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Research Article

Asymptotic Behavior of a Class of Degenerate Parabolic Equations

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We investigate the asymptotic behavior of solutions of a class of degenerate parabolic equations in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ with a polynomial growth nonlinearity of arbitrary order. The existence of global attractors is proved in $L^2(\Omega)$, $L^p(\Omega)$, and $H_0^{1,a}(\Omega)$, respectively, when $H_0^{1,a}(\Omega)$ can be just compactly embedded into $L^r(\Omega)$ $(r < 2)$ but not $L^2(\Omega)$.

1. Introduction

Let us consider the following degenerate parabolic equations:

$$\begin{gathered}\frac{\partial u}{\partial t} - \operatorname{div}(a(x)\nabla u) + f(u) = g(x) \quad \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0 \quad \text{in } \Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geqslant 2)$, with smooth boundary $\partial\Omega$, a is a given nonnegative function, and f is a C^1 function satisfying

$$f'(s) \geqslant -l, \tag{1.2}$$

$$C_1|s|^p - C_0 \leqslant f(s)s \leqslant C_2|s|^p + C_0, \quad p \geqslant 2, \tag{1.3}$$

both for all $s \in \mathbb{R}$.

For the long-time behavior problems of the classical evolutionary equations, especially, the classical reaction-diffusion equation, much has been accomplished in recent years (see, e.g., [1–9] and the references therein), whereas for degenerated evolutionary equations such information is by comparison very incomplete. The main feature of the problem (1.1) is that the differential operator $-\operatorname{div}(a(x)\nabla u)$ is degenerate because of the presence of a nonnegative diffusion coefficient $a(x)$ which is allowed to vanish somewhere (the physical meaning, see [10–12]). Actually, in order to handle media which have possibly somewhere "perfect" insulators (see [10]) the coefficient a is allowed to have "essential" zeroes at some points or even to be unbounded. In [13], the authors considered the existence of positive solutions when nonlinearity is superlinear and subcritical function for a semilinear degenerate elliptic equation under the assumption that $a \in L^1_{\mathrm{loc}}(\Omega)$, for some $\alpha \in (0,2]$, satisfies

$$\liminf_{x \to z} |x - z|^{-\alpha} a(x) > 0, \quad \text{for every } z \in \overline{\Omega}. \tag{1.4}$$

Recently, motivated by [13], under the same assumption as in [13], the authors of [11, 12, 14–20] proved the existence of global attractors of a class of degenerate evolutionary equations for the case of $\alpha \in (0,2)$.

The present paper is devoted to the case of $\alpha = 2$ which is essentially different from the case of $\alpha \in (0,2)$, and which will cause some technical difficulties. In [13], the authors pointed out that the number $2^*_\alpha = 2n/(n-2+\alpha)$ plays the role of critical exponent. It is well known that some kind of compactness of the semigroup associated with (1.1) is necessary to prove the existence of the global attractor in $L^2(\Omega)$. However, there is no corresponding compact embedding result in this case since $H_0^{1,a}(\Omega)$ is compactly embedded only into $L^r(\Omega)$ $(r < 2)$ but not $L^2(\Omega)$. Hence, the existence of the global attractor in $L^2(\Omega)$ cannot be obtained by usual methods.

In this paper, we assume the weighted function a satisfies the following.

(A_1) $a \in L^\infty(\Omega)$ and $\liminf_{x \to z} |x - z|^2 a(x) > 0$ for every $z \in \overline{\Omega}$.

We will firstly obtain the existence and uniqueness of weak global solutions by use of the singular perturbation then use the asymptotic a priori estimate (see [9]) to verify that the semigroup associated with our problem is asymptotically compact and establish the existence of the global attractor in $L^2(\Omega)$, $L^p(\Omega)$ $(p \geqslant 2)$ and $H_0^{1,a}(\Omega)$, respectively.

2. Preliminary Results

In this section, we firstly present some notation and preliminary facts on functional spaces then review some necessary concepts and theorems that will be used to prove compactness of the semigroup. For convenience, hereafter let $\|\cdot\|_p$ be the norm of $L^p(\Omega)$ $(p \geqslant 1)$, $|u|$ the modular (or the absolute value) of u, and C an arbitrary positive constant, which may vary from line to line and even in the same line.

2.1. Functional Spaces

The appropriate Sobolev space for (1.1) is $H_0^{1,a}(\Omega)$, defined as a completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H_0^{1,a}} = \left(\int_\Omega a(x)|\nabla u|^2 dx\right)^{1/2}. \tag{2.1}$$

The dual space is denoted by $H^{-1,a}(\Omega)$, that is, $(H_0^{1,a}(\Omega))^* = H^{-1,a}(\Omega)$.

The next proposition refers to continuous and compact inclusion of $H_0^{1,a}(\Omega)$.

Proposition 2.1 (see [13]). *Let Ω be bounded domain in $\mathbb{R}^n$ ($n \geqslant 2$) and let $a \in L^1_{loc}(\Omega)$ satisfy* (1.4) *for some $\alpha \in (0,2]$. Then the following embeddings hold:*

(i) *$H_0^{1,a}(\Omega)$ is continuously embedded in $W_0^{1,2n/(n+\alpha)}(\Omega)$;*

(ii) *$H_0^{1,a}(\Omega)$ is continuously embedded in $L^{2^*_\alpha}(\Omega)$;*

(iii) *$H_0^{1,a}(\Omega)$ is compactly embedded in $L^r(\Omega)$ as $1 \leqslant r < 2^*_\alpha = 2n/(n-2+\alpha)$.*

Remark 2.2. $2^*_\alpha \geqslant 2$ when $\alpha \in (0,2)$, $2^*_\alpha = 2$ when $\alpha = 2$, which plays the role of the critical exponent in the Sobolev embedding.

In this paper we only consider the case of $\alpha = 2$ when $n \geqslant 2$.

2.2. Some Results on Existence of Global Attractors

In this subsection, we review briefly some basic concepts and results on the existence of global attractors; see [2, 5, 7, 9] for more details.

Definition 2.3. Let $\{S(t)\}_{t\geqslant 0}$ be a semigroup on Banach space X. $\{S(t)\}_{t\geqslant 0}$ is called asymptotically compact if for any bounded sequence $\{x_n\}_{n=1}^\infty$ and $t_n \geqslant 0$, $t_n \to \infty$ as $n \to \infty$, and $\{S(t_n)x_n\}_{n=1}^\infty$ has a convergent subsequence in X.

Theorem 2.4. *Suppose $\{S(t)\}_{t\geqslant 0}$ is a semigroup on $L^p(\Omega)$ ($p \geqslant 1$). Assume further $\{S(t)\}_{t\geqslant 0}$ is a continuous or weak continuous semigroup on $L^q(\Omega)$ for some $q \leqslant p$ and possesses a global attractor in $L^q(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is bounded. Then $\{S(t)\}_{t\geqslant 0}$ possesses a global attractor in $L^p(\Omega)$ if and only if*

(i) *$\{S(t)\}_{t\geqslant 0}$ has a bounded absorbing set B_0 in $L^p(\Omega)$, and*

(ii) *for any $\varepsilon > 0$ and any bounded subset $B \subset L^p(\Omega)$, there exist positive constants $T = T(\varepsilon, B)$ and $M = M(\varepsilon, B)$ such that*

$$\int_{\Omega(|S(t)u_0|\geqslant M)} |S(t)u_0|^p dx < \varepsilon \quad \text{for any } u_0 \in B,\ t \geqslant T. \tag{2.2}$$

Theorem 2.5. *Let $\{S(t)\}_{t\geqslant 0}$ be a semigroup on $L^p(\Omega)$ $(p \geqslant 1)$ and have a bounded absorbing set in $L^p(\Omega)$. Then for any $\varepsilon > 0$ and any bounded subset $B \subset L^p(\Omega)$, there exist positive constants $T = T_B$ and $M = M(\varepsilon)$ such that*

$$m(\Omega(|S(t)u_0| \geqslant M)) \leqslant \varepsilon \quad \text{for any } t \geqslant T,\ u_0 \in B, \tag{2.3}$$

where $m(e)$ (sometimes we also write it as $|e|$) denotes the Lebesgue measure of $e \subset \Omega$ and $\Omega(|u| \geqslant M) \triangleq \{x \in \Omega \mid |u(x)| \geqslant M\}$.

Theorem 2.6. *For any $\varepsilon > 0$, the bounded subset B of $L^p(\Omega)$ $(p \geqslant 1)$ has a finite ε-net in $L^p(\Omega)$ if there exists a positive constant $M = M(\varepsilon)$ which depends on ε such that*

(i) *B has a finite $(3M)^{(q-p)/q}(\varepsilon/2)^{p/q}$-net in $L^q(\Omega)$ for some q, $q \geqslant 1$;*

(ii)

$$\int_{\Omega(|u|\geqslant M)} |u|^p dx \leqslant 2^{-(2p+2)/p}\varepsilon \quad \text{for any } u \in B. \tag{2.4}$$

3. Existence and Uniqueness of the Weak Global Solutions

In this paper, throughout we denote $\Omega_T = \Omega \times [0,T]$, $V = L^2(0,T;H_0^{1,a}(\Omega)) \cap L^p(\Omega_T)$ and $V^* = L^2(0,T;H^{-1,a}(\Omega)) + L^q(\Omega_T)$, respectively, where q is the conjugate exponent of p, that is, $1/p + 1/q = 1$. In addition, we always assume that f satisfies (1.2)-(1.3) and the external forcing term g belongs only to $L^2(\Omega)$.

Definition 3.1. A function $u(x,t)$ is called a weak solution of (1.1) on $[0,T]$ if and only if

$$u \in C\Big([0,T];L^2(\Omega)\Big) \cap L^2\Big(0,T;H_0^{1,a}(\Omega)\Big) \cap L^p(0,T;L^p(\Omega)) \tag{3.1}$$

and $u|_{t=0} = u_0$ almost everywhere in Ω such that

$$\int_{\Omega_T}\left(\frac{\partial u}{\partial t}\xi + a\nabla u\nabla\xi + f(u)\xi\right) = \int_{\Omega_T} g\xi \tag{3.2}$$

holds for all test functions $\xi \in V$.

The following lemma makes the initial condition in problem (1.1) meaningful.

Lemma 3.2 (see [16]). *If $u \in V$ and $du/dt \in V^*$, then $u \in C([0,T];L^2(\Omega))$.*

Theorem 3.3. *Assume $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ is a bounded open domain with smooth boundary, f satisfies* (1.2)-(1.3), *and $g \in L^2(\Omega)$. Then for any $u_0 \in L^2(\Omega)$ and $T > 0$ there exists a unique solution u of* (1.1) *which satisfies*

$$u \in C\Big([0,T];L^2(\Omega)\Big) \cap L^2\Big(0,T;H_0^{1,a}(\Omega)\Big) \cap L^p(\Omega_T). \tag{3.3}$$

The mapping $u_0 \mapsto u(t)$ is continuous in $L^2(\Omega)$.

Proof. For any $0 < \varepsilon < 1$, we choose $u_{\varepsilon,0} \in C_c^\infty(\Omega)$ such that $\|u_{\varepsilon,0}\|_{L^\infty(\Omega)}$ are uniformly bounded with respect to ε, and

$$u_{\varepsilon,0} \longrightarrow u_0 \quad \text{in } L^2(\Omega). \tag{3.4}$$

Consider the problem

$$\begin{gathered}\frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}(a_\varepsilon(x)\nabla u_\varepsilon) + f(u_\varepsilon) = g \quad \text{in } \Omega \times \mathbb{R}^+,\\ u_\varepsilon = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+,\\ u_\varepsilon(x,0) = u_{\varepsilon 0} \quad \text{in } \Omega,\end{gathered} \tag{3.5}$$

where

$$a_\varepsilon(x) = a(x) + \varepsilon, \quad x \in \Omega. \tag{3.6}$$

According to the standard Galerkin methods (see, e.g., [2, 6, 7]), we know the problem (3.5) admits a unique weak solution $u_\varepsilon \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap L^p(0,T; L^p(\Omega))$. Here u_ε is called a weak solution of the problem (3.5), if, for any $\varphi \in C_0^\infty(\Omega_T)$, we have

$$\int_0^T \int_\Omega \left(\frac{\partial u_\varepsilon}{\partial t}\varphi + a_\varepsilon \nabla u_\varepsilon \nabla\varphi + f(u_\varepsilon)\varphi \right) dx\, dt = \int_0^T \int_\Omega g\varphi\, dx\, dt \tag{3.7}$$

and $u_\varepsilon|_{t=0} = u_{\varepsilon,0}$ almost everywhere in Ω.

Now we do some estimates on u_ε in the following.

Multiplying (3.5) by u_ε and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon\|_2^2 + \int_\Omega a_\varepsilon(x)|\nabla u_\varepsilon|^2 dx + \int_\Omega f(u_\varepsilon)u_\varepsilon dx = \int_\Omega g u_\varepsilon dx. \tag{3.8}$$

By (1.3) and the Hölder's inequality, we can deduce that

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon\|_2^2 + \int_\Omega a_\varepsilon(x)|\nabla u_\varepsilon|^2 dx + C_1 \int_\Omega |u_\varepsilon|^p dx \leqslant C_0|\Omega| + \frac{1}{2C_1}\|g\|_2^2 + \frac{C_1}{2}\|u_\varepsilon\|_2^2, \tag{3.9}$$

where $|\Omega| = \int_\Omega 1\, dx$.

Using the Gronwall lemma, for any $T > 0$, we have the following:

$$u_\varepsilon \text{ is uniformly bounded in } L^\infty\left(0,T; L^2(\Omega)\right) \text{ with respect to } \varepsilon. \tag{3.10}$$

Integrating (3.8) and (3.9), both sides between 0 and T, and using the Young's inequality, we may get by a standard procedure (see, e.g., [2, 6, 7]) that

$$\int_{\Omega_T} a|\nabla u_\varepsilon|^2 dx\,dt \leqslant \int_{\Omega_T} a_\varepsilon |\nabla u_\varepsilon|^2 dx\,dt = \int_{\Omega_T} a|\nabla u_\varepsilon|^2 dx\,dt + \varepsilon \int_{\Omega_T} |\nabla u_\varepsilon|^2 dx\,dt \leqslant C,$$
$$\int_{\Omega_T} |u_\varepsilon|^p dx\,dt \leqslant C, \tag{3.11}$$

with C independent of ε.

Noting that (1.3), we obtain

$$\begin{aligned}
\|f(u_\varepsilon)\|^q_{L^q(\Omega_T)} &= \int_0^T \left(\int_\Omega |f(u_\varepsilon)|^q dx \right) dt \\
&\leqslant C \int_0^T \left(\int_\Omega \left(1 + |u_\varepsilon|^{p-1}\right)^q dx \right) dt \\
&\leqslant C \int_0^T \left(\int_\Omega 1 + |u_\varepsilon|^{q(p-1)} dx \right) dt.
\end{aligned} \tag{3.12}$$

So we have the following:

$$f(u_\varepsilon) \text{ is uniformly bounded in } L^q(0,T;L^q(\Omega)) \text{ with respect to } \varepsilon. \tag{3.13}$$

We now extract a weakly convergent subsequence, denoted also by u_ε for convenience, with

$$\begin{gathered}
u_\varepsilon \rightharpoonup u \quad \text{in } L^2\left(0,T;H_0^{1,a}(\Omega)\right), \\
u_\varepsilon \rightharpoonup u \quad \text{in } L^p(0,T;L^p(\Omega)), \\
f(u_\varepsilon) \rightharpoonup \chi \quad \text{in } L^q(0,T;L^q(\Omega)), \\
a_\varepsilon \nabla u_\varepsilon \rightharpoonup \vec{\vartheta} \quad \text{in } L^2(\Omega_T, \mathbb{R}^n).
\end{gathered} \tag{3.14}$$

Since $f \in C(\mathbb{R})$, it follows that

$$f(u_\varepsilon) \rightharpoonup f(u) \quad \text{in } L^q(0,T;L^q(\Omega)). \tag{3.15}$$

Now we show that u is a weak solution of Problem (1.1). Multiply (3.5) by φ and let $\varepsilon \to 0^+$ to derive

$$\int_0^T \int_\Omega \frac{\partial u}{\partial t}\varphi + \vec{\vartheta} \cdot \nabla\varphi + f(u)\varphi\,dx\,dt = \int_0^T \int_\Omega g\varphi\,dx\,dt \tag{3.16}$$

for $\varphi \in V$.

Therefore, in order to obtain the existence we need only to prove

$$\int_0^T \int_\Omega \vec{\vartheta} \cdot \nabla\varphi \, dx \, dt = \int_0^T \int_\Omega a\nabla u \nabla\varphi \, dx \, dt, \quad \varphi \in C_0^\infty(\Omega_T) \tag{3.17}$$

for $C_0^\infty(\Omega_T)$ is dense in V.

From (3.8) we can obtain

$$\int_{\Omega_T} a_\varepsilon |\nabla u_\varepsilon|^2 dx \, dt = -\int_{\Omega_T} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon dx \, dt - \int_{\Omega_T} f(u_\varepsilon) u_\varepsilon dx \, dt + \int_{\Omega_T} g u_\varepsilon dx \, dt. \tag{3.18}$$

Let $v \in C([0,T], L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$. It is obvious that

$$\int_0^T \int_\Omega a_\varepsilon (\nabla u_\varepsilon - \nabla v) \cdot (\nabla u_\varepsilon - \nabla v) dx \, dt \geqslant 0. \tag{3.19}$$

Therefore,

$$\begin{aligned} &-\int_{\Omega_T} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon dx \, dt - \int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \nabla v \, dx \, dt - \int_{\Omega_T} a\nabla v (\nabla u_\varepsilon - \nabla v) dx \, dt \\ &\quad + \varepsilon \int_{\Omega_T} \nabla v (\nabla u_\varepsilon - \nabla v) dx \, dt - \int_{\Omega_T} f(u_\varepsilon) u_\varepsilon dx \, dt + \int_{\Omega_T} g u_\varepsilon dx \, dt \geqslant 0. \end{aligned} \tag{3.20}$$

Taking $\varepsilon \longrightarrow 0^+$ in the above inequality and noticing that

$$\varepsilon \left| \int_{\Omega_T} \nabla v (\nabla u_\varepsilon - \nabla v) dx \, dt \right| \leqslant \varepsilon \int_{\Omega_T} |\nabla v| |\nabla u_\varepsilon| dx \, dt + \varepsilon \int_{\Omega_T} |\nabla v|^2 dx \, dt \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0^+, \tag{3.21}$$

we arrive at

$$-\int_{\Omega_T} \frac{\partial u}{\partial t} u - \int_{\Omega_T} \vec{\vartheta} \cdot \nabla v - \int_{\Omega_T} a\nabla v (\nabla u - \nabla v) - \int_{\Omega_T} f(u) u + \int_{\Omega_T} g u \geqslant 0. \tag{3.22}$$

On the other hand, choosing $\varphi = u$ in (3.7) leads to

$$\int_0^T \int_\Omega \vec{\vartheta} \cdot \nabla u \, dx \, dt = -\int_{\Omega_T} \frac{\partial u}{\partial t} u - \int_0^T \int_\Omega f(u) u \, dx \, dt + \int_0^T \int_\Omega g u \, dx \, dt. \tag{3.23}$$

Then, it follows from (3.22) and (3.23) that

$$\int_0^T \int_\Omega \left(\vec{\vartheta} - a\nabla v \right) \cdot (\nabla u - \nabla v) dx \, dt \geqslant 0. \tag{3.24}$$

Choosing $v = u - \lambda\varphi$ with $\lambda > 0$ in the above inequality, we get

$$\int_0^T \int_\Omega \left(\vec{\vartheta} - a\nabla(u - \lambda\varphi)\right) \cdot \nabla\varphi \, dx \, dt \geqslant 0, \tag{3.25}$$

which implies by letting $\lambda \to 0^+$ that

$$\int_0^T \int_\Omega \left(\vec{\vartheta} - a\nabla u\right) \cdot \nabla\varphi \, dx \, dt \geqslant 0. \tag{3.26}$$

If we choose $\lambda < 0$, we achieve the inequality with opposite sign. Thus

$$\int_0^T \int_\Omega \left(\vec{\vartheta} - a\nabla u\right) \cdot \nabla\varphi \, dx \, dt = 0, \tag{3.27}$$

which leads to (3.17). Then $u \in C([0,T]; L^2(\Omega))$ follows from Lemma 3.2.

Now we will show that $u(0) = u_0$. Choosing some $\phi \in C^1([0,T]; H_0^{1,a}(\Omega) \cap L^p(\Omega))$ with $\phi(T) = 0$ as a test function and integrating by parts in the t variable we have

$$\int_0^T -\langle u, \phi' \rangle + \langle Au, \phi \rangle + \langle f(u), \phi \rangle ds = \int_0^T \langle g, \phi \rangle ds + (u(0), \phi(0)). \tag{3.28}$$

Doing the approximations as above yields

$$\int_0^T -\langle u_\varepsilon, \phi' \rangle + (Au_\varepsilon, \phi) + \langle f(u_\varepsilon(s)), \phi \rangle ds = \int_0^T \langle g, \phi \rangle ds + \langle u_\varepsilon(0), \phi(0) \rangle. \tag{3.29}$$

taking limits to conclude that

$$\int_0^T -\langle u, \phi' \rangle + \langle Au, \phi \rangle + \langle f(u), \phi \rangle ds = \int_0^T \langle g, \phi \rangle ds + (u_0, \phi(0)), \tag{3.30}$$

since $u_{\varepsilon 0} \to u_0$. Thus $u(0) = u_0$.

Thanks to (1.2), uniqueness and continuous dependence on initial conditions can easily be obtained. □

We can therefore use these solutions to define a semigroup $\{S(t)\}_{t \geqslant 0}$ on $L^2(\Omega)$ by setting

$$S(t)u_0 = u(t), \tag{3.31}$$

which is continuous on u_0 in $L^2(\Omega)$.

4. Existence of Global Attractors

In this section, we prove the existence of the global attractors in $L^2(\Omega)$, $L^p(\Omega)$, and $H_0^{1,a}(\Omega)$, respectively. The following result is the existence of bounded absorbing sets which has been established in [18].

Theorem 4.1. *The semigroup $\{S(t)\}_{t\geqslant 0}$ possesses bounded absorbing sets in $L^2(\Omega)$, $L^p(\Omega)$, and $H_0^{1,a}(\Omega)$, respectively; that is, for any bounded subset B in $L^2(\Omega)$, there exists a constant $T(\|u_0\|_2)$, such that*

$$\begin{gathered}\|u(t)\|_2^2 \leqslant \rho_0,\\ \|u(t)\|_p^p + \int_\Omega a(x)|\nabla u(t)|^2 \leqslant \rho_1,\end{gathered} \tag{4.1}$$

for all $t \geqslant T$ and $u_0 \in B$, where both ρ_0 and ρ_1 are positive constants independent of B, $u(t) = S(t)u_0$.

In order to obtain the existence of a global attractor in $L^2(\Omega)$ we need to verify that $\{S(t)\}_{t\geqslant 0}$ possesses some kind of compactness in $L^2(\Omega)$, which, however, we cannot obtain by usual methods for lack of the corresponding Sobolev compact embedding results for this case. Here, the new method introduced in [9] is used.

Let B_0 be the bounded absorbing set in $H_0^{1,a}(\Omega)$, then we can consider our problem only in B_0. For $H_0^{1,a}(\Omega)$ is compactly continuous into $L^r(\Omega)$ for some $1 \leqslant r < 2$, we know that B_0 is compact in $L^r(\Omega)$, and B_0 has a finite ε-net in $L^r(\Omega)$.

Firstly, we give the following useful a priori estimate.

Theorem 4.2. *For any $\varepsilon > 0$ and bounded subsets $B \subset L^2(\Omega)$, there exist $T = T(\varepsilon, B)$ and $M = M(\varepsilon)$ such that*

$$\int_{\Omega(|u(t)|\geqslant M)} |u(t)|^2 dx < C\varepsilon \quad \text{for any } u_0 \in B,\ t \geqslant T. \tag{4.2}$$

Proof. For any fixed $\varepsilon > 0$, there exists $\delta > 0$ such that for any $e \subset \Omega$ and $m(e) \leqslant \delta$ we have

$$\int_e \left|g(x)\right|^2 dx < \varepsilon. \tag{4.3}$$

Moreover, from Theorem 2.5, we know that there exist $T = T(B, \varepsilon)$ and $M_1 = M(\varepsilon)$ such that

$$m(\Omega(|u(t)| \geqslant M_1)) \leqslant \min\{\varepsilon, \delta\} \quad \text{for } u_0 \in B,\ t \geqslant T. \tag{4.4}$$

In addition, thanks to (1.3), we know $f(s) \geqslant 0$ when $s > (C_0/C_1)^{1/p}$. In the following we assume $M = \max\{M_1, (C_0/C_1)^{1/p}\}$ and $t \geqslant T$.

Multiplying (1.1) by $(u - M)_+$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|(u-M)_+\|_2^2 + \int_\Omega a(x)|\nabla(u-M)_+|^2 dx + \int_\Omega f(u)(u-M)_+ dx = \int_\Omega (u-M)_+ dx, \tag{4.5}$$

where $(u-M)_+$ denotes the positive part of $u-M$, that is,

$$(u-M)_+ = \begin{cases} u-M, & u \geqslant M, \\ 0, & u \leqslant M. \end{cases} \tag{4.6}$$

Let $\Omega_1 = \Omega(u(t) \geqslant M)$, then

$$\frac{1}{2}\frac{d}{dt}\|(u-M)_+\|_2^2 + \int_{\Omega_1} a(x)|\nabla u|^2 dx + \int_{\Omega_1} f(u)(u-M)dx = \int_{\Omega_1} g(u-M)dx. \tag{4.7}$$

By the Cauchy's and Hölder's inequality, we deduce that

$$\frac{d}{dt}\|(u-M)_+\|_2^2 + C\int_{\Omega_1} (u-M)^p dx \leqslant C\left(\int_{\Omega_1} |g|^2 dx + \int_{\Omega_1} |u-M|^2 dx\right) + C_0|\Omega_1|. \tag{4.8}$$

Combining with (4.3)-(4.4) and $L^p(\Omega) \hookrightarrow L^2(\Omega)(p \geqslant 2)$, we get

$$\frac{d}{dt}\|(u-M)_+\|_2^2 + C\int_{\Omega_1} (u-M)^2 dx \leqslant C\varepsilon. \tag{4.9}$$

We apply the Gronwall lemma to infer

$$\|(u-M)_+\|_2^2 \leqslant C\varepsilon. \tag{4.10}$$

Replacing $(u-M)_+$ with $(u+M)_-$ and using the same method as above, we obtain

$$\|(u+M)_-\|_2^2 \leqslant C\varepsilon. \tag{4.11}$$

Hence, by (4.10) and (4.11), we have (4.2). □

According to Theorem 2.6, we know B_0 is compact in $L^2(\Omega)$; hence, Theorem 4.1 implies the existence of an attractor in $L^2(\Omega)$, immediately.

Theorem 4.3. *The semigroup $\{S(t)\}_{t\geqslant 0}$ associated with* (1.1) *possesses a global attractor $\mathcal{A}_2$ in $L^2(\Omega)$, that is, $\mathcal{A}_2$ is compact and invariant in $L^2(\Omega)$ and attracts the bounded sets of $L^2(\Omega)$ in the topology of $L^2(\Omega)$.*

We now establish the existence of global attractor in $L^p(\Omega)$.

Theorem 4.4. *The semigroup $\{S(t)\}_{t\geqslant 0}$ associated with* (1.1) *possesses a global attractor $\mathcal{A}_p$ in $L^p(\Omega)$, that is, $\mathcal{A}_p$ is compact and invariant in $L^p(\Omega)$ and attracts the bounded sets of $L^2(\Omega)$ in the topology of $L^p(\Omega)$.*

Proof . From Theorems 2.4, 4.1, and 4.3, we need only to verify that for any $\varepsilon > 0$ and bounded subset $B \subset L^2(\Omega)$ there exist $T = T(\varepsilon, B)$ and $M = M(\varepsilon)$ such that

$$\int_{\Omega(|u(t)|\geqslant M)} |u(t)|^p dx < C\varepsilon \quad \text{for } u_0 \in B,\ t \geqslant T. \tag{4.12}$$

Letting $F(s) = \int_0^s f(\tau)d\tau$, from (1.3), we deduce that

$$\widetilde{C}_1|s|^p - k \leqslant F(s) \leqslant k + \widetilde{C}_2|s|^p. \tag{4.13}$$

So,

$$\widetilde{C}_1 \int_\Omega |u|^p dx - k|\Omega| \leqslant \int_\Omega F(u)dx \leqslant k|\Omega| + \widetilde{C}_2 \int_\Omega |u|^p dx. \tag{4.14}$$

On account of the standard Cauchy's and Hölder's inequalities, it follows from (4.7) that

$$\begin{aligned} &\frac{d}{dt}\|(u-M)_+\|_2^2 + C\left(\int_{\Omega_1} a(x)|\nabla u|^2 dx + \int_{\Omega_1} f(u)(u-M)dx\right) \\ &\quad \leqslant C\left(\int_{\Omega_1} |u-M|^2 dx + \int_{\Omega_1} |g|^2 dx\right). \end{aligned} \tag{4.15}$$

Taking $t \geqslant T$, integrating the last equality between t and $t+1$, and combining with (4.2)–(4.4), we have

$$\int_t^{t+1} \left(\int_{\Omega_1} a(x)|\nabla u|^2 dx + \int_{\Omega_1} f(u)(u-M)dx\right) ds \leqslant C\varepsilon. \tag{4.16}$$

On the other hand, let $\Omega_2 = \Omega(u \geqslant 2M)$, multiplying (1.1) by $[(u-2M)_+]_t$ and integrating over Ω, then we have

$$\frac{d}{dt}\left(\int_{\Omega_2} a(x)|\nabla u|^2 dx + \int_{\Omega_2} F(u)dx\right) \leqslant C\varepsilon. \tag{4.17}$$

From (4.16) and (4.17) we apply the uniform Gronwall lemma to obtain

$$\int_{\Omega_2} a(x)|\nabla u|^2 dx + \int_{\Omega_2} F(u)dx \leqslant C\varepsilon. \tag{4.18}$$

Hence

$$\int_{\Omega_2} F(u)dx \leqslant C\varepsilon. \tag{4.19}$$

Replacing $(u-M)_+$ and $[(u-2M)_+]_t$ with $(u+M)_-$ and $[(u+2M)_-]_t$, respectively, and repeating the same steps as above we obtain

$$\int_{\Omega(u\leqslant -2M)} F(u)dx \leqslant C\varepsilon. \tag{4.20}$$

Then, from (4.19)-(4.20), we have

$$\int_{\Omega(|u(t)|\geqslant 2M)} F(u)dx \leqslant C\varepsilon. \tag{4.21}$$

Thus, by (4.4) and (4.14), (4.21) implies (4.12). The proof is finished. □

4.1. Global Attractor in $H_0^{1,a}(\Omega)$

In order to prove the existence of a global attractor in $H_0^{1,a}(\Omega)$, we need the following lemma.

Lemma 4.5. *For any bounded subset B in $L^2(\Omega)$, there exists a constant $T=T(B)$ such that*

$$\|u_t\|_2^2 \leqslant M \quad \text{for any } u_0 \in B,\ s \geqslant T, \tag{4.22}$$

where $u_t(s) = (d/dt)(S(t)u_0)|_{t=s}$, M is independent of B.

Proof. Multiplying (1.1) by u_t and integrating over Ω, we get

$$\|u_t\|_2^2 + \frac{1}{2}\frac{d}{dt}\left(\int_\Omega a(x)|\nabla u|^2 dx + 2\int_\Omega F(u)dx\right) = \int_\Omega g u_t dx. \tag{4.23}$$

Let $v=u_t$ and differentiate (1.1) with respect to t to get

$$v_t - \operatorname{div}(a(x)\nabla v) + f'(u)v = 0. \tag{4.24}$$

Multiplying the above equality by v and integrating over Ω, by (1.2), we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \int_\Omega a(x)|\nabla v|^2 dx \leqslant l\|v\|_2^2. \tag{4.25}$$

Taking $t \geqslant T$, integrating (4.23) from t to $t+1$, and considering Theorem 4.1, we get

$$\int_t^{t+1} \|v\|_2^2 dt \leqslant C\left(\|g\|_2^2, |\Omega|, \rho_0^2\right) \tag{4.26}$$

as t is large enough.

Combing with (4.25) and (4.26) and using the uniform Gronwall lemma, we have

$$\|v(t+1)\|_2^2 \leqslant Ce^l \tag{4.27}$$

as t is large enough. □

Next, we verify $\{S(t)\}_{t\geqslant 0}$ is asymptotically compact in $H_0^{1,a}(\Omega)$.

Theorem 4.6. *The semigroup $\{S(t)\}_{t\geqslant 0}$ is asymptotically compact in $H_0^{1,a}(\Omega)$.*

Proof. Let B_0 be an absorbing set in $H_0^{1,a}(\Omega)$ obtained in Theorem 4.1, then we need only to verify that $\{u_n(t_n)\}_{n=1}^{\infty}$ possesses a convergent subsequence in $H_0^{1,a}(\Omega)$ for any sequence $\{u_{0n}\}_{n=1}^{\infty} \subset B_0$.

In fact, by Theorems 4.3 and 4.4, we know $\{u_n(t_n)\}_{n=1}^{\infty}$ is precompact in $L^2(\Omega)$ and $L^p(\Omega)$. So we can assume that the subsequence $\{u_{n_k}(t_{n_k})\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega)$ and $L^p(\Omega)$. Now we prove that $\{u_{n_k}(t_{n_k})\}_{k=1}^{\infty}$ is a Cauchy sequence in $H_0^{1,a}(\Omega)$.

Noticing that the prime operator $A_a u = -\operatorname{div}(a(x)\nabla u)$ is strong monotone, that is,

$$\left\|u_{n_k}(t_{n_k}) - u_{n_j}\left(t_{n_j}\right)\right\|_{H_0^{1,a}(\Omega)}^2 \leqslant \left\langle A_a u_{n_k}(t_{n_k}) - A_a u_{n_j}\left(t_{n_j}\right), u_{n_k}(t_{n_k}) - u_{n_j}\left(t_{n_j}\right)\right\rangle, \tag{4.28}$$

we have

$$\begin{aligned}
&\left\|u_{n_k}(t_{n_k}) - u_{n_j}\left(t_{n_j}\right)\right\|_{H_0^{1,a}(\Omega)} \\
&\quad\leqslant \left\langle -\frac{d}{dt}u_{n_k}(t_{n_k}) - f(u_{n_k}(t_{n_k})) + \frac{d}{dt}u_{n_j}\left(t_{n_j}\right) + f\left(u_{n_j}\left(t_{n_j}\right)\right), u_{n_k}(t_{n_k}) - u_{n_j}\left(t_{n_j}\right)\right\rangle \\
&\quad\leqslant \int_\Omega \left|\frac{d}{dt}u_{n_k}(t_{n_k}) - \frac{d}{dt}u_{n_j}\left(t_{n_j}\right)\right|\left|u_{n_k}(t_{n_k}) - u_{n_j}\left(t_{n_j}\right)\right| \\
&\qquad + \int_\Omega \left|f(u_{n_k}(t_{n_k})) - f\left(u_{n_j}\left(t_{n_j}\right)\right)\right|\left|u_{n_k}(t_{n_k}) - u_{n_j}\left(t_{n_j}\right)\right| \\
&\quad\leqslant \left\|\frac{d}{dt}u_{n_k}(t_{n_k}) - \frac{d}{dt}u_{n_j}\left(t_{n_j}\right)\right\|_2 \left\|u_{n_k}(t_{n_k}) - u_{n_j}\left(t_{n_j}\right)\right\|_2 \\
&\qquad + C\left(1 + \|u_{n_k}(t_{n_k})\|_p^p + \left\|u_{n_j}\left(t_{n_j}\right)\right\|_p^p\right)\left\|u_{n_k}(t_{n_k}) - u_{n_j}\left(t_{n_j}\right)\right\|_p,
\end{aligned} \tag{4.29}$$

which, combining with Lemma 4.5, yields the respected result immediately. □

In order to establish the existence of global attractor in $H_0^{1,a}(\Omega)$ we need some continuity of the semigroup to guarantee the invariance of global attractor. However, it is difficult to obtain the continuity of semigroup in $H_0^{1,a}(\Omega)$ since we do not impose any restriction on p. Here we use the norm-to-weak continuity instead of the norm-to-norm (or weak-to-weak) continuity of the semigroup in the usual criterions for the existence of global attractors.

Theorem 4.7. *The semigroup* $\{S(t)\}_{t\geqslant 0}$ *possesses a global attractor* $\mathcal{A}$ *in* $H_0^{1,a}(\Omega)$*, that is,* $\mathcal{A}$ *is compact and invariant in* $H_0^{1,a}(\Omega)$ *and attracts every bounded subset of* $L^2(\Omega)$ *in the topology of* $H_0^{1,a}(\Omega)$*.*

Proof. Let B_0 be an absorbing set in $H_0^{1,a}(\Omega)$ obtained in Theorem 4.1. Set

$$\mathcal{A} = \bigcap_{s\geqslant 0} \overline{\bigcup_{t\geqslant s} S(t)B_0}^{H_0^{1,a}(\Omega)}. \tag{4.30}$$

Then from Theorems 4.1 and 4.6 we know that $\mathcal{A}$ is nonempty and compact in $H_0^{1,a}(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the topology of $H_0^{1,a}(\Omega)$. In addition, it is easy to obtain the norm-to-weak continuity of the semigroup from Theorem 3.2 in [9] which can guarantee that $\mathcal{A}$ is invariant.

Therefore, the desired claim follows immediately. □

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