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Research Article

Global Robust Exponential Stability Analysis for Interval Neural Networks with Mixed Delays

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A class of interval neural networks with time-varying delays and distributed delays is investigated. By employing H-matrix and M-matrix theory, homeomorphism techniques, Lyapunov functional method, and linear matrix inequality approach, sufficient conditions for the existence, uniqueness, and global robust exponential stability of the equilibrium point to the neural networks are established and some previously published results are improved and generalized. Finally, some numerical examples are given to illustrate the effectiveness of the theoretical results.

1. Introduction

In recent years, great attention has been paid to the neural networks due to their applications in many areas such as signal processing, associative memory, pattern recognition, parallel computation, and optimization. It should be pointed out that the successful applications heavily rely on the dynamic behaviors of neural networks. Stability, as one of the most important properties for neural networks, is crucially required when designing neural networks. For example, in order to solve problems in the fields of optimization, neural control, and signal processing, neural networks have to be designed such that there is only one equilibrium point, and it is globally asymptotically stable so as to avoid the risk of having spurious equilibria and local minima.

We should point out that neural networks have recently been implemented on electronic chips. In electronic implementation of neural networks, time delays are unavoidably encountered during the processing and transmission of signals, which can cause oscillation and instability of a neural network. On the other hand, there exist inevitably some uncertainties caused by the existence of modeling errors, external disturbance, and parameter fluctuation, which would lead to complex dynamic behaviors. Thus, a good neural network should have robustness against such uncertainties. If the uncertainties of a system are due to the deviations and perturbations of parameters and if these deviations and perturbations are assumed to be bounded, then this system is called an interval system. Recently, global robust stability of interval neural networks with time delays are widely investigated (see [1–22] and references therein). In particular, Faydasicok and Arik [3, 4] proposed two criteria for the global asymptotical robust stability to a class of neural networks with constant delays by utilizing the Lyapunov stability theorems and homeomorphism theorem. The obtained conditions are independent of time delays and only rely on the network parameters of the neural system. Employing Lyapunov-Krasovskii functionals, Balasubramaniam et al. [10, 11] derived two passivity criteria for interval neural networks with time-varying delays in terms of linear matrix inequalities (LMI), which are dependent on the size of the time delays. In practice, to achieve fast response, it is often expected that the designed neural networks can converge fast enough. Thus, it is not only theoretically interesting but also practically important to establish some sufficient conditions for global robust exponential stability of neural networks. In [8], Zhao and Zhu established some sufficient conditions for the global robust exponential stability for interval neural networks with constant delays. In [18], Wang et al. obtained some criteria for the global robust exponential stability for interval Cohen-Grossberg neural networks with time-varying delays using LMI, matrix inequality, matrix norm, and Halanay inequality techniques. In [15–17], employing homeomorphism techniques, Lyapunov method, H-matrix and M-matrix theory, and LMI approach, Shao et al. established some sufficient conditions for the existence, uniqueness, and global robust exponential stability of the equilibrium point for interval Hopfield neural networks with time-varying delays. Recently, the stability of neural networks with time-varying delays has been extensively investigated and various sufficient conditions have been established for the global asymptotic and exponential stability in [10–27]. Generally, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. It is desired to model them by introducing continuously distributed delays over a certain duration of time such that the distant past has less influence compared to the recent behavior of the state (see [28–30]). However, the distributed delays were not taken into account in [15–17] and most of the above references. To the best of our knowledge, there are fewer robust stability results about the interval neural networks with both the time-varying delays and the distributed delays (see [21, 22]).

Motivated by the works of [15–17] and the discussions above, the purpose of this paper is to present some new sufficient conditions for the global robust exponential stability of neural networks with time-varying and distributed delays. The obtained results can be easily checked. Comparisons are made with some previous works by some remarks and numerical examples, which show that our results effectually improve and generalize some existing works. The neural network can be described by the following differential equations:

$$\begin{aligned}\dot{x}_i(t) = &- d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j\left(t-\tau_j(t)\right)\right) \\ &+ \sum_{j=1}^{n} c_{ij} \int_{t-\sigma}^{t} k_j(t-s) f_j\left(x_j(s)\right) \mathrm{d}s + J_i, \quad i = 1, 2, \ldots, n,\end{aligned} \tag{1.1}$$

or equivalently

$$\dot{x}(t) = -Dx(t) + Af(x(t)) + Bf(x(t-\tau(t))) + C\int_{t-\sigma}^{t} K(t-s)f(x(s))\mathrm{d}s + J, \tag{1.2}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^{\mathrm{T}} \in \mathbb{R}^n$ denotes the state vector associated with the neurons, $D = \mathrm{diag}(d_1, d_2, \dots, d_n)$ is a positive diagonal matrix, and $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, and $C = (c_{ij})_{n\times n}$ are the interconnection weight matrix and the time-varying delayed interconnection weight matrix and the distributed delayed interconnection weight matrix, respectively. $f(x) = (f_1(x_1), f_2(x_2), \dots, f_n(x_n))^{\mathrm{T}} \in \mathbb{R}^n$, and $f_j(x_j)$ denotes the activation function, $\tau_j(t)$ denotes the time-varying delay associated with the jth neuron, $f(x(t-\tau(t))) = (f_1(x_1(t-\tau_1(t))), f_2(x_2(t-\tau_2(t))), \dots, f_n(x_n(t-\tau_n(t))))^{\mathrm{T}}$, $K(t) = \mathrm{diag}(k_1(t), k_2(t), \dots, k_n(t))$, $k_j(t) > 0$ represents the delay kernel function, which is a real-valued continuous function defined in $[0,\sigma]$ satisfying $\int_0^{\sigma} k_j(s)\mathrm{d}s = 1$, $j = 1, 2, \dots, n$. $J = (J_1, J_2, \dots, J_n)^{\mathrm{T}}$ is the constant input vector. The coefficients d_i, a_{ij}, b_{ij}, and c_{ij} can be intervalised as follows:

$$\begin{aligned}
D_I &= \left[\underline{D}, \overline{D}\right] = \left\{D = \mathrm{diag}(d_i) : 0 < \underline{d}_i \le d_i \le \overline{d}_i,\ i = 1, 2, \dots, n\right\}, \\
A_I &= \left[\underline{A}, \overline{A}\right] = \left\{A = (a_{ij})_{n\times n} : \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij},\ i, j = 1, 2, \dots, n\right\}, \\
B_I &= \left[\underline{B}, \overline{B}\right] = \left\{B = (b_{ij})_{n\times n} : \underline{b}_{ij} \le b_{ij} \le \overline{b}_{ij},\ i, j = 1, 2, \dots, n\right\}, \\
C_I &= \left[\underline{C}, \overline{C}\right] = \left\{C = (c_{ij})_{n\times n} : \underline{c}_{ij} \le c_{ij} \le \overline{c}_{ij},\ i, j = 1, 2, \dots, n\right\},
\end{aligned} \tag{1.3}$$

where $\underline{D} = \mathrm{diag}(\underline{d}_i)$, $\overline{D} = \mathrm{diag}(\overline{d}_i)$, $\underline{X} = (\underline{x}_{ij})_{n\times n}$, $\overline{X} = (\overline{x}_{ij})_{n\times n}$, and $\mathrm{X} = A, B, C$. Denote $B^* = (\overline{B} + \underline{B})/2$ and $B_* = (\overline{B} - \underline{B})/2$. Clearly, B_* is a nonnegative matrix and the interval matrix $[\underline{B}, \overline{B}] = [B^* - B_*, B^* + B_*]$. Consequently, $B = B^* + \Delta B$, $\Delta B \in [-B_*, B_*]$. C^* and C_* are defined correspondingly.

Throughout this paper, we make the following assumptions.

(H1) $f_i(x)$ $(i = 1, 2, \dots, n)$ are Lipschitz continuous and monotonically nondecreasing, that is, there exist constants $l_i > 0$ such that

$$0 \le \left(f_i(x) - f_i(y)\right)(x - y) \le l_i (x - y)^2, \quad \forall x, y \in \mathbb{R}. \tag{1.4}$$

(H2) $\tau_i(t)$ $(i = 1, 2, \dots, n)$ are bounded differential functions of time t and satisfy $0 \le \tau_i(t) \le \tau, 0 \le \dot{\tau}_i(t) \le h_i < 1$.

Denote $L = \mathrm{diag}(l_1, l_2, \dots, l_n)$, $L_M = \max_{1\le i\le n}\{l_i\}$, $h = \max_{1\le i\le n}\{h_i\}$, and $\delta = \max\{\tau, \sigma\}$.

The organization of this paper is as follows. In Section 2, some preliminaries are given. In Section 3, sufficient conditions for the existence, uniqueness, and global robust exponential stability of the equilibrium point for system (1.1) are presented. In Section 4, some numerical examples are provided to illustrate the effectiveness of the obtained results and comparisons are made between our results and the previously published ones. A concluding remark is given in Section 5 to end this work.

2. Preliminaries

We give some preliminaries in this section. For a vector $x = (x_1, x_2, \ldots, x_n)$, $\|x\|_2 = (\sum_{i=1}^{n} x_i^2)^{1/2}$. For a matrix $A = (a_{ij})_{n\times n}$, A^{T} denotes the transpose; A^{-1} denotes the inverse; $A > (\geq) 0$ means that A is a symmetric positive definite (semidefinite) matrix; $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the largest and the smallest eigenvalues of A, respectively; $\|A\|_2 = \sqrt{\lambda_{\max}(A^{\mathrm{T}}A)}$ denotes the spectral norm of A. I denotes the identity matrix. $*$ denotes the symmetric block in a symmetric matrix.

Definition 2.1 (see [20]). The neural network (1.1) with the parameter ranges defined by (1.3) is globally robustly exponentially stable if for each $D \in D_I$, $A \in A_I$, $B \in B_I$, $C \in C_I$, and J, system (1.1) has a unique equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^{\mathrm{T}}$, and there exist constants $b > 0$ and $\alpha \geq 1$ such that

$$\|x(t) - x^*\| \leq \alpha \|\phi(\theta) - x^*\| \mathrm{e}^{-bt}, \quad \forall t > 0, \tag{2.1}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$ is a solution of system (1.1) with the initial value $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \ldots, \phi_n(\theta))$, $\theta \in [-\delta, 0]$.

Definition 2.2 (see [31]). Let $Z_n = \{A = (a_{ij})_{n\times n} \in M_n(\mathbb{R}) : a_{ij} \leq 0 \text{ if } i \neq j,\ i, j = 1, 2, \ldots, n\}$, where $M_n(\mathbb{R})$ denotes the set of all $n \times n$ matrices with entries from $\mathbb{R}$. Then a matrix A is called an M-matrix if $A \in Z_n$ and all successive principal minors of A are positive.

Definition 2.3 (see [31]). An $n\times n$ matrix $A = (a_{ij})_{n\times n}$ is said to be an H-matrix if its comparison matrix $M(A) = (m_{ij})_{n\times n}$ is an M-matrix, where

$$m_{ij} = \begin{cases} |a_{ii}| & \text{if } i = j, \\ -|a_{ij}| & \text{if } i \neq j. \end{cases} \tag{2.2}$$

Lemma 2.4 (see [19]). *For any vectors $x, y \in \mathbb{R}^n$ and positive definite matrix $G \in \mathbb{R}^{n\times n}$, the following inequality holds: $2x^{\mathrm{T}}y \leq x^{\mathrm{T}}Gx + y^{\mathrm{T}}G^{-1}y$.*

Lemma 2.5 (see [31]). *Let $A, B \in Z_n$. If A is an M-matrix and the elements of matrices A, B satisfy the inequalities $a_{ij} \leq b_{ij}$, $i, j = 1, 2, \ldots, n$, then B is an M-matrix.*

Lemma 2.6 (see [31]). *The following LMI: $\begin{pmatrix} Q(x) & S(x) \\ S^{\mathrm{T}}(x) & R(x) \end{pmatrix} > 0$, where $Q(x) = Q^{\mathrm{T}}(x)$, $R(x) = R^{\mathrm{T}}(x)$, is equivalent to $R(x) > 0$ and $Q(x) - S(x)R^{-1}(x)S^{\mathrm{T}}(x) > 0$ or $Q(x) > 0$ and $R(x) - S^{\mathrm{T}}(x)Q^{-1}(x) S(x) > 0$.*

Lemma 2.7 (see [1]). *$H(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism if $H(x)$ satisfies the following conditions:*

(1) *$H(x)$ is injective, that is, $H(x) \neq H(y)$, $\forall x \neq y$;*

(2) *$H(x)$ is proper, that is, $\|H(x)\| \to +\infty$ as $x \to +\infty$.*

Lemma 2.8. *Suppose that the neural network parameters are defined by* (1.3), *and*

$$\Xi = \begin{pmatrix} \Phi - S & -PB_* & -PC_* \\ * & (1-h)R & 0 \\ * & * & Q \end{pmatrix} > 0, \tag{2.3}$$

where $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$, $R = \mathrm{diag}(r_1, r_2, \ldots, r_n)$, *and* $Q = \mathrm{diag}(q_1, q_2, \ldots, q_n)$ *are positive diagonal matrices,* $\Phi = (\Phi_{ij})_{n\times n} = 2P\underline{D}L^{-1} - ((2-h)/(1-h))\|PB^*\|_2 I - 2\|PC^*\|_2 I - R - Q$, $S = (s_{ij})_{n\times n}$ *with*

$$s_{ij} = \begin{cases} 2p_i\overline{a}_{ii} & \text{if } i = j, \\ \max\left\{\left|p_i\underline{a}_{ij} + p_j\underline{a}_{ji}\right|, \left|p_i\overline{a}_{ij} + p_j\overline{a}_{ji}\right|\right\} & \text{if } i \neq j. \end{cases} \tag{2.4}$$

Then, for all $A \in A_I$, $B \in B_I$, *and* $C \in C_I$, *we have*

$$\Theta = \begin{pmatrix} \Phi - S' & -P\Delta B & -P\Delta C \\ * & (1-h)R & 0 \\ * & * & Q \end{pmatrix} > 0, \tag{2.5}$$

where $S' = (s'_{ij})_{n\times n} = PA + A^{\mathrm{T}}P$.

Proof. Denote

$$\begin{aligned} T &= (t_{ij})_{n\times n} = \frac{1}{1-h}PB_*R^{-1}B_*^{\mathrm{T}}P + PC_*Q^{-1}C_*^{\mathrm{T}}P, \\ T' &= \left(t'_{ij}\right)_{n\times n} = \frac{1}{1-h}P\Delta BR^{-1}\Delta B^{\mathrm{T}}P + P\Delta CQ^{-1}\Delta C^{\mathrm{T}}P. \end{aligned} \tag{2.6}$$

By Lemma 2.6, $\Xi > 0$ is equivalent to

$$\Omega = (\Omega_{ij})_{n\times n} = \Phi - S - T > 0. \tag{2.7}$$

Obviously, $\Omega \in Z_n$, and it follows by Definition 2.2 that Ω is an M-matrix.

By Lemma 2.6, $\Theta > 0$ is equivalent to $\Lambda = (\Lambda_{ij})_{n\times n} = \Phi - S' - T' > 0$. Therefore, we need only to verify that $\Lambda > 0$. Noting that $t_{ij} \geq t'_{ij} \geq 0$ $(i, j = 1, 2, \ldots, n)$ and $s'_{ii} = 2p_i a_{ii} \leq 2p_i\overline{a}_{ii} = s_{ii}$, we have

$$\Lambda_{ii} = \Phi_{ii} - s'_{ii} - t'_{ii} \geq \Phi_{ii} - s_{ii} - t_{ii} = \Omega_{ii} > 0, \quad i = 1, 2, \ldots, n. \tag{2.8}$$

Denote the comparison matrix of Λ by $M(\Lambda) = (m_{ij})_{n\times n}$, where

$$m_{ij} = \begin{cases} |\Lambda_{ii}| & \text{if } i = j, \\ -\left|\Lambda_{ij}\right| & \text{if } i \neq j. \end{cases} \tag{2.9}$$

Considering $s_{ij} \geq |s'_{ij}|$ $(i, j = 1, 2, \ldots, n,\ i \neq j)$, we can obtain that

$$m_{ij} = -\left|s'_{ij} + t'_{ij}\right| \geq -\left|s'_{ij}\right| - \left|t'_{ij}\right| \geq -s_{ij} - t_{ij} = \Omega_{ij}, \quad i, j = 1, 2, \ldots, n,\ i \neq j. \tag{2.10}$$

It follows from (2.8) and (2.10) that $m_{ij} \geq \Omega_{ij},\ i, j = 1, 2, \ldots, n$. From Lemma 2.5, we deduce that $M(\Lambda)$ is an M-matrix, that is, Λ is an H-matrix with positive diagonal elements. It is well known that a symmetric H-matrix with positive diagonal entries is positive definite, then $\Lambda > 0$, which implies that $\Theta > 0$ for all $A \in A_I$, $B \in B_I$, and $C \in C_I$. The proof is complete. □

3. Global Robust Exponential Stability

In this section, we will give a new sufficient condition for the existence and uniqueness of the equilibrium point for system (1.1) and analyze the global robust exponential stability of the equilibrium point.

Theorem 3.1. *Under assumptions (H1) and (H2), if there exist positive diagonal matrices* $P = \operatorname{diag}(p_1, p_2, \ldots, p_n)$*,* $R = \operatorname{diag}(r_1, r_2, \ldots, r_n)$*, and* $Q = \operatorname{diag}(q_1, q_2, \ldots, q_n)$ *such that* $\Xi > 0$*, or equivalently* $\Omega > 0$*, where* Ξ *and* Ω *are defined by* (2.3) *and* (2.7)*, respectively, then system* (1.1) *is globally robustly exponentially stable.*

Proof. We will prove the theorem in two steps.

Step 1. We will prove the existence and uniqueness of the equilibrium point of system (1.1).

Define a map: $H(x) = -Dx + (A + B + C)f(x) + J$. We will prove that $H(x)$ is a homeomorphism of $\mathbb{R}^n$ into itself.

First, we prove that $H(x)$ is an injective map on $\mathbb{R}^n$. For $x, y \in R^n$, $x \neq y$, we have

$$H(x) - H(y) = -D(x - y) + (A + B + C)(f(x) - f(y)). \tag{3.1}$$

If $f(x) = f(y)$, then $H(x) \neq H(y)$ for $x \neq y$. If $f(x) \neq f(y)$, multiplying both sides of (3.1) by $2(f(x) - f(y))^{\mathrm{T}}P$, and utilizing Lemma 2.4, assumptions (H1), (H2) and the compatibility of vector 2-norm and matrix spectral norm, we deduce that

$$\begin{aligned}
&2(f(x) - f(y))^{\mathrm{T}}P(H(x) - H(y)) \\
&\quad = -2(f(x) - f(y))^{\mathrm{T}}PD(x - y) + 2(f(x) - f(y))^{\mathrm{T}}P(A + B + C)(f(x) - f(y)) \\
&\quad \leq -2(f(x) - f(y))^{\mathrm{T}}P\underline{D}L^{-1}(f(x) - f(y)) + 2(f(x) - f(y))^{\mathrm{T}}PA(f(x) - f(y)) \\
&\qquad + 2(f(x) - f(y))^{\mathrm{T}}P(B^* + \Delta B + C^* + \Delta C)(f(x) - f(y))
\end{aligned}$$

$$
\begin{aligned}
&\leq -2(f(x)-f(y))^{\mathrm{T}} P\underline{D}L^{-1}(f(x)-f(y)) + 2(f(x)-f(y))^{\mathrm{T}} PA(f(x)-f(y)) \\
&\quad + \frac{2-h}{1-h}\|PB^{*}\|_2 \|f(x)-f(y)\|_2^2 + \frac{1}{1-h}(f(x)-f(y))^{\mathrm{T}} R(f(x)-f(y)) \\
&\quad + (f(x)-f(y))^{\mathrm{T}} P\Delta B R^{-1}\Delta B^{\mathrm{T}} P(f(x)-f(y)) + 2\|PC^{*}\|_2 \|f(x)-f(y)\|_2^2 \\
&\quad + (f(x)-f(y))^{\mathrm{T}} P\Delta C Q^{-1}\Delta C^{\mathrm{T}} P(f(x)-f(y)) + (f(x)-f(y))^{\mathrm{T}} Q(f(x)-f(y)) \\
&= -(f(x)-f(y))^{\mathrm{T}} \Lambda(f(x)-f(y)).
\end{aligned}
\tag{3.2}
$$

By Lemma 2.8, we have shown that $\Lambda > 0$ if $\Xi > 0$, which leads to

$$
(f(x)-f(y))^{\mathrm{T}} P(H(x)-H(y)) < 0, \quad f(x) \neq f(y). \tag{3.3}
$$

Therefore, $H(x) \neq H(y)$ for all $x \neq y$, that is, $H(x)$ is injective.

Next, we prove that $\|H(x)\|_2 \to +\infty$ as $\|x\|_2 \to +\infty$. Letting $y = 0$ in (3.2), we get

$$
\begin{aligned}
&2(f(x)-f(0))^{\mathrm{T}} P(H(x)-H(0)) \\
&\quad \leq -(f(x)-f(0))^{\mathrm{T}} \Lambda(f(x)-f(0)) \leq -\lambda_{\min}(\Lambda)\|f(x)-f(0)\|_2^2.
\end{aligned}
\tag{3.4}
$$

It follows that

$$
2\|P\|_2 \|H(x)-H(0)\|_2 \geq \lambda_{\min}(\Lambda)\|f(x)-f(0)\|_2, \tag{3.5}
$$

which yields

$$
\|H(x)\|_2 + \|H(0)\|_2 \geq \frac{\lambda_{\min}(\Lambda)}{2\|P\|_2}\left(\|f(x)\|_2 - \|f(0)\|_2\right). \tag{3.6}
$$

Since $\|H(0)\|_2$ and $\|f(0)\|_2$ are finite, it is obvious that $\|H(x)\|_2 \to +\infty$ as $\|f(x)\|_2 \to +\infty$. On the other hand, for unbounded activation functions, by (H1), $\|f(x)\|_2 \to +\infty$ implies $\|x\|_2 \to +\infty$. For bounded activation functions, it is not difficult to derive from (3.1) that $\|H(x)\|_2 \to +\infty$ as $\|x\|_2 \to +\infty$.

By Lemma 2.7, we know that $H(x)$ is a homeomorphism on $\mathbb{R}^n$. Thus, system (1.1) has a unique equilibrium point $x^{*} = (x_1^{*}, x_2^{*}, \ldots, x_n^{*})^{\mathrm{T}}$.

Step 2. We prove that the unique equilibrium point x^{*} is globally robustly exponentially stable.

Let $y(t) = x(t) - x^*$; one can transform system (1.1) into the following system:

$$\begin{aligned}\dot{y}_i(t) = &- d_i y_i(t) + \sum_{j=1}^{n} a_{ij} g_j\left(y_j(t)\right) + \sum_{j=1}^{n} b_{ij} g_j\left(y_j\left(t - \tau_j(t)\right)\right) \\ &+ \sum_{j=1}^{n} c_{ij} \int_{t-\sigma}^{t} k_j(t-s) g_j\left(y_j(s)\right) \mathrm{d}s, \quad i = 1, 2, \ldots, n,\end{aligned} \tag{3.7}$$

or equivalently

$$\dot{y}(t) = -Dy(t) + Ag\left(y(t)\right) + Bg\left(y(t - \tau(t))\right) + C \int_{t-\sigma}^{t} K(t-s) g\left(y(s)\right) \mathrm{d}s, \tag{3.8}$$

where $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^{\mathrm{T}}$, $g(y(t)) = (g_1(y_1(t)), g_2(y_2(t)), \ldots, g_n(y_n(t)))^{\mathrm{T}}$, $g(y(t - \tau(t))) = (g_1(y_1(t - \tau_1(t))), g_2(y_2(t - \tau_2(t))), \ldots, g_n(y_n(t - \tau_n(t))))^{\mathrm{T}}$ with $g_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*)$.

We define a Lyapunov functional: $V(t) = \sum_{i=1}^{4} V_i(t)$, where

$$\begin{aligned}&V_1(t) = s\mathrm{e}^{bt} y^{\mathrm{T}}(t) y(t), \qquad V_2(t) = 2\mathrm{e}^{bt} \sum_{i=1}^{n} p_i \int_{0}^{y_i(t)} g_i(\xi) \mathrm{d}\xi, \\ &V_3(t) = \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \left(r_i + \|PB^*\|_2\right) \mathrm{e}^{b(\xi+\tau)} g_i^2\left(y_i(\xi)\right) \mathrm{d}\xi, \\ &V_4(t) = \sum_{i=1}^{n} \left(q_i + \|PC^*\|_2\right) \int_{0}^{\sigma} k_i(v) \mathrm{e}^{bv} \int_{t-v}^{t} \mathrm{e}^{b\xi} g_i^2\left(y_i(\xi)\right) \mathrm{d}\xi \mathrm{d}v.\end{aligned} \tag{3.9}$$

Calculating the derivative of $V(t)$ along the trajectories of system (3.7), we obtain that

$$\begin{aligned}\dot{V}_1(t) &= sb\mathrm{e}^{bt} y^{\mathrm{T}}(t) y(t) + 2s\mathrm{e}^{bt} y^{\mathrm{T}}(t) \dot{y}(t) \\ &= sb\mathrm{e}^{bt} y^{\mathrm{T}}(t) y(t) + 2s\mathrm{e}^{bt} y^{\mathrm{T}}(t) \Bigg[- Dy(t) + Ag\left(y(t)\right) + Bg\left(y(t - \tau(t))\right) \\ &\qquad + C \int_{t-\sigma}^{t} K(t-s) g\left(y(s)\right) \mathrm{d}s \Bigg], \\ \dot{V}_2(t) &= 2b\mathrm{e}^{bt} \sum_{i=1}^{n} p_i \int_{0}^{y_i(t)} g_i(\xi) \mathrm{d}\xi \\ &\quad + 2\mathrm{e}^{bt} \sum_{i=1}^{n} p_i g_i\left(y_i(t)\right) \Bigg[-d_i y_i(t) + \sum_{j=1}^{n} a_{ij} g_j\left(y_j(t)\right) \\ &\qquad + \sum_{j=1}^{n} b_{ij} g_j\left(y_j\left(t - \tau_j(t)\right)\right) + \sum_{j=1}^{n} c_{ij} \int_{t-\sigma}^{t} k_j(t-s) g_j\left(y_j(s)\right) \mathrm{d}s \Bigg]\end{aligned}$$

$$
\begin{aligned}
&\leq b\mathrm{e}^{bt}y^{\mathrm{T}}(t)PLy(t)-2\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))P\underline{D}L^{-1}g(y(t))+2\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))PAg(y(t))\\
&\quad+\mathrm{e}^{bt}\|PB^{*}\|_2\left(\frac{1}{1-h}\|g(y(t))\|_2^2+(1-h)\|g(y(t-\tau(t)))\|_2^2\right)\\
&\quad+2\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))P\Delta Bg(y(t-\tau(t)))+2\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))P\Delta C\int_0^{\sigma}K(s)g(y(t-s))\mathrm{d}s\\
&\quad+\mathrm{e}^{bt}\|PC^{*}\|_2\left(\|g(y(t))\|_2^2+\left\|\int_0^{\sigma}K(s)g(y(t-s))\mathrm{d}s\right\|_2^2\right),\\
\dot{V}_3(t)&=\sum_{i=1}^{n}(r_i+\|PB^{*}\|_2)\left[\mathrm{e}^{b(t+\tau)}g_i^2(y_i(t))-(1-\dot{\tau}_i(t))\mathrm{e}^{b(t-\tau_i(t)+\tau)}g_i^2(y_i(t-\tau_i(t)))\right]\\
&\leq \mathrm{e}^{b(t+\tau)}g^{\mathrm{T}}(y(t))Rg(y(t))+\mathrm{e}^{b(t+\tau)}\|PB^{*}\|_2\|g(y(t))\|_2^2\\
&\quad-(1-h)\mathrm{e}^{bt}g^{\mathrm{T}}(y(t-\tau(t)))Rg(y(t-\tau(t)))-(1-h)\mathrm{e}^{bt}\|PB^{*}\|_2\|g(y(t-\tau(t)))\|_2^2,\\
\dot{V}_4(t)&=\sum_{i=1}^{n}(q_i+\|PC^{*}\|_2)\int_0^{\sigma}k_i(v)\mathrm{e}^{bv}\left[\mathrm{e}^{bt}g_i^2(y_i(t))-\mathrm{e}^{b(t-v)}g_i^2(y_i(t-v))\right]\mathrm{d}v\\
&\leq \mathrm{e}^{b(t+\sigma)}g^{\mathrm{T}}(y(t))(Q+\|PC^{*}\|_2I)g(y(t))-\mathrm{e}^{bt}\left[\int_0^{\sigma}K(s)g(y(t-s))\mathrm{d}s\right]^{\mathrm{T}}\\
&\quad\times(Q+\|PC^{*}\|_2I)\left[\int_0^{\sigma}K(s)g(y(t-s))\mathrm{d}s\right]\quad(\text{Schwarz's inequality}).
\end{aligned}
\tag{3.10}
$$

Therefore, one can deduce that

$$
\begin{aligned}
\dot{V}(t)&\leq sb\mathrm{e}^{bt}y^{\mathrm{T}}(t)y(t)+b\mathrm{e}^{bt}y^{\mathrm{T}}(t)PLy(t)-2s\mathrm{e}^{bt}y^{\mathrm{T}}(t)Dy(t)+2s\mathrm{e}^{bt}y^{\mathrm{T}}(t)Ag(y(t))\\
&\quad+2s\mathrm{e}^{bt}y^{\mathrm{T}}(t)Bg(y(t-\tau(t)))+2s\mathrm{e}^{bt}y^{\mathrm{T}}(t)C\int_{t-\sigma}^{t}K(t-s)g(y(s))\mathrm{d}s\\
&\quad-2\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))P\underline{D}L^{-1}g(y(t))+2\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))PAg(y(t))\\
&\quad+\mathrm{e}^{bt}\frac{1}{1-h}\|PB^{*}\|_2\|g(y(t))\|_2^2+2\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))P\Delta Bg(y(t-\tau(t)))\\
&\quad+2\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))P\Delta C\int_0^{\sigma}K(s)g(y(t-s))\mathrm{d}s+\mathrm{e}^{bt}\|PC^{*}\|_2\|g(y(t))\|_2^2\\
&\quad+\mathrm{e}^{b(t+\tau)}g^{\mathrm{T}}(y(t))Rg(y(t))+\mathrm{e}^{b(t+\tau)}\|PB^{*}\|_2\|g(y(t))\|_2^2+\mathrm{e}^{bt}\|PB^{*}\|_2\|g(y(t))\|_2^2\\
&\quad-\mathrm{e}^{bt}\|PB^{*}\|_2\|g(y(t))\|_2^2+\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))Rg(y(t))-\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))Rg(y(t))\\
&\quad-(1-h)\mathrm{e}^{bt}g^{\mathrm{T}}(y(t-\tau(t)))Rg(y(t-\tau(t)))+\mathrm{e}^{b(t+\sigma)}g^{\mathrm{T}}(y(t))(Q+\|PC^{*}\|_2I)g(y(t))\\
&\quad-\mathrm{e}^{bt}\left[\int_0^{\sigma}K(s)g(y(t-s))\mathrm{d}s\right]^{\mathrm{T}}Q\left[\int_0^{\sigma}K(s)g(y(t-s))\mathrm{d}s\right]\\
&\quad+\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))(Q+\|PC^{*}\|_2I)g(y(t))-\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))(Q+\|PC^{*}\|_2I)g(y(t))
\end{aligned}
$$

$$\begin{aligned}
&\leq b\mathrm{e}^{bt}y^{\mathrm{T}}(t)(sI+PL)y(t)+\left(\mathrm{e}^{b\tau}-1\right)\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))(R+\|PB^{*}\|_{2}I)g(y(t)) \\
&\quad+\left(\mathrm{e}^{b\sigma}-1\right)\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))(Q+\|PC^{*}\|_{2}I)g(y(t))-\mathrm{e}^{bt}z^{\mathrm{T}}(t)\Psi z(t),
\end{aligned} \tag{3.11}$$

where

$$\begin{gathered}
\Psi=\begin{pmatrix} 2sD & -sA & -sB & -sC \\ * & \Phi-S' & -P\Delta B & -P\Delta C \\ * & * & (1-h)R & 0 \\ * & * & * & Q \end{pmatrix}, \\
z(t)=\left(y^{\mathrm{T}}(t)\ \ g^{\mathrm{T}}(y(t))\ \ g^{\mathrm{T}}(y(t-\tau(t)))\ \ \left[\int_0^{\sigma}K(s)g(y(t-s))\mathrm{d}s\right]^{\mathrm{T}}\right)^{\mathrm{T}}.
\end{gathered} \tag{3.12}$$

Denote $\Upsilon=(A\ \ B\ \ C)$, from Lemma 2.6, $\Psi>0$ is equivalent to $\Theta-(s/2)\Upsilon^{\mathrm{T}}D^{-1}\Upsilon>0$, where Θ is defined by (2.5). By Lemma 2.8, we have $\Theta>0$. Letting $0<s<2\min_{1\leq i\leq n}\{d_i\}\lambda_{\min}(\Theta)/\lambda_{\max}(\Upsilon^{\mathrm{T}}\Upsilon)$, we can derive that

$$\Theta-\frac{s}{2}\Upsilon^{\mathrm{T}}D^{-1}\Upsilon\geq\Theta-\frac{s}{2\min_{1\leq i\leq n}\{d_i\}}\Upsilon^{\mathrm{T}}\Upsilon>0, \tag{3.13}$$

which yields $\Psi>0$. Choosing $0<b<\min_{1\leq i\leq 3}\{b_i\}$ with

$$\begin{aligned}
b_1&=\frac{\lambda_{\min}(\Psi)}{s+\max_{1\leq i\leq n}\{p_i l_i\}}, \qquad b_2=\frac{1}{\tau}\ln\left(\frac{\lambda_{\min}(\Psi)}{2\max_{1\leq i\leq n}\{r_i+\|PB^{*}\|_2\}}+1\right), \\
b_3&=\frac{1}{\sigma}\ln\left(\frac{\lambda_{\min}(\Psi)}{2\max_{1\leq i\leq n}\{q_i+\|PC^{*}\|_2\}}+1\right),
\end{aligned} \tag{3.14}$$

we can get

$$\begin{aligned}
\dot{V}(t)&\leq b_1\mathrm{e}^{bt}y^{\mathrm{T}}(t)(sI+PL)y(t)+\left(\mathrm{e}^{b_2\tau}-1\right)\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))(R+\|PB^{*}\|_{2}I)g(y(t)) \\
&\quad+\left(\mathrm{e}^{b_3\sigma}-1\right)\mathrm{e}^{bt}g^{\mathrm{T}}(y(t))(Q+\|PC^{*}\|_{2}I)g(y(t))-\mathrm{e}^{bt}z^{\mathrm{T}}(t)\Psi z(t) \\
&<\mathrm{e}^{bt}\lambda_{\min}(\Psi)\left[y^{\mathrm{T}}(t)y(t)+g^{\mathrm{T}}(y(t))g(y(t))\right]-\mathrm{e}^{bt}z^{\mathrm{T}}(t)\Psi z(t)\leq 0.
\end{aligned} \tag{3.15}$$

Consequently, $V(t)\leq V(0)$ for all $t\geq 0$.

On the other hand,

$$
\begin{aligned}
V(0) = {} & s y^{\mathrm{T}}(0) y(0) + 2\sum_{i=1}^{n} p_i \int_0^{y_i(0)} g_i(\xi)\mathrm{d}\xi + \sum_{i=1}^{n} \int_{-\tau_i(0)}^{0} \left(r_i + \|PB^*\|_2\right) \mathrm{e}^{b(\xi+\tau)} g_i^2(y_i(\xi))\mathrm{d}\xi \\
& + \sum_{i=1}^{n} \left(q_i + \|PC^*\|_2\right) \int_0^{\sigma} k_i(v)\mathrm{e}^{bv} \int_{-v}^{0} \mathrm{e}^{b\xi} g_i^2(y_i(\xi))\mathrm{d}\xi\mathrm{d}v \\
\le {} & s\left\|y(0)\right\|_2^2 + \max_{1\le i\le n}\{p_i l_i\}\left\|y(0)\right\|_2^2 + \frac{1}{b}\left(\max_{1\le i\le n}\{r_i\} + \|PB^*\|_2\right) L_M^2\left(\mathrm{e}^{b\tau} - 1\right) \sup_{-\tau\le\theta\le 0}\left\|y(\theta)\right\|_2^2 \\
& + \frac{1}{b}\left(\max_{1\le i\le n}\{q_i\} + \|PC^*\|_2\right) L_M^2\left(\mathrm{e}^{b\sigma} - 1\right) \sup_{-\sigma\le\theta\le 0}\left\|y(\theta)\right\|_2^2 \\
\le {} & \alpha \sup_{-\delta\le\theta\le 0}\left\|y(\theta)\right\|_2^2,
\end{aligned}
\tag{3.16}
$$

where $\alpha = s + \max_{1\le i\le n}\{p_i l_i\} + (1/b)(\max_{1\le i\le n}\{r_i\} + \max_{1\le i\le n}\{q_i\} + \|PB^*\|_2 + \|PC^*\|_2) L_M^2(\mathrm{e}^{b\delta} - 1)$. Hence, $s\mathrm{e}^{bt}\|y(t)\|_2^2 \le V(t) \le V(0) \le \alpha \sup_{-\delta\le\theta\le 0}\|y(\theta)\|_2^2$, that is,

$$
\|x(t) - x^*\| \le \sqrt{\frac{\alpha}{s}}\left\|\phi(\theta) - x^*\right\|\mathrm{e}^{-bt/2}, \quad t > 0. \tag{3.17}
$$

Thus, system (1.1) is globally robustly exponentially stable. The proof is complete. □

Remark 3.2. For the case of infinite distributed delays, that is, letting $\sigma = \infty$ in (1.1), assume that the delay kernels $k_j(\cdot)$ $(j = 1, 2, \ldots, n)$ satisfy

(H3) $\int_0^{\infty} k_j(s)\mathrm{d}s = 1$ and $\int_0^{\infty} k_j(s)\mathrm{e}^{\mu s}\mathrm{d}s < \infty$

for some positive constant μ. A typical example of such delay kernels is given by $k_j(s) = s^r/r!\gamma_j^{r+1}\mathrm{e}^{-\gamma_j s}$ for $s \in [0, \infty)$, where $\gamma_j \in [0, \infty)$, $r \in \{0, 1, \ldots, n\}$, which are called the Gamma Memory Filter in [32]. From assumption (H3), we can choose a constant $b_3' : 0 < b_3' < \mu$ satisfying the following requirement:

$$
\int_0^{\infty} k_j(s)\mathrm{e}^{b_3' s}\mathrm{d}s \le \frac{\lambda_{\min}(\Psi)}{2\max_{1\le i\le n}\{q_i + \|PC^*\|_2\}} + 1, \quad 1 \le j \le n. \tag{3.18}
$$

In a similar argument as the proof of Theorem 3.1, Under the conditions of Theorem 3.1 and assumption (H3), we can derive that

$$
\|x(t) - x^*\| \le \sqrt{\frac{\alpha'}{s}}\left\|\phi(\theta) - x^*\right\|\mathrm{e}^{-bt/2}, \quad t > 0, \tag{3.19}
$$

where $0 < b < \min\{b_1, b_2, b_3'\}$, $\alpha' = s + \max_{1\le i\le n}\{p_i l_i\} + (1/b)(\max_{1\le i\le n}\{r_i\} + \max_{1\le i\le n}\{q_i\} + \|PB^*\|_2 + \|PC^*\|_2) L_M^2 \max\{\mathrm{e}^{b\tau} - 1, \lambda_{\min}(\Psi)/2\max_{1\le i\le n}\{q_i + \|PC^*\|_2\}\}$. Hence, for the case of $\sigma = \infty$, system (1.1) is also globally robustly exponentially stable.

Remark 3.3. Letting $P = pI$ be a positive scalar matrix in Theorem 3.1, we can get a robust exponential stability criterion based on LMI.

Remark 3.4. In [8, 13, 15–18], the authors have dealt with the robust exponential stability of neural networks with time-varying delays. However, the distributed delays were not taken into account in models. Therefore, our results in this paper are more general than those reported in [8, 13, 15–18]. It should be noted that the main results in [15] are a special case of Theorem 3.1 when $C = 0$. Also, our results generalize some previous ones in [2, 6, 7] as mentioned in [15].

Remark 3.5. In previous works such as [2, 6, 7, 17], $\|B_*\|_2$ is often used as a part to estimate the bounds for $\|B\|_2$. Considering that B_* is a nonnegative matrix, we develop a new approach based on H-matrix theory. The obtained robust stability criterion is in terms of the matrices B_* and B_*^{T}, which can reduce the conservativeness of the robust results to some extent.

4. Numerical Simulations and Comparisons

In what follows, we give some examples to illustrate the results above and make comparisons between our results and the previously published ones.

Example 4.1. Consider system (1.1) with the following parameters:

$$\begin{gathered}
\underline{A} = \begin{pmatrix} -0.3 & -0.2 \\ -0.5 & -0.6 \end{pmatrix}, \qquad \overline{A} = \begin{pmatrix} 0.3 & 0.2 \\ 0.2 & 0.1 \end{pmatrix}, \qquad \underline{B} = \begin{pmatrix} -0.8 & -0.9 \\ -0.4 & -1 \end{pmatrix}, \\
\overline{B} = \begin{pmatrix} 0.5 & 0.6 \\ 0.7 & 1 \end{pmatrix}, \qquad \underline{C} = \begin{pmatrix} -0.5 & -0.5 \\ -0.3 & -0.8 \end{pmatrix}, \qquad \overline{C} = \begin{pmatrix} 0.5 & 0.6 \\ 0.3 & 1 \end{pmatrix}, \\
\underline{d_1} = 3.6, \qquad \underline{d_2} = 5.8, \qquad J_1 = J_2 = 0, \qquad f_1(x) = f_2(x) = \frac{1}{2}(|x+1| + |x-1|), \\
\tau_1(t) = \tau_2(t) = 1 - \frac{\mathrm{e}^{-t}}{2}, \qquad \sigma = +\infty, \qquad k_j(t) = t\mathrm{e}^{-t}.
\end{gathered} \tag{4.1}$$

It is clear that $l_1 = l_2 = 1$, $\tau = 1$, $h = 0.5$, $\mu = 1$. Using the optimization toolbox of Matlab and solving the optimization problem (2.3), we can obtain a feasible solution:

$$P = \operatorname{diag}(1.3970, 1.0335), \qquad R = \operatorname{diag}(1.9323, 2.0000), \qquad Q = \operatorname{diag}(1.7742, 1.9982). \tag{4.2}$$

In this case,

$$\Omega = \begin{pmatrix} 1.7869 & -2.8928 \\ -2.8928 & 4.6886 \end{pmatrix} > 0. \tag{4.3}$$

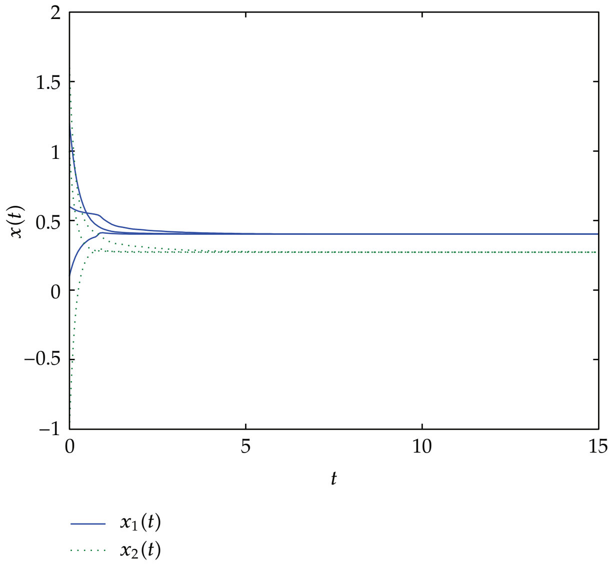

Figure 1: Time responses of the state variables $x(t)$ with different initial values in Example 4.1.

By Theorem 3.1, system (1.1) with above parameters is globally robustly exponentially stable. To illustrate the theoretical result, we present a simulation with the following parameters:

$$A = \begin{pmatrix} 0.2 & 0.1 \\ -0.1 & -0.4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.1 & 0.5 \\ 0.6 & 0.5 \end{pmatrix}, \qquad C = \begin{pmatrix} 0.2 & 0.36 \\ 0.2 & 0.8 \end{pmatrix}, \qquad D = \begin{pmatrix} 3.6 & 0 \\ 0 & 5.8 \end{pmatrix}. \tag{4.4}$$

We can find that the neuron vector $x = (x_1(t), x_2(t))^{\mathrm{T}}$ converges to the unique equilibrium point $x^* = (0.4024, 0.2723)^{\mathrm{T}}$ (see Figure 1). Further, from (3.14) and (3.18), we can deduce that $b_1 = 0.086$, $b_2 = 0.0445$, and $b'_3 = 0.014$. Thus the exponential convergence rate is $b = 0.014$.

Next, we will compare our results with the previously robust stability results derived in the literature. If $c_{ij} = 0$, $\tau_j(t) \equiv \tau_j$, τ_j is a constant, $i, j = 1, 2, \ldots, n$, system (1.1) reduces to the following interval neural networks:

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_j)) + J_i, \quad i = 1, 2, \ldots, n, \tag{4.5}$$

which was studied in [3, 4, 8] and the main results are restated as follows.

Theorem 4.2 (see [3]). *Let $f \in \mathcal{L}$. Then neural network model* (4.5) *is globally asymptotically robustly stable, if the following condition holds:*

$$\delta = d_m - L_M \sqrt{\|A^*\|_2^2 + \|A_*\|_2^2 + 2\left\|A_*^T |A^*|\right\|_2} - L_M \sqrt{\left\|\widehat{B}\right\|_1 \left\|\widehat{B}\right\|_\infty} > 0, \tag{4.6}$$

where $L_M = \max(l_i)$, $d_m = \min(\underline{d}_i)$, $A^* = (1/2)(\overline{A} + \underline{A})$, $A_* = (1/2)(\overline{A} - \underline{A})$, *and* $\widehat{B} = (\widehat{b}_{ij})_{n\times n}$ *with* $\widehat{b}_{ij} = \max\{|\underline{b}_{ij}|, |\overline{b}_{ij}|\}$.

Theorem 4.3 (see [4]). *For the neural network defined by* (4.5), *assume that* $f \in \mathcal{L}$. *Then, neural network model* (4.5) *is globally asymptotically stable if the following condition holds:*

$$\varepsilon = d_m - L_M\|Q\|_2 - L_M\sqrt{n}\left((1-p)\sqrt{\|R\|_\infty} + p\sqrt{\|R\|_1}\right) > 0, \tag{4.7}$$

where $d_m = \min(\underline{d}_i)$, $L_M = \max(l_i)$, $0 \le p \le 1$,

$$\|Q\|_2 = \min\left\{\|A^*\|_2 + \|A_*\|_2, \sqrt{\|A^*\|_2^2 + \|A_*\|_2^2 + 2\left\|A_*^T|A^*|\right\|_2}, \left\|\widehat{A}\right\|_2\right\} \tag{4.8}$$

with $A^* = (1/2)(\overline{A}+\underline{A})$, $A_* = (1/2)(\overline{A}-\underline{A})$, $\widehat{A} = (\widehat{a}_{ij})_{n\times n}$ *with* $\widehat{a}_{ij} = \max\{|\underline{a}_{ij}|, |\overline{a}_{ij}|\}$, $R = (r_{ij})_{n\times n}$ *with* $r_{ij} = \widehat{b}_{ij}^2$ *and* $\widehat{b}_{ij} = \max\{|\underline{b}_{ij}|, |\overline{b}_{ij}|\}$.

Theorem 4.4 (see [8]). *Under assumption (H1), if there exists a positive definite diagonal matrix* $P = \operatorname{diag}(p_1, p_2, \ldots, p_n)$, *such that*

$$\Pi = 2P\underline{D}L^{-1} - S - 2\|P\|_2(\|B^*\|_2 + \|B_*\|_2)I > 0, \tag{4.9}$$

where S is defined in Lemma 2.5, then system (4.5) *is globally robustly exponentially stable.*

Example 4.5. In system (4.5), we choose

$$\underline{A} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \qquad \underline{B} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \overline{A} = -\underline{A}, \qquad \overline{B} = -\underline{B},$$
$$\underline{D} = \operatorname{diag}(8, 3, 7, 7.5), \quad J_i = 0, \quad f_i(x) = \frac{1}{2}(|x+1| + |x-1|), \quad \tau_i = 1 \quad (i = 1, 2, 3, 4). \tag{4.10}$$

It is clear that $l_i = 1$ $(i = 1, 2, 3, 4)$. Solving the optimization problem (2.3), we obtain

$$P = \operatorname{diag}(0.4556, 0.9154, 0.5650, 0.5271), \qquad R = \operatorname{diag}(0.8122, 0.5392, 1.1254, 1.1470),$$
$$Q = \operatorname{diag}(0.4983, 0.1274, 0.8477, 0.8710),$$
$$\Omega = \begin{pmatrix} 3.0558 & -1.3709 & -1.0206 & -0.9826 \\ -1.3709 & 2.9948 & -1.4804 & -1.4424 \\ -1.0206 & -1.4804 & 4.8070 & -1.0921 \\ -0.9826 & -1.4424 & -1.0921 & 4.8338 \end{pmatrix} > 0. \tag{4.11}$$

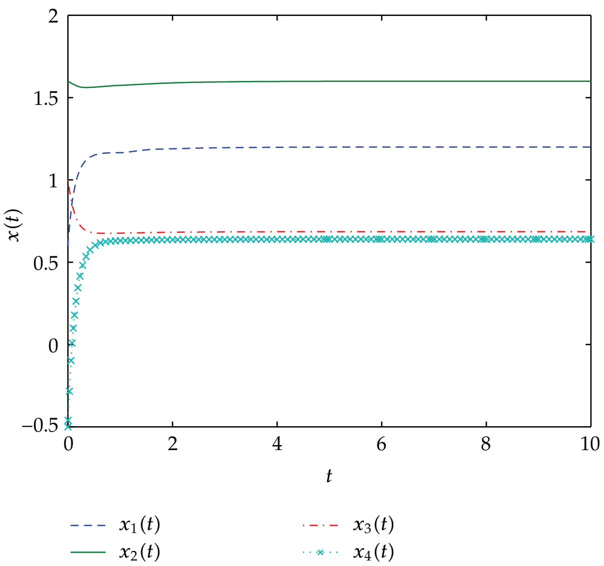

Figure 2: Time responses of the state variables $x(t)$ with initial value $\phi(s) = (0.6, 1.6, 1, -0.5)^{\mathrm{T}}$ for $s \in [-1, 0]$ in Example 4.5.

By Theorem 3.1, system (4.5) is globally robustly exponentially stable. To illustrate the theoretical result, we present a simulation with $A = \overline{A}$, $B = \overline{B}$, and $D = \underline{D}$. We can find that the neuron vector $x(t)$ converges to the unique equilibrium point $x^* = (1.2000, 1.6000, 0.6857, 0.6400)^{\mathrm{T}}$ (see Figure 2).

Now, applying the result of Theorem 4.2 to this example yields

$$\delta = d_m - L_M\sqrt{\|A^*\|_2^2 + \|A_*\|_2^2 + 2\left\|A_*^{\mathrm{T}}|A^*|\right\|_2} - L_M\sqrt{\left\|\widehat{B}\right\|_1\left\|\widehat{B}\right\|_\infty} = d_m - 6. \tag{4.12}$$

The choice $d_m > 6$ ensures the global robust stability of system (4.5).

If we apply the result of Theorem 4.3 to this example and choose $p = 1$, which can guarantee d_m reaches the minimum value when $\varepsilon > 0$, then we have

$$\varepsilon = d_m - L_M\|Q\|_2 - L_M\sqrt{n}\left((1-p)\sqrt{\|R\|_\infty} + p\sqrt{\|R\|_1}\right) = d_m - 6, \tag{4.13}$$

from which we can obtain that system (4.5) is globally robustly exponentially stable for $d_m > 6$.

In this example, noting that system (4.5) is globally robustly exponentially stable for $d_2 = 3$ in our result, hence, for the network parameters of this example, our result derived in Theorem 3.1 imposes a less restrictive condition on d_m than those imposed by Theorems 4.2 and 4.3.

Example 4.6. In system (4.5), we choose

$$\underline{A} = \begin{pmatrix} 0.8 & 0.6 \\ 0.8 & 1 \end{pmatrix}, \quad \overline{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \underline{B} = \begin{pmatrix} -0.9 & -0.8 \\ -0.2 & -0.6 \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} 0.6 & 0.5 \\ 1 & 0.8 \end{pmatrix},$$
$$\underline{D} = \operatorname{diag}(3.8, 3.8), \quad J_i = 0, \quad f_i(x) = \frac{1}{2}(|x+1| + |x-1|), \quad \tau_i = 1 \quad (i = 1, 2). \tag{4.14}$$

Using Theorem 4.4, one can obtain

$$\begin{aligned}\Pi &= \begin{pmatrix} 7.6p_1 & 0 \\ 0 & 7.6p_2 \end{pmatrix} - \begin{pmatrix} 2p_1 & p_1 + p_2 \\ p_1 + p_2 & 2p_2 \end{pmatrix} - 3.6077 \times \max\{p_1, p_2\} \times I \\ &\leq \begin{pmatrix} 1.9923p_1 & -(p_1 + p_2) \\ -(p_1 + p_2) & 1.9923p_2 \end{pmatrix}.\end{aligned} \tag{4.15}$$

Clearly, there do not exist suitable positive constants p_1 and p_2 such that $\Pi > 0$. As a result, Theorem 4.4 cannot be applied to this example.

Solving the optimization problem (2.3), we obtain that

$$P = \operatorname{diag}(1.7909, 1.5070), \qquad R = \operatorname{diag}(2.0000, 1.9893), \qquad Q = \operatorname{diag}(0.0001, 0.0001),$$
$$\Omega = \begin{pmatrix} 5.0299 & -4.5224 \\ -4.5224 & 4.0661 \end{pmatrix} > 0. \tag{4.16}$$

By Theorem 3.1, system (4.5) is globally robustly exponentially stable.

5. Conclusion

In this paper, we discussed a class of interval neural networks with time-varying delays and finite as well as infinite distributed delays. By employing H-matrix and M-matrix theory, homeomorphism techniques, Lyapunov functional method, and LMI approach, sufficient conditions for the existence, uniqueness, and global robust exponential stability of the equilibrium point for the neural networks were established. It was shown that the obtained results improve and generalize the previously published results. Therefore, our results extend the application domain of neural networks to a larger class of engineering problems. Numerical simulations demonstrated the main results. At last, in order to guide the readers to future works of robust stability of neural networks, we would like to point out that the key factor should be the determination of new upper bound norms for the intervalized connection matrices. Such new upper bound estimations for intervalized connection matrices might help us to derive new sufficient conditions for robust stability of delayed neural networks.

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