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Research Article

Differential Subordination Results for Certain Integrodifferential Operator and Its Applications

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We introduce an integrodifferential operator $J_{s,b}(f)$ which plays an important role in the *Geometric Function Theory*. Some theorems in differential subordination for $J_{s,b}(f)$ are used. Applications in *Analytic Number Theory* are also obtained which give new results for Hurwitz-Lerch Zeta function and Polylogarithmic function.

1. Introduction

Let A denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Also, let μ denote the class of analytic functions in the form

$$r(z) = 1 + \sum_{k=1}^{\infty} a_k z^k. \tag{1.2}$$

We begin by recalling that a general Hurwitz-Lerch Zeta function $\Phi(z,s,b)$ defined by (cf., e.g., [1, P. 121 et seq.])

$$\Phi(z,s,b)=\sum_{k=0}^{\infty}\frac{z^k}{(k+b)^s}, \tag{1.3}$$

$(b\in\mathbb{C}\setminus\mathbb{Z}_0^-,\ \mathbb{Z}_0^-=\mathbb{Z}^-\cup\{0\}=\{0,-1,-2,\ldots\},\ s\in\mathbb{C}$ when $z\in\mathbb{U}$, $\operatorname{Re}(s)>1$ when $|z|=1)$

which contains important functions of *Analytic Number Theory*, as the Polylogarithmic function:

$$Li_s(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^s}=z\Phi(z,s,1), \qquad (s\in\mathbb{C}\text{ when } z\in\mathbb{U},\ \operatorname{Re}(s)>1\text{ when } |z|=1). \tag{1.4}$$

Several properties of $\Phi(z,s,b)$ can be found in the recent papers, for example Choi et al. [2], Ferreira and López [3], Gupta et al. [4], and Luo and Srivastava [5]. See, also [6–16].

Recently, Srivastava and Attiya [8] introduced the operator $J_{s,b}(f)$ which makes a connection between *Geometric Function Theory* and *Analytic Number Theory*, defined by

$$J_{s,b}(f)(z)=G_{s,b}(z)*f(z), \qquad \left(z\in\mathbb{U};f\in A;b\in\mathbb{C}\setminus\mathbb{Z}_0^-;s\in\mathbb{C}\right), \tag{1.5}$$

where

$$G_{s,b}(z)=(1+b)^s\left[\Phi(z,s,b)-b^{-s}\right] \tag{1.6}$$

and $*$ denotes the Hadamard product (or convolution).

Furthermore, Srivastava and Attiya [8] showed that

$$J_{s,b}(f)(z)=z+\sum_{k=2}^{\infty}\left(\frac{1+b}{k+b}\right)^s a_k z^k. \tag{1.7}$$

As special cases of $J_{s,b}(f)$, Srivastava and Attiya [8] introduced the following identities:

$$\begin{aligned}
J_{0,b}(f)(z) &= f(z),\\
J_{1,0}(f)(z) &= \int_0^z \frac{f(t)}{t}dt = A(f)(z),\\
J_{1,1}(f)(z) &= \frac{2}{z}\int_0^z f(t)dt = \mathcal{L}(f)(z),\\
J_{1,\gamma}(f)(z) &= \frac{1+\gamma}{z^{\gamma}}\int_0^z f(t)t^{\gamma-1}dt = \mathcal{L}_{\gamma}(f)(z) \quad (\gamma \text{ real}; \gamma > -1),\\
J_{\sigma,1}(f)(z) &= \frac{2^{\sigma}}{z\Gamma(\sigma)}\int_0^z \left(\log\left(\frac{z}{t}\right)\right)^{\sigma-1} f(t)dt = I^{\sigma}(f)(z) \quad (\sigma \text{ real}; \sigma > 0),
\end{aligned} \tag{1.8}$$

where, the operators $A(f)$ and $\mathcal{L}(f)$ are the integral operators introduced earlier by Alexander [17] and Libera [18], respectively, $\mathcal{L}_{\gamma}(f)$ is the generalized Bernardi operator, $\mathcal{L}_{\gamma}(f)(\gamma \in \mathbb{N} = \{1,2,\ldots\})$ introduced by Bernardi [19], and $I^{\sigma}(f)$ is the Jung-Kim-Srivastava integral operator introduced by Jung et al. [20].

Moreover, in [8], Srivastava and Attiya defined the operator $J_{s,b}(f)$ for $b \in \mathbb{C} \setminus \mathbb{Z}^-$, by using the following relationship:

$$J_{s,0}(f)(z) = \lim_{b\to 0} J_{s,b}(f)(z). \tag{1.9}$$

Some applications of the operator $J_{s,b}(f)$ to certain classes in *Geometric Function Theory* can be found in [21, 22].

In our investigations we need the following definitions and lemma.

Definition 1.1. Let $f(z)$ and $F(z)$ be analytic functions. The function $f(z)$ is said to be subordinate to $F(z)$, written $f(z) \prec F(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| \le 1$, and such that $f(z) = F(w(z))$. If $F(z)$ is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

Definition 1.2. Let $\Psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ be analytic in domain $\mathbb{D}$, and let $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ with $(p(z), zp'(z)) \in \mathbb{D}$ when $z \in \mathbb{U}$, then we say that $p(z)$ satisfies a first order differential subordination if

$$\Psi(p(z), zp'(z); z) \prec h(z) \quad (z \in U). \tag{1.10}$$

The univalent function $q(z)$ is called dominant of the differential subordination (1.10), if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.10), if $\widetilde{q}(z) \prec q(z)$ for all dominant of (1.10), then we say that $\widetilde{q}(z)$ is the best dominant of (1.10).

Lemma 1.3 (see [8]). *If $z \in \mathbb{U}$, $f \in A$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$ and $s \in \mathbb{C}$, then*

$$zJ'_{s+1,b}(f)(z) = (1+b)J_{s,b}(f)(z) - bJ_{s+1,b}(f)(z). \tag{1.11}$$

The purpose of the present paper is to extend the use of $J_{s,b}(f)$ as integrodifferential operator, and some theorems in differential subordination for $J_{s,b}(f)$ are used. Applications in *Analytic Number Theory* are also obtained which give new results for Hurwitz-Lerch Zeta function and Polylogarithmic function.

2. Making Use of $J_{s,b}(f)$ as a Differential Operator

From the definition of $J_{s,b}(f)$ in (1.5) and using (1.7), we obtain the following identities.

For $z \in \mathbb{U}$, $f \in A$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $b \in \mathbb{C} \setminus \mathbb{Z}^-$, we have

$$\begin{aligned} J_{-1,0}(f)(z) &= zf'(z), \\ J_{-1,1}(f)(z) &= \frac{1}{2}\{f(z) + zf'(z)\}, \\ J_{-1,1/(1-\lambda)}(f)(z) &= \lambda f(z) + (1-\lambda)zf'(z) \quad (\lambda \neq 1), \\ J_{-n,0}(f)(z) &= D^n(f)(z), \\ J_{-n,(1/\lambda)-1}(f)(z) &= D_\lambda^n(f)(z) \quad (\lambda \neq 0), \\ J_{-n,\lambda}(f)(z) &= I_\lambda^n(f)(z) \quad (\lambda > -1), \\ J_{-n,1}(f)(z) &= I_n(f)(z), \end{aligned} \tag{2.1}$$

where $D^n(f)$ is the Sălăgean differential operator which introduced by Sălăgean [23], $D_\lambda^n(f)$ is the generalized of operator, $D_\lambda^n(f)$ $(\lambda > 0$; real) introduced by Al-Oboudi [24], $I_\lambda^n(f)$ was studied by Cho and Srivastava [25] and by Cho and Kim [26], and the operator $I_n(f)$ was studied by Uralegaddi and Somanatha [27].

Also, we note that

$$\begin{aligned} J_{-n,0}(f)(z) &= Li_{-n}(z) * f(z) \quad (n \in \mathbb{N}_0; f \in A), \\ J_{-n,1}(f)(z) &= \frac{Li_{-n}(z)}{z} * f(z) \quad (n \in \mathbb{N}_0; f \in A), \end{aligned} \tag{2.2}$$

where $Li_s(z)$ is the Polylogarithmic function defined by (1.4).

Now, we prove the following lemma.

Lemma 2.1. *If $z \in \mathbb{U}$, $f \in A$, $n \in \mathbb{N}_0$ and $b \in \mathbb{C} \setminus \mathbb{Z}^-$, then*

$$J_{-n,b}(f)(z) = \frac{1}{(1+b)^n}(zD+b)^n f(z)\left(D := \frac{d}{dz}\right), \tag{2.3}$$

where $(zD+b)^n = (zD+b) \circ (zD+b) \circ \cdots \circ (zD+b)$ *to* n*-times, and* $\circ$ *denotes the composition* $(I \circ J)(f)(z) = I(J(f(z)))$.

Proof. Putting $s = -n$ $(n \in \mathbb{N}_0)$ in (1.11), we have

$$\begin{aligned}(1+b)(J_{-n,b})(f)(z) &= \left[z\frac{d}{dz}J_{-n+1,b}(f)(z) + bJ_{-n+1,b}(f)(z)\right] \\ &= (zD+b)J_{-n+1,b}(f)(z) \quad \left(D := \frac{d}{dz}\right),\end{aligned} \tag{2.4}$$

therefore,

$$J_{-n,b}(f)(z) = \frac{1}{(1+b)}(zD+b)J_{-n+1,b}(f)(z). \tag{2.5}$$

Noting that the relation (2.5) is a recurrence relation, by using mathematical induction, we get (2.3), which completes the proof of the lemma. □

Putting $f(z) = f_0(z) = z/(1-z)$ in Lemma 2.1, we obtain the following properties for both Hurwitz-Lerch Zeta function $\Phi(z,s,b)$ and Polylogarithmic function $Li_s(z)$.

Corollary 2.2. *Let* $\Phi(z,s,b)$ *and* $Li_s(z)$ *be the Hurwitz-Lerch Zeta function and Polylogarithmic function defined by* (1.3) *and* (1.4)*, respectively, then we have*

$$\begin{aligned}\Phi(z,-n,b) &= b^n + \left(z\frac{d}{dz}+b\right)^n\left(\frac{z}{1-z}\right) \quad (|z|<1), \\ Li_{-n}(z) &= z\left\{1 + \left(z\frac{d}{dz}+1\right)^n\left(\frac{z}{1-z}\right)\right\} \quad (|z|<1),\end{aligned} \tag{2.6}$$

where $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$ *and* $n \in \mathbb{N}_0$.

Example 2.3. Using Corollary 2.2, we have the following well known results for $z(z \in \mathbb{C}; |z| < 1)$.

(i) $\Phi(z,0,b) = 1/(1-z)$.

(ii) $\Phi(z,-1,b) = b + ((1+b)z - bz^2)/(1-z)^2$.

(iii) $\Phi(z,-2,b) = b^2 + ((1+b)^2 z + (1-2b-2b^2)z^2 + b^2z^3)/(1-z)^3$.

(iv) $Li_0(z) = z/(1-z)$.

(v) $Li_{-1}(z) = z/(1-z)^2$.

(vi) $Li_{-2}(z) = z(1+z)/(1-z)^3$.

3. Applications of Differential Subordination for $J_{s,b}(f)$

To prove our results, we need the following lemmas due to Hallenbeck and Ruscheweyh [28] and Miller and Mocanu [29], respectively, see also Miller and Mocanu [30].

Lemma 3.1. *Let $h(z)$ be convex univalent in $\mathbb{U}$, with $h(0) = 1$, $\gamma \neq 0$ and $\mathrm{Re}(\gamma) \geq 0$. If $q(z) \in \mu$ and*

$$q(z) + \frac{zq'(z)}{\gamma} \prec h(z), \tag{3.1}$$

then

$$q(z) \prec S(z) \prec h(z), \tag{3.2}$$

where

$$S(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t) t^{\gamma-1} dt. \tag{3.3}$$

The function $S(z)$ is convex univalent and is the best dominant.

Lemma 3.2. *Let $\lambda > 0$, and let $\beta_0 = \beta_0(\lambda)$ be the root of the equation as follows:*

$$\beta\pi = \frac{3\pi}{2} - \tan^{-1}(\lambda\beta). \tag{3.4}$$

In addition, let $\alpha = \alpha(\beta, \lambda) = \beta + (2/\pi)\tan^{-1}(\lambda\pi)$, for $0 < \beta \leq \beta_0$.
If $p(z) \in \mu$ and

$$p(z) + \lambda z p'(z) \prec \left[\frac{1+z}{1-z}\right]^{\alpha} \tag{3.5}$$

then

$$p(z) \prec \left[\frac{1+z}{1-z}\right]^{\beta}. \tag{3.6}$$

Now, we define the function $L(f)(z) := L_{(s,b,\lambda)}(f)(z)$ as the following:

$$\begin{gathered} L(f)(z) = (1-\lambda-\lambda b) J_{s,b}(f)(z) + \lambda(1+b) J_{s-1,b}(f)(z) \quad (z \in \mathbb{U}), \\ \left(z \in \mathbb{U}; f \in A; b \in \mathbb{C} \setminus \mathbb{Z}^-; \{s, \lambda \in \mathbb{C}; \lambda \neq 0; \mathrm{Re}\,\lambda \geq 0\}\right). \end{gathered} \tag{3.7}$$

Theorem 3.3. *Let the function $L(f)(z)$ defined by* (3.7) *and for some $\alpha (0 \leq \alpha < 1)$. If*

$$\mathrm{Re}\left\{\frac{L(f)(z)}{z}\right\} > \alpha, \tag{3.8}$$

then

$$\operatorname{Re}\left\{\frac{J_{s,b}(f)(z)}{z}\right\} > (2\alpha-1)+2(1-\alpha)\,{}_2F_1\left(1,\frac{1}{\lambda};\frac{1}{\lambda}+1,-1\right). \tag{3.9}$$

The constant $(2\alpha-1)+2(1-\alpha)\ {}_2\mathrm{F}_1(1,1/\lambda;(1/\lambda)+1,-1)$ is the best estimate.

Proof. Defining the function $q(z)=J_{s,b}(f)(z)/z$, then we have $q(z)\in\mu$.

If we take $\gamma=1/\lambda$, and the convex univalent function $h(\mathrm{z})$ defined by

$$h(z)=\frac{1+(2\alpha-1)z}{1+z},\quad 0\le\alpha<1, \tag{3.10}$$

then, we have

$$q(z)+\frac{zq'(z)}{\gamma}=(1-\lambda)\frac{J_{s,b}(f)(z)}{z}+\lambda J'_{s,b}(f)(z). \tag{3.11}$$

Using Lemma 1.3 and (3.7), therefore (3.11) can be written as

$$q(z)+\frac{zq'(z)}{\gamma}=\frac{L(f)(z)}{z}, \tag{3.12}$$

then,

$$q(z)+\frac{zq'(z)}{\gamma}\prec h(z), \tag{3.13}$$

where $h(\mathrm{z})$ is defined by (3.10) satisfying $h(0)=1$.

Applying Lemma 3.1, we obtain that $J_{s,b}(f)(z)/z\prec S(z)$, where the convex univalent function $S(z)$ defined by

$$S(z)=\frac{1}{\lambda z^{1/\lambda}}\int_0^z\frac{1+(2\alpha-1)t}{1+t}t^{((1/\lambda)-1)}dt. \tag{3.14}$$

Since $\operatorname{Re}\{h(\mathrm{z})\}>0$ and $S(z)\prec h(z)$, we have $\operatorname{Re}\{S(z)\}>0$.

This implies that

$$\begin{aligned}\inf_{z\in\mathbb{U}}\operatorname{Re}\{S(z)\}=S(1)&=(2\alpha-1)+\frac{2}{\lambda}(1-\alpha)\int_0^1\frac{u^{((1/\lambda)-1)}}{1+u}du\\&=(2\alpha-1)+2(1-\alpha)\int_0^1\frac{dt}{1+t^{\lambda}}\\&=(2\alpha-1)+2(1-\alpha)\,{}_2F_1\left(1,\frac{1}{\lambda};\frac{1}{\lambda}+1,-1\right).\end{aligned} \tag{3.15}$$

Hence, the constant $(2\alpha-1)+2(1-\alpha)\ {}_2\mathrm{F}_1(1,1/\lambda;(1/\lambda)+1,-1)$ cannot be replace by any larger one.

This completes the proof of Theorem 3.3. □

Theorem 3.4. *Let the function $L(f)(z)$ with $\lambda > 0$; real, defined by* (3.7), *and let β_0 satisfy the following equation:*

$$\beta_0\pi + \tan^{-1}\left(\frac{\beta_0}{2}\right) = \frac{3\pi}{2}. \tag{3.16}$$

If

$$\frac{L(f)(z)}{z} \prec \left[\frac{1+z}{1-z}\right]^{\beta+(2/\pi)\tan^{-1}(\lambda\beta)}, \tag{3.17}$$

then

$$\frac{J_{s,b}(f)(z)}{z} \prec \left[\frac{1+z}{1-z}\right]^{\beta} \quad (0<\beta\le\beta_0). \tag{3.18}$$

Proof. Defining the function $p(z) = J_{s,b}(f)(z)/z \in \mu$, then we have

$$p(z)+\lambda zp'(z) = (1-\lambda)\frac{J_{s,b}(f)(z)}{z} + \lambda J'_{s,b}(f)(z). \tag{3.19}$$

Using Lemma 1.3 and (3.7), therefore (3.11) can be written as

$$p(z)+\lambda zp'(z) = \frac{L(f)(z)}{z}. \tag{3.20}$$

This completes the proof of Theorem 3.4 after applying Lemma 3.2 □

4. Applications in Analytic Number Theory

Putting $f(z) = f_0(z) = z/(1-z)$ in Theorem 3.3, then we have the following property of Hurwitz-Lerch Zeta function.

Corollary 4.1. *Let the function $G_{s,b}(z)$ defined by* (1.6). *If*

$$\mathrm{Re}\left\{\frac{(1-\lambda-\lambda b)G_{s,b}(z)+\lambda(1+b)G_{s-1,b}(z)}{z}\right\} > \alpha, \tag{4.1}$$

then

$$\mathrm{Re}\left\{\frac{G_{s,b}(z)}{z}\right\} > (2\alpha-1)+2(1-\alpha)\ {}_2F_1\left(1,\frac{1}{\lambda};\frac{1}{\lambda}+1,-1\right), \tag{4.2}$$

where $z\in\mathbb{U}$, $0\le\alpha<1$, $b\in\mathbb{C}\setminus\mathbb{Z}^-$ and $\{s,\lambda\in\mathbb{C};\lambda\ne0;\ \mathrm{Re}\ \lambda\ge0\}$.

The constant $(2\alpha - 1) + 2(1 - \alpha)\ {}_2\mathrm{F}_1(1, 1/\lambda; (1/\lambda) + 1, -1)$ is the best estimate.

Putting $f(z) = f_0(z) = z/(1 - z)$ in Theorem 3.4, then we have another property of Hurwitz-Lerch Zeta function.

Corollary 4.2. *Let the function* $G_{s,b}(z)$ *defined by* (1.6)*, and let* β_0 *satisfy the following equation:*

$$\beta_0\pi + \tan^{-1}(\lambda\beta_0) = \frac{3\pi}{2}. \tag{4.3}$$

If

$$\frac{(1 - \lambda - \lambda b)G_{s,b}(z) + \lambda(1 + b)G_{s-1,b}(z)}{z} \prec \left[\frac{1 + z}{1 - z}\right]^{\beta + (2/\pi)\tan^{-1}(\lambda\beta)}, \tag{4.4}$$

then

$$\frac{G_{s,b}(z)}{z} \prec \left[\frac{1 + z}{1 - z}\right]^{\beta} \quad (0 < \beta \leq \beta_0), \tag{4.5}$$

where $z \in \mathbb{U}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$, $s \in \mathbb{C}$ *and* $\lambda > 0$*; real.*

Putting $f(z) = f_0(z) = z/(1 - z)$ and $b = 1$ in Theorem 3.3, then we have the following property of Polylogarithmic function.

Corollary 4.3. *Let the function* $H_s(z)$ *defined by*

$$H_s(z) = 2^s\left[\frac{Li_s(z)}{z} - 1\right]. \tag{4.6}$$

If

$$\operatorname{Re}\left\{\frac{(1 - 2\lambda)H_s(z) + 2\lambda\, H_{s-1}(z)}{z}\right\} > \alpha, \tag{4.7}$$

then

$$\operatorname{Re}\left\{\frac{H_s(z)}{z}\right\} > (2\alpha - 1) + 2(1 - \alpha)\ {}_2F_1\left(1, \frac{1}{\lambda}; \frac{1}{\lambda} + 1, -1\right), \tag{4.8}$$

where $z \in \mathbb{U}$, $0 \leq \alpha < 1$ *and* $\{s, \lambda \in \mathbb{C};\ \lambda \neq 0;\ \operatorname{Re}\lambda \geq 0\}$.

The constant $(2\alpha - 1) + 2(1 - \alpha)\,{}_2F_1(1, 1/\lambda; (1/\lambda) + 1, -1)$ *is the best estimate.*

Putting $f(z) = f_0(z) = z/(1 - z)$ and $b = 1$ in Theorem 3.4, then we have the following property of Polylogarithmic function.

Corollary 4.4. *Let the functions $G_{s,b}(z)$ and $H_s(z)$ defined by* (1.6) *and* (4.6)*, respectively, and let β_0 satisfy the following:*

$$\beta_0\pi + \tan^{-1}(\lambda\beta_0) = \frac{3\pi}{2}. \tag{4.9}$$

If

$$\frac{(1-2\lambda)H_s(z) + 2\lambda H_{s-1}(z)}{z} \prec \left[\frac{1+z}{1-z}\right]^{\beta+(2/\pi)\tan^{-1}(\lambda\beta)}, \tag{4.10}$$

then

$$\frac{G_{s,b}(z)}{z} \prec \left[\frac{1+z}{1-z}\right]^{\beta} \quad (0 < \beta \leq \beta_0), \tag{4.11}$$

where $z \in \mathbb{U}$, $s \in \mathbb{C}$ and $\lambda > 0$; real.

Setting $f(z) = f_0(z) = z/(1-z)$, $b = 1$ and $\lambda = 1/2$ in Theorem 3.3, then we have the following property of Polylogarithmic function.

Corollary 4.5. *Let the function $H_s(z)$ defined by* (4.6).

If

$$\operatorname{Re}\left\{\frac{H_{s-1}(z)}{z}\right\} > \alpha, \tag{4.12}$$

then

$$\operatorname{Re}\left\{\frac{H_s(z)}{z}\right\} > 2(2\ln 2 - 1)\alpha + (3 - 4\ln 2), \tag{4.13}$$

where $z \in \mathbb{U}$, $0 \leq \alpha < 1$ and $s \in \mathbb{C}$.

The constant $2(2\ln 2 - 1)\alpha + (3 - 4\ln 2)$ is the best estimate.

Taking $f(z) = f_0(z) = z/(1-z)$, $b = 1$ and $\lambda = 1/2$ in Theorem 3.4, then we have the following property of polylogarithmic function.

Corollary 4.6. *Let the function $H_s(z)$ defined by* (4.6).

If

$$\frac{H_{s-1}(z)}{z} \prec \left[\frac{1+z}{1-z}\right]^{\beta+(2/\pi)\tan^{-1}(\beta)}, \tag{4.14}$$

then

$$\frac{H_s(z)}{z} \prec \left[\frac{1+z}{1-z}\right]^{\beta} \quad (0 < \beta \leq 1.3148754023\ldots), \tag{4.15}$$

where $z \in \mathbb{U}$ and $s \in \mathbb{C}$.

Corollary 4.7. *Let the function $H_s(z)$ defined by* (4.6) *as follows:*

If

$$\frac{H_{s-1}(z)}{z} \prec \left[\frac{1+z}{1-z}\right]^{3/2}, \tag{4.16}$$

then

$$\operatorname{Re}\left\{\frac{H_{s+n}(z)}{z}\right\} > 1-(4\ln 2-2)^n \quad (n \in \mathbb{N}_0), \tag{4.17}$$

where $z \in \mathbb{U}$ and $s \in \mathbb{C}$.

Proof. Let $H_{s-1}(z)$ satisfy the condition (4.16). Also, putting $f(z) = f_0(z) = z/(1-z)$, $b = 1$, $\lambda = 1/2$ and $\beta = 1$ in Theorem 3.4.

Using (4.16), then we have

$$\frac{H_s(z)}{z} \prec \left[\frac{1+z}{1-z}\right], \tag{4.18}$$

therefore

$$\operatorname{Re}\left\{\frac{H_s(z)}{z}\right\} > 0. \tag{4.19}$$

Corollary 4.5, gives

$$\operatorname{Re}\left\{\frac{H_{s+1}(z)}{z}\right\} > 3-4\ln 2. \tag{4.20}$$

Applied (4.11) again and to n-times, which gives (4.17). This completes the proof of Corollary 4.7. □

Finally, we can put Corollary 4.7 in the following form.

Corollary 4.8. *Let the function $H_s(z)$ defined by* (4.6)*.*

If

$$\left|\operatorname{Arg}\left(\frac{H_{s-1}(z)}{z}\right)\right| < \frac{3\pi}{4}, \tag{4.21}$$

then

$$\operatorname{Re}\left\{\frac{H_{s+n}(z)}{z}\right\} > 1-(4\ln 2-2)^n \quad (n \in \mathbb{N}_0), \tag{4.22}$$

where $z \in \mathbb{U}$ and $s \in \mathbb{C}$.

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