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Research Article

Some Properties of Solutions for the Sixth-Order Cahn-Hilliard-Type Equation

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We study the initial boundary value problem for a sixth-order Cahn-Hilliard-type equation which describes the separation properties of oil-water mixtures, when a substance enforcing the mixing of the phases is added. We show that the solutions might not be classical globally. In other words, in some cases, the classical solutions exist globally, while in some other cases, such solutions blow up at a finite time. We also discuss the existence of global attractor.

1. Introduction

We consider the following equation:

$$u_t = \gamma \Delta^3 u + \Delta \left[-a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + f(u) \right], \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \le 3)$ with smooth boundary and $\gamma > 0$. Equation (1.1) is supplemented by the boundary value conditions:

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial \Delta u}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial \Delta^2 u}{\partial n}\right|_{\partial\Omega} = 0, \quad t > 0, \tag{1.2}$$

and the initial value condition:

$$u(x,0) = u_0(x). \tag{1.3}$$

Equation (1.1) describes dynamics of phase transitions in ternary oil-water-surfactant systems [1–3]. The surfactant has a character that one part of it is hydrophilic and the other lipophilic is called amphiphile. In the system, almost pure oil, almost pure water, and microemulsion which consist of a homogeneous, isotropic mixture of oil and water can coexist in equilibrium. Here $u(x,t)$ is the scalar order parameter which is proportional to the local difference between oil and water concentrations. The amphiphile concentration $a(u)$ is approximated by the quadratic function [1]

$$a(u) = a_2 u^2 + a_0. \tag{1.4}$$

From the physical consideration, we prefer to consider a typical case of the volumetric free energy $F(u)$, that is, $F'(u) = f(u)$, in the following form:

$$F(u) = \int_0^u f(s)ds = \gamma_1 (u+1)^2 \left(u^2 + h_0\right)(u-1)^2. \tag{1.5}$$

During the past years, many authors have paid much attention to the sixth-order-parabolic equation, such as the existence, uniqueness, and regularity of the solutions [4–8]. However, as far as we know, there are few investigations concerned with the sixth order Cahn-Hilliard equation. Pawłow and Zajączkowski [9] proved that the initial-boundary value problem (1.1)–(1.5) with $\gamma_1 = 1$ admits a unique global smooth solution which depends continuously on the initial datum. Schimperna and Pawłow [10] studied (1.1) with viscous term Δu_t and logarithmic potential:

$$F(r) = (1-r)\log(1-r) + (1+r)\log(1+r) - \frac{\lambda}{2}r^2, \quad \lambda > 0. \tag{1.6}$$

They investigated the behavior of the solutions to the sixth-order system as the parameter γ tends to 0. The uniqueness and regularization properties of the solutions have been discussed. Liu studied the following equation:

$$\frac{\partial u}{\partial t} - \operatorname{div}\left[m(u)\left(k\nabla\Delta^2 u + \nabla\left(-a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + f(u)\right)\right)\right] = 0, \tag{1.7}$$

and he proved the existence of classical solutions for two dimensions [11]. Korzec et al. [12] established the stationary solutions of the sixth-order convective Cahn-Hilliard type equation, which arose from epitaxial growth of nanostructures on crystal surfaces.

The dynamic properties of (1.1), such as the global asymptotical behaviors of solutions and existence of global attractors, are important for the study of higher order parabolic system. During the past years, many authors have paid much attention to the attractors. Nicolaenko et al. [13] proved the existence of global compact and finite-dimensional attractors for Cahn-Hilliard equation (see also [14–16]).

In this paper, we consider the problem (1.1)–(1.3). The purpose of the present paper is devoted to the investigation of properties of solutions with γ_1 not restricted to be positive. We first discuss the regularity. We show that the solutions might not be classical globally. In other words, in some cases, the classical solutions exist globally, while in some other cases, such solutions blow up at a finite time. We also discuss the existence of global attractor. We will use the regularity estimates for the linear semigroups, combining with the iteration technique and the classical existence theorem of global attractors, to prove that the problem (1.1)–(1.3) possesses a global attractor in H^k $(k \geq 0)$ space.

The plan of the paper is as follows. In Section 2, we investigate the global existence of the solution when $\gamma_1 > 0$. The blowup of the solution is obtained in Section 3 when $\gamma_1 < 0$. In Section 4, we obtain the existence of the global attractor in $H^k(k \geq 0)$ space.

Throughout the paper, we use Q_T to denote $\Omega \times (0, T)$, and

$$H^{6,1}(Q_T) = \left\{ u;\ \frac{\partial u}{\partial t} \in L^2(Q_T),\ D^i u \in L^2(Q_T),\ 0 \leq i \leq 6 \right\}. \tag{1.8}$$

The norms of $L^\infty(\Omega)$, $L^2(\Omega)$, and $H^s(\Omega)$ are denoted by $\|\cdot\|_\infty$, $\|\cdot\|$, and $\|\cdot\|_s$.

2. Global Existence

Now, we deal with problem (1.1)–(1.3) for $n = 2, 3$. The one-dimensional case is similar. From the classical approach, it is not difficult to conclude that the problem admits a unique classical solution local in time. So it is sufficient to make a priori estimates.

Theorem 2.1. *For the initial data $u_0 \in H^3(\Omega)$, $(\partial u_0/\partial n)|_{\partial\Omega} = 0$, and $T > 0$,*

(i) *if $\gamma_1 > 0$ and $a_2 > 0$, then the problem* (1.1)–(1.3) *exists a unique global solution $u \in H^{6,1}(Q_T)$;*

(ii) *if $\gamma_1 > 0$, $a_2 < 0$, and γ are sufficiently large, then the problem* (1.1)–(1.3) *also admits a unique global solution $u \in H^{6,1}(Q_T)$.*

Proof. (i) First, we set

$$E(u) = \int_\Omega \left(\frac{\gamma}{2}|\Delta u|^2 + \frac{a(u)}{2}|\nabla u|^2 + F(u) \right) dx. \tag{2.1}$$

Integrating by parts and using (1.1) itself and the boundary value condition (1.2), we see that

$$\begin{aligned} \frac{dE(t)}{dt} &= \int_\Omega \left[\gamma \Delta u \Delta u_t + a(u) \nabla u \nabla u_t + \frac{a'(u)}{2}|\nabla u|^2 u_t + f(u) u_t \right] dx \\ &= \int_\Omega \left[\gamma \Delta^2 u - a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + f(u) \right] u_t dx \\ &= -\int_\Omega \left[\gamma \nabla \Delta^2 u + \nabla\left(-a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + f(u) \right) \right]^2 dx \\ &\leq 0. \end{aligned} \tag{2.2}$$

This implies

$$E(t) \leq E(0), \tag{2.3}$$

where

$$E(0) = \int_\Omega \left(\frac{\gamma}{2} |\Delta u_0|^2 + \frac{a(u_0)}{2} |\nabla u_0|^2 + F(u_0) \right) dx. \tag{2.4}$$

On the other hand, we have

$$\int_\Omega |\nabla u|^2 dx \leq \varepsilon \int_\Omega |\Delta u|^2 dx + C(\varepsilon) \int_\Omega u^2 dx. \tag{2.5}$$

By Young's inequality, we derive

$$u^2 \leq \varepsilon u^6 + C_{1\varepsilon}, \qquad u^4 \leq \varepsilon u^6 + C_{2\varepsilon}. \tag{2.6}$$

Combining the above inequalities, we get

$$\sup_{0<t<T} \int_\Omega |\Delta u|^2 dx \leq C, \tag{2.7}$$

$$\sup_{0<t<T} \int_\Omega |\nabla u|^2 dx \leq C, \tag{2.8}$$

$$\sup_{0<t<T} \int_\Omega u^6 dx \leq C. \tag{2.9}$$

From (2.9), we know

$$\sup_{0<t<T} \int_\Omega u^2 dx \leq C. \tag{2.10}$$

By (2.7), (2.8), and (2.10), we obtain

$$\|u\|_{H^2} \leq C. \tag{2.11}$$

By Sobolev embedding theorem, it follows from (2.11) that

$$\begin{gathered} \|u\|_\infty \leq C, \quad (n = 2, 3), \\ \|\nabla u\|_{L^q} \leq C, \quad \text{for any } q < \infty, \ (n = 2), \\ \|\nabla u\|_{L^6} \leq C, \quad (n = 3). \end{gathered} \tag{2.12}$$

The second step: multiplying (1.1) by Δu and integrating with respect to x, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \gamma\int_\Omega \left|\Delta^2 u\right|^2 dx \\
&\quad = -\int_\Omega \Delta f(u)\Delta u\, dx + \int_\Omega a(u)\Delta u \Delta^2 u\, dx + \int_\Omega \frac{a'(u)}{2}|\nabla u|^2 \Delta^2 u\, dx.
\end{aligned} \tag{2.13}$$

Thus, it follows from (2.11) and (2.12) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \gamma\int_\Omega \left|\Delta^2 u\right|^2 dx \\
&\quad = -\int_\Omega \left(f'(u)\Delta u + f''(u)|\nabla u|^2\right)\Delta u\, dx + \int_\Omega \left(a_2 u^2 + a_0\right)\Delta u \Delta^2 u\, dx \\
&\qquad + \int_\Omega \frac{a'(u)}{2}|\nabla u|^2 \Delta^2 u\, dx \\
&\quad \le C\int_\Omega |\Delta u|^2 dx + C\int_\Omega |\nabla u|^4 dx + C\int_\Omega |\Delta u|^2 dx \\
&\qquad + C\int_\Omega |\Delta u|^2 dx + \frac{\gamma}{4}\int_\Omega \left|\Delta^2 u\right|^2 dx + C\int_\Omega |\nabla u|^4 dx + \frac{\gamma}{4}\int_\Omega \left|\Delta^2 u\right|^2 dx \\
&\quad \le \frac{\gamma}{2}\int_\Omega \left|\Delta^2 u\right|^2 dx + C.
\end{aligned} \tag{2.14}$$

By the Gronwall's inequality, (2.14) implies

$$\iint_{Q_T} \left|\Delta^2 u\right|^2 dx\, dt \le C. \tag{2.15}$$

The third step: multiplying (1.1) by $\Delta^2 u$ and integrating with respect to x, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta u|^2 dx + \gamma\int_\Omega \left|\nabla\Delta^2 u\right|^2 dx \\
&\quad = -\int_\Omega f'(u)\nabla u \nabla\Delta^2 u\, dx + \int_\Omega \nabla(a(u)\Delta u)\nabla\Delta^2 u\, dx \\
&\qquad + \int_\Omega \nabla\left(\frac{a'(u)}{2}|\nabla u|^2\right)\nabla\Delta^2 u\, dx \\
&\quad = -\int_\Omega f'(u)\nabla u \nabla\Delta^2 u\, dx + \int_\Omega a(u)\nabla\Delta u \nabla\Delta^2 u\, dx \\
&\qquad + 2\int_\Omega a'(u)\nabla u \Delta u \nabla\Delta^2 u\, dx + \frac{1}{2}\int_\Omega a''(u)|\nabla u|^2 \nabla u \nabla\Delta^2 u\, dx.
\end{aligned} \tag{2.16}$$

On the other hand, by the Nirenberg's inequality and (2.12), we have

$$\begin{aligned}
&\|\nabla u\|_\infty \le C\left\|\nabla \Delta^2 u\right\|^{a}\|\nabla u\|_{L^q}^{1-a}, \quad \text{where } a=\frac{1}{1+3q/2}, \ (n=2),\\
&\|\nabla u\|_\infty \le C\left\|\nabla \Delta^2 u\right\|^{1/6}\|\nabla u\|_{L^6}^{5/6}, \quad (n=3),\\
&\|\nabla \Delta u\| \le \left\|\nabla \Delta^2 u\right\|^{1/2}\|\nabla u\|^{1/2}, \quad (n=2),\\
&\|\nabla \Delta u\| \le \left\|\nabla \Delta^2 u\right\|^{1/3}\|\nabla u\|_{L^6}^{2/3}, \quad (n=3).
\end{aligned} \tag{2.17}$$

Using (2.7), (2.12), and the above inequality, we derive

$$\begin{gathered}
\left|-\int_\Omega f'(u)\nabla u\nabla \Delta^2 u\,dx\right| \le C\int_\Omega |\nabla u|^2 dx+\frac{\gamma}{8}\int_\Omega \left|\nabla \Delta^2 u\right|^2 dx,\\
\left|\int_\Omega a(u)\nabla \Delta u\nabla \Delta^2 u\,dx\right| \le C\int_\Omega |\nabla \Delta u|^2 dx+\frac{\gamma}{16}\int_\Omega \left|\nabla \Delta^2 u\right|^2 dx \le \frac{\gamma}{8}\int_\Omega \left|\nabla \Delta^2 u\right|^2 dx+C,\\
\left|2\int_\Omega a'(u)\nabla u\Delta u\nabla \Delta^2 u\,dx\right| \le C\|u\|_\infty\|\Delta u\|\left\|\nabla \Delta^2 u\right\| \le \frac{\gamma}{8}\int_\Omega \left|\nabla \Delta^2 u\right|^2 dx+C,\\
\left|\frac{1}{2}\int_\Omega a''(u)|\nabla u|^2\nabla u\nabla \Delta^2 u\,dx\right| \le C\int_\Omega |\nabla u|^6 dx+\frac{\gamma}{8}\int_\Omega \left|\nabla \Delta^2 u\right|^2 dx \le \frac{\gamma}{8}\int_\Omega \left|\nabla \Delta^2 u\right|^2 dx+C.
\end{gathered} \tag{2.18}$$

From these inequalities we finally arrive at

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta u|^2 dx+\gamma\int_\Omega \left|\nabla \Delta^2 u\right|^2 dx \le \frac{\gamma}{2}\int_\Omega \left|\nabla \Delta^2 u\right|^2 dx+C. \tag{2.19}$$

A Gronwall's argument now gives

$$\iint_{Q_T} \left|\nabla \Delta^2 u\right|^2 dx\,dt \le C. \tag{2.20}$$

Similarly, multiplying (1.1) by $\Delta^3 u$ and using

$$\begin{aligned}
&\|\nabla u\|_\infty \le \left\|\Delta^3 u\right\|^b \|\nabla u\|_{L^q}^{1-b}, \quad b = \frac{1}{1+2q}, \ (n=2),\\
&\|\nabla u\|_\infty \le \left\|\Delta^3 u\right\|^{1/8} \|\nabla u\|_{L^6}^{7/8}, \quad (n=3),\\
&\|\Delta u\|_\infty \le \left\|\Delta^3 u\right\|^{2/5} \|\nabla u\|^{3/5}, \quad (n=2),\\
&\|\Delta u\|_\infty \le \left\|\Delta^3 u\right\|^{3/8} \|\nabla u\|_{L^6}^{5/8}, \quad (n=3),\\
&\left\|\Delta^2 u\right\|_\infty \le \left\|\Delta^3 u\right\|^{4/5} \|\nabla u\|^{1/5}, \quad (n=2),\\
&\left\|\Delta^2 u\right\|_\infty \le \left\|\Delta^3 u\right\|^{7/8} \|\nabla u\|_{L^6}^{1/8}, \quad (n=3),
\end{aligned} \tag{2.21}$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \Delta u|^2 dx + \gamma \int_\Omega \left|\Delta^3 u\right|^2 dx \le \frac{\gamma}{2}\int_\Omega \left|\Delta^3 u\right|^2 dx + C. \tag{2.22}$$

So that

$$\sup_{0<t<T} \int_\Omega |\nabla \Delta u|^2 dx \le C, \tag{2.23}$$

$$\iint_{Q_T} \left|\Delta^3 u\right|^2 dx\, dt \le C. \tag{2.24}$$

Define the linear spaces

$$X = \left\{ u \in H^{6,1}(Q_T);\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial \Delta u}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial \Delta^2 u}{\partial n}\right|_{\partial\Omega} = 0,\ u(x,0) = u_0(x) \right\} \tag{2.25}$$

and the associated operator $W : X \to X,\ u \to w$, where w is determined by the following linear problem:

$$\begin{aligned}
&\frac{\partial w}{\partial t} - \gamma \Delta^3 w = -\Delta\left(a(u)\Delta u + \frac{a'(u)}{2}|\nabla u|^2 \right) + \Delta f(u), \quad x \in \Omega,\\
&\left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial \Delta w}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial \Delta^2 w}{\partial n}\right|_{\partial\Omega} = 0, \quad t > 0,\\
&w(x,0) = u_0(x).
\end{aligned} \tag{2.26}$$

From the discussions above and by the contraction mapping principle, W has a unique fixed point u, which is the desired solution of the problem (1.1)–(1.3).

Now, we show the uniqueness. Assume u and v are two solutions of the problem (1.1)–(1.3). Let $w = u - v$. Then w satisfies

$$w_t = \gamma\Delta^3 w + \Delta\varphi(u) - \Delta\varphi(v), \quad \text{in } \Omega\times(0,T),$$
$$\left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial \Delta w}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial \Delta^2 w}{\partial n}\right|_{\partial\Omega} = 0, \tag{2.27}$$
$$w(x,0) = 0,$$

where $\varphi(u) = -a(u)\Delta u - (a'(u)/2)|\nabla u|^2 + f(u)$.

Multiplying the above equation by w and integrating with respect to x, integrating by parts, and using the boundary value condition, we have

$$\begin{aligned}\frac{d}{dt}\|w\|^2 + 2\gamma\|\nabla\Delta w\|^2 &\le \left|\int_\Omega \varphi(u)\Delta w\,dx\right| + \left|\int_\Omega \varphi(v)\Delta w\,dx\right| \\ &\le C\|\Delta w\|^2 \le \frac{\gamma}{2}\|\nabla\Delta w\|^2 + C\|w\|^2.\end{aligned} \tag{2.28}$$

The Gronwall inequality yields

$$\|w\|^2 \le C\|w_0\|^2 = 0. \tag{2.29}$$

Therefore, $\|w\|^2 = 0$, that is, $w = 0$.

(ii) From the proof of (i), we know

$$E(t) \le E(0), \tag{2.30}$$

which together with the Young inequality gives

$$\int_\Omega \left(\frac{\gamma}{2}|\Delta u|^2 + \frac{\gamma_1}{2}u^6\right)dx \le \int_\Omega \frac{|a_0|}{2}|\nabla u|^2 dx - \int_\Omega \frac{a_2}{2}u^2|\nabla u|^2 dx + E(u_0) + C_0. \tag{2.31}$$

Note that

$$\int_\Omega u^2|\nabla u|^2 dx = -\int_\Omega \left(2u^2|\nabla u|^2 + u^3\Delta u\right)dx, \tag{2.32}$$

that is,

$$\int_\Omega u^2|Du|^2 dx = -\frac{1}{3}\int_\Omega u^3\Delta u dx. \tag{2.33}$$

Hence,

$$\left|\frac{a_2}{2}\int_\Omega u^2|\nabla u|^2dx\right| = \left|\frac{a_2}{6}\int_\Omega u^3\Delta u\,dx\right| \le \frac{|a_2|}{6}\left(\varepsilon\int_\Omega u^6dx + C(\varepsilon)\int_\Omega|\Delta u|^2dx\right). \tag{2.34}$$

Taking into account (2.31) and (2.34), we see

$$\begin{aligned}\int_\Omega\left(\frac{\gamma}{2}|\Delta u|^2+\frac{\gamma_1}{2}u^6\right)dx &\le \int_\Omega\frac{|a_0|}{2}|\nabla u|^2dx - \int_\Omega\frac{|a_2|}{2}u^2|\nabla u|^2dx + E(u_0) + C_0\\ &\le \varepsilon_1\int_\Omega\frac{|a_0|}{2}|\Delta u|^2dx\\ &\quad + C(\varepsilon_1)\int_\Omega u^2dx + \frac{|a_2|}{6}\left(\varepsilon\int_\Omega u^6dx + C(\varepsilon)\int_\Omega|\Delta u|^2dx\right) + E(u_0) + C\\ &\le C_1\int_\Omega|\Delta u|^2dx + \frac{\gamma_1}{4}\int_\Omega u^6dx + E(u_0) + C.\end{aligned} \tag{2.35}$$

Hence, when γ is sufficiently large such that $\gamma/2 - C_1 > 0$, we obtain the estimates (2.7)–(2.9). The other steps are similar to the proof of (i), so the details are omitted here. □

3. Blow Up

In the previous sections, we have seen that the solution of the problem is globally existent, provided that $\gamma_1 > 0$. The following theorem shows that the solution of the problem blows up at a finite time for $\gamma_1 < 0$.

Theorem 3.1. *If $\gamma_1 < 0$, $-\int_\Omega((\gamma/2)|\Delta u_0|^2 + a((u_0)/2)|\nabla u_0|^2 + F(u_0)dx)$ is sufficiently large.*

(i) *If $a_2 < 0$, then the solution u of* (1.1)–(1.3) *blows up in finite time, that is, for $T > 0$,*

$$\lim_{t\to T}\|u(t)\| = +\infty. \tag{3.1}$$

(ii) *If $a_2 > 0$ and $a_0 > 0$, the solution must blow up in a finite time.*

(iii) *If $a_2 > 0$, $a_0 < 0$, and γ are large enough, the solution blows up in finite time.*

Proof. (i) From the proof of Theorem 2.1, we know

$$E(t) = \int_\Omega\left(\frac{\gamma}{2}|\Delta u|^2 + \frac{a(u)}{2}|\nabla u|^2 + F(u)\right)dx \le E(0). \tag{3.2}$$

Hence,

$$-\int_\Omega\gamma|\Delta u|^2dx \ge \int_\Omega a(u)|\nabla u|^2dx + 2\int_\Omega F(u)dx - 2E(0). \tag{3.3}$$

Let w be the unique solution of the following problem:

$$\begin{gathered}\Delta w = u, \quad x \in \Omega, \\ \frac{\partial w}{\partial n} = 0, \quad \text{on } \partial\Omega, \\ \int_\Omega w\,dx = 0.\end{gathered} \tag{3.4}$$

It is easily seen that

$$\|\nabla w\|^2 \le C\|u\|^2. \tag{3.5}$$

Multiplying (1.1) by w and integrating with respect to x, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx &= -2\gamma\int_\Omega (\Delta u)^2 dx + 2\int_\Omega a(u)u\Delta u\,dx + \int_\Omega a'(u)|\nabla u|^2 u\,dx - 2\int_\Omega f(u)u dx \\ &\ge 2\int_\Omega a(u)|\nabla u|^2 dx + 2\int_\Omega a(u)u\Delta u\,dx + \int_\Omega a'(u)|\nabla u|^2 u\,dx \\ &\quad + 4\int_\Omega F(u)dx - 2\int_\Omega f(u)u\,dx - 4E(0) \\ &= \int_\Omega 4a_2u^2|\nabla u|^2 dx + \int_\Omega 2a_0|\nabla u|^2 dx + \int_\Omega 2a_2u^3\Delta u\,dx + 2a_0\int_\Omega u\Delta u\,dx \\ &\quad + 4\int_\Omega F(u)dx - 2\int_\Omega f(u)u\,dx - 4E(0) \\ &= -2a_2\int_\Omega u^2|\nabla u|^2 dx + 4\int_\Omega F(u)dx - 2\int_\Omega f(u)u\,dx - 4E(0) \\ &= -2a_2\int_\Omega u^2|\nabla u|^2 dx + \gamma_1\int_\Omega \left(-8u^6 - 4(h_0-2)u^4 + 4h_0\right)dx - 4E(0).\end{aligned} \tag{3.6}$$

Owing to $a_2 < 0$, $\gamma_1 < 0$, and (3.5), it follows from the above inequality that

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx &\ge -\gamma_1\int_\Omega u^6 dx - 4E(0) - C(h_0, \gamma_1, |\Omega|) \\ &\ge -\gamma_1\left(\int_\Omega u^2 dx\right)^3 - 4E(0) - C(h_0, \gamma_1, |\Omega|) \\ &\ge -\gamma_1\left(\int_\Omega |\nabla w|^2 dx\right)^3 - 4E(0) - C(h_0, \gamma_1, |\Omega|),\end{aligned} \tag{3.7}$$

where $C_1 > 0$. Hence, when $-E(0)$ is sufficiently large, such that $-4E(0) - C(h_0, \gamma_1, |\Omega|) \ge 0$, then by $\gamma_1 < 0$, we know that $\|u\|$ has to blow up.

From the proof of (i), we obtain

$$\frac{d}{dt}\int_{\Omega}|\nabla w|^2dx \ge -2a_2\int_{\Omega}u^2|\nabla u|^2dx + 4\int_{\Omega}F(u)dx - 2\int_{\Omega}f(u)u\,dx - 4E(0). \tag{3.8}$$

On the other hand, we have

$$\int_{\Omega}u^2|\nabla u|^2dx = -\frac{1}{3}\int_{\Omega}u^3\Delta u\,dx \le \frac{1}{3}\left(\varepsilon\int_{\Omega}u^6dx + C(\varepsilon)\int_{\Omega}(\Delta u)^2dx\right). \tag{3.9}$$

From the above inequality, we know

$$C(\varepsilon)\int_{\Omega}(\Delta u)^2dx \ge 3\int_{\Omega}u^2|\nabla u|^2dx - \varepsilon\int_{\Omega}u^6dx. \tag{3.10}$$

Using (3.3) and (3.10), we see that

$$\begin{aligned}
&-\frac{\gamma}{C(\varepsilon)}\left(3\int_{\Omega}u^2|\nabla u|^2dx - \varepsilon\int_{\Omega}u^6dx\right)\\
&\quad\ge -\gamma\int_{\Omega}(\Delta u)^2dx \ge \int_{\Omega}a(u)|\nabla u|^2dx + 2\int_{\Omega}F(u)dx - 2E(0)\\
&\quad= \int_{\Omega}\left(a_2u^2|\nabla u|^2 + a_0|\nabla u|^2\right)dx + 2\int_{\Omega}F(u)dx - 2E(0).
\end{aligned} \tag{3.11}$$

It follows that

$$-\left(\frac{3\gamma}{C(\varepsilon)} + a_2\right)\int_{\Omega}u^2|\nabla u|^2dx \ge \int_{\Omega}a_0|\nabla u|^2dx + \frac{\varepsilon\gamma}{C(\varepsilon)}\int_{\Omega}u^6dx + 2\int_{\Omega}F(u)dx - 2E(0). \tag{3.12}$$

Again by (3.12) and (3.3), we get

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega}|\nabla w|^2dx &\ge -2a_2\int_{\Omega}u^2|\nabla u|^2dx + 4\int_{\Omega}F(u)dx - 2\int_{\Omega}f(u)u\,dx - 4E(0)\\
&\ge \frac{2a_2C(\varepsilon)}{3\gamma + a_2C(\varepsilon)}\left(\int_{\Omega}a_0|\nabla u|^2dx + \frac{\varepsilon\gamma}{C(\varepsilon)}\int_{\Omega}u^6dx + 2\int_{\Omega}F(u)dx - 2E(0)\right)\\
&\quad + 4\int_{\Omega}F(u)dx - 2\int_{\Omega}f(u)u\,dx - 4E(0).
\end{aligned} \tag{3.13}$$

By $a_0 > 0$, $a_2 > 0$, $C(\varepsilon) > 0$, $\gamma > 0$, choosing $C(\varepsilon)$ large enough, we obtian

$$\frac{d}{dt}\int_{\Omega}|\nabla w|^2dx \ge -\gamma_1\int_{\Omega}u^6dx - CE(0) - C(h_0, \gamma_1, |\Omega|). \tag{3.14}$$

Similar to the proof of (i), we easily see that (ii) holds.

(iii) The crucial term is

$$\int_\Omega a_0 |\nabla u|^2 dx. \tag{3.15}$$

By the Young inequality, we have

$$\int_\Omega |\nabla u|^2 dx \le \frac{1}{2}\int_\Omega (\Delta u)^2 dx + \frac{1}{2}\int_\Omega u^2 dx. \tag{3.16}$$

By (3.3) and (3.16), we have

$$\begin{aligned}
-\gamma\left(2\int_\Omega |\nabla u|^2 dx - \int_\Omega u^2 dx\right) &\ge -\int_\Omega \gamma|\Delta u|^2 dx \\
&\ge \int_\Omega a(u)|\nabla u|^2 dx + 2\int_\Omega F(u)dx - 2E(0) \\
&= \int_\Omega a_2 u^2 |\nabla u|^2 dx + \int_\Omega a_0|\nabla u|^2 dx + 2\int_\Omega F(u)dx - 2E(0) \\
&\ge \int_\Omega a_0|\nabla u|^2 dx + 2\int_\Omega F(u)dx - 2E(0).
\end{aligned} \tag{3.17}$$

It follows that

$$(-2\gamma - a_0)\int_\Omega |\nabla u|^2 dx \ge -\gamma\int_\Omega u^2 dx + 2\int_\Omega F(u)dx - 2E(0). \tag{3.18}$$

If $2\gamma + a_0 > 0$, substituting the (3.18) into (3.13), similarly, we know that the solution must blow up in finite time. □

4. Global Attractor in H^k Space

In this section, we will give the existence of the global attractors of the problem (1.1)–(1.3) in any kth order space $H^k(\Omega)$.

First of all, we will prove the existence of attractor for $\gamma_1 > 0$. We define the operator semigroup $\{S(t)\}_{t\ge 0}$ in $H^3(\Omega)$ space by

$$S(t)u_0 = u(t), \quad t \ge 0, \tag{4.1}$$

where $u(t)$ is the solution of the problem (1.1)–(1.3).

We let

$$X_m = \left\{ u \mid u \in H^3(\Omega), \left| \int_\Omega u\,dx \right| \le m \right\}, \tag{4.2}$$

where $m > 0$ is a constant, and the $\{S(t)\}$ on X_m is a well-defined semigroup.

Theorem 4.1. *For every m chosen as above, the semiflow associated with the solution u of the problem* (1.1)–(1.3) *possesses in X_m a global attractor A_m, which attracts all the bounded set in X_m.*

In order to prove Theorem 4.1, we need to establish some a priori estimates for the solution u of problem (1.1)–(1.3). In what follows, we always assume that $\{S(t)\}_{t\ge 0}$ is the semigroup generated by the weak solutions of (1.1) with initial data $u_0 \in H^3(\Omega)$.

Lemma 4.2. *There exists a bounded set B_m whose size depends only on m and Ω in X_m such that for all the orbits staring from any bounded set B in X_m, $\exists t_1 = t_1(X) \ge 0$, such that for all $t \ge t_1$ all the orbits will stay in B_m.*

Proof. It suffices to prove that there is a positive constant C such that for large t, then the following holds:

$$\|u\|_3 \le C. \tag{4.3}$$

From (2.22), we have

$$\frac{d}{dt}\int_\Omega |\nabla \Delta u|^2 dx + \gamma \int_\Omega \left|\Delta^3 u\right|^2 dx \le C. \tag{4.4}$$

On the other hand, we know that

$$\begin{gathered} \int_\Omega |\nabla \Delta u|^2 dx = -\int_\Omega u \Delta^3 u\,dx \le C\int_\Omega |u|^2 dx + \frac{\gamma}{2}\int_\Omega \left|\Delta^3 u\right|^2 dx, \\ \int_\Omega |u|^2 dx \le \varepsilon \int_\Omega |\nabla \Delta u|^2 dx. \end{gathered} \tag{4.5}$$

Hence, we see that

$$\frac{d}{dt}\int_\Omega |\nabla \Delta u|^2 dx + C_1 \int_\Omega |\nabla \Delta u|^2 dx \le C_2, \tag{4.6}$$

which gives

$$\int_\Omega |\nabla \Delta u|^2 dx \le e^{-C_1 t}\int_\Omega |\nabla \Delta u_0|^2 dx + \frac{C_2}{C_1}. \tag{4.7}$$

The proof is completed. □

Lemma 4.3. *For any initial data u_0 in any bounded set $B \subset X_m$, there exists $t_2 = t_2(B) > 0$ such that*

$$\|u(t)\|_{H^4} \leq C, \quad \forall t \geq t_2 > 0, \tag{4.8}$$

which turns out that $\bigcup_{t \geq t_2} u(t)$ is relatively compact in X_m.

Proof. From Theorem 2.1, we know when $\gamma_1 > 0,\ a_2 > 0$ or $\gamma_1 > 0,\ a_2 < 0$ that (2.14) holds. Integrating (2.14) over $(t, t+1)$, we obtain

$$\int_t^{t+1} \int_\Omega \left(\Delta^2 u\right)^2 dx\, dt \leq C. \tag{4.9}$$

On the other hand, Differentiating (1.1) gives

$$\Delta u_t = \gamma \Delta^4 u + \Delta^2 \left[-a(u)\Delta u - \frac{a'(u)}{2} |\nabla u|^2 + f(u) \right]. \tag{4.10}$$

Multiplying (4.10) by $\Delta^3 u$ and integrating on Ω, using the boundary conditions, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\Delta^2 u\right|^2 dx + \gamma \int_\Omega \left|\nabla \Delta^3 u\right|^2 dx \leq \frac{\gamma}{2}\int_\Omega \left|\nabla \Delta^3 u\right|^2 dx + C. \tag{4.11}$$

From (4.11), (4.9), and the uniform Gronwall inequality, we have

$$\left\|\Delta^2 u\right\| \leq C, \qquad t_2 \geq t_2(B). \tag{4.12}$$

□

Then by [17, Theorem I.1.1], we immediately conclude that $A_m = \omega(B_m)$; the ω-limit set of absorbing set B_m is a global attractor in X_m. By Lemma 4.3, this global attractor is a bounded set in $H^3(\Omega)$. Thus, we complete the proof of Theorem 4.1.

Secondly, we consider the existence of the global attractors of the problem (1.1)–(1.3) in any kth order space $H^k(\Omega)$. Because both $\gamma_1 > 0,\ a_2 > 0$ and $\gamma_1 > 0,\ a_2 < 0$ lead to Theorem 4.1, the following proofs are based on Theorem 4.1, hence for simplicity, we let $\gamma = 1$.

In order to consider the global attractor for (1.1) in H^k space, we introduce the definition as follows:

$$\begin{aligned} H &= \left\{ u \in L^2(\Omega); \left| \int_\Omega u dx \right| \leq m \right\}, \\ H_{1/2} &= \left\{ u \in H^3(\Omega) \cap H;\ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = \frac{\partial \Delta u}{\partial n}\bigg|_{\partial\Omega} = 0 \right\}, \\ H_1 &= \left\{ u \in H^6(\Omega) \cap H;\ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = \frac{\partial \Delta u}{\partial n}\bigg|_{\partial\Omega} = \frac{\partial \Delta^2 u}{\partial n}\bigg|_{\partial\Omega} = 0 \right\}, \end{aligned} \tag{4.13}$$

where $m > 0$ is a constant.

We rewrite (1.1) as

$$u_t - \Delta^3 u = \Delta g(u), \tag{4.14}$$

where

$$g(u) = \left(-a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + f(u)\right). \tag{4.15}$$

Let

$$L = \Delta^3 : H_1 \longrightarrow H, \qquad G = \Delta g : H_1 \longrightarrow H. \tag{4.16}$$

Then, (4.14) and (4.15) can be rewritten as

$$\frac{du}{dt} = Lu + Gu. \tag{4.17}$$

The linear operator L is a sectorial operator which generates an analytic semigroup e^{tL}. Without loss of generality, we assume that L generates the fractional power operators $\mathcal{L}^\alpha$ and the fractional order spaces H_α as follows:

$$\mathcal{L}^\alpha = (-L)^\alpha : H_\alpha \longrightarrow H, \quad \alpha \in R, \tag{4.18}$$

where $H_\alpha = D(\mathcal{L}^\alpha)$ is the domain of $\mathcal{L}^\alpha$ and $H_\beta \subset H_\alpha$ is a compact inclusion for any $\beta > \alpha$ [see Pazy [18]].

The space $H_{1/6}$ is given by $H_{1/6}$ = the closure of $H_{1/2}$ in $H^1(\Omega)$ and $H_k = H^{6k} \cap H_1$ for $k \geq 1$.

The following lemmas which can be found in [19, 20] are crucial to our proof.

Lemma 4.4. *Let $u(t,u_0) = S(t)u_0$ $(u_0 \in H,\ t \geq 0)$ be the solution of* (4.17) *and $S(t)$ the semigroup generated by* (4.17). *Assume H_α is the fractional order space generated by L and*

(1) *for some $\alpha \geq 0$ there is a bounded set $\widetilde{B} \subset H_\alpha$, for any $u_0 \in H_\alpha$ there exists $t_\alpha > 0$ such that*

$$u(t,u_0) \in \widetilde{B}, \quad \forall t > t_{u_0}; \tag{4.19}$$

(2) *there is a $\beta > \alpha$, for any bounded set $U \subset H_\beta$, $\exists T > 0$ and $C > 0$ such that*

$$\|u(t,u_0)\|_{H_\beta} \leq C, \quad \forall t > T,\ u_0 \in U. \tag{4.20}$$

Then (4.17) *has a global attractor A which attracts any bounded set H_α in the H_α-norm.*

Lemma 4.5. *Let $L : H_1 \to H$ be a sectorial operator which generates an analytic semigroup $T(t) = e^{tL}$. If all eigenvalues λ of L satisfy $\operatorname{Re}\lambda < -\lambda_0$ for some real number $\lambda_0 > 0$, then for $\mathcal{L}^\alpha$ $(\mathcal{L} = L)$ we have*

(1) *$T(t) : H_1 \to H$ is bounded for all $\alpha \in \mathbb{R}$ and $t > 0$;*

(2) *$T(t)\mathcal{L}^\alpha x = \mathcal{L}^\alpha T(t)x$, for all $x \in H_\alpha$;*

(3) *for each $t > 0$, $T(t)\mathcal{L}^\alpha : H \to H$ is bounded, and*

$$\|\mathcal{L}^\alpha T(t)\| \le C_\alpha t^{-\alpha} e^{-\sigma t}, \tag{4.21}$$

where some $\sigma > 0$, $C_\alpha > 0$ is a constant only depending on α;

(4) *the H_α-norm can be defined by*

$$\|x\|_{H_\alpha} = \|\mathcal{L}^\alpha x\|_H. \tag{4.22}$$

Now, we give the main theorem.

Theorem 4.6. *For any $\alpha \ge 0$ the problem* (1.1)–(1.3) *has a global attractor A in H_α, and A attracts any bounded set of H_α in the H_α-norm.*

Proof. Owing to (4.17), the solution of the system (1.1)–(1.3) can be written as

$$u(t, u_0) = e^{tL} + \int_0^t e^{(t-\tau)L} G(u)\,d\tau. \tag{4.23}$$

First of all, we are going to prove that for any $\alpha \ge 0$, the solution $u(t, u_0)$ of the problem is uniformly bounded in H_α, that is, for any bounded set $U \subset H_\alpha$, there exists C such that

$$\|u(t, u_0)\|_{H_\alpha} \le C, \quad \forall t \ge 0,\ u_0 \in U \subset H_\alpha,\ \alpha \ge 0. \tag{4.24}$$

From Theorem 4.1, we have known that, for any bounded set $U \subset H_{1/2}$ there is a constant $C > 0$ such that

$$\|u\|_{H_{1/2}} \le C, \quad \forall t \ge 0,\ u_0 \in U \subset H_{1/2}. \tag{4.25}$$

Next, according to Lemma 4.4, we prove (4.24) for any $\alpha > 1/2$ in the following steps.

Step 1. We are going to show that for any bounded set $U \subset H_\alpha$ $(0 < \alpha < 1)$, there exists $C > 0$ such that

$$\|u(t, u_0)\|_{H_\alpha} \le C, \quad \forall t \ge 0,\ u_0 \in U \subset H_\alpha,\ \alpha < 1. \tag{4.26}$$

In fact, by the embedding theorems of fractional order spaces [18], we have

$$H^3(\Omega) \hookrightarrow W^{2,2}(\Omega), \qquad H^3(\Omega) \hookrightarrow W^{1,4}(\Omega), \qquad H_{1/2} \hookrightarrow C^0(\Omega) \cap H^3(\Omega). \tag{4.27}$$

From (4.23), we have

$$\begin{aligned}\|u(t,u_0)\|_{H_\alpha} &= \left\| e^{tL} + \int_0^t e^{(t-\tau)L} G(u)\, d\tau \right\|_{H_\alpha} \\ &\le \|u_0\|_{H_\alpha} + \int_0^t \left\| L^\alpha e^{(t-\tau)L} G(u) \right\|_H d\tau \\ &\le \|u_0\|_{H_\alpha} + \int_0^t \left\| L^{\alpha+(1/3)} e^{(t-\tau)L} \right\| \|g(u)\|_H d\tau.\end{aligned} \tag{4.28}$$

We deduce that

$$\begin{aligned}\|g(u)\| &= \int_\Omega \left| \Delta u - a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + f(u) \right|^2 dx \\ &\le \int_\Omega \left| a_2 u^2 \Delta u + (a_0+1)\Delta u + a_2 |\nabla u|^2 + Cu^6 + C \right|^2 dx \\ &\le C \int_\Omega \left| u^4 |\Delta u|^2 + |\Delta u|^2 + |\nabla u|^4 + u^{12} \right| dx + C \\ &\le C\left(\|u\|_{H_{1/2}}^4 \|u\|_{H_{1/2}}^2 + \|u\|_{H_{1/2}}^2 + \|u\|_{H_{1/2}}^4 + \|u\|_{H_{1/2}}^{12} \right) + C,\end{aligned} \tag{4.29}$$

which means that $g : H_{1/2} \to H$ is bounded.

Hence, from (4.25), (4.28), (4.29), and Lemma 4.5, we have

$$\|u\|_{H_\alpha} \le \|u_0\|_{H_\alpha} + C\int_0^t \tau^{-\alpha-(1/3)} e^{-\sigma\tau} d\tau \le C, \tag{4.30}$$

where $0 < \alpha < 2/3$.

Step 2. We prove that for any bounded set $U \subset H_\alpha$ $(2/3 \le \alpha < 5/6)$, there is a constant $C > 0$ such that

$$\|u(t,u_0)\|_{H_\alpha} \le C, \quad \forall t \ge 0,\ u_0 \in U \subset H_\alpha,\ \alpha < \frac{5}{6}. \tag{4.31}$$

In fact, by the embedding theorems, we have

$$\begin{gathered} H^3(\Omega) \hookrightarrow W^{1,4}(\Omega), \qquad H^3(\Omega) \hookrightarrow W^{2,4}(\Omega), \qquad H^3(\Omega) \hookrightarrow W^{1,6}(\Omega), \\ H^3(\Omega) \hookrightarrow W^{1,2}(\Omega), \qquad H_\alpha(\Omega) \hookrightarrow C^0(\Omega) \cap H^3(\Omega), \qquad \alpha > \frac{1}{2}. \end{gathered} \tag{4.32}$$

We deduce that

$$\begin{aligned}
\|g(u)\|_{H_{1/6}} &= \int_\Omega \left|\nabla\left(\Delta u - a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + f(u)\right)\right|^2 dx \\
&\leq C\int_\Omega \left(u^2|\nabla u|^2|\Delta u|^2 + u^4|\nabla\Delta u|^2 + |\nabla\Delta u|^2 + |\nabla u|^6 + u^{10}|\nabla u|^2\right)dx + C \\
&\leq C\int_\Omega \left(\left(\sup_{x\in\Omega}|u|^2\right)|\nabla u|^2|\Delta u|^2 + \left(\sup_{x\in\Omega}|u|^4\right)|\nabla\Delta u|^2\right. \\
&\quad \left. +|\nabla\Delta u|^2 + |\nabla u|^6 + \left(\sup_{x\in\Omega}|u|^{10}\right)|\nabla u|^2\right)dx + C \\
&\leq C\int_\Omega \left(\left(\sup_{x\in\Omega}|u|^2\right)|\nabla u|^4 + \left(\sup_{x\in\Omega}|u|^2\right)|\Delta u|^4 + \left(\sup_{x\in\Omega}|u|^4\right)|\nabla\Delta u|^2\right. \\
&\quad \left. +|\nabla\Delta u|^2 + |\nabla u|^6 + \left(\sup_{x\in\Omega}|u|^{10}\right)|\nabla u|^2\right)dx + C \\
&\leq \|u\|_{H_\alpha}^2\|u\|_{H_\alpha}^4 + \|u\|_{H_\alpha}^2\|u\|_{H_\alpha}^2 + \|u\|_{H_\alpha}^4\|u\|_{H_\alpha}^2 + \|u\|_{H_\alpha}^2 + \|u\|_{H_\alpha}^6 + \|u\|_{H_\alpha}^{10}\|u\|_{H_\alpha}^4 + C.
\end{aligned} \tag{4.33}$$

It implies that

$$g : H_\alpha \longrightarrow H_{1/6} \text{ is bounded for } \frac{1}{2} \leq \alpha < \frac{2}{3}. \tag{4.34}$$

Therefore, it follows from (4.26) and (4.33) that

$$\|g(u(t,u_0))\|_{H_{1/6}} \leq C, \quad \forall t \geq 0,\ u_0 \in U,\ \frac{1}{2} \leq \alpha < \frac{2}{3}. \tag{4.35}$$

Thus, by the same method as that in Step 1, we get from (4.35) that

$$\begin{aligned}
\|u(t,u_0)\|_{H_\alpha} &= \left\|e^{tL} + \int_0^t e^{(t-\tau)L}G(u)d\tau\right\|_{H_\alpha} \\
&\leq \|u_0\|_{H_\alpha} + \int_0^t \left\|L^\alpha e^{(t-\tau)L}G(u)\right\|_H d\tau \\
&\leq \|u_0\|_{H_\alpha} + \int_0^t \left\|L^{(\alpha+(1/3)-(1/6))}e^{(t-\tau)L}\right\| \|\nabla g(u)\|_H d\tau \\
&\leq \|u_0\|_{H_\alpha} + \int_0^t \left\|L^{(\alpha+(1/3)-(1/6))}e^{(t-\tau)L}\right\| \|g(u)\|_{H_{(1/6)}} d\tau \\
&\leq \|u_0\|_{H_\alpha} + C\int_0^t \tau^{-\beta}e^{-\sigma\tau}d\tau \\
&\leq C, \quad \forall t \geq 0,\ u_0 \in U \subset H_\alpha,
\end{aligned} \tag{4.36}$$

where $\beta = \alpha + 1/3 - 1/6$ $(0 < \beta < 1)$.

Step 3. We prove that for any bounded set $U \subset H_\alpha$ $(5/6 \leq \alpha < 1)$, there is a constant $C > 0$ such that

$$\|u(t,u_0)\|_{H_\alpha} \leq C, \quad \forall t \geq 0,\ u_0 \in U \subset H_\alpha,\ \alpha < 1. \tag{4.37}$$

In fact, by the embedding theorems, we have

$$\begin{gathered} H^4(\Omega) \hookrightarrow W^{1,8}(\Omega), \qquad H^4(\Omega) \hookrightarrow W^{2,4}(\Omega), \qquad H^4(\Omega) \hookrightarrow W^{1,4}(\Omega), \\ H^4(\Omega) \hookrightarrow W^{3,4}(\Omega), \qquad H^4(\Omega) \hookrightarrow W^{2,2}(\Omega), \qquad H_\alpha(\Omega) \hookrightarrow C^0(\Omega) \cap H^4(\Omega), \quad \alpha \geq \frac{2}{3}. \end{gathered} \tag{4.38}$$

We deduce

$$\begin{aligned} \|g(u)\|_{H_{1/3}} &= \int_\Omega \left|\Delta\left(\Delta u - a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + f(u)\right)\right|^2 dx \\ &\leq C\int_\Omega \left||\nabla u|^2\Delta u + u|\Delta u|^2 + u\nabla u\nabla\Delta u + u^2\Delta^2 u + \Delta^2 u + u^4|\nabla u|^2 + u^5\Delta u\right|^2 dx + C \\ &\leq C\int_\Omega \left(|\nabla u|^4|\Delta u|^2 + \left(\sup_{x\in\Omega}|u|^2\right)|\Delta u|^4\right. \\ &\qquad + \left(\sup_{x\in\Omega}|u|^2\right)\left(|\nabla u|^2\right)|\nabla\Delta u|^2 dx + \left(\sup_{x\in\Omega}|u|^4\right)\left|\Delta^2 u\right|^2 + \left|\Delta^2 u\right|^2 \\ &\qquad \left.+\left(\sup_{x\in\Omega}|u|^8\right)|\nabla u|^4 + \left(\sup_{x\in\Omega}|u|^{10}\right)|\Delta u|^2\right) dx + C \\ &\leq \|u\|_{W^{1,8}}^8 + \|u\|_{W^{2,4}}^4 + \|u\|_{H_\alpha}^2\|u\|_{W^{2,4}}^4 + \|u\|_{H_\alpha}^2\|u\|_{W^{1,4}}^4 + \|u\|_{H_\alpha}^2\|u\|_{W^{3,4}}^2 \\ &\qquad + \|u\|_{H_\alpha}^4\|u\|_{W^{4,2}}^2 + \|u\|_{W^{4,2}}^2 + \|u\|_{H_\alpha}^8\|u\|_{W^{1,4}}^4 + \|u\|_{H_\alpha}^{10}\|u\|_{W^{2,2}}^2 + C \\ &\leq \|u\|_{H_\alpha}^8 + \|u\|_{H_\alpha}^4 + \|u\|_{H_\alpha}^2\|u\|_{H_\alpha}^4 + \|u\|_{H_\alpha}^2\|u\|_{H_\alpha}^4 + \|u\|_{H_\alpha}^2\|u\|_{H_\alpha}^2 \\ &\qquad + \|u\|_{H_\alpha}^4\|u\|_{H_\alpha}^2 + \|u\|_{H_\alpha}^2 + \|u\|_{H_\alpha}^8\|u\|_{H_\alpha}^4 + \|u\|_{H_\alpha}^{10}\|u\|_{H_\alpha}^2 + C. \end{aligned} \tag{4.39}$$

It implies that

$$g : H_\alpha \longrightarrow H_{1/3} \text{ is bounded for } \frac{2}{3} \leq \alpha < \frac{5}{6}. \tag{4.40}$$

Therefore, it follows from (4.31) and (4.40) that

$$\|g(u(t,u_0))\|_{H_{1/3}} \leq C, \quad \forall t \geq 0,\ u_0 \in U,\ \frac{2}{3} \leq \alpha < \frac{5}{6}. \tag{4.41}$$

Thus, by the same method as those in Steps 1 and 2, we get from (4.41) that

$$\begin{aligned}
\|u(t,u_0)\|_{H_\alpha} &= \left\| e^{tL} + \int_0^t e^{(t-\tau)L} G(u)\,d\tau \right\|_{H_\alpha} \\
&\le \|u_0\|_{H_\alpha} + \int_0^t \left\| L^\alpha e^{(t-\tau)L} G(u) \right\|_H d\tau \\
&\le \|u_0\|_{H_\alpha} + \int_0^t \left\| L^\alpha e^{(t-\tau)L} \right\| \|\Delta g(u)\|_H d\tau \\
&\le \|u_0\|_{H_\alpha} + \int_0^t \left\| L^\alpha e^{(t-\tau)L} \right\| \|g(u)\|_{H_{1/3}} d\tau \\
&\le \|u_0\|_{H_\alpha} + C\int_0^t \tau^{-\beta} e^{-\sigma\tau}\,d\tau \\
&\le C, \quad \forall t \ge 0,\ u_0 \in U \subset H_\alpha,
\end{aligned} \tag{4.42}$$

where $\beta = \alpha$ $(0 < \beta < 1)$.

Step 4. We prove that for any bounded set $U \subset H_\alpha$ $(1 \le \alpha < (7/6))$, there is a constant $C > 0$ such that

$$\|u(t,u_0)\|_{H_\alpha} \le C, \quad \forall t \ge 0,\ u_0 \in U \subset H_\alpha,\ 1 \le \alpha < \frac{7}{6}. \tag{4.43}$$

In fact, by the embedding theorems, we have

$$\begin{gathered}
H^5(\Omega) \hookrightarrow W^{1,4}(\Omega), \qquad H^5(\Omega) \hookrightarrow W^{2,8}(\Omega), \qquad H^5(\Omega) \hookrightarrow W^{1,8}(\Omega), \\
H^5(\Omega) \hookrightarrow W^{3,4}(\Omega), \qquad H^5(\Omega) \hookrightarrow W^{2,4}(\Omega), \qquad H^5(\Omega) \hookrightarrow W^{4,4}(\Omega), \\
H^5(\Omega) \hookrightarrow W^{1,6}(\Omega), \qquad H^5(\Omega) \hookrightarrow W^{3,2}(\Omega), \qquad H_\alpha(\Omega) \hookrightarrow C^0(\Omega) \cap H^5(\Omega), \quad \alpha \ge \frac{5}{6}.
\end{gathered} \tag{4.44}$$

Hence, similar to the above, we have

$$\begin{aligned}
\|g(u)\|_{H_{1/2}} &= \int_\Omega \left|\nabla\Delta\left(\Delta u - a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + f(u)\right)\right|^2 dx \\
&\le C\int_\Omega \Big(\nabla u|\Delta u|^2 + |\nabla u|^2\nabla\Delta u + u\Delta u\nabla\Delta u + u\nabla u\Delta^2 u \\
&\quad + u^2\nabla\Delta^2 u + \nabla\Delta^2 u + u^3|\nabla u|^2\nabla u + u^4\nabla u\Delta u + u^5\nabla\Delta u\Big)^2 dx + C \\
&\le \|u\|_{H_\alpha}^4 + \|u\|_{H_\alpha}^8 + \|u\|_{H_\alpha}^8 + \|u\|_{H_\alpha}^4 + \|u\|_{H_\alpha}^2\|u\|_{H_\alpha}^4 + \|u\|_{H_\alpha}^2\|u\|_{H_\alpha}^4 \\
&\quad + \|u\|_{H_\alpha}^2\|u\|_{H_\alpha}^4 + \|u\|_{H_\alpha}^2\|u\|_{H_\alpha}^4 + \|u\|_{H_\alpha}^4\|u\|_{H_\alpha}^2 + \|u\|_{H_\alpha}^2 + \|u\|_{H_\alpha}^6\|u\|_{H_\alpha}^6 \\
&\quad + \|u\|_{H_\alpha}^8\|u\|_{H_\alpha}^4 + \|u\|_{H_\alpha}^8\|u\|_{H_\alpha}^4 + \|u\|_{H_\alpha}^{10}\|u\|_{H_\alpha}^4 + C.
\end{aligned} \tag{4.45}$$

It implies that

$$g : H_\alpha \longrightarrow H_{1/2} \quad \text{is bounded for } \frac{5}{6} \le \alpha < 1. \tag{4.46}$$

Therefore, it follows from (4.37) and (4.46) that

$$\left\|g(u(t,u_0))\right\|_{H_{1/2}} \le C, \quad \forall t \ge 0,\ u_0 \in U,\ \frac{5}{6} \le \alpha < 1. \tag{4.47}$$

Thus, by the same method as that in Steps 1, 2, and 3, we get from (4.47) that

$$\begin{aligned}
\|u(t,u_0)\|_{H_\alpha} &= \left\| e^{tL} + \int_0^t e^{(t-\tau)L}G(u)d\tau \right\|_{H_\alpha} \\
&\le \|u_0\|_{H_\alpha} + \int_0^t \left\|L^\alpha e^{(t-\tau)L}G(u)\right\|_H d\tau \\
&\le \|u_0\|_{H_\alpha} + \int_0^t \left\|L^{(\alpha+1/3-1/2)}e^{(t-\tau)L}\right\| \left\|\nabla\Delta g(u)\right\|_H d\tau \\
&\le \|u_0\|_{H_\alpha} + \int_0^t \left\|L^{(\alpha+1/3-1/2)}e^{(t-\tau)L}\right\| \left\|g(u)\right\|_{H_{1/2}} d\tau \\
&\le \|u_0\|_{H_\alpha} + C\int_0^t \tau^{-\beta}e^{-\sigma\tau}d\tau \\
&\le C, \quad \forall t \ge 0,\ u_0 \in U \subset H_\alpha,
\end{aligned} \tag{4.48}$$

where $\beta = \alpha + 1/3 - 1/2$ $(0 < \beta < 1)$.

In the same fashion as in the proof of (4.43), by iteration we can prove that for any bounded set $U \subset H_\alpha$ $(\alpha > 0)$, there is a constant $C > 0$ such that

$$\|u\|_{H_\alpha} \leq C, \quad \forall t \geq 0,\ u_0 \in U \subset H_\alpha,\ \alpha \geq 0. \tag{4.49}$$

That is, for all $\alpha \geq 0$ the semigroup $S(t)$ generated by the problem (1.1)–(1.3) is uniformly compact in H_α.

Secondly, we are going to show that for any $\alpha \geq 0$, the problem (1.1)–(1.3) has a bounded absorbing set in H_α; that is, for any bounded set $U \subset H_\alpha$ $(\alpha \geq 0)$ there are $T > 0$ and a constant $C > 0$ independent of u_0, such that

$$\|u(t,u_0)\|_{H_\alpha} \leq C, \quad \forall t \geq T,\ u_0 \in U \subset H_\alpha. \tag{4.50}$$

For $\alpha = 1/2$, this follows from Theorem 4.1. Now, we will prove (4.13) for any $\alpha \geq 1/2$ in the following steps.

Step 1. We will prove that for any $1/2 \leq \alpha < 4/3$, the problem (1.1)–(1.3) has a bounded absorbing set in H_α.

By (4.23), we have

$$u(t,u_0) = e^{(t-T)L}u(T,u_0) + \int_T^t e^{(t-\tau)L} g(u)\,d\tau. \tag{4.51}$$

Let $D \subset H_{1/2}$ be the bounded absorbing set of the problem (1.1)–(1.3) in $H_{1/2}$, and $T_0 > 0$ is the time such that

$$u(t,u_0) \in D, \quad \forall t > T_0,\ u_0 \in U \subset H_\alpha,\ \alpha \geq \frac{1}{2}. \tag{4.52}$$

On the other hand, it is known that

$$\left\| e^{tL} \right\| \leq Ce^{-\lambda_1^3 t}, \tag{4.53}$$

where $\lambda_1 > 0$ is the first eigenvalue of

$$\begin{gathered} -\Delta u = \lambda u, \\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \\ \int_\Omega u\,dx = 0. \end{gathered} \tag{4.54}$$

For any given $T > 0$ and $u_0 \in U \subset H_\alpha$ $(\alpha \geq 1/2)$, we have

$$\left\| e^{(t-T)L} u(T,u_0) \right\|_{H_\alpha} \longrightarrow 0, \quad \text{as } t \longrightarrow \infty. \tag{4.55}$$

By assertion (3) of Lemma 4.5, it follows from (4.51) and (4.29) that for any $1/2 \leq \alpha < 2/3$, we have

$$\begin{aligned}
\|u(t,u_0)\|_{H_\alpha} &= \left\|e^{(t-T_0)L}u(T_0,u_0)\right\|_{H_\alpha} + \int_{T_0}^{t}\left\|L^{\alpha}e^{(t-\tau)L}\right\| \cdot \left\|\Delta g(x,u)\right\|_H d\tau \\
&= \left\|e^{(t-T_0)L}u(T_0,u_0)\right\|_{H_\alpha} + \int_{T_0}^{t}\left\|L^{\alpha+1/3}e^{(t-\tau)L}\right\| \cdot \left\|g(x,u)\right\|_H d\tau \\
&\leq \left\|e^{(t-T_0)L}u(T_0,u_0)\right\|_{H_\alpha} + \int_{T_0}^{t}\left\|L^{\alpha+1/3}e^{(t-\tau)L}\right\| \cdot \left\|g(x,u)\right\|_H d\tau \\
&\leq \left\|e^{(t-T_0)L}u(T_0,u_0)\right\|_{H_\alpha} + C\int_{T_0}^{t}\left\|L^{\alpha+1/3}e^{(t-\tau)L}\right\| d\tau \\
&\leq \left\|e^{(t-T_0)L}u(T_0,u_0)\right\|_{H_\alpha} + C\int_{0}^{t-T_0}\tau^{-(\alpha+1/3)}e^{-\sigma\tau} d\tau \\
&\leq \left\|e^{(t-T_0)L}u(T_0,u_0)\right\|_{H_\alpha} + C,
\end{aligned} \tag{4.56}$$

where $C > 0$ is a constant independent of u_0. Then, we infer from (4.55) and (4.56) that (4.50) holds for all $1/2 \leq \alpha < 2/3$.

Step 2. We will show that for any $2/3 \leq \alpha < 5/6$, the problem (1.1)–(1.3) has a bounded absorbing set in H_α.

By (4.33) and (4.51), we deduce that

$$\begin{aligned}
\|u(t,u_0)\|_{H_\alpha} &= \left\|e^{(t-T_0)L}u(T_0,u_0)\right\|_{H_\alpha} + \int_{T_0}^{t}\left\|L^{\alpha}e^{(t-\tau)L}\right\| \cdot \left\|\Delta g(x,u)\right\|_H d\tau \\
&\leq \left\|e^{(t-T_0)L}u(T_0,u_0)\right\|_{H_\alpha} + \int_{T_0}^{t}\left\|L^{\alpha+1/3-1/6}e^{(t-\tau)L}\right\| \cdot \left\|g(x,u)\right\|_{H_{1/6}} d\tau \\
&\leq \left\|e^{(t-T_0)L}u(T_0,u_0)\right\|_{H_\alpha} + C\int_{T_0}^{t}\left\|L^{\alpha+1/3-1/6}e^{(t-\tau)L}\right\| d\tau \\
&\leq \left\|e^{(t-T_0)L}u(T_0,u_0)\right\|_{H_\alpha} + C\int_{0}^{t-T_0}\tau^{-(\alpha+1/3-1/6)}e^{-\sigma\tau} d\tau \\
&\leq \left\|e^{(t-T_0)L}u(T_0,u_0)\right\|_{H_\alpha} + C,
\end{aligned} \tag{4.57}$$

where $C > 0$ is a constant independent of u_0. Thus, we verify from (4.55) and (4.57) that (4.50) is true for all $2/3 \leq \alpha < 5/6$.

By iteration, we can obtain (4.50) for all $\alpha \geq 0$. Hence, (1.1)–(1.3) has a bounded absorbing set in H_α for all $\alpha \geq 0$.

Finally, this theorem follows from (4.24), (4.50), and Lemma 4.4. The proof is completed. □

Remark 4.7. The attractors $A_\alpha \subset H_\alpha$ in Theorem 4.6 are the same for all $\alpha \geq 0$, that is, $A_\alpha = A$, for all $\alpha \geq 0$. Hence, $A \subset C^\infty(\Omega)$. Theorem 4.6 implies that for any $u_0 \in H$, the solution $u(t, u_0)$ of the problem (1.1)–(1.3) satisfies that

$$\lim_{t\to\infty} \inf_{v\in A} \|u(t, u_0) - v\|_{C^k} = 0, \quad \forall k \geq 1. \tag{4.58}$$

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References

[1] G. Gompper and J. Goos, "Fluctuating interfaces in microemulsion and sponge phases," *Physical Review E*, vol. 50, no. 2, pp. 1325–1335, 1994.
[2] G. Gompper and M. Kraus, "Ginzburg-Landau theory of ternary amphiphilic systems. I. Gaussian interface fluctuations," *Physical Review E*, vol. 47, no. 6, pp. 4289–4300, 1993.
[3] G. Gompper and M. Kraus, "Ginzburg-Landau theory of ternary amphiphilic systems. II. Monte Carlo simulations," *Physical Review E*, vol. 47, no. 6, pp. 4301–4312, 1993.
[4] J. D. Evans, V. A. Galaktionov, and J. R. King, "Unstable sixth-order thin film equation. I. Blow-up similarity solutions," *Nonlinearity*, vol. 20, no. 8, pp. 1799–1841, 2007.
[5] J. D. Evans, V. A. Galaktionov, and J. R. King, "Unstable sixth-order thin film equation. II. Global similarity patterns," *Nonlinearity*, vol. 20, no. 8, pp. 1843–1881, 2007.
[6] Z. Li and C. Liu, "On the nonlinear instability of travelingwaves for a sixth order parabolic equation," *Abstract and Applied Analysis*, vol. 2012, Article ID 739156, 17 pages, 2012.
[7] C. Liu, "Qualitative properties for a sixth-order thin film equation," *Mathematical Modelling and Analysis*, vol. 15, no. 4, pp. 457–471, 2010.
[8] C. Liu and Y. Tian, "Weak solutions for a sixth-order thin film equation," *The Rocky Mountain Journal of Mathematics*, vol. 41, no. 5, pp. 1547–1565, 2011.
[9] I. Pawłow and W. M. Zajączkowski, "A sixth order Cahn-Hilliard type equation arising in oil-water-surfactant mixtures," *Communications on Pure and Applied Analysis*, vol. 10, no. 6, pp. 1823–1847, 2011.
[10] G. Schimperna and I. Pawłow, "On a class of Cahn-Hilliardmodels with nonlinear diffusion," http://arxiv.org/abs/1106.1581.
[11] C. Liu, "Regularity of solutions for a sixth order nonlinear parabolic equation in two spacedimensions," *Annales Polonici Mathematici*. In press.
[12] M. D. Korzec, P. L. Evans, A. Münch, and B. Wagner, "Stationary solutions of driven fourth- and sixth-order Cahn-Hilliard-type equations," *SIAM Journal on Applied Mathematics*, vol. 69, no. 2, pp. 348–374, 2008.
[13] B. Nicolaenko, B. Scheurer, and R. Temam, "Some global dynamical properties of a class of pattern formation equations," *Communications in Partial Differential Equations*, vol. 14, no. 2, pp. 245–297, 1989.
[14] T. Dłotko, "Global attractor for the Cahn-Hilliard equation in H^2 and H^3," *Journal of Differential Equations*, vol. 113, no. 2, pp. 381–393, 1994.
[15] D. Li and C. Zhong, "Global attractor for the Cahn-Hilliard system with fast growing nonlinearity," *Journal of Differential Equations*, vol. 149, no. 2, pp. 191–210, 1998.
[16] H. Wu and S. Zheng, "Global attractor for the 1-D thin film equation," *Asymptotic Analysis*, vol. 51, no. 2, pp. 101–111, 2007.
[17] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, vol. 68 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1988.
[18] A. Pazy, "Semigroups of linear operators and applications to partial differential equations," in *Applied Mathematical Sciences*, vol. 44, Springer, 1983.
[19] L. Song, Y. Zhang, and T. Ma, "Global attractor of the Cahn-Hilliard equation in H^k spaces," *Journal of Mathematical Analysis and Applications*, vol. 355, no. 1, pp. 53–62, 2009.
[20] L. Song, Y. Zhang, and T. Ma, "Global attractor of a modified Swift-Hohenberg equation in H^k spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 1, pp. 183–191, 2010.