Research Article

Characterization of Eigenvalues in Spectral Gap for Singular Differential Operators

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The spectral properties for n order differential operators are considered. When given a spectral gap (a, b) of the minimal operator T_0 with deficiency index r, arbitrary m points β_i $(i = 1, 2, \ldots, m)$ in (a, b), and a positive integer function p such that $\sum_{i=1}^{m} p(\beta_i) \leq r$, T_0 has a self-adjoint extension $\widetilde{T}$ such that each β_i $(i = 1, 2, \ldots, m)$ is an eigenvalue of $\widetilde{T}$ with multiplicity at least $p(\beta_i)$.

1. Introduction

In this paper, we consider the following nth-order formal symmetric differential expression:

$$\tau y = w^{-1} \left\{ \sum_{j=0}^{[n/2]} (-1)^j \left(p_j y^{(j)} \right)^{(j)} + \sum_{j=0}^{[n-1/2]} (-1)^j \left[\left(q_j y^{(j)} \right)^{(j+1)} - \left(q_j^* y^{(j+1)} \right)^{(j)} \right] \right\}, \tag{1.1}$$

on (α, β), where $-\infty \leq \alpha < \beta \leq +\infty$, $y = y(t)$ is a complex-valued m-vector function, $p_j(t), q_j(t)$ and $w(t)$ are measurable and locally integrable $m \times m$ matrices, $p_j(t), w(t)$ are Hermitian, and $w(t) > 0$. *a.e.* $t \in (\alpha, \beta)$.

The spectral properties of nth-order differential expression, particularly the distribution of eigenvalues, have been widely researched in these years (see [1–8] and references cited therein).

Let us recall some known results due to Neumann [1], Stone [2], Friedrichs [3] and Krein [4]. For convenience of the reader in addition to these original sources we will also give text book references [7].

An open interval (a, b) with $-\infty < a < b < \infty$ is called a spectral gap of a symmetric operator A if

$$\left\|\left(A - \frac{a+b}{2}\right)f\right\|_w \geq \frac{b-a}{2}\|f\|_w \quad \forall f \in D(A). \tag{1.2}$$

$\|\cdot\|_w$ and $\langle\cdot,\cdot\rangle_w$ are defined in Section 2. It is easy to find that $(a+b)/2$ is a real regular point of A. If $\langle Af, f\rangle_w \geq b\|f\|_w^2$, we shall also say that $(-\infty, b)$ is a spectral gap of A (the latter definition is not generally used but convenient for our purpose; and it is justified since each interval (a, b) with $a < b$ is a spectral gap of A). Let (a, b) be a spectral gap of A, then there exists a self-adjoint extension $\tilde{A}$ of A (for instance the famous Friedrichs extension) such that $(a, b) \subset \Gamma(\tilde{A})$ (here $\Gamma(\tilde{A})$ is the regular-form domain of $\tilde{A}$ which will be defined in Definition 2.4), this is the reason why we call (a, b) a spectral gap of A.

Suppose in addition that the deficiency index of a symmetric operator A is equal to n $(n \in \mathbb{N})$. Let $\tilde{A}$ be a self-adjoint extension of A. The sum of the multiplicities of the eigenvalues of $\tilde{A}$ within the interval (a, b) is at most n, and no point of the continuous spectrum of $\tilde{A}$ lies in the interval (a, b) (cf. [5, Theorem 8.19 and Corollary 2 in Section 8.3]). Conversely, given any finite subset $\mathcal{E}$ of (a, b) and positive integers $p(\lambda)$, $\lambda \in \mathcal{E}$, such that $\sum_{\lambda \in \mathcal{E}} p(\lambda) \leq n$ there exists a self-adjoint extension $\tilde{A}$ of A such that $\sigma(\tilde{A}) \cap (a, b) = \mathcal{E}$ and the multiplicities of λ as an eigenvalue of $\tilde{A}$ are equal $p(\lambda)$ for each $\lambda \in \mathcal{E}$ (cf. [4]). If one only requires that (a, b) is some interval within the set of real regular points of A, then the corresponding statement is false. For instance, one can give a symmetric operator $\hat{A}$, such that each $\lambda \in \mathbb{R}$ is a regular point of $\hat{A}$, the deficiency index of $\hat{A}$ is equal one and each self-adjoint extension $\tilde{A}$ of $\hat{A}$ has a periodic point spectrum with period p independent of $\tilde{A}$ (cf. [4]).

This paper consists of three sections including the introduction. In Section 2, we present some preliminary materials that include definitions and theorems needed for the rest of the paper. In Section 3, we give three main results in the study of self-adjoint extensions of a minimal operator generated by differential expression τ. Firstly, we present a partial self-adjointness of the minimal operator T_0. Secondly, if (a, b) is a spectral gap of T_0, for all $\beta_1, \beta_2, \ldots \beta_m \in (a, b)$ $(m \leq r, r$ is the deficiency index of $T_0)$ and positive integer function p satisfying $\sum_{i=1}^m p(\beta_i) = r$, T_0 has a self-adjoint extension $\tilde{T}$ with the following properties:

(i) $\sigma_p(\tilde{T}) \cap (a, b) = \{\beta_1, \beta_2, \ldots \beta_m\}$;

(ii) each β_i $(i = 1, 2, \ldots m)$ is an eigenvalue of $\tilde{T}$ with multiplicity equal to $p(\beta_i)$;

(iii) $\tilde{T}$ has pure point spectrum within (a, b).

Finally, given a symmetric operator T_0 with the deficiency index being equal to r, $(a, b) \subset \mathbb{R}$, $\beta_1, \beta_2, \ldots \beta_m \in (a, b)$ $(m \leq r)$ being real regular points of T_0, (the interval (a, b) need not be a spectral gap of T_0) and positive integer function p satisfying $\sum_{i=1}^m p(\beta_i) \leq r$, T_0 has a self-adjoint extension $\tilde{T}$ with the properties that each β_i $(i = 1, 2, \ldots m)$ is an eigenvalue of $\tilde{T}$ with multiplicity at least β_i.

2. Preliminaries

In this section, we introduce notations, definitions, and some theorems that are needed in this paper.

First, we define the following space:

$$L_2(\alpha, \beta; w) := \left\{ f : \int_\alpha^\beta f^*(t) w(t) f(t) < +\infty \right\}, \tag{2.1}$$

with the inner product

$$\langle f, g \rangle_w = \int_\alpha^\beta g^*(t) w(t) f(t) dt, \tag{2.2}$$

where the weight function $w(t)$ is the same as that in (1.1). Denote $\|f\|_w = (\langle f, f \rangle_w)^{1/2}$ for $f \in L_2(\alpha, \beta; w)$. If $f \in L_2(\alpha, \beta; w)$, then f is called square integrable. Here, we note that if w is singular, $L_2(\alpha, \beta; w)$ is a quotient space in the sense that $y = z$ if and only if $\|y - z\|_w = 0$. In this case, $L_2(\alpha, \beta; w)$ is a Hilbert space.

Now we introduce the maximal operator T and minimal operator T_0 generated by the expression τ.

Definition 2.1. The maximal operator T generated by τ is defined by

$$D(T) = \left\{ f \in L_2(\alpha, \beta; w) : f^{\{0\}}, f^{\{1\}}, \dots f^{\{n-1\}} \in AC_{\text{loc}}(\alpha, \beta), \ \tau f \in L_2(\alpha, \beta; w) \right\},$$
$$Tf = \tau f \quad \text{for } f \in D(T), \tag{2.3}$$

where $AC_{\text{loc}}(\alpha, \beta)$ denotes the collection of functions on (α, β) which are absolutely continuous locally. Roughly speaking the jth quasi-derivative $u^{\{j\}}$ will be collection of terms that, if differentiated $(n - j)$ times, is "part" of the differential expression $r\tau u$ (see [7] for details). We know that T is densely defined and closed.

Definition 2.2. The preminimal operator T_{00} generated by τ is defined by

$$D(T_{00}) = \{ f \in D(T) : f \text{ has compact support in } (\alpha, \beta) \},$$
$$T_{00} f = \tau f = Tf \quad \text{for } f \in D(T_{00}). \tag{2.4}$$

Obviously, $T_{00} \subset T$ and T_{00} is Hermitian. It is easy to know that $D(T_{00})$ is dense, so T_{00} is symmetric, and T_{00} is not closed [8]. The closure of T_{00} is called the (closed) minimal operator denoted by T_0.

Definition 2.3. Given a linear operator A with domain and range in a Hilbert space $\mathbb{H}$, the resolvent set of A is

$$\rho(A) = \left\{ \lambda \in \mathbb{C} \mid (\lambda - A)^{-1} \in B(\mathbb{H}) \right\}, \tag{2.5}$$

the spectrum of A is the set $\sigma(A) = \mathbb{C} \setminus \rho(A)$. Denote

$$\sigma_p = \{\lambda : (\lambda - A)^{-1} \text{ does not exist}\},$$

$$\sigma_c = \{\lambda : (\lambda - A)^{-1} \text{ exists and } \overline{R(\lambda - A)} = \mathbb{H}, \text{ but } (\lambda - A)^{-1} \text{ is not continuous}\},$$

$$\sigma_r = \{\lambda : (\lambda - A)^{-1} \text{ exists and } \overline{R(\lambda - A)} \neq \mathbb{H}\}.$$

Obviously, $\sigma(A) = \sigma_p \cup \sigma_c \cup \sigma_r$ (see [8]).

Definition 2.4. Let A be a linear operator in a Hilbert space $\mathbb{H}$. The set of regular points of A, is called the regular-form domain of A, denoted by $\gamma(A)$:

$$\Gamma(A) = \{z \in \mathbb{C} : A - z \text{ is continuously invertible}\}$$

$$= \{z \in \mathbb{C} : \text{there exists } a \quad v = v(z) > 0 \text{ such that } \|(A - z)f\| \geq v\|f\|, \forall f \in D(A)\} \tag{2.6}$$

it is an open subset of $\mathbb{C}$ containing $\mathbb{C} \setminus \mathbb{R}$.

Definition 2.5. For a closed operator A in a Hilbert space $\mathbb{H}$, the essential spectrum of A is defined as

$$\sigma_e(A) = \left\{\lambda \in \sigma(A) : R(\lambda - A) \neq \overline{R(\lambda - A)}\right\}. \tag{2.7}$$

We assume that T_0 has real regular points, that is, $\Gamma(T_0) \cap \mathbb{R} \neq \emptyset$. In this case the deficiency index

$$n(T_0, z) = n(T_0) := \dim \ker(T - z) = \dim R(T_0 - \overline{z})^{\perp} \tag{2.8}$$

does not depend on the special choice of the regular point of T_0 and consequently T_0 has self-adjoint extensions.

The kernel $N(T - z)$ is the vector space of those solutions of the differential equation $(\tau - z)u = 0$ which are elements of $L_2(\alpha, \beta; w)$. Since the space of all solutions of $(\tau - z)u = 0$ (which in general is not contained in $L_2(\alpha, \beta; w)$) has dimension $p := n \times m$, this implies

$$0 \leq n(T_0, z) \leq p. \tag{2.9}$$

Definition 2.6. We say that an operator A has pure point spectrum within (a, b) if $\sigma(A) \cap (a, b) = \overline{\sigma_p(A)} \cap (a, b)$.

Proposition 2.7. *Let A be a self-adjoint operator in a Hilbert space $\mathbb{H}$. One has $\rho(A) = \Gamma(A)$.*

Proof. From Definitions 2.3–2.5, we have $\rho(A) \subseteq \Gamma(A)$ and σ_p, σ_c do not belong to $\Gamma(A)$, so $\rho(A) \cup (\sigma_r \cap \Gamma(A)) = \Gamma(A)$, A is densely defined since it is a self-adjoint operator, so $\sigma_r = \emptyset$, then we get $\rho(A) = \Gamma(A)$.   □

The following results give the relation between deficiency index and self-adjoint extension.

Proposition 2.8. *If T_{00}, T_0, T are the preminimal operator, minimal operator and maximal operator generated by τ, and $\widetilde{T}$ is one of the self-adjoint extension of T_0, then One has:*

$$T_{00} \subset \overline{T_{00}} = T_0 \subset \widetilde{T} = \left(\widetilde{T}\right)^* \subset T = T_0^*. \tag{2.10}$$

Proposition 2.9. *If A is Hermitian, then $\mathbb{C} \setminus \mathbb{R} \subset \Gamma(A)$ (cf. [5, Proposition 2, page 229]).*

Proposition 2.10. *The deficiency index is constant on each connected subset of $\Gamma(A)$. If A is Hermitian, then the deficiency index is constant in the upper and lower half-planes (cf. [5, Theorem 8.1]).*

Proposition 2.11. *A closed symmetric operator is self-adjoint if and only if its deficiency index is equal to 0.*

Proposition 2.12. *Let A be a symmetric operator,*

(a) *A has self-adjoint extension if and only if its deficiency indices are equal,*

(b) *$\Gamma(A) \cap \mathbb{R} \neq \phi$, then A has self-adjoint extensions,*

(c) *if A is semibounded, then A has self-adjoint extensions,*

We can find the proof from Theorem 8.8 in [7].

Proposition 2.13 (see [8]). *Let A be a closed symmetric operator in $L_2(\alpha, \beta; w)$ and $A_1 = A^*$. Set $N_0(\lambda - A_1) = N(\lambda - A_1) \cap D(A)$. If there exists a $\lambda \in \mathbb{R}$ such that $\lambda \notin \sigma_e(A)$, then A has a self-adjoint extension and*

$$\dim N(\lambda - A_1) = n(A) + \dim N_0(\lambda - A_1). \tag{2.11}$$

Furthermore, there exists a self-adjoint extension $\widehat{A}$ of A such that

$$[N(\lambda - A_1) \ominus N_0(\lambda - A_1)] \cap D\left(\widehat{A}\right) = \{0\}. \tag{2.12}$$

3. Main Results

Let T_0, T be the minimal operator and maximal operator generated by τ in (1.1) and let the deficiency index of T_0 be equal to r $(0 < r \leq p)$. In this section we assume that T_0 has real regular points. That is, $\Gamma(T_0) \cap \mathbb{R} \neq \emptyset$.

Definition 3.1. A closed subspace $\mathcal{M}$ of $L_2(\alpha, \beta; w)$ is called a reducing subspace of T_0 if $Pf \in D(T_0)$, and $T_0 Pf = P T_0 f$ for all $f \in D(T_0)$, where P denotes the orthogonal projection in $D(T_0)$ onto $\mathcal{M}$.

Obviously along with $\mathcal{M}$ the orthogonal complement $\mathcal{M}^\perp$ of $\mathcal{M}$ is also a reducing subspace of T_0. It is easy to see that the closed span of an orthogonal system of eigenvectors of T_0 is a reducing subspace of T_0. If $\mathcal{M}$ is a reducing subspace of T_0, then the part of T_0 in $\mathcal{M}$, that is, the restriction of T_0 to $D(T_0) \cap \mathcal{M}$ may be (and in the following will be) regarded as an operator in the Hilbert space $\mathcal{M}$.

Since $\Gamma(T_0) \cap \mathbb{R} \neq \emptyset$, there exists a real point λ of T_0, and the positive and negative deficiency indices of T_0 are equal. We choose a one-dimensional subspace $\mathcal{L}_\lambda$ of $\ker(T_0^* - \lambda)$ $(= \ker(T - \lambda))$, set

$$D(T_0') := D(T_0) \dotplus \mathcal{L}_\lambda,$$
$$T_0'(f + g) := T_0 f + \lambda g \quad \forall f \in D(T_0) \text{ and } \forall g \in \mathcal{L}_\lambda. \tag{3.1}$$

Obviously T_0' is a symmetric extension of T_0, and $n(T_0', z) = n(T_0, z) - 1$ for each regular point z of T_0' and $\mathcal{L}_\lambda$ the corresponding eigenspace. Since the graph of T_0' is a one-dimensional extension of the closed graph of T_0, therefore the operator T_0' is closed [6].

Theorem 3.2. *Assume that the deficiency index of T_0 is equal to r, let (a, b) be a spectral gap of T_0, and let T_0' be defined as above.*

(1) *If T_0' has an eigenvalue $\mu \in (a, b)$, then (a, b) is a spectral gap of the restriction of T_0' to the Hilbert space $\ker (T_0' - \mu)^\perp$.*

(2) *The deficiency index of the restriction of T_0' to the Hilbert space $\ker(T_0' - \mu)^\perp$ is equal to $r - 1$.*

Proof. (1) First we consider the case that $-\infty < a < b < +\infty$. Without loss of generality, we may assume that $(a, b) = (-v, v)$. We have to show that

$$\|T_0' f\| \geq v \|f\| \quad \forall f \in D(T_0') \cap \ker (T_0' - \mu)^\perp. \tag{3.2}$$

Assume that $\|T_0' f\| < v \|f\|$ for some $f \in D(T_0') \cap \ker(T_0' - \mu)^\perp$. Let $g \in \ker(T_0' - \mu)$ and $g \neq 0$. Since $\ker(T_0' - \mu)$ reduces T_0',

$$\begin{aligned}
\|T_0'(c_1 f + c_2 g)\|^2 &= |c_1|^2 \|T_0' f\|^2 + |c_2|^2 \|T_0' g\|^2 \\
&= |c_1|^2 \|T_0' f\|^2 + |c_2|^2 \mu \|g\|^2 \\
&< |c_1|^2 v^2 \|f\|^2 + |c_2|^2 v^2 \|g\|^2 \\
&= v^2 \|c_1 f + c_2 g\|^2,
\end{aligned} \tag{3.3}$$

for all $(c_1, c_2) \in \mathbb{C}^2 \setminus (0, 0)$. Since f and g span a two-dimensional subspace of $D(T_{0'})$ and $\dim D(T_0') \setminus D(T_0) = 1$, it follows that $\|T_0' h\| < v \|h\|$ for some $h \in D(T_0)$, this is a contradiction to the hypothesis that $(-v, v)$ is a spectral gap of T_0. This completes the proof in the case that $-\infty < a < b < +\infty$. $\qquad \square$

The proof in the semibounded case is similar. One only has to replace expression of the form $\|T_0 f\|$ by those of the form $\langle T_0 f, f \rangle$, here we omit the details.

(2) Since $n(T_{0'}, z) = n(T_0, z) - 1$, that is, $n(T_0', z) = r - 1$, and the restriction of T_0' to the Hilbert space $\ker(T_0' - \mu)$ is self-adjoint, so the deficiency index of the restriction of T_0' onto Hilbert space $\ker(T_0' - \mu)^\perp$ is equal to $r - 1$.

Theorem 3.3. *Suppose the deficiency index of T_0 is equal to r and λ is a real regular point of T_0. Let $f_1, f_2, \ldots f_m \in L_2(\alpha, \beta; w)$ be linearly independent solutions of $(\tau - \lambda)f = 0$. One has the following.*

(1) *T_0 has a j dimensional symmetric extension T_0^j $(j = 1, 2, \ldots m)$ with the following properties:*

$$D\left(T_0^j\right) := D(T_0) + \mathrm{span}\{f_1\} + \cdots + \mathrm{span}\{f_m\},$$

$$T_0^j(f + g_1 + g_2 + \cdots + g_m) = T_0 f + \lambda(g_1 + g_2 + \cdots + g_m), \tag{3.4}$$

$$\forall f \in D(T_0) \text{ and } \forall g_j \in \mathrm{span}\{f_j\} \ (j = 1, 2 \ldots m).$$

(2) *$m \leq r$.*

(3) *If $m = r$, then T_0 has a self-adjoint extension.*

Proof. (1) Since $\mathrm{span}\{f_1\}$ is a one-dimensional subspace of $\ker(T - \lambda)$, we define

$$D\left(T_0^1\right) := D(T_0) + \mathrm{span}\{f_1\},$$

$$T_0^1(f + g) := Tf + \lambda g \quad \forall f \in D(T_0) \text{ and } \forall g \in \mathrm{span}\{f_1\}. \tag{3.5}$$

Obviously, T_0^1 is a symmetric extension of T_0, $n(T_0^1, z) = n(T_0, z) - 1 = r - 1$, and λ is an eigenvalue of T_0^1, and span $\{f_1\}$ is the corresponding eigenspace, and T_0^1 is also closed. By induction, we get

$$n\left(T_0^j, z\right) = n(T_0, z) - j = r - j, \tag{3.6}$$

and T_0^j $(j = 1, 2 \ldots m)$ is a closed symmetric operator.

(2) When $m > r$, then $n(T_0^m, z) = n(T_0, z) - m = r - m < 0$, that is a contradiction, so we have $m \leq r$.

(3) When $m = r$, then $n(T_0^m, z) = n(T_0, z) - m = r - r = 0$, so from **Proposition 2.11**, we have that T_0^m is a self-adjoint extension of T_0. $\square$

Theorem 3.4. *Assume that the deficiency index of T_0 is equal to r. If (a, b) is a spectral gap of T_0, then for all $\beta_1, \beta_2, \ldots \beta_m \in (a, b)$ (where $m \leq r$) and positive integer function p satisfying $\sum_{i=1}^m p(\beta_i) = r$, the minimal operator T_0 has a self-adjoint extension $\widetilde{T}$ with the following properties:*

(i) *$\sigma_p(\widetilde{T}) \cap (a, b) = \{\beta_1, \beta_2, \ldots \beta_m\}$,*

(ii) *each β_j $(j = 1, 2, \ldots m)$ is an eigenvalue of $\widetilde{T}$ with multiplicity $p(\beta_j)$,*

(iii) *$\widetilde{T}$ has pure point spectrum within (a, b).*

Proof. We choose $\lambda_1, \lambda_2, \ldots \lambda_r$ in (a, b), such that $\#\{l \leq r : \lambda_l = \beta_j\} = p(\beta_j)$ $(j = 1, \ldots, m)$ ($\#M$ denote the cardinality of the set M), in other words each $\beta_j \in (a, b)$ occurs exactly $p(\beta_j)$ times in the sequence $\{\lambda_1, \lambda_2, \ldots \lambda_r\}$.

We choose a one-dimensional space $\mathcal{L}_{\lambda_1}$ of $\ker(T - \lambda_1)$ and define

$$D(T_1') := D(T_0) \dotplus \mathcal{L}_{\lambda_1},$$

$$T_1'(f + g) := T_0 f + \lambda g \quad \forall f \in D(T_0) \text{ and } \forall g \in \mathcal{L}_{\lambda_1}. \tag{3.7}$$

Then T_1' is a closed symmetric extension of T_0 with deficiency index being equal to $r - 1$ and λ_1 is an eigenvalue of T_1' with eigensubspace being $\mathcal{L}_{\lambda_1}$.

Set $\mathcal{H}_1 = \mathcal{L}_{\lambda_1}^\perp$ and denote by M_1 and T_1 the restriction of T_1' to $\mathcal{L}_{\lambda_1}$ and $\mathcal{H}_1$, respectively. Obviously M_1 is a self-adjoint operator in the one-dimensional Hilbert space $\mathcal{L}_{\lambda_1}$ with $\sigma(M_1) = \{\lambda_1\}$. T_1 is a closed symmetric operator in Hilbert space $\mathcal{H}_1$, with deficiency index being equal to $r - 1$, and $T_1' = M_1 \dotplus T_1$. Moreover, by Lemma 3.1, (a, b) is a spectral gap of T_1. Thus we can replace $L_2(\alpha, \beta; W)$ by $\mathcal{H}_1$, T_0 by T_1, and λ_1 by λ_2 in the above conclusions.

Proceeding further in this way by induction, we obtain sequences $\{\mathcal{L}_{\lambda_1}, \ldots \mathcal{L}_{\lambda_r}\}$ and $\{\mathcal{H}_1, \ldots \mathcal{H}_r\}$ of Hilbert spaces and sequence $\{M_1, \ldots M_r\}$ and $\{T_1', \ldots T_r'\}$ of the operators with the following properties:

(i) T_l' are closed symmetric extension of T_0 which satisfy: $T_1' \subset T_2' \subset \cdots \subset T_r'$, T_l' with deficiency indices $r - l$ $(l = 1, 2, \ldots r)$,

(ii) M_l is a self-adjoint operator on a one-dimensional subspace $\mathcal{L}_{\lambda_l}$ of $\mathcal{H}_{l-1}$ ($\mathcal{H}_0 := L_2(\alpha, \beta; w)$), $\mathcal{H}_l = \mathcal{H}_{l-1} \ominus \mathcal{L}_{\lambda_l}$, with $\sigma(M_l) = \{\lambda_l\}$ for each λ_l $(l = 1, 2, \ldots r)$,

(iii) T_l is a closed symmetric operator in the Hilbert space $\mathcal{H}_l$ with spectral gap (a, b) for all $l = 1, 2 \ldots r$.

(iv) $T_l' = \{\oplus_{j=1}^l M_j\} \oplus T_l$ $(l = 1, \ldots r)$.

Then we have that T_r' is a closed symmetric extension of T_0 with deficiency index 0. So $\widetilde{T} = T_r'$ is a self-adjoint extension of T_0 with the required properties. $\qquad \square$

Theorem 3.5. *Assume the deficiency index of T_0 is equal to r. Let $\beta_1, \beta_2, \ldots \beta_m \in (a, b)$ (where $m \le r$, (a, b) need not be a spectral gap of T_0) be real regular points of T_0 and let positive integer function p satisfy $\sum_{j=1}^m p(\beta_j) \le r$. Moreover, one assumes that $(\tau - \lambda_i)f = 0$ has at least $p(\beta_i)$ square integrable solutions. Then T_0 has a self-adjoint extension $\widetilde{T}$ such that each β_j $(j = 1, 2, \ldots m)$ is an eigenvalue of $\widetilde{T}$ with multiplicity at least equal to $p(\beta_j)$.*

Proof. Choose a sequence $\lambda_1, \ldots \lambda_r$ in (a, b) such that each β_i occurs at least $p(\beta_i)$ times in this sequence (the times k_i of the λ_i appears in the sequence satisfies $p(\beta_i) \le k \le \widetilde{k}_i$, $\widetilde{k}_i$ is the number of square integrable solutions of $(\tau - \lambda_i)f = 0$; the λ_i which is not in $\{\beta_1, \ldots, \beta_m\}$ will be selected as a regular point of T satisfying $\ker(T - \lambda_i) \ne \emptyset$). Since the deficiency index of T_0 is equal to r, the negative and positive deficiency indices are equal to r, so we select a normalized orthogonal basis $f_1^+ \cdots f_r^+ \in R(T_0 + i)^\perp$ and $f_1^- \cdots f_r^- \in R(T_0 - i)^\perp$. Now, we define by induction, orthonormal sequences $g_1, \ldots g_r \in D(T)$ as follows:

for $k = 1$, we choose normalized element $g_1 \in \ker(T - \lambda_1)$ with

$$\langle f_1^+, g_1 \rangle = 0 = \langle f_1^-, g_1 \rangle. \tag{3.8}$$

Suppose for given $k < r-1$, the $g_1, \ldots, g_k$ $(g_j \in \ker(T-\lambda_j))$ are well defined with the properties

$$\langle f_i^+, g_j \rangle = 0 = \langle f_i^-, g_j \rangle \quad \forall 1 \leq i, j \leq k, \tag{3.9}$$

(those g_i are eigenfunctions of T relating to λ_j, so they are orthogonal if $\lambda_i \neq \lambda_j$, and if $\lambda_i = \lambda_j$, we select different orthogonal eigenfunctions of T relating to λ_j), we choose a normalized element $g_{k+1} \in \ker(T - \lambda_{k+1})$ with

$$\left\langle g_{k+1}, f_j^+ \right\rangle = 0 = \left\langle g_{k+1}, f_j^- \right\rangle \quad \forall 1 \leq j \leq k+1. \tag{3.10}$$

When $k = r-1$, we stop the induction process. So $\{g_1, \ldots g_r\}$ is an orthonormal sequence with the following properties:

(i) $g_k \in \ker(T - \lambda_k)$ for all $k = 1, \ldots r$,

(ii) $f_k^+ \in R(T_0 + i)^\perp$ and $f_k^- \in R(T_0 - i)^\perp$ for all $k = 1, \ldots r$,

(iii) $\langle g_j, f_k^+ \rangle = \langle g_j, f_k^- \rangle = 0$ for all $j, k \in \{1, \ldots r\}$.

Set

$$D\left(\tilde{T}\right) := D(T_0) + \mathrm{span}\{g_1, \ldots g_r\},$$
$$\tilde{T}h := Th \quad \forall h \in D\left(\tilde{T}\right). \tag{3.11}$$

Obviously $\tilde{T}$ is a symmetric extension of T_0 and $\tilde{T}g_j = Tg_j = \lambda_j g_j$ for $j = 1, \ldots m$, so each β_j is an eigenvalue of $\tilde{T}$ with multiplicity at least equal to $p(\beta_j)$. For every $k \in \{1, \ldots, r\}$, we have

$$\left\langle f_k^+, \left(\tilde{T} + i\right)f \right\rangle = \langle f_k^+, (T_0 + i)f \rangle = 0 \quad \forall f \in D(T_0),$$
$$\left\langle f_k^+, \left(\tilde{T} + i\right)g_j \right\rangle = \langle f_k^+, (\lambda_j + i)g_j \rangle = 0 \quad \forall j \in \{1, \ldots r\}. \tag{3.12}$$

It follows that $f_k^+ \in R(\tilde{T} + i)^\perp$ for all $k = 1, \ldots, r$. Thus we have

$$\dim R\left(\tilde{T} + i\right)^\perp = \dim R(T_0 + i)^\perp - r = 0. \tag{3.13}$$

Analogously, it can be shown that

$$\dim R\left(\tilde{T} - i\right)^\perp = \dim R(T_0 - i)^\perp - r = 0. \tag{3.14}$$

Thus $\dim R(\tilde{T} + i)^\perp = \dim R(\tilde{T} - i)^\perp = 0$ and consequently $\tilde{T}$ is a self-adjoint extensions which has the properties mentioned above. $\qquad \square$

Acknowledgments

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