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Research Article

Minimum-Norm Fixed Point of Pseudocontractive Mappings

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Let K be a closed convex subset of a real Hilbert space H and let $T : K \to K$ be a continuous pseudocontractive mapping. Then for $\beta \in (0,1)$ and each $t \in (0,1)$, there exists a sequence $\{y_t\} \subset K$ satisfying $y_t = \beta P_K[(1-t)y_t] + (1-\beta)T(y_t)$ which converges strongly, as $t \to 0^+$, to the minimum-norm fixed point of T. Moreover, we provide an explicit iteration process which converges strongly to a minimum-norm fixed point of T provided that T is Lipschitz. Applications are also included. Our theorems improve several results in this direction.

1. Introduction

Let K be a nonempty subset of a real Hilbert space H. A mapping $T : K \to H$ is called *Lipschitz* if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in K. \tag{1.1}$$

If $L \in [0,1)$, then T is called *a contraction*; if $L = 1$ then T is called *a nonexpansive*. It is easy to see from (1.1) that every contraction mapping is nonexpansive, and every nonexpansive mapping is Lipschitz.

A mapping T is called *strongly pseudocontractive* if there exists $\alpha \in (0,1)$ such that inequality

$$\langle Tx - Ty, x - y\rangle \leq \alpha\|x - y\|^2, \tag{1.2}$$

holds for all $x, y \in K$. T is called *pseudocontractive* if the inequality

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \tag{1.3}$$

holds for all $x, y \in K$. Note that inequality (1.3) can be equivalently written as

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in K. \tag{1.4}$$

It is easy to see that nonexpansive and strongly pseudocontractive mappings are pseudocontractive mappings. However, the converse may not be true (see [1, 2] for details).

Interest in pseudocontractive mappings stems mainly from their firm connection with the important class of nonlinear *monotone* mappings, where a mapping A with domain $D(A)$ and range $R(A)$ in H is called *monotone* if the inequality

$$\langle Ax - Ay, x - y \rangle \geq 0, \tag{1.5}$$

holds for every $x, y \in D(A)$. We note that A is monotone if and only if $T := I - A$ is pseudocontractive, and hence a zero of A, $N(A) := \{x \in D(A) : Ax = 0\}$ is a fixed point of T, $F(T) := \{x \in D(T) : Tx = x\}$.

Let K be a nonempty closed convex subset of a real Hilbert space H and $T : K \to K$ a pseudcontractive mapping. Assume that the set of fixed points of T is nonempty. It is known from [3] that $F(T)$ is closed and convex.

Let the variational inequality (VI) be given as finding a point x^* with the property that

$$x^* \in F(T) \text{ such that } \langle x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{1.6}$$

Then, x^* is the minimum-norm fixed point of T which exists uniquely and is exactly the (nearest point or metric) projection of the origin onto $F(T)$, that is, $x^* = P_{F(T)}(0)$. We also observe that the minimum-norm fixed point of pseudocontractive T is the minimum-norm solution of a monotone operator equation $Ax = 0$, where $A = (I - T)$.

It is quite often to seek the minimum-norm solution of a given nonlinear problem. In an abstract way, we may formulate such problems as finding a point x^* with the property

$$x^* \in K, \quad \|x^*\| = \min_{x \in K} \|x\|. \tag{1.7}$$

In other words, x^* is the projection of the origin onto K, that is,

$$x^* = P_K(0). \tag{1.8}$$

A typical example is the split feasibility problem (SFP), formulated as finding a point x^* with the property that

$$x^* \in K, \quad Ax^* \in Q, \tag{1.9}$$

where K and Q are nonempty closed convex subsets of the infinite-dimension real Hilbert spaces H_1 and H_2, respectively, and A is bounded linear mapping from H_1 to H_2. Equation (1.9) models many applied problems arising from image reconstructions and learning theory (see, e.g., [4]). Some works on the finite dimensional setting with relevant projection methods for solving image recovery problems can be found in [5–7]. Defining the proximity function f by

$$f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2, \tag{1.10}$$

we consider the convex optimization problem:

$$\min_{x\in K} f(x) := \min_{x\in K} \frac{1}{2}\|Ax - P_Q Ax\|^2. \tag{1.11}$$

It is clear that x^* is a solution to the split feasibility problem (1.9) if and only if $x^* \in K$ and $Ax^* - P_Q Ax^* = 0$ which is the minimum-norm solution of the minimization problem (1.11).

Motivated by the above split feasibility problem, we study the general case of finding the minimum-norm fixed point of a pseudocontractive mapping $T : K \to K$, that is, we find minimum norm fixed point of T which satisfies

$$x^* \in F(T) \quad \text{such that } \|x^*\| = \min\{\|x\| : x \in F(T)\}. \tag{1.12}$$

Let $T : K \to K$ be a nonexpansive self-mapping on *closed convex* subset K of a Banach space E. For a given $u \in K$ and for a given $t \in (0,1)$ define a contraction $T_t : K \to K$ by

$$T_t x = (1-t)u + tTx, \quad x \in K. \tag{1.13}$$

By Banach contraction principle, it yields a fixed point $z_t \in K$ of T_t, that is, z_t is the unique solution of the equation:

$$z_t = (1-t)u + tTz_t. \tag{1.14}$$

Browder [8] proved that as $t \to 1$, z_t converges strongly to a fixed point of T which is closer to u, that is, the nearest point projection of u onto $F(T)$. In 1980, Reich [9] extended the result of Browder to a more general Banach spaces. Furthermore, Takahashi and Ueda [10] and Morales and Jung [11] improved results of Reich [9] to the class of continuous pseudocontractive mappings. For other results on pseudocontractive mappings, we refer to [12–15].

We note that the above methods can be used to find the minimum-norm fixed point x^* of T if $0 \in K$. However, if $0 \notin K$ neither Browder's, Reich's, Takahashi and Ueda's, nor Morales and Jung's method works to find minimum-norm fixed point of T.

Our concern is now the following: is it possible to construct a scheme, implicit or explicit, which converges strongly to the minimum-norm fixed point of T for any closed convex domain K of T?

In this direction, Yang et al. [4] introduced an implicit and explicit iteration processes which converge strongly to the minimum-norm fixed point of nonexpansive self-mapping T, in real Hilbert spaces. In fact, they proved the following theorems.

Theorem YLY1 (see [4]). *Let K be a nonempty closed convex subset of a real Hilbert space H and $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$. For $\beta \in (0,1)$ and each $t \in (0,1)$, let y_t be defined as the unique solution of fixed point equation:*

$$y_t = \beta T y_t + (1-\beta) P_K\left[(1-t)y_t\right], \quad t \in (0,1). \tag{1.15}$$

Then the net $\{y_t\}$ converges strongly, as $t \to 0$, to the minimum-norm fixed point of T.

Theorem YLY2 (see [4]). *Let K be a nonempty closed convex subset of a real Hilbert space H, and let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For a given $x_0 \in K$, define a sequence $\{x_n\}$ iteratively by*

$$x_{n+1} = \beta T x_n + (1-\beta) P_K[(1-\alpha_n)x_n], \quad n \geq 1, \tag{1.16}$$

where $\beta \in (0,1)$ and $\alpha_n \in (0,1)$, satisfying certain conditions. Then the sequence $\{x_n\}$ converges strongly to the minimum-norm fixed point of T.

A natural question arises whether the above theorems can be extended to a more general class of pseudocontractive mappings or not.

Let K be a closed convex subset a real Hilbert space H and let $T : K \to K$ be continuous pseudocontractive mapping.

It is our purpose in this paper to prove that for $\beta \in (0,1)$ and each $t \in (0,1)$, there exists a sequence $\{y_t\} \subset K$ satisfying $y_t = \beta P_K[(1-t)y_t] + (1-\beta)T(y_t)$ which converges strongly, as $t \to 0^+$, to the minimum-norm fixed point of T. Moreover, we provide an explicit iteration process which converges strongly to the minimum-norm fixed point of T provided that T is Lipschitz. Our theorems improve Theorem YLY1 and Theorem YLY2 of Yang et al. [4] and Theorems 3.1, and 3.2 of Cai et al. [16].

2. Preliminaries

In what follows, we shall make use of the following lemmas.

Lemma 2.1 (see [11]). *Let H be a real Hilbert space. Then, for any given $x, y \in H$, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle. \tag{2.1}$$

Lemma 2.2 (see [17]). *Let K be a closed and convex subset of a real Hilbert space H. Let $x \in H$. Then $x_0 = P_K x$ if and only if*

$$\langle z - x_0, x - x_0\rangle \leq 0, \quad \forall z \in K. \tag{2.2}$$

Lemma 2.3 (see [18]). *Let $\{\lambda_n\}$, $\{\alpha_n\}$, and $\{\gamma_n\}$ be sequences of nonnegative numbers satisfying the conditions:* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, *and* $\gamma_n/\alpha_n \to 0$, *as* $n \to \infty$. *Let the recursive inequality:*

$$\lambda_{n+1} \leq \lambda_n - \alpha_n\psi(\lambda_{n+1}) + \gamma_n, \quad n = 1, 2, \ldots, \tag{2.3}$$

be given where $\psi : [0,\infty) \to [0,\infty)$ *is a strictly increasing function such that it is positive on* $(0,\infty)$ *and* $\psi(0) = 0$. *Then* $\lambda_n \to 0$, *as* $n \to \infty$.

Lemma 2.4 (see [3]). *Let H be a real Hilbert space, K be a closed convex subset of H and $T : K \to K$ be a continuous pseudocontractive mapping, then*

(i) *$F(T)$ is closed convex subset of K;*

(ii) *$(I - T)$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in K such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$, as $n \to \infty$, then $x = T(x)$.*

Lemma 2.5 (see [19]). *Let H be a real Hilbert space. Then for all $x, y \in H$ and $\alpha \in [0,1]$, the following equality holds:*

$$\|\alpha x + (1-\alpha)x\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2. \tag{2.4}$$

3. Main Results

Theorem 3.1. *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $T : K \to K$ be a continuous pseudocontractive mapping with $F(T) \neq \emptyset$. Then for $\beta \in (0,1)$ and each $t \in (0,1)$, there exists a sequence $\{y_t\} \subset K$ satisfying the following condition:*

$$y_t = \beta P_K[(1-t)y_t] + (1-\beta)T(y_t) \tag{3.1}$$

and the net $\{y_t\}$ converges strongly, as $t \to 0^+$, to the minimum-norm fixed point of T.

Proof. For $\beta \in (0,1)$ and each $t \in (0,1)$ let $T_t(y) := \beta P_K[(1-t)y] + (1-\beta)T(y)$. Then using nonexpansiveness of P_K and pseudocontractivity of T, for $x, y \in K$, we have that

$$\begin{aligned}
\langle T_t x - T_t y, x - y\rangle &= \beta\langle P_K[(1-t)x] - P_K[(1-t)y], x - y\rangle \\
&\quad + (1-\beta)\langle T(x) - T(y), x - y\rangle \\
&\leq \beta(1-t)\|x - y\|^2 + (1-\beta)\|x - y\|^2 \\
&\leq (1 - t\beta)\|x - y\|^2.
\end{aligned} \tag{3.2}$$

This implies that T_t is strongly pseudocontractive on K. Thus, by Corollary 1 of [20] T_t has a unique fixed point, y_t, in K. This means that the equation:

$$y_t = \beta P_K[(1-t)y_t] + (1-\beta)T(y_t) \tag{3.3}$$

has a unique solution for each $t \in (0,1)$. Furthermore, since $F(T) \neq \emptyset$, for $y^* \in F(T)$, we have that

$$\begin{aligned} \|y_t - y^*\|^2 &= \langle \beta P_K[(1-t)y_t] + (1-\beta)Ty_t - y^*, y_t - y^* \rangle \\ &= \beta \langle P_K[(1-t)y_t] - P_K y^*, y_t - y^* \rangle + (1-\beta)\langle Ty_t - Ty^*, y_t - y^* \rangle \\ &\leq \beta \|(1-t)y_t - y^*\| \cdot \|y_t - y^*\| + (1-\beta)\|y_t - y^*\|^2 \\ &\leq \beta[(1-t)\|y_t - y^*\| + t\|y^*\|]\|y_t - y^*\| + (1-\beta)\|y_t - y^*\|^2, \end{aligned} \tag{3.4}$$

which implies that

$$\|y_t - y^*\| \leq \beta(1-t)\|y_t - y^*\| + \beta t\|y^*\| + (1-\beta)\|y_t - y^*\|, \tag{3.5}$$

and hence $\|y_t - y^*\| \leq \|y^*\|$. Therefore, $\{y_t\}$ and hence $\{Ty_t\}$ is bounded.

Furthermore, from (3.3) and using nonexpansiveness of P_K we get that

$$\begin{aligned} \|y_t - Ty_t\| &= \|\beta P_K[(1-t)y_t] + (1-\beta)T(y_t) - Ty_t\| \\ &= \beta \|P_K[(1-t)y_t] - P_K Ty_t\| \\ &\leq \beta \|(1-t)y_t - Ty_t\| \\ &\leq \beta \|y_t - Ty_t\| + \beta t\|y_t\|, \end{aligned} \tag{3.6}$$

which implies that

$$\|y_t - Ty_t\| \leq \frac{\beta}{(1-\beta)} t\|y_t\| \longrightarrow 0, \quad \text{as t} \longrightarrow 0. \tag{3.7}$$

Furthermore, from (3.3), convexity of $\|\cdot\|^2$, (1.4), and (3.7), we get that

$$\begin{aligned} \|y_t - y^*\|^2 &= \|(1-\beta)(Ty_t - y^*) + \beta(P_K[(1-t)y_t] - P_K y^*)\|^2 \\ &= (1-\beta)\|Ty_t - y^*\|^2 + \beta\|P_K[(1-t)y_t] - P_K y^*\|^2 \\ &\leq (1-\beta)\left[\|y_t - y^*\|^2 + \|Ty_t - y_t\|^2\right] + \beta\|(1-t)y_t - y^*\|^2 \\ &\leq (1-\beta)\|y_t - y^*\|^2 + (1-\beta)\|Ty_t - y_t\|^2 + \beta\|(1-t)y_t - y^*\|^2 \\ &\leq (1-\beta)\|y_t - y^*\|^2 + \frac{\beta^2}{(1-\beta)} t^2\|y_t\|^2 \\ &\quad + \beta\left[\|y_t - y^*\|^2 - 2t\|y_t - y^*\|^2 - 2t\langle y^*, y_t - y^* \rangle + t^2\|y_t\|^2\right]. \end{aligned} \tag{3.8}$$

This implies that

$$\|y_t - y^*\|^2 \le \langle y^*, y^* - y_t\rangle + tM, \quad \text{for some } M > 0. \tag{3.9}$$

Now, for $t_n \to 0$, as $n \to \infty$, let $\{y_n := y_{t_n}\}$ be a subsequence of $\{y_t\}$ such that $y_n \rightharpoonup y'$. Then, we have from (3.7) and Lemma 2.4 that $y' \in F(T)$. Furthermore, replacing y^* by y' in (3.9) and the fact that $y_n \rightharpoonup y'$ imply that

$$\|y_n - y'\|^2 \le \langle y', y' - y_n\rangle + t_n M \longrightarrow 0 \quad \text{, as n} \longrightarrow \infty, \tag{3.10}$$

which implies that

$$y_n \longrightarrow y', \quad \text{as n} \longrightarrow \infty. \tag{3.11}$$

Thus, from (3.9) and (3.11), we have that

$$\|y' - y^*\|^2 \le \langle y^*, y^* - y'\rangle, \quad \text{as n} \longrightarrow \infty, \tag{3.12}$$

which is equivalent to the inequality:

$$\langle y', y^* - y'\rangle \ge 0 \qquad \text{and hence} \quad y' = P_F 0. \tag{3.13}$$

If there is another subsequence $\{y_m\}$ of $\{y_t\}$ such that $y_m \rightharpoonup y''$, similar argument gives that $y'' = P_F 0$, which implies, by uniqueness of $P_F 0$, that $y'' = y'$. Therefore, the net $y_t \longrightarrow y' = P_F 0$ which is the minimum-norm of fixed point of T. The proof is complete. □

We now state and prove a convergence theorem for the minimum-norm zero of a monotone mapping A.

Theorem 3.2. *Let H be a real Hilbert space. Let $A : H \to H$ be a continuous monotone mapping with $N(A) \neq \emptyset$. Then for $\beta \in (0,1)$ and each $t \in (0,1)$, there exists a sequence $\{y_t\} \subset H$ satisfying the following condition:*

$$y_t = \beta(1 - t)y_t + (1 - \beta)(I - A)y_t, \tag{3.14}$$

and the net $\{y_t\}$ converges strongly, as $t \to 0^+$, to the minimum-norm zero of A.

Proof. Let $Tx := (I - A)x$. Then, we get that T is continuous pseudocontractive mapping with $F(T) = N(A) \neq \emptyset$. Moreover, since P_H is an identity mapping on H, when A is replaced with $(I - T)$ scheme (3.14) reduces to scheme (3.1), and hence the conclusion follows from Theorem 3.1. □

If in Theorem 3.1, we consider $\{t_n\}, \{\beta_n\} \subset (0,1)$ such that $t_n \to 0$, $\beta_n \to 0$ and $y_n := y_{t_n}$, the method of proof of Theorem 3.1 provides the following corollary.

Corollary 3.3. *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $T : K \to K$ be continuous pseudocontractive mapping with $F(T) \neq \emptyset$. Then the sequence $\{y_n\} \subset K$ defined by*

$$y_n = \beta_n P_K[(1-t_n)y_n] + (1-\beta_n)T(y_n), \tag{3.15}$$

where $\{t_n\}, \{\beta_n\} \subset (0,1)$ such that $t_n \to 0$, $\beta_n \to 0$, as $n \to \infty$, converges strongly, as $n \to \infty$, to the minimum-norm fixed point of T.

The following proposition and lemma play an important role in proving the next theorem.

Proposition 3.4. *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $T : K \to K$ be continuous pseudocontractive mapping. Then the sequence $\{y_n\}$ in* (3.15) *satisfies the following inequality:*

$$\|y_n - y_{n-1}\| \leq \frac{|\theta_{n-1}-\theta_n|}{\theta_n t_n}\left[\|y_n\| + \|P_K[(1-t_n)y_{n-1}]\|\right] + \frac{\theta_{n-1}}{\theta_n}\frac{|t_n - t_{n-1}|}{t_n}\|y_{n-1}\|, \tag{3.16}$$

where $\theta_n := \beta_n/(1-\beta_n)$ for $\{\beta_n\}$ decreasing sequence.

Proof. If we put $\theta_n := \beta_n/(1-\beta_n)$, (3.15) reduces to

$$y_n = Ty_n + \theta_n(P_K[(1-t_n)y_n] - y_n). \tag{3.17}$$

Thus, using pseudocontractivity of T and nonexpansiveness of P_K we get that

$$\begin{aligned}
\|y_n - y_{n-1}\|^2 &= \|Ty_n + \theta_n(P_K[(1-t_n)y_n] - y_n) - Ty_{n-1} - \theta_{n-1}(P_K[(1-t_{n-1})y_{n-1}] - y_{n-1})\|^2 \\
&= \|Ty_n - Ty_{n-1} + \theta_{n-1}y_{n-1} - \theta_n y_n + \theta_{n-1}y_n - \theta_{n-1}y_n \\
&\qquad + \theta_n P_K[(1-t_n)y_n] - \theta_{n-1}P_K[(1-t_{n-1})y_{n-1}]\|^2 \\
&= \langle Ty_n - Ty_{n-1} + \theta_{n-1}(y_{n-1} - y_n) + (\theta_{n-1}-\theta_n)\ y_n, y_n - y_{n-1}\rangle \\
&\quad + \langle \theta_n P_K[(1-t_n)y_n] - \theta_n P_K[(1-t_n)y_{n-1}], y_n - y_{n-1}\rangle \\
&\quad + \langle \theta_n P_K[(1-t_n)y_{n-1}] - \theta_{n-1}P_K[(1-t_n)y_{n-1}], y_n - y_{n-1}\rangle \\
&\quad + \langle \theta_{n-1}P_K[(1-t_n)y_{n-1}] - \theta_{n-1}P_K[(1-t_{n-1})y_{n-1}], y_n - y_{n-1}\rangle \\
&\leq \|y_n - y_{n-1}\|^2 - \theta_{n-1}\|y_n - y_{n-1}\|^2 + (\theta_{n-1}-\theta_n)\|y_n\| \\
&\quad \times \|y_n - y_{n-1}\| + \theta_n(1-t_n)\|y_n - y_{n-1}\|^2 \\
&\quad + (\theta_n - \theta_{n-1})\|P_K[(1-t_n)y_{n-1}]\| \cdot \|y_{n-1} - y_n\| \\
&\quad + \theta_{n-1}|t_n - t_{n-1}| \cdot \|y_{n-1}\|\|y_n - y_{n-1}\|,
\end{aligned} \tag{3.18}$$

which implies, using the fact that θ_n is decreasing, that

$$\begin{aligned}\|y_n - y_{n-1}\| &\le [1-\theta_{n-1}+\theta_n(1-t_n)]\|y_n - y_{n-1}\| + |\theta_{n-1}-\theta_n|[\|y_n\| + \|P_K[(1-t_n)y_{n-1}]\|] \\ &\quad + \theta_{n-1}|t_n - t_{n-1}|\cdot\|y_{n-1}\| \\ &\le (1-t_n\theta_n)\|y_n - y_{n-1}\| + |\theta_{n-1}-\theta_n|[\|y_n\| + \|P_K[(1-t_n)y_{n-1}]\|] \\ &\quad + \theta_{n-1}|t_n - t_{n-1}|\cdot\|y_{n-1}\|,\end{aligned} \tag{3.19}$$

and hence

$$\|y_n - y_{n-1}\| \le \frac{|\theta_{n-1}-\theta_n|}{\theta_n t_n}[\|y_n\| + \|P_K[(1-t_n)y_{n-1}]\|] + \frac{\theta_{n-1}}{\theta_n}\frac{|t_n - t_{n-1}|}{t_n}\|y_{n-1}\|. \tag{3.20}$$

The proof is complete. □

For the rest of this paper, let $\{\lambda_n\}$, $\{\theta_n\}$ (decreasing) and $\{t_n\}$ be real sequences in $(0,1]$ satisfying the following conditions: (i) $lim_{n\to\infty}\theta_n = 0 = lim_{n\to\infty}t_n$; (ii) $\lambda_n(1+\theta_n) \le 1$, $\sum \lambda_n\theta_n t_n = \infty$, $lim_{n\to\infty}\lambda_n/\theta_n t_n = 0$; (iii) $lim_{n\to\infty}[\theta_{n-1}-\theta_n]/\lambda_n\theta_n^2 t_n^2 = 0$ and $lim_{n\to\infty}[t_{n-1} - t_n]/\lambda_n\theta_n t_n^2 = 0$. Examples of real sequences which satisfy these conditions are $\lambda_n = 1/(n+1)^{1/2}$, $\theta_n = 1/(n+1)^{1/3}$ and $t_n = 1/(n+1)^{1/14}$.

Lemma 3.5. *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \to K$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \ge 0$ and $F(T) \ne \emptyset$. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n\theta_n(x_n - P_K[(1-t_n)x_n]), \tag{3.21}$$

for all positive integers $n \ge 1$. Then $\{x_n\}$ is bounded.

Proof. We follow the method of proof of Chidume and Zegeye [21]. Since $\lambda_n/(\theta_n t_n) \to 0$, there exists $N_0 > 0$ such that $\lambda_n/(\theta_n t_n) \le d := 1/(2(3+L)^2)$, for all $n \ge N_0$. Let $x^* \in F(T)$ and $r > 0$ be sufficiently large such that $x_{N_0} \in B_r(x^*)$ and $\|x^*\| \le r/(2(4+L))$. Now, we show by induction that $\{x_n\}$ belongs to $B := \overline{B_r(x^*)}$ for all integers $n \ge N_0$. By construction, we have $x_{N_0} \in B$. Assume that $x_n \in B$ for any $n > N_0$. Then, we prove that $x_{n+1} \in B$. Suppose x_{n+1} is not in B. Then $\|x_{n+1}-x^*\| > r$, and thus from the recursion formula (1.2) and Lemma 2.1 we get that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|x_n - x^* - \lambda_n((x_n - Tx_n) + \theta_n(x_n - P_K[(1-t_n)x_n]))\|^2 \\
&= \|x_n - x^*\|^2 - 2\lambda_n\langle (x_n - Tx_n) \\
&\quad + \theta_n(x_n - P_K[(1-t_n)x_n]), j(x_{n+1}-x^*)\rangle \\
&= \|x_n - x^*\|^2 - 2\lambda_n\theta_n\langle x_{n+1}-x^*, x_{n+1}-x^*\rangle \\
&\quad + 2\lambda_n\langle \theta_n(x_{n+1}-x_n) - (x_n - Tx_n) + \theta_n(P_K[(1-t_n)x_n] - x^*) \\
&\quad + (x_{n+1} - Tx_{n+1}) - (x_{n+1} - Tx_{n+1}), j(x_{n+1}-x^*)\rangle.
\end{aligned} \tag{3.22}$$

Since T is pseudocontractive we have $\langle x_{n+1} - Tx_{n+1}, j(x_{n+1}-x^*)\rangle \geq 0$. Thus, (3.22) gives

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \|x_n - x^*\|^2 - 2\lambda_n\theta_n\|x_{n+1}-x^*\|^2 \\
&\quad + 2\lambda_n(2+L)\|x_{n+1}-x_n\| \cdot \|x_{n+1}-x^*\| \\
&\quad + 2\lambda_n\theta_n\langle P_K[(1-t_n)x_n] - x^*, j(x_{n+1}-x^*)\rangle \\
&= \|x_n - x^*\|^2 - 2\lambda_n\theta_n\|x_{n+1}-x^*\|^2 \\
&\quad + 2\lambda_n(2+L)\|x_{n+1}-x_n\| \cdot \|x_{n+1}-x^*\| \\
&\quad + 2\lambda_n\theta_n\langle P_K[(1-t_n)x_n] - P_K[(1-t_n)x_{n+1}] + P_K[(1-t_n)x_{n+1}] \\
&\quad - P_K[(1-t_n)x^*] + P_K[(1-t_n)x^*] - x^*, j(x_{n+1}-x^*)\rangle,
\end{aligned} \tag{3.23}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \|x_n - x^*\|^2 - 2\lambda_n\theta_n t_n\|x_{n+1}-x^*\|^2 \\
&\quad + 2\lambda_n(2+L+(1-t_n))\|x_{n+1}-x_n\| \cdot \|x_{n+1}-x^*\| \\
&\quad + 2\lambda_n\theta_n\|P_K[(1-t_n)x^*] - x^*\| \cdot \|x_{n+1}-x^*\| \\
&= \|x_n - x^*\|^2 - 2\lambda_n\theta_n t_n\|x_{n+1}-x^*\|^2 \\
&\quad + 2\lambda_n(3+L)[\lambda_n\|\theta_n(P_K[(1-t_n)x_n] - P_K[(1-t_n)x^*] \\
&\quad + P_K[(1-t_n)x^*] - x^* + x^* - x_n) + Tx_n - Tx^* + x^* - x_n\|] \\
&\quad \times \|x_{n+1}-x^*\| + 2\lambda_n\theta_n t_n\|x^*\| \cdot \|x_{n+1}-x^*\| \\
&\leq \|x_n - x^*\|^2 - 2\lambda_n\theta_n t_n\|x_{n+1}-x^*\|^2 \\
&\quad + 2\lambda_n^2(3+L)^2\|x_n - x^*\| \cdot \|x_{n+1}-x^*\| \\
&\quad + 2\lambda_n\theta_n t_n(4+L)\|x^*\|\|x_{n+1}-x^*\|.
\end{aligned} \tag{3.24}$$

Since $\|x_{n+1} - x^*\| > \|x_n - x^*\|$, from (3.24) we get that

$$\|x_{n+1} - x^*\| \le \frac{\lambda_n}{\theta_n t_n}(3+L)^2\|x_n - x^*\| + (4+L)\|x^*\|, \tag{3.25}$$

and hence $\|x_{n+1} - x^*\| \le r$, since $x_n \in B$, $\|x^*\| \le r/(2(4+L))$ and $\lambda_n/\theta_n t_n \le 1/2(3+L)^2$ for all $n \ge N_0$. But this is a contradiction. Therefore, $x_n \in B$ for all positive integers $n \ge N_0$, and hence the sequence $\{x_n\}$ is bounded. □

For the next theorem, let $\{y_n\}$ denotes the sequence defined by $y_n := y_{s_n} = s_n T y_{s_n} + (1-s_n)P_K[(1-t_n)y_n], s_n = 1/(1+\theta_n)$, for all $n \ge 1$, guaranteed by Corollary 3.3 (which reduces to $\theta_n(P_K[(1-t_n)y_n] - y_n) - (y_n - Ty_n) = 0$).

Theorem 3.6. *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \to K$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \ge 0$ and $F(T) \ne \emptyset$. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n\theta_n(x_n - P_K[(1-t_n)x_n]), \tag{3.26}$$

for all positive integers $n \ge 1$. Then $\{x_n\}$ converges strongly to the minimum-norm fixed point of T, as $n \to \infty$.

Proof. By Lemma 3.5, we have that the sequence $\{x_n\}$ is bounded. Now, we show that it converges strongly to a minimum-norm fixed point of T. But from (3.26) and Lemma 2.1, we have that

$$\begin{aligned}
\|x_{n+1} - y_n\|^2 \le{}& \|x_n - y_n\|^2 - 2\lambda_n\theta_n\langle (x_{n+1} - y_n), j(x_{n+1} - y_n)\rangle \\
&+ 2\lambda_n\langle \theta_n(x_{n+1} - y_n) - (x_n - Tx_n) \\
&-\theta_n(x_n - P_K[(1-t_n)x_n]), j(x_{n+1} - y_n)\rangle \\
={}& \|x_n - y_n\|^2 - 2\lambda_n\theta_n\|x_{n+1} - y_n\|^2 + 2\lambda_n\langle \theta_n(x_{n+1} - x_n) \\
&+ [\theta_n(P_K[(1-t_n)y_n] - y_n) - (y_n - Ty_n)] - [(x_{n+1} - Tx_{n+1}) \\
&-(y_n - Ty_n)] + \theta_n(P_K[(1-t_n)x_n] - P_K[(1-t_n)y_n]) \\
&+[(x_{n+1} - Tx_{n+1}) - (x_n - Tx_n)], j(x_{n+1} - y_n)\rangle.
\end{aligned} \tag{3.27}$$

Observe that by the property of y_n and pseudocontractivity of T we have $\theta_n(P_K[(1-t_n)y_n] - y_n) - (y_n - Ty_n) = 0$ (see (3.17)) and $\langle (x_{n+1} - Tx_{n+1}) - (y_n - Ty_n), j(x_{n+1} - y_n)\rangle \ge 0$ for all $n \ge 1$. Thus, we have from (3.27) that

$$\begin{aligned}
\|x_{n+1} - y_n\|^2 \le{}& \|x_n - y_n\|^2 - 2\lambda_n\theta_n\|x_{n+1} - y_n\|^2 + 2\lambda_n\langle \theta_n(x_{n+1} - x_n) \\
&+ \theta_n(P_K[(1-t_n)x_n] - P_K[(1-t_n)x_{n+1}] \\
&+P_K[(1-t_n)x_{n+1}] - P_K[(1-t_n)y_n]) \\
&+(x_{n+1} - Tx_{n+1}) - (x_n - Tx_n), j(x_{n+1} - y_n)\rangle
\end{aligned}$$

$$\leq \left\|x_n - y_n\right\|^2 - 2\lambda_n\theta_n t_n\left\|x_{n+1} - y_n\right\|^2 + 2\lambda_n(3+L)\|x_{n+1} - x_n\| \cdot \left\|x_{n+1} - y_n\right\|. \tag{3.28}$$

But by Corollary 3.3, we have that $\{y_n\}$ is bounded. Therefore, there exists $M_1 > 0$ such that $\max\{(3+L)\|x_{n+1} - y_n\| \cdot \|x_n - Tx_n + \theta_n(x_n - P_K[(1-t_n)x_n])\|\} \leq M_1$. Thus from (3.28), we get that

$$\left\|x_{n+1} - y_n\right\|^2 \leq \left\|x_n - y_n\right\|^2 - 2\lambda_n\theta_n t_n\left\|x_{n+1} - y_n\right\|^2 + 2\lambda_n^2 M_1. \tag{3.29}$$

But using triangle inequality and Proposition 3.4, we have that

$$\begin{aligned} \left\|x_n - y_n\right\|^2 &\leq \left[\left\|x_n - y_{n-1}\right\| + \left\|y_{n-1} - y_n\right\|\right]^2 \\ &\leq \left\|x_n - y_{n-1}\right\|^2 + \left\|y_{n-1} - y_n\right\| M_2 \\ &\leq \left\|x_n - y_{n-1}\right\|^2 + \frac{|\theta_{n-1} - \theta_n|}{\theta_n t_n} M_3 + \frac{|t_n - t_{n-1}|}{t_n} M_3, \end{aligned} \tag{3.30}$$

for some $M_2, M_3 > 0$, and for all $n \geq N_0$. Now, substituting (3.30) in (3.29) we obtain that

$$\begin{aligned} \left\|x_{n+1} - y_n\right\|^2 \leq{}& \left\|x_n - y_{n-1}\right\|^2 - 2\lambda_n\theta_n t_n\left\|x_{n+1} - y_n\right\|^2 \\ &+ 2\lambda_n^2 M_4 + \frac{\theta_{n-1} - \theta_n}{\theta_n t_n} M_4 + \frac{|t_n - t_{n-1}|}{t_n} M_4, \end{aligned} \tag{3.31}$$

for some constant $M_4 > 0$. Now, by Lemma 2.3 and the conditions on $\{\lambda_n\}$, $\{\theta_n\}$, and $\{t_n\}$ we get $x_{n+1} - y_n \to 0$. Consequently, $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Therefore, since by Corollary 3.3 we have that $y_t \to y^* \in F(T)$, where y^* is with the minimum-norm in $F(T)$, we get that $\{x_n\}$ converges strongly to the minimum-norm of fixed point of T. □

Corollary 3.7. *Let H be a real Hilbert space. Let $A : H \to H$ be a Lipschitz monotone mapping with Lipschitz constant $L \geq 0$ and $N(A) \neq \emptyset$. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in H$ by*

$$x_{n+1} = x_n - \lambda_n A x_n + \lambda_n\theta_n t_n x_n, \tag{3.32}$$

for all positive integers n. Then $\{x_n\}$ converges strongly to the minimum-norm solution of the equation $Ax = 0$.

Proof. Let $T := (I - A)$. Then T is a Lipschitz pseudocontractive mapping with Lipschitz constant $L' := (L + 1)$, and the minimum-norm solution of the equation $Ax = 0$ is the minimum-norm fixed point of T. Moreover, if we replace T by $(I - A)$ in (3.26), then the equation reduces to (3.32). Thus, the conclusion follows from Theorem 3.6. □

4. Applications

For the rest of this paper, let H be a Hilbert space and $A : H \to H$ a bounded linear operator. Consider the convexly constrained linear inverse problem, which has extensively been discussed in the literature (see, e.g., [22]), given by:

$$x \in K, \quad Ax = b, \tag{4.1}$$

where K is closed and convex subset of H and $b \in H$, which is a special case of the SFP problem (1.9). Set

$$\varphi(x) := \frac{1}{2}\|Ax - b\|^2. \tag{4.2}$$

The least-square solution of (4.1) is the least-norm minimizer of the minimization problem (4.2). Let Ω denote the solution set of (4.2). It is known that Ω is nonempty if and only if $P_{\overline{A(K)}}(b) \in A(\text{K})$. In this case, Ω has a unique element with minimum norm which is a least-square solution of (4.1), that is, there exists a unique point $x^* \in \Omega$ such that

$$\|x^*\| = \min\{\|x\| : x \in \Omega\}. \tag{4.3}$$

We note that $\varphi(x)$ is a quadratic function with gradient:

$$\nabla\varphi(x) = A^*(Ax - b), \tag{4.4}$$

where A^* is adjoint of A. Let $\gamma > 0$ and $x^* \in \Omega$. Thus, x^* is the minimum-norm solution of the minimization problem (4.2) if and only if x^* a solution of

$$\gamma\nabla\varphi(x) = \gamma A^*(Ax - b) = 0. \tag{4.5}$$

Now, we state applications of our theorems.

Theorem 4.1. *Assume that the solution set of convexly constrained linear inverse problem* (4.1) *with $K := H$, a real Hilbert space, is nonempty and that $\nabla\varphi$ is monotone. Then for $\beta \in (0,1)$ and each $t \in (0,1)$, there exists a sequence $\{y_t\} \subset H$ satisfying the following condition:*

$$y_t = \beta(1-t)y_t + (1-\beta)\left(y_t - \gamma A^*(Ay_t - b)\right), \tag{4.6}$$

where A^* is adjoint of A, and the net $\{y_t\}$ converges strongly, as $t \to 0^+$, to the minimum-norm solution of the split feasibility problem (4.1).

Proof. We note that $\varphi(x)$ is continuously differentiable function with gradient:

$$\nabla\varphi(x) = A^*(Ax - b), \tag{4.7}$$

where A^* is adjoint of A, which is Lipschitz (see Lemma 8.1 of [5]) and monotone (by hypothesis). Thus, the conclusion follows from Theorem 3.2. □

Theorem 4.2. *Assume that the solution set of split feasibility problem* (4.1) *is nonempty and that* $\nabla\varphi$ *with* $K := H$*, a real Hilbert space, is monotone. Let a sequence* $\{x_n\}$ *be generated from arbitrary* $x_1 \in E$ *by*

$$x_{n+1} = x_n - \lambda_n \gamma A^*(Ax_n - b) + \lambda_n \theta_n t_n x_n, \tag{4.8}$$

for all positive integers n*, where* $\gamma > 0$ *and* A^* *is adjoint of* A*. Then,* $\{x_n\}$ *converges strongly to the minimum-norm solution of the split feasibility problem* (4.1)*.*

Remark 4.3. Theorem 3.1 improves Theorem YLY1 and Theorem 3.1 of Cai et al. [16] to a more general class of pseudocontractive mappings. Moreover, Theorem 3.6 improves Theorem YLY1 and Theorem 3.2 of Cai et al. [16] in the sense that our scheme provides a minimum-norm fixed point of pseudocontractive mapping T.

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