5/4}\left(\varphi_p\left(\mathscr{D}_{\mathbf{t}}{}^{3/2}x\right)\right)(t) = \frac{1}{t^{1/4}\sqrt[3]{x^2(t)}} + \sin t, \quad t \in (0,1), \\
x(0) = 0, \qquad x(1) = \frac{1}{8}x\left(\frac{1}{4}\right), \\
\mathscr{D}_{\mathbf{t}}{}^{3/2}(0) = 0, \qquad \mathscr{D}_{\mathbf{t}}{}^{3/2}x(1) = \frac{1}{3}\mathscr{D}_{\mathbf{t}}{}^{3/2}x\left(\frac{3}{4}\right).
\end{gathered} \tag{66}$$

Let $\alpha = 3/2$, $\beta = 5/4$, and

$$f(t,x) = \frac{1}{t^{1/4}\sqrt[3]{x^2}} + \sin t, \quad e(t) = t^{1/2}. \tag{67}$$

Obviously, (S1) holds.

For any $\kappa > 0$,

$$\begin{aligned}
0 &< \int_0^1 \sigma_2(s)\varphi_q\left(\int_0^1 H(s,\tau) f(\tau,\kappa e(\tau))\,d\tau\right)ds \\
&= \int_0^1 \sigma_2(s)\varphi_q\left(\int_0^1 H(s,\tau)\left(\kappa^{-2/3}\tau^{-7/12} + \sin\tau\right)d\tau\right)ds \\
&< +\infty,
\end{aligned} \tag{68}$$

which implies that (S2) holds.

By Theorem 11, that the boundary value problem (66) has at least one positive solution.

At the end of this work we also remark that the extension of the pervious results to the nonlinearities depending on the time delayed differential system for energy price adjustment or impulsive differential equation in financial field requires some further nontrivial modifications, and the reader can try to obtain results in our direction. We also anticipate that the methods and concepts here can be extended to the systems with economic processes such as risk model, the CIR model, and the Gaussian model as considered by Almeida and Vicente [1].

References

[1] C. Almeida and J. Vicente, "Are interest rate options important for the assessment of interest rate risk?" *Journal of Banking & Finance*, vol. 33, pp. 1376–1387, 2009.

[2] X. Zhang and L. Liu, "Positive solutions of fourth-order four-point boundary value problems with p-Laplacian operator," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 2, pp. 1414–1423, 2007.

[3] J. Wang, H. Xiang, and Z. Liu, "Upper and lower solutions method for a class of singular fractional boundary value problems with p-Laplacian operator," *Abstract and Applied Analysis*, vol. 2010, Article ID 971824, 12 pages, 2010.

[4] D.-X. Ma and X.-Z. Yang, "Upper and lower solution method for fourth-order four-point boundary value problems," *Journal of Computational and Applied Mathematics*, vol. 223, no. 2, pp. 543–551, 2009.

[5] S. Chen, W. Ni, and C. Wang, "Positive solution of fourth order ordinary differential equation with four-point boundary conditions," *Applied Mathematics Letters*, vol. 19, no. 2, pp. 161–168, 2006.

[6] Z. Bai and H. Lv, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[7] K. S. Miller and B. Ross, *Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, NY, USA, 1993.

[8] M. Jia, X. Liu, and X. Gu, "Uniqueness and asymptotic behavior of positive solutions for a fractional-order integral boundary value problem," *Abstract and Applied Analysis*, vol. 2012, Article ID 294694, 21 pages, 2012.

[9] M. Jia, X. Zhang, and X. Gu, "Nontrivial solutions for a higher fractional differential equation with fractional multi-point boundary conditions," *Boundary Value Problems*, vol. 2012, article 70, 2012.

[10] X. Zhang and Y. Han, "Existence and uniqueness of positive solutions for higher order nonlocal fractional differential equations," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 555–560, 2012.

[11] X. Zhang, L. Liu, and Y. Wu, "The eigenvalue problem for a singular higher order fractional differential equation involving fractional derivatives," *Applied Mathematics and Computation*, vol. 218, no. 17, pp. 8526–8536, 2012.

[12] X. Zhang, L. Liu, and Y. Wu, "The uniqueness of positive solution for a singular fractional differential system involving derivatives," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, pp. 1400–1409, 2013.

[13] X. Zhang, L. Liu, B. Wiwatanapataphee, and Y. Wu, "Positive solutions of eigenvalue problems for a class of fractional differential equations with derivatives," *Abstract and Applied Analysis*, vol. 2012, Article ID 512127, 16 pages, 2012.

[14] X. Zhang, L. Liu, and Y. Wu, "Existence results formultiple positive solutions of nonlinear higher order perturbed fractional differential equations with derivatives," *Applied Mathematics and Computation*, vol. 219, no. 4, pp. 1420–1433, 2012.

[15] X. Zhang, L. Liu, Y. Wu, and Y. Lu, "The iterative solutions of nonlinear fractional differential equations," *Applied Mathematics and Computation*, vol. 219, no. 9, pp. 4680–4691, 2013.

[16] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1999.

[17] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science, Yverdon, Switzerland, 1993.