

Research Article

Explicit Formulas Involving q -Euler Numbers and Polynomials

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We deal with q -Euler numbers and q -Bernoulli numbers. We derive some interesting relations for q -Euler numbers and polynomials by using their generating function and derivative operator. Also, we derive relations between the q -Euler numbers and q -Bernoulli numbers via the p -adic q -integral in the p -adic integer ring.

1. Preliminaries

Imagine that p is a fixed odd prime number. Throughout this paper we use the following notations, where $\mathbb{Z}_p$ denotes the ring of p -adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p -adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

The p -adic absolute value is defined by

$$|p|_p = \frac{1}{p}. \quad (1.1)$$

In this paper, we will assume that $|q - 1|_p < 1$ as an indeterminate. $[x]_q$ is a q -extension of x , which is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}. \quad (1.2)$$

We note that $\lim_{q \rightarrow 1} [x]_q = x$ (see [1–12]).

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y}, \quad (1.3)$$

has a limit $f'(a)$ as $(x, y) \rightarrow (a, a)$ and denote this by $f \in \text{UD}(\mathbb{Z}_p)$.

Let $\text{UD}(\mathbb{Z}_p)$ be the set of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in \text{UD}(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \leq \xi < p^N} f(\xi) q^\xi = \sum_{0 \leq \xi < p^N} f(\xi) \mu_q(\xi + p^N \mathbb{Z}_p), \quad (1.4)$$

which represents p -adic q -analogue of Riemann sums for f . The integral of f on $\mathbb{Z}_p$ will be defined as the limit ($N \rightarrow \infty$) of these sums, when it exists. The p -adic q -integral of function $f \in \text{UD}(\mathbb{Z}_p)$ is defined by Kim

$$I_q(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_q(\xi) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{\xi=0}^{p^N-1} f(\xi) q^\xi. \quad (1.5)$$

The bosonic integral is considered as a bosonic limit $q \rightarrow 1$, $I_1(f) = \lim_{q \rightarrow 1} I_q(f)$. Similarly, the fermionic p -adic integral on $\mathbb{Z}_p$ is introduced by Kim as follows:

$$I_{-q}(f) = \lim_{q \rightarrow -q} I_q(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_{-q}(\xi) \quad (1.6)$$

(for more details, see [9–12]).

In [6], the q -Euler polynomials with weight 0 are introduced as

$$\tilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x + y)^n d\mu_{-q}(y). \quad (1.7)$$

From (1.7), we have

$$\tilde{E}_{n,q}(x) = \sum_{l=0}^n \binom{n}{l} x^l \tilde{E}_{n-l,q}, \quad (1.8)$$

where $\tilde{E}_{n,q}(0) = \tilde{E}_{n,q}$ are called q -Euler numbers with weight 0. Then, q -Euler numbers are defined as

$$q(\tilde{E}_q + 1)^n + \tilde{E}_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases} \quad (1.9)$$

where the usual convention about replacing $(\tilde{E}_q)^n$ by $\tilde{E}_{n,q}$ is used.

Similarly, the q -Bernoulli polynomials and numbers with weight 0 are defined, respectively, as

$$\begin{aligned}\tilde{B}_{n,q}(x) &= \lim_{n \rightarrow \infty} \frac{1}{[p^n]_q} \sum_{y=0}^{p^n-1} (x+y)^n q^y \\ &= \int_{\mathbb{Z}_p} (x+y)^n d\mu_q(y), \\ \tilde{B}_{n,q} &= \int_{\mathbb{Z}_p} y^n d\mu_q(y)\end{aligned}\tag{1.10}$$

(for more information, see [4]).

We, by using the Kim et al. method in [2], will investigate some interesting identities on the q -Euler numbers and polynomials arising from their generating function and derivative operator. Consequently, we derive some properties on the q -Euler numbers and polynomials and q -Bernoulli numbers and polynomials by using q -Volkenborn integral and fermionic p -adic q -integral on $\mathbb{Z}_p$.

2. On the q -Euler Numbers and Polynomials

Let us consider Kim's q -Euler polynomials as follows:

$$F_x^q = F_x^q(t) = \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!}.\tag{2.1}$$

Here x is a fixed parameter. Thus, by expression of (2.1), we can readily see the following:

$$qe^t F_x^q + F_x^q = [2]_q e^{xt}.\tag{2.2}$$

Last from equality, taking derivative operator D as $D = d/dt$ on the both sides of (2.2). Then, we easily see that

$$qe^t (D + I)^k F_x^q + D^k F_x^q = [2]_q x^k e^{xt},\tag{2.3}$$

where $k \in \mathbb{N}^*$ and I is identity operator. By multiplying e^{-t} on both sides of (2.3), we get

$$q(D + I)^k F_x^q + e^{-t} D^k F_x^q = [2]_q x^k e^{(x-1)t}.\tag{2.4}$$

Let us take derivative operator D^m ($m \in \mathbb{N}$) on both sides of (2.4). Then we get

$$qe^t D^m (D + I)^k F_x^q + D^k (D - I)^m F_x^q = [2]_q x^k (x - 1)^m e^{xt}.\tag{2.5}$$

Let $G[0]$ (not $G(0)$) be the constant term in a Laurent series of $G(t)$. Then, from (2.5), we get

$$\sum_{j=0}^k \binom{k}{j} \left(q e^t D^{k+m-j} F_x^q(t) \right) [0] + \sum_{j=0}^m \binom{m}{j} (-1)^j \left(D^{k+m-j} F_x^q(t) \right) [0] = [2]_q x^k (x-1)^m. \quad (2.6)$$

By (2.1), we see

$$\left(D^N F_x^q(t) \right) [0] = \tilde{E}_{N,q}(x), \quad \left(e^t D^N F_x^q(t) \right) [0] = \tilde{E}_{N,q}(x). \quad (2.7)$$

By expressions of (2.6) and (2.7), we see that

$$\sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \tilde{E}_{k+m-j,q}(x) = [2]_q x^k (x-1)^m. \quad (2.8)$$

From (2.1), we note that

$$\frac{d}{dx} \left(\tilde{E}_{n,q}(x) \right) = n \sum_{l=0}^{n-1} \binom{n-1}{l} \tilde{E}_{l,q} x^{n-1-l} = n \tilde{E}_{n-1,q}(x). \quad (2.9)$$

By (2.9), we easily see

$$\int_0^1 \tilde{E}_{n,q}(x) dx = \frac{\tilde{E}_{n+1,q}(1) - \tilde{E}_{n+1,q}}{n+1} = -\frac{[2]_{q^{-1}}}{n+1} \tilde{E}_{n+1,q}. \quad (2.10)$$

Now, let us consider definition of integral from 0 to 1 in (2.8), then we have

$$\begin{aligned} & -[2]_{q^{-1}} \sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \frac{\tilde{E}_{k+m-j+1,q}}{k+m-j+1} \\ &= [2]_q (-1)^m B(k+1, m+1) \\ &= [2]_q (-1)^m \frac{\Gamma(k+1) \Gamma(m+1)}{\Gamma(k+m+2)}, \end{aligned} \quad (2.11)$$

where $B(m, n)$ is beta function which is defined by

$$\begin{aligned} B(m, n) &= \int_0^1 x^{m-1} (1-x)^{n-1} dx \\ &= \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}, \quad m > 0, \quad n > 0. \end{aligned} \quad (2.12)$$

As a result, we obtain the following theorem.

Theorem 2.1. For $n \in \mathbb{N}$, one has

$$\begin{aligned} & \sum_{j=1}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \frac{\tilde{E}_{k+m-j+1,q}}{k+m-j+1} \\ &= q \frac{(-1)^{m+1}}{(k+m+1) \binom{k+m}{k}} - [2]_q \frac{\tilde{E}_{k+m+1,q}}{k+m+1}. \end{aligned} \quad (2.13)$$

Substituting $m = k + 1$ into Theorem 2.1, we readily get

$$\begin{aligned} & \sum_{j=1}^{k+1} \left[q \binom{k}{j} + (-1)^j \binom{k+1}{j} \right] \frac{\tilde{E}_{2k+2-j,q}}{2k+2-j} \\ &= q \frac{(-1)^k}{(2k+2) \binom{2k+1}{k}} - [2]_q \frac{\tilde{E}_{2k+2,q}}{2k+2}. \end{aligned} \quad (2.14)$$

By (2.1), it follows that

$$\begin{aligned} & \sum_{j=0}^{\max\{k,m\}} (k+m-j) \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \tilde{E}_{k+m-j-1,q}(x) \\ &= [2]_q x^{k-1} (x-1)^{m-1} ((k+m)x - k). \end{aligned} \quad (2.15)$$

Let $m = k$ in (2.1), we see that

$$\sum_{j=0}^k \left[q \binom{k}{j} + (-1)^j \binom{k}{j} \right] \tilde{E}_{2k-j,q}(x) = [2]_q x^k (x-1)^k. \quad (2.16)$$

Last from equality, we discover the following:

$$[2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \tilde{E}_{2k-2j,q}(x) + (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \tilde{E}_{2k-2j-1,q}(x) = [2]_q x^k (x-1)^k. \quad (2.17)$$

Here $[\cdot]$ is Gauss' symbol. Then, taking integral from 0 to 1 in both sides of last equality, we get

$$\begin{aligned} & -[2]_{q^{-1}} [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \frac{\tilde{E}_{2k-2j+1,q}}{2k-2j+1} + [2]_{q^{-1}} (1-q) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \frac{\tilde{E}_{2k-2j,q}}{2k-2j} \\ &= [2]_q (-1)^k B(k+1, k+1) \\ &= \frac{[2]_q (-1)^k}{(2k+1) \binom{2k}{k}}. \end{aligned} \quad (2.18)$$

Consequently, we derive the following theorem.

Theorem 2.2. *The following identity*

$$\begin{aligned} & [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \frac{\tilde{E}_{2k-2j+1,q}}{2k-2j+1} + (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \frac{\tilde{E}_{2k-2j,q}}{2k-2j} \\ &= \frac{q(-1)^{k+1}}{(2k+1) \binom{2k}{k}} \end{aligned} \quad (2.19)$$

is true.

In view of (2.1) and (2.17), we discover the following applications:

$$\begin{aligned} &= \sum_{j=0}^{k+1} \left[q \binom{k}{j} + (-1)^j \binom{k+1}{j} \right] \tilde{E}_{2k+1-j,q}(x) \\ &= [2]_q \tilde{E}_{2k+1,q}(x) + \sum_{j=1}^{[(k+1)/2]} \left[q \binom{k}{2j} + \binom{k}{2j} + \binom{k}{2j-1} \right] \tilde{E}_{2k+1-2j,q}(x) \\ &\quad + \sum_{j=0}^{[(k+1)/2]} \left[q \binom{k}{2j+1} - \binom{k}{2j+1} - \binom{k}{2j} \right] \tilde{E}_{2k-2j,q}(x) \\ &= - \left[\sum_{j=0}^{[k/2]} \binom{k}{2j} \tilde{E}_{2k-2j,q}(x) + \frac{q-1}{1+q} \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \tilde{E}_{2k-2j+1}(x) \right] \\ &\quad + [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \tilde{E}_{2k+1-2j,q}(x) + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \tilde{E}_{2k+1-2j,q}(x) \\ &\quad + (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \tilde{E}_{2k-2j,q}(x) + \frac{q-1}{1+q} \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \tilde{E}_{2k-2j+1}(x). \end{aligned} \quad (2.20)$$

By expressions (2.17) and (2.20), we have the following theorem.

Theorem 2.3. *For $k \in \mathbb{N}$, one has*

$$\begin{aligned} & [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \tilde{E}_{2k+1-2j,q}(x) + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \tilde{E}_{2k+1-2j,q}(x) \\ &\quad + (q-1) \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left[\tilde{E}_{2k-2j,q}(x) + \frac{1}{1+q} \tilde{E}_{2k-2j+1}(x) \right] \\ &= x^k (x-1)^k ([2]_q x - q). \end{aligned} \quad (2.21)$$

3. p -adic Integral on $\mathbb{Z}_p$ Associated with Kim's q -Euler Polynomials

In this section, we consider Kim's q -Euler polynomials by means of p -adic q -integral on $\mathbb{Z}_p$. Now we start with the following assertion.

Let $m, k \in \mathbb{N}$. Then by (2.8),

$$\begin{aligned} I_1 &= [2]_q \int_{\mathbb{Z}_p} x^k (x-1)^m d\mu_{-q}(x) \\ &= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_p} x^{l+k} d\mu_{-q}(x) \\ &= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \tilde{E}_{l+k,q}. \end{aligned} \quad (3.1)$$

On the other hand, in right hand side of (2.8),

$$\begin{aligned} I_2 &= \sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\ &= \sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{E}_{l,q}. \end{aligned} \quad (3.2)$$

Equating I_1 and I_2 , we get the following theorem.

Theorem 3.1. For $m, k \in \mathbb{N}$, one has

$$\begin{aligned} &\sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{E}_{l,q} \\ &= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \tilde{E}_{l+k,q}. \end{aligned} \quad (3.3)$$

Let us take fermionic p -adic q -integral on $\mathbb{Z}_p$ in left hand side of (2.21), we get

$$\begin{aligned} I_3 &= \int_{\mathbb{Z}_p} x^k (x-1)^k ([2]_q x - q) d\mu_{-q}(x) \\ &= [2]_q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l+1} d\mu_{-q}(x) - q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_{-q}(x) \\ &= [2]_q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \tilde{E}_{k+l+1,q} - q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \tilde{E}_{k+l,q}. \end{aligned} \quad (3.4)$$

In other words, we consider right hand side of (2.21) as follows:

$$\begin{aligned}
I_4 &= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\
&\quad + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\
&\quad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left[\begin{aligned} &(q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \\ &+ \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \int_{\mathbb{Z}_p} x^l d\mu_{-q}(x) \end{aligned} \right] \\
&= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{E}_{l,q} \\
&\quad + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{E}_{l,q} \\
&\quad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left[\begin{aligned} &(q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \tilde{E}_{l,q} \\ &+ \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \tilde{E}_{l,q} \end{aligned} \right]. \tag{3.5}
\end{aligned}$$

Equating I_3 and I_4 , we get the following theorem.

Theorem 3.2. For $k \in \mathbb{N}$, one has

$$\begin{aligned}
&\sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \{ [2]_q \tilde{E}_{k+l+1,q} - q \tilde{E}_{k+l,q} \} \\
&= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{E}_{l,q} \\
&\quad + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{E}_{l,q} \\
&\quad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left\{ \begin{aligned} &(q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \tilde{E}_{l,q} \\ &+ \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \tilde{E}_{l,q} \end{aligned} \right\}. \tag{3.6}
\end{aligned}$$

Now, we consider (2.8) and (2.1) by means of q -Volkenborn integral. Then, by (2.8), we see

$$\begin{aligned}
 [2]_q \int_{\mathbb{Z}_p} x^k (x-1)^m d\mu_q(x) \\
 &= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \int_{\mathbb{Z}_p} x^{l+k} d\mu_q(x) \\
 &= [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \tilde{B}_{l+k,q}.
 \end{aligned} \tag{3.7}$$

On the other hand,

$$\begin{aligned}
 \sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\
 = \sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{B}_{l,q}.
 \end{aligned} \tag{3.8}$$

Therefore, we get the following theorem.

Theorem 3.3. For $m, k \in \mathbb{N}$, one has

$$\begin{aligned}
 [2]_q \sum_{l=0}^m \binom{m}{l} (-1)^{m-l} \tilde{B}_{l+k,q} \\
 = \sum_{j=0}^{\max\{k,m\}} \left[q \binom{k}{j} + (-1)^j \binom{m}{j} \right] \sum_{l=0}^{k+m-j} \binom{k+m-j}{l} \tilde{E}_{k+m-j-l,q} \tilde{B}_{l,q}.
 \end{aligned} \tag{3.9}$$

By using fermionic p -adic q -integral on $\mathbb{Z}_p$ in left hand side of (2.21), we get

$$\begin{aligned}
 I_5 &= [2]_q \int_{\mathbb{Z}_p} x^k (x-1)^k ([2]x - q) d\mu_q(x) \\
 &= [2]_q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l+1} d\mu_q(x) - q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_q(x) \\
 &= [2]_q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \tilde{B}_{k+l+1,q} - q \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \tilde{B}_{k+l,q}.
 \end{aligned} \tag{3.10}$$

Also, we consider right hand side of (2.21) as follows:

$$\begin{aligned}
I_6 &= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\
&\quad + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\
&\quad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left[\begin{aligned} &(q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \\ &+ \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \int_{\mathbb{Z}_p} x^l d\mu_q(x) \end{aligned} \right] \quad (3.11) \\
&= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{B}_{l,q} \\
&\quad + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{B}_{l,q} \\
&\quad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left[\begin{aligned} &(q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \tilde{B}_{l,q} \\ &+ \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \tilde{B}_{l,q} \end{aligned} \right].
\end{aligned}$$

Equating I_5 and I_6 , we get the following corollary.

Corollary 3.4. *For $k \in \mathbb{N}$, one gets*

$$\begin{aligned}
&\sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \{ [2]_q \tilde{B}_{k+l+1,q} - q \tilde{B}_{k+l,q} \} \\
&= [2]_q \sum_{j=0}^{[k/2]} \binom{k}{2j} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{B}_{l,q} \\
&\quad + \sum_{j=1}^{[k/2]} \binom{k}{2j-1} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k+1-2j-l,q} \tilde{B}_{l,q} \quad (3.12) \\
&\quad + \sum_{j=0}^{[k/2]} \binom{k}{2j+1} \left\{ \begin{aligned} &(q-1) \sum_{j=0}^{2k-2j} \binom{2k-2j}{l} \tilde{E}_{2k-2j-l,q} \tilde{B}_{l,q} \\ &+ \frac{q-1}{1+q} \sum_{l=0}^{2k-2j+1} \binom{2k-2j+1}{l} \tilde{E}_{2k-2j-l+1} \tilde{B}_{l,q} \end{aligned} \right\}.
\end{aligned}$$

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