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Research Article

On t-Derivations of BCI-Algebras

G. Muhiuddin and Abdullah M. Al-roqi

Department of Mathematics, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia

Correspondence should be addressed to G. Muhiuddin, chishtygm@gmail.com

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We introduce the notion of t-derivation of a BCI-algebra and investigate related properties. Moreover, we study t-derivations in a p-semisimple BCI-algebra and establish some results on t-derivations in a p-semisimple BCI-algebra.

1. Introduction

The notion of BCK-algebra was proposed by Imai and Iséki in 1966 [1]. In the same year, Iséki introduced the notion of a BCI-algebra [2], which is a generalization of a BCK-algebra. A series of interesting notions concerning BCI-algebras were introduced and studied, several papers have been written on various aspects of these algebras [3–5]. Recently, in the year 2004 [6], Jun and Xin have applied the notion of derivation in BCI-algebras which is defined in a way similar to the notion of derivation in rings and near-rings theory which was introduced by Posner in 1957 [7]. In fact, the notion of derivation in ring theory is quite old and plays a significant role in analysis, algebraic geometry and algebra.

After the work of Jun and Xin (2004) [6], many research articles have appeared on the derivations of BCI-algebras in different aspects as follows: in 2005 [8], Zhan and Liu have given the notion of f-derivation of BCI-algebras and studied p-semisimple BCI-algebras by using the idea of regular f-derivation in BCI-algebras. In 2006 [9], Abujabal and Al-Shehri have extended the results of BCI-algebras. Further, in the next year 2007 [10], they defined and studied the notion of left derivation of BCI-algebras and investigated some properties of left derivation in p-semisimple BCI-algebras. In 2009 [11], Öztürk and Çeven have defined the notion of derivation and generalized derivation determined by a derivation for a complicated subtraction algebra and discussed some related properties. Also, in 2009 [12], Öztürk et al. have introduced the notion of generalized derivation in BCI-algebras and established some results. Further, they have given the idea of torsion free BCI-algebra and explored some properties. In 2010 [13], Al-Shehri has applied the notion of left-right (resp., right-left) derivation in BCI-algebra to B-algebra and obtained some of its properties. In 2011 [14], Ilbira et al. have studied the notion of left-right (resp., right-left) symmetric biderivation in BCI-algebras.

Motivated by a lot of work done on derivations of BCI-algebras and on derivations of other related abstract algebraic structures, in this paper we introduce the notion of t-derivations on BCI-algebras and obtain some of its related properties. Further, we characterize the notion of p-semisimple BCI-algebra X by using the notion of t-derivation and show that if d_t and d'_t are t-derivations on X, then $d_t \circ d'_t$ is also a t-derivation and $d_t \circ d'_t = d'_t \circ d_t$. Finally, we prove that $d_t * d'_t = d'_t * d_t$, where d_t and d'_t are t-derivations on a p-semisimple BCI-algebra.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

Definition 2.1 (see [2]). Let X be a set with a binary operation "$*$" and a constant 0. Then $(X, *, 0)$ is called a BCI algebra if the following axioms are satisfied for all $x, y, z \in X$:

(i) $((x * y) * (x * z)) * (z * y) = 0$,

(ii) $(x * (x * y)) * y = 0$,

(iii) $x * x = 0$,

(iv) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$.

Define a binary relation $\leq$ on X by letting $x * y = 0$ if and only if $x \leq y$. Then $(X, \leq)$ is a partially ordered set. A BCI-algebra X satisfying $0 \leq x$ for all $x \in X$, is called BCK-algebra (see [1]).

In any BCI-algebra X for all $x, y \in X$, the following properties hold.

(1) $(x * y) * z = (x * z) * y$.

(2) $x * 0 = x$.

(3) $(x * z) * (y * z) \leq x * y$.

(4) $x * 0 = 0$ implies $x = 0$.

(5) $x \leq y \Leftrightarrow x * z \leq y * z$ and $z * y \leq z * x$. A BCI-algebra X is said to be associative if for all $x, y, z \in X$, the following holds:

(6) $(x*y)*z = x*(y*z)$ [4]. Let X be a BCI-algebra, we denote $X_+ = \{x \in X \mid 0 \leq x\}$, the BCK-part of X and by $G(X) = \{x \in X \mid 0 * x = x\}$, the BCI-G part of X. If $X_+ = \{0\}$, then X is called a p-semisimple BCI-algebra. In a p-semisimple BCI-algebra X, the following properties hold.

(7) $x * (x * y) = y$.

(8) $x * (0 * y) = y * (0 * x)$.

(9) $x * y = 0$ implies $x = y$.

(10) $(x * z) * (y * z) = x * y$.

(11) $x * a = x * b$ implies $a = b$ that is left cancelable.

(12) $a * x = b * x$ implies $a = b$ that is right cancelable.

Definition 2.2 (see [6]). A subset S of a BCI-algebra X is called subalgebra of X if $x * y \in S$ whenever $x, y \in S$.

For a BCI-algebra X, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$ [6]. For more details we refer to [3, 5, 6].

3. t-Derivations in a BCI-Algebra/p-Semisimple BCI-Algebra

The following definitions introduce the notion of t-derivation for a BCI-algebra.

Definition 3.1. Let X be a-BCI-algebra. Then for any $t \in X$, we define a self map $d_t : X \rightarrow X$ by $d_t(x) = x * t$ for all $x \in X$.

Definition 3.2. Let X be a BCI-algebra. Then for any $t \in X$, a self map $d_t : X \rightarrow X$ is called a left-right t-derivation or (l,r)-t-derivation of X if it satisfies the identity $d_t(x * y) = (d_t(x) * y) \wedge (x * d_t(y))$ for all $x, y \in X$.

Similarly, we get the following.

Definition 3.3. Let X be a BCI-algebra. Then for any $t \in X$, a self map $d_t : X \rightarrow X$ is called a right-left t-derivation or (r,l)-t-derivation of X if it satisfies the identity $d_t(x * y) = (x * d_t(y)) \wedge (d_t(x) * y)$ for all $x, y \in X$.

Moreover, if d_t is both a (l,r)- and a (r,l)-t-derivation on X, we say that d_t is a t-derivation on X.

Example 3.4. Let $X = \{0, 1, 2\}$ be a BCI-algebra with the following Cayley table:

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

(3.1)

For any $t \in X$, define a self map $d_t : X \rightarrow X$ by $d_t(x) = x * t$ for all $x \in X$. Then it is easily checked that d_t is a t-derivation of X.

Proposition 3.5. *Let d_t be a self map of an associative BCI-algebra X. Then d_t is a (l,r)-t-derivation of X.*

Proof. Let X be an associative BCI-algebra, then we have

$$\begin{aligned} d_t(x * y) &= (x * y) * t \\ &= \{x * (y * t)\} * 0 \quad \text{by Property (6) and (2)} \\ &= \{x * (y * t)\} * [\{x * (y * t)\} * \{x * (y * t)\}] \quad \text{by Property (iii)} \\ &= \{x * (y * t)\} * [\{x * (y * t)\} * \{(x * y) * t\}] \quad \text{by Property (6)} \end{aligned}$$

$$
\begin{aligned}
&= \{x * (y * t)\} * [\{x * (y * t)\} * \{(x * t) * y\}] \quad \text{by Property (1)} \\
&= ((x * t) * y) \wedge (x * (y * t)) \\
&= (d_t(x) * y) \wedge (x * d_t(y)).
\end{aligned} \tag{3.2}
$$

□

Proposition 3.6. *Let d_t be a self map of an associative BCI-algebra X. Then, d_t is a (r, l)-t-derivation of X.*

Proof. Let X be an associative BCI-algebra, then we have

$$
\begin{aligned}
d_t(x * y) &= (x * y) * t \\
&= \{(x * t) * y\} * 0 \quad \text{by Property (1) and (2)} \\
&= \{(x * t) * y\} * [\{(x * t) * y\} * \{(x * t) * y\}] \quad (\text{as } x * x = 0) \\
&= \{(x * t) * y\} * [\{(x * t) * y\} * \{(x * y) * t\}] \quad \text{by Property (1)} \\
&= \{(x * t) * y\} * [\{(x * t) * y\} * \{x * (y * t)\}] \quad \text{by Property (6)} \\
&= (x * (y * t)) \wedge ((x * t) * y) \quad (\text{as } y * (y * x) = x \wedge y) \\
&= (x * d_t(y)) \wedge (d_t(x) * y).
\end{aligned} \tag{3.3}
$$

Combining Propositions 3.5 and 3.6, we get the following Theorem. □

Theorem 3.7. *Let d_t be a self map of an associative BCI-algebra X. Then, d_t is a t-derivation of X.*

Definition 3.8. A self map d_t of a BCI-algebra X is said to be t-regular if $d_t(0) = 0$.

Example 3.9. Let $X = \{0, a, b\}$ be a BCI-algebra with the following Cayley table:

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

(3.4)

(i) For any $t \in X$, define a self map $d_t : X \to X$ by

$$
d_t(x) = x * t = \begin{cases} b & \text{if } x = 0, a \\ 0 & \text{if } x = b. \end{cases} \tag{3.5}
$$

Then it is easily checked that d_t is (l, r) and (r, l)-t-derivations of X, which is not t-regular.

(ii) For any $t \in X$, define a self map $d'_t : X \to X$ by

$$d'_t(x) = x * t = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b. \end{cases} \tag{3.6}$$

Then it is easily checked that d'_t is (l, r) and (r, l)-t-derivations of X, which is t-regular.

Proposition 3.10. *Let d_t be a self map of a BCI-algebra X. Then*

(i) *If d_t is a (l, r)-t-derivation of X, then $d_t(x) = d_t(x) \wedge x$ for all $x \in X$.*

(ii) *If d_t is a (r, l)-t-derivation of X, then $d_t(x) = x \wedge d_t(x)$ for all $x \in X$ if and only if d_t is t-regular.*

Proof of (i). Let d_t be a (l, r)-t-derivation of X, then

$$\begin{aligned} d_t(x) &= d_t(x * 0) \\ &= (d_t(x) * 0) \wedge (x * d_t(0)) \\ &= d_t(x) \wedge \{x * d_t(0)\} \\ &= \{x * d_t(0)\} * [\{x * d_t(0)\} * d_t(x)] \\ &= \{x * d_t(0)\} * [\{x * d_t(x)\} * d_t(0)] \\ &\leq x * \{x * d_t(x)\} \quad \text{by Property (3)} \\ &= d_t(x) \wedge x. \end{aligned} \tag{3.7}$$

But $d_t(x) \wedge x \leq d_t(x)$ is trivial so (i) holds. □

Proof of (ii). Let d_t be a (r, l)-t-derivation of X. If $d_t(x) = x \wedge d_t(x)$ then

$$\begin{aligned} d_t(0) &= 0 \wedge d_t(0) \\ &= d_t(0) * \{d_t(0) * 0\} \\ &= d_t(0) * d_t(0) \\ &= 0 \end{aligned} \tag{3.8}$$

thereby implying d_t is t-regular. Conversely, suppose that d_t is t-regular, that is $d_t(0) = 0$, then we have

$$\begin{aligned} d_t(x) &= d_t(x * 0) \\ &= (x * d_t(0)) \wedge (d_t(x) * 0) \\ &= (x * 0) \wedge d_t(x) \\ &= x \wedge d_t(x). \end{aligned} \tag{3.9}$$

This completes the proof. □

Theorem 3.11. *Let d_t be a (l,r)-t-derivation of a p-semisimple BCI-algebra X. Then the following hold:*

(i) $d_t(0) = d_t(x) * x$ *for all* $x \in X$.

(ii) d_t *is one-one.*

(iii) *If* d_t *is t-regular, then it is an identity map.*

(iv) *if there is an element* $x \in X$ *such that* $d_t(x) = x$, *then* d_t *is identity map.*

(v) *if* $x \leq y$, *then* $d_t(x) \leq d_t(y)$ *for all* $x, y \in X$.

Proof of (i). Let d_t be a (l,r)-t-derivation of a p-semisimple BCI-algebra X. Then for all $x \in X$, we have $x * x = 0$ and so

$$\begin{aligned} d_t(0) &= d_t(x * x) \\ &= (d_t(x) * x) \wedge (x * d_t(x)) \\ &= \{x * d_t(x)\} * [\{x * d_t(x)\} * \{d_t(x) * x\}] \\ &= d_t(x) * x \quad \text{by property (7)}. \end{aligned} \tag{3.10}$$

□

Proof of (ii). Let $d_t(x) = d_t(y) \Longrightarrow x * t = y * t$, then by property (12), we have $x = y$ and so d_t is one-one. □

Proof of (iii). Let d_t be t-regular and $x \in X$. Then, $0 = d_t(0)$ so by the above part (i), we have $0 = d_t(x) * x$ and hence by property (9), we obtain $d_t(x) = x$ for all $x \in X$. Therefore, d_t is the identity map. □

Proof of (iv). It is trivial and follows from the above part (iii). □

Proof of (v). Let $x \leq y$ implying $x * y = 0$. Now,

$$\begin{aligned} d_t(x) * d_t(y) &= (x * t) * (y * t) \\ &= x * y \quad \text{by property (10)} \\ &= 0. \end{aligned} \tag{3.11}$$

Therefore, $d_t(x) \leq d_t(y)$. This completes the proof. □

Definition 3.12. Let d_t be a t-derivation of a BCI-algebra X. Then, d_t is said to be an isotone t-derivation if $x \leq y \Longrightarrow d_t(x) \leq d_t(y)$ for all $x, y \in X$.

Example 3.13. In Example 3.9(ii), d_t' is an isotone t-derivation, while in Example 3.9(i), d_t is not an isotone t-derivation.

Proposition 3.14. *Let X be a BCI-algebra and d_t be a t-derivation on X. Then for all $x, y \in X$, the following hold:*

(i) *If* $d_t(x \wedge y) = d_t(x) \wedge d_t(y)$, *then* d_t *is an isotone t-derivation.*

(ii) *If* $d_t(x * y) = d_t(x) * d_t(y)$, *then* d_t *is an isotone t-derivation.*

Proof of (i). Let $d_t(x \wedge y) = d_t(x) \wedge d_t(y)$. If $x \leq y \Longrightarrow x \wedge y = x$ for all $x, y \in X$. Therefore, we have

$$\begin{aligned} d_t(x) &= d_t(x \wedge y) \\ &= d_t(x) \wedge d_t(y) \\ &\leq d_t(y). \end{aligned} \tag{3.12}$$

Henceforth $d_t(x) \leq d_t(y)$ which implies that d_t is an isotone t-derivation. □

Proof of (ii). Let $d_t(x * y) = d_t(x) * d_t(y)$. If $x \leq y \Longrightarrow x * y = 0$ for all $x, y \in X$. Therefore, we have

$$\begin{aligned} d_t(x) &= d_t(x * 0) \\ &= d_t\{x * (x * y)\} \\ &= d_t(x) * d_t(x * y) \\ &= d_t(x) * \{d_t(x) * d_t(y)\} \\ &\leq d_t(y) \quad \text{by property (ii).} \end{aligned} \tag{3.13}$$

Thus, $d_t(x) \leq d_t(y)$. This completes the proof. □

Theorem 3.15. *Let d_t be a t-regular (r, l)-t-derivation of a BCI-algebra X. Then, the following hold:*

(i) $d_t(x) \leq x$ *for all* $x \in X$.

(ii) $d_t(x) * y \leq x * d_t(y)$ *for all* $x, y \in X$.

(iii) $d_t(x * y) = d_t(x) * y \leq d_t(x) * d_t(y)$ *for all* $x, y \in X$.

(iv) $\ker(d_t) := \{x \in X : d_t(x) = 0\}$ *is a subalgebra of* X.

Proof of (i). For any $x \in X$, we have $d_t(x) = d_t(x * 0) = (x * d_t(0)) \wedge (d_t(x) * 0) = (x * 0) \wedge (d_t(x) * 0) = x \wedge d_t(x) \leq x$. □

Proof of (ii). Since $d_t(x) \leq x$ for all $x \in X$, then $d_t(x) * y \leq x * y \leq x * d_t(y)$ and hence the proof follows. □

Proof of (iii). For any $x, y \in X$, we have

$$\begin{aligned} d_t(x * y) &= (x * d_t(y)) \wedge (d_t(x) * y) \\ &= \{d_t(x) * y\} * [\{d_t(x) * y\} * \{x * d_t(y)\}] \\ &= \{d_t(x) * y\} * 0 \\ &= d_t(x) * y \leq d_t(x) * d_t(y). \end{aligned} \tag{3.14}$$

□

Proof of (iv). Let $x, y \in \ker(d_t) \Longrightarrow d_t(x) = 0 = d_t(y)$. From (iii), we have $d_t(x * y) \leq d_t(x) * d_t(y) = 0 * 0 = 0$ implying $d_t(x * y) \leq 0$ and so $d_t(x * y) = 0$. Therefore, $x * y \in \ker(d_t)$. Consequently $\ker(d_t)$ is a subalgebra of X. This completes the proof. □

Definition 3.16. Let X be a BCI-algebra and let d_t, d'_t be two self maps of X. Then we define $d_t \circ d'_t : X \to X$ by $(d_t \circ d'_t)(x) = d_t(d'_t(x))$ for all $x \in X$.

Example 3.17. Let $X = \{0, a, b\}$ be a BCI algebra which is given in Example 3.4. Let d_t and d'_t be two self maps on X as defined in Example 3.9(i) and Example 3.9(ii), respectively.

Now, define a self map $d_t \circ d'_t : X \to X$ by

$$(d_t \circ d'_t)(x) = \begin{cases} 0 & \text{if } x = a, b \\ b & \text{if } x = 0. \end{cases} \tag{3.15}$$

Then, it is easily checked that $(d_t \circ d'_t)(x) = d_t(d'_t(x))$ for all $x \in X$.

Proposition 3.18. *Let X be a p-semisimple BCI-algebra X and let d_t, d'_t be (l, r)-t-derivations of X. Then, $d_t \circ d'_t$ is also a (l, r)-t-derivation of X.*

Proof. Let X be a p-semisimple BCI-algebra. d_t and d'_t are (l, r)-t-derivations of X. Then for all $x, y \in X$, we get

$$\begin{aligned} (d_t \circ d'_t)(x * y) &= d_t(d'_t(x * y)) \\ &= d_t\left[(d'_t(x) * y) \ \wedge (x * d'_t(y))\right] \\ &= d_t\left[(x * d'_t(y)) * \{(x * d'_t(y)) * (d'_t(x) * y)\}\right] \\ &= d_t(d'_t(x) * y) \quad \text{by property (7)} \\ &= \{x * d_t(d'_t(y))\} * \left[\{x * d_t(d'_t(y))\} * \{d_t(d'_t(x) * y)\}\right] \\ &= \{d_t(d'_t(x) * y)\} \wedge \{x * d_t(d'_t(y))\} \\ &= ((d_t \circ d'_t)(x) * y) \wedge (x * (d_t \circ d'_t)(y)). \end{aligned} \tag{3.16}$$

Therefore, $(d_t \circ d'_t)$ is a (l, r)-t-derivation of X.

Similarly, we can prove the following. □

Proposition 3.19. *Let X be a p-semisimple BCI-algebra and let d_t, d'_t be (r, l)-t-derivations of X. Then $d_t \circ d'_t$ is also a (r, l)-t-derivation of X.*

Combining Propositions 3.18 and 3.19, we get the following.

Theorem 3.20. *Let X be a p-semisimple BCI-algebra and let d_t, d'_t be t-derivations of X. Then, $d_t \circ d'_t$ is also a t-derivation of X.*

Now, we prove the following theorem.

Theorem 3.21. *Let X be a p-semisimple BCI-algebra and let d_t, d'_t be t-derivations of X. Then $d_t \circ d'_t = d'_t \circ d_t$.*

Proof. Let X be a p-semisimple BCI-algebra. d_t and d'_t, t-derivations of X. Suppose d'_t is a (l,r)-t-derivation, then for all $x, y \in X$, we have

$$\begin{aligned} (d_t \circ d'_t)(x * y) &= d_t(d'_t(x * y)) \\ &= d_t[(d'_t(x) * y) \wedge (x * d'_t(y))] \\ &= d_t[(x * d'_t(y)) * \{(x * d'_t(y)) * (d'_t(x) * y)\}] \\ &= d_t(d'_t(x) * y) \quad \text{by property (7)}. \end{aligned} \tag{3.17}$$

As d_t is a (r,l)-t-derivation, then

$$\begin{aligned} &= (d'_t(x) * d_t(y)) \wedge (d_t(d'_t(x)) * y) \\ &= d'_t(x) * d_t(y). \end{aligned} \tag{3.18}$$

Again, if d_t is a (r,l)-t-derivation, then we have

$$\begin{aligned} (d'_t \circ d_t)(x * y) &= d'_t[d_t(x * y)] \\ &= d'_t[(x * d_t(y)) \wedge (d_t(x) * y)\ ] \\ &= d'_t[x * d_t(y)] \quad \text{by property (7)} \end{aligned} \tag{3.19}$$

But d'_t is a (l,r)-t-derivation, then

$$\begin{aligned} &= (d'_t(x) * d_t(y)) \wedge (x * d'_t(d_t(y))) \\ &= d'_t(x) * d_t(y). \end{aligned} \tag{3.20}$$

Therefore from (3.18) and (3.20), we obtain

$$(d_t \circ d'_t)(x * y) = (d'_t \circ d_t)(x * y). \tag{3.21}$$

By putting $y = 0$, we get

$$(d_t \circ d'_t)(x) = (d'_t \circ d_t)(x) \quad \forall x \in X. \tag{3.22}$$

Hence, $d_t \circ d'_t = d'_t \circ d_t$. This completes the proof. □

Definition 3.22. Let X be a BCI-algebra and let d_t, d'_t be two self maps of X. Then we define $d_t * d'_t : X \to X$ by $(d_t * d'_t)(x) = d_t(x) * d'_t(x)$ for all $x \in X$.

Example 3.23. Let $X = \{0, a, b\}$ be a BCI algebra which is given in Example 3.4. Let d_t and d_t' be two self maps on X as defined in Example 3.9(i) and Example 3.9(ii), respectively.

Now, define a self map $d_t * d_t' : X \to X$ by

$$(d_t * d_t')(x) = \begin{cases} 0 & \text{if } x = a, b \\ b & \text{if } x = 0. \end{cases} \tag{3.23}$$

Then, it is easily checked that $(d_t * d_t')(x) = d_t(x) * d_t'(x)$ for all $x \in X$.

Theorem 3.24. *Let X be a p-semisimple BCI-algebra and let d_t, d_t' be t-derivations of X. Then $d_t * d_t' = d_t' * d_t$.*

Proof. Let X be a p-semisimple BCI-algebra. d_t and d_t', t-derivations of X.

Since d_t' is a (r, l)-t-derivation of X, then for all $x, y \in X$, we have

$$\begin{aligned} (d_t \circ d_t')(x * y) &= d_t[d_t'(x * y)] \\ &= d_t[(x * d_t'(y)) \wedge (d_t'(x) * y)] \\ &= d_t[x * d_t'(y)] \quad \text{by property (7).} \end{aligned} \tag{3.24}$$

But d_t is a (l, r)-t-derivation, so

$$\begin{aligned} &= (d_t(x) * d_t'(y)) \wedge (x * d_t(d_t'(y))) \\ &= d_t(x) * d_t'(y). \end{aligned} \tag{3.25}$$

Again, if d_t' is a (l, r)-t-derivation of X, then for all $x, y \in X$, we have

$$\begin{aligned} (d_t \circ d_t')(x * y) &= d_t[d_t'(x * y)] \\ &= d_t[(d_t'(x) * y) \wedge (x * d_t'(y))] \\ &= d_t[(x * d_t'(y)) * \{(x * d_t'(y)) * (d_t'(x) * y)\}] \\ &= d_t(d_t'(x) * y) \quad \text{by property (7).} \end{aligned} \tag{3.26}$$

As d_t is a (r, l)-t-derivation, then

$$\begin{aligned} &= (d_t'(x) * d_t(y)) \wedge (d_t(d_t'(x)) * y) \\ &= d_t'(x) * d_t(y). \end{aligned} \tag{3.27}$$

Henceforth from (3.25) and (3.27), we conclude

$$d_t(x) * d'_t(y) = d'_t(x) * d_t(y) \tag{3.28}$$

By putting $y = x$, we get

$$\begin{aligned} d_t(x) * d'_t(x) &= d'_t(x) * d_t(x) \\ (d_t * d'_t)(x) &= (d'_t * d_t)(x) \quad \forall x \in X. \end{aligned} \tag{3.29}$$

Hence, $d_t * d'_t = d'_t * d_t$. This completes the proof. □

4. Conclusion

Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. The theory of derivations of algebraic structures is a direct descendant of the development of classical Galois theory (namely, Suzuki [15] and Van der Put and Singer [16, 17]) and the theory of invariants. An extensive and deep theory has been developed for derivations in algebraic structures viz. BCI-algebras, C^*-algebras, commutative Banach algebras and Galois theory of linear differential equations (see, e.g., Jun and Xin [6], Ara and Mathieu [18], Bonsall and Duncan [19], Murphy [20] and Villena [21] where further references can be found). It plays a significant role in functional analysis; algebraic geometry; algebra and linear differential equations.

In the present paper, we have considered the notion of t-derivations in BCI-algebras and investigated the useful properties of the t-derivations in BCI-algebras. Finally, we investigated the notion of t-derivations in a p-semisimple BCI-algebra and established some results on t-derivations in a p-semisimple BCI-algebra. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as subtraction algebras [11], B-algebras [13], MV-algebras [22], d-algebras, Q-algebras and so forth. In future we can study the notion of t-derivations on various algebraic structures which may have a lot of applications in different branches of theoretical physics, engineering and computer science. It is our hope that this work would serve as a foundation for the further study in the theory of derivations of BCK/BCI-algebras.

In our future study of t-derivations in BCI-algebras, may be the following topics should be considered:

(1) to find the generalized t-derivations of BCI-algebras,

(2) to find more results in t-derivations of BCI-algebras and its applications,

(3) to find the t-derivations of B-algebras, Q-algebras, subtraction algebras, d-algebra and so forth.

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