

## Review Article

# A Survey on Extremal Problems of Eigenvalues

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Given an integrable potential  $q \in L^1([0, 1], \mathbb{R})$ , the Dirichlet and the Neumann eigenvalues  $\lambda_n^D(q)$  and  $\lambda_n^N(q)$  of the Sturm-Liouville operator with the potential  $q$  are defined in an implicit way. In recent years, the authors and their collaborators have solved some basic extremal problems concerning these eigenvalues when the  $L^1$  metric for  $q$  is given;  $\|q\|_{L^1} = r$ . Note that the  $L^1$  spheres and  $L^1$  balls are nonsmooth, noncompact domains of the Lebesgue space  $(L^1([0, 1], \mathbb{R}), \|\cdot\|_{L^1})$ . To solve these extremal problems, we will reveal some deep results on the dependence of eigenvalues on potentials. Moreover, the variational method for the approximating extremal problems on the balls of the spaces  $L^\alpha([0, 1], \mathbb{R})$ ,  $1 < \alpha < \infty$  will be used. Then the  $L^1$  problems will be solved by passing  $\alpha \downarrow 1$ . Corresponding extremal problems for eigenvalues of the one-dimensional  $p$ -Laplacian with integrable potentials have also been solved. The results can yield optimal lower and upper bounds for these eigenvalues. This paper will review the most important ideas and techniques in solving these difficult and interesting extremal problems. Some open problems will also be imposed.

## 1. Introduction

Minimization and maximization problems for eigenvalues are important in applied sciences like optimal control theory, population dynamics [1–3], and propagation speeds of traveling waves [4, 5]. They are also interesting mathematical problems [6, 7] because the solutions to them involve many different branches of mathematics. In recent years, some fundamental properties of eigenvalues have been revealed [8–16], such as strong continuity of eigenvalues in potentials/weights in the sense of weak topologies and continuous differentiability of eigenvalues in potentials/weights in the sense of the usual  $L^\alpha$  norms. Based on such eigenvalue properties and some topological facts on  $L^\alpha$  spaces, several interesting extremal problems for eigenvalues with  $L^1$  potentials/weights have been solved via variational methods and limiting approaches [17–23]. This paper will give a brief survey of the

papers mentioned above and outline the ideas wherein to solve the extremal problems of eigenvalues.

To illustrate the problems and the ideas more explicitly, let us first focus on one typical model. It is well known that for any integrable potential  $q \in \mathcal{L}^1 := L^1([0, 1], \mathbb{R})$ , all eigenvalues of the Sturm-Liouville operator

$$x'' + (\lambda + q(t))x = 0, \quad t \in [0, 1], \quad (1.1)$$

associated with the Dirichlet boundary conditions  $x(0) = x(1) = 0$  are given by a sequence

$$\lambda_1(q) < \lambda_2(q) < \cdots < \lambda_n(q) < \cdots, \quad (1.2)$$

with  $\lim_{n \rightarrow \infty} \lambda_n(q) = +\infty$ . Let us consider such extremal problems as

$$I_n(r) := \inf_{q \in B_1[r]} \lambda_n(q), \quad M_n(r) := \sup_{q \in B_1[r]} \lambda_n(q) \quad (1.3)$$

for any  $n \in \mathbb{N}$ , where  $B_1[r]$  is a ball of radius  $r$  in  $\mathcal{L}^1$ . These problems are interesting and of difficulty, because  $\lambda_n(\cdot)$  are implicit functionals of  $q \in \mathcal{L}^1$ , while, topologically,  $B_1[r]$  is not compact or sequentially compact even in the weak topology  $w_1$  of  $\mathcal{L}^1$  [24, 25], and, geometrically,  $B_1[r]$  is also nonsmooth in the space  $(\mathcal{L}^1, \|\cdot\|_1)$ . Therefore, they cannot be solved directly by using the standard variational method.

The main ideas developed in recent papers [17–23] to solve such extremal problems for implicit functionals on noncompact nonsmooth sets are in two steps.

Firstly, for any  $\alpha > 1$ , we consider the counterparts of  $I_n$  and  $M_n$  with potentials confined to  $B_\alpha[r]$  (balls in  $\mathcal{L}^\alpha := L^\alpha([0, 1], \mathbb{R})$  of radius  $r$ ), denoted by  $I_{n,\alpha}$  and  $M_{n,\alpha}$ , respectively. Since the functional  $\lambda_n(\cdot)$  is continuous (in weak topology) and continuously differentiable (in the usual  $L^\alpha$  norm) in potential  $q \in \mathcal{L}^1$  (and hence in  $q \in \mathcal{L}^\alpha$ ,  $\alpha > 1$ ), and the balls  $B_\alpha[r]$ ,  $\alpha > 1$ , are compact in weak topology and smooth in the usual  $\mathcal{L}^\alpha$  norm  $\|\cdot\|_\alpha$ , both the minimum and the maximum can be obtained, and one can study the critical potentials via standard variational methods. In this step, one can find a critical equation, in which the critical potential, denoted by  $q_\alpha$ , the critical eigenfunction, denoted by  $y_\alpha$ , and the extremal value  $I_{n,\alpha}$  or  $M_{n,\alpha}$  are all involved.

Secondly, we will employ the limiting approach  $\alpha \downarrow 1$  to obtain  $I_n$  and  $M_n$ . This step is based on some topological facts that balls and spheres in  $\mathcal{L}^1$  space can be approximated by balls and spheres in  $\mathcal{L}^\alpha$  spaces as  $\alpha \downarrow 1$ . Then the strong continuity of eigenvalue  $\lambda_n(\cdot)$  in potentials ensures that  $\lim_{\alpha \downarrow 1} I_{n,\alpha} = I_n$  and  $\lim_{\alpha \downarrow 1} M_{n,\alpha} = M_n$ . In this step, besides critical equations in  $\mathcal{L}^\alpha$  spaces, properties of critical potentials and critical eigenfunctions should also be sufficiently utilized to get a final solution to  $I_n$  and  $M_n$ .

The final solution to problems in (1.3) can yield optimal lower and upper bounds of eigenvalues. Actually, from (1.3), we have

$$I_n(\|q\|_1) \leq \lambda_n(q) \leq M_n(\|q\|_1) \quad \forall q \in \mathcal{L}^1, \quad (1.4)$$

which are optimal.

Several extremal problems on (Dirichlet, Neumann, and periodic) eigenvalues (of the Sturm-Liouville operator and the  $p$ -Laplace operator) have been studied recently, where the potentials are confined to different sets such as balls or spheres in  $\mathcal{L}^1$ . For definition of these problems, see (4.1). In the following sections, we give a slightly detailed description on these results. Some topological facts about  $\mathcal{L}^\alpha$ ,  $\alpha \geq 1$ , are listed in Section 2. The essential of this section is that those “bad” balls or spheres in  $\mathcal{L}^1$  (neither smooth nor weak compact) can be approximated by “good” balls or spheres in  $\mathcal{L}^\alpha$ ,  $\alpha > 1$  (smooth and weak compact). Section 3 is devoted to some properties of eigenvalues, including the scaling results which enable us to consider only those integrable potentials defined on the interval  $[0, 1]$ , the relationship between the first and higher-order eigenvalues which plays a role to enable us to consider only the first order, and the strong continuity (in weak topology) together with the continuous differentiability (in strong topology) which enable us to apply the variational method to problems confined on those “good” balls or spheres in  $\mathcal{L}^\alpha$ ,  $\alpha > 1$ . In Section 4 we introduce how variational method is applied to get critical equations for the extremal problems for cases  $\alpha > 1$ . After analysis on the critical equations, these extremal values are determined by some singular integrals. However, they cannot be expressed by elementary functions. Section 5 deals with the extremal problems in  $L^1$  spaces via limiting approaches. Final results of extremal values in  $\mathcal{L}^1$  case are stated in this section. For the Sturm-Liouville operator, these extremal values can be expressed by elementary functions of the radius  $r$ . In Section 6, the corresponding extremal problems for eigenvalues of measure differential equations [26, 27] are discussed briefly. The minimizing measures for the problems in (1.3) are explained. In Section 7, two open problems for further study are imposed. One is on the eigenvalue gaps and the other is on the corresponding extremal problems of eigenvalues of the beam equation with integrable potentials.

## 2. Some Topological Facts on $\mathcal{L}^\alpha$ Spaces

In the Lebesgue space  $\mathcal{L}^\alpha$ ,  $1 \leq \alpha \leq \infty$ , the usual  $\mathcal{L}^\alpha$  topology is induced by  $\mathcal{L}^\alpha$  norm  $\|\cdot\|_\alpha$ . Besides this strong topology, one has also the weak topology  $w_\alpha$  which is defined as follows [24, 25].

*Definition 2.1.* Let  $q_n, q \in \mathcal{L}^\alpha$ . We say that  $q_n$  is weakly convergent to  $q$ , written as  $q_n \xrightarrow{w_\alpha} q$  in  $\mathcal{L}^\alpha$ , or  $q_n \rightarrow q$  in  $(\mathcal{L}^\alpha, w_\alpha)$ , if

$$\lim_{n \rightarrow \infty} \int_0^1 q_n(t)u(t)dt = \int_0^1 q(t)u(t)dt \quad \forall u \in \mathcal{L}^{\alpha^*}. \quad (2.1)$$

Here  $\alpha^*$  is the conjugate exponent of  $\alpha$ :  $\alpha^* = \alpha/(\alpha - 1) \in [1, \infty]$ .

For  $\alpha \in [1, \infty)$ ,  $r \in [0, \infty)$  and  $h \in [-r, r]$ , let us take the following notations:

$$\begin{aligned} S_\alpha[r] &= \{q \in \mathcal{L}^\alpha : \|q\|_\alpha = r\}, \\ B_\alpha[r] &= \{q \in \mathcal{L}^\alpha : \|q\|_\alpha \leq r\}, \\ S_\alpha[r, h] &= \{q \in \mathcal{L}^\alpha : \|q\|_\alpha = r, \bar{q} = h\}, \\ B_\alpha[r, h] &= \{q \in \mathcal{L}^\alpha : \|q\|_\alpha \leq r, \bar{q} = h\}, \end{aligned} \tag{2.2}$$

where  $\bar{q} = \int_0^1 q(t)dt$  is the mean value of  $q$ .

### 2.1. Approximation to $\mathcal{L}^1$ Balls/Spheres in Strong Topology

**Lemma 2.2** (see [21, Lemma 2.1]). *Given that  $r > 0$  and  $q \in S_1[r]$ , there exists  $q_\alpha \in S_\alpha[r]$  such that  $\lim_{\alpha \downarrow 1} \|q_\alpha - q\|_1 = 0$ .*

**Lemma 2.3** (see [19, Lemma 2.3]). *Let  $r > 0$  and  $h \in (-r, r)$ . For any  $q \in S_1[r, h]$ , there exists  $\{q_\alpha\} \in S_\alpha[r, h]$ ,  $1 < \alpha < \infty$ , such that  $\lim_{\alpha \downarrow 1} \|q_\alpha - q\|_1 = 0$ .*

Lemmas 2.2 and 2.3 give very nice topological approximation to spheres in  $\mathcal{L}^1$  space because the spheres in  $\mathcal{L}^\alpha$  ( $1 < \alpha < \infty$ ) space have nicer topological and geometric properties.

*Remark 2.4.* A direct consequence from Lemmas 2.2 and 2.3 is that,  $\mathcal{L}^1$  balls  $B_1[r]$  and  $B_1[r, h]$  can be approximated, in the sense of  $\mathcal{L}^1$  norm, by  $\mathcal{L}^\alpha$  balls  $B_\alpha[r]$  and  $B_\alpha[r, h]$ , respectively.

### 2.2. Approximation to $\mathcal{L}^1$ Balls/Spheres in Weak Topology

For any  $\alpha \in (1, \infty)$ , it is well known that  $B_\alpha[r]$  is compact and sequentially compact in the space  $(\mathcal{L}^\alpha, w_\alpha)$  [24, 25].

**Lemma 2.5** (see [19, Lemma 2.5]). *Suppose that  $\alpha \in (1, \infty)$ . Then, for any  $r > 0$  and  $h \in [-r, r]$ ,  $B_\alpha[r, h]$  is compact and sequentially compact in the space  $(\mathcal{L}^\alpha, w_\alpha)$ .*

The balls  $B_1[r]$  and  $B_1[r, h]$  are not compact or sequentially compact even in weak topology. Remark 2.4 tells us that such “bad” balls can be approximated, in the sense of  $\mathcal{L}^1$  norm, by “good” balls  $B_\alpha[r]$  and  $B_\alpha[r, h]$ , which are compact in weak topology  $w_\alpha$  and smooth in geometry. In fact, there hold the following stronger topological facts.

**Lemma 2.6** (see [19, Lemma 2.6]). *(i) Let  $\alpha \in [1, \infty)$ ,  $r \in (0, \infty)$  and  $h \in (-r, r)$ . For any  $q \in B_\alpha[r, h]$ , there exists a sequence  $\{q_n\} \subset S_\alpha[r, h]$  such that  $q_n \rightarrow q$  in  $(\mathcal{L}^\alpha, w_\alpha)$ . In other words, the closure of  $S_\alpha[r, h]$  in the space  $(\mathcal{L}^\alpha, w_\alpha)$  is  $B_\alpha[r, h]$ .*

*(ii) Consequently, for any  $q \in S_\alpha[r, h]$ ,  $1 \leq \alpha < \infty$ , there exists a sequence  $\{q_n\} \subset S_1[r, h]$  such that  $q_n \rightarrow q$  in  $(\mathcal{L}^1, w_1)$ .*

### 3. Properties of Eigenvalues on Potentials/Weights

Denote by  $\phi_p(\cdot)$  the scalar  $p$ -Laplacian, that is,  $\phi_p(x) = |x|^{p-2}x$  for any  $x \neq 0$  and  $\phi_p(0) = 0$ . For any integrable potential  $q \in L^1([a, b], \mathbb{R})$ , it is well known that all eigenvalues of

$$(\phi_p(x'))' + (\lambda + q(t))\phi_p(x) = 0, \quad t \in [a, b] \tag{3.1}$$

associated with the Dirichlet boundary condition  $x(a) = x(b) = 0$  consist of a sequence

$$\lambda_1^D(q) < \lambda_2^D(q) < \dots < \lambda_n^D(q) < \dots, \tag{3.2}$$

and all eigenvalues associated with the Neumann boundary condition  $x'(a) = x'(b) = 0$  consist of a sequence as follows:

$$\lambda_0^N(q) < \lambda_1^N(q) < \dots < \lambda_n^N(q) < \dots, \tag{3.3}$$

see [28, 29]. To emphasize the dependence on the interval  $[a, b]$  and the boundary conditions, we also write the eigenvalues as  $\lambda_n^\sigma(q, [a, b])$ , where the superscript  $\sigma$  can be chosen as  $D$  (for the Dirichlet eigenvalues) or  $N$  (for the Neumann eigenvalues).

#### 3.1. Scaling Results on Eigenvalues

Let  $I = [a, b]$  be a finite interval of length  $|I| = b - a$ . Given that  $\alpha \in [1, \infty]$  and  $q \in \mathcal{L}^\alpha := L^\alpha([0, 1], \mathbb{R})$ , define a potential  $q_I \in L^\alpha(I, \mathbb{R})$  by the following:

$$q_I(t) := |I|^{-p}q\left(\frac{t-a}{|I|}\right), \quad t \in I. \tag{3.4}$$

**Lemma 3.1** (see [17, Lemma 2.1]). *Let  $q$  and  $q_I$  be as in (3.4). Then for any admissible  $n$ , there hold*

$$\|q_I\|_{L^\alpha(I)} = |I|^{-p+1/\alpha}\|q\|_{\alpha'}, \quad \lambda_n^\sigma(q_I, I) = |I|^{-p}\lambda_n^\sigma(q, [0, 1]), \tag{3.5}$$

where the superscript  $\sigma$  can be chosen to be  $D$  or  $N$ .

By this lemma, we need only to consider eigenvalues for those potentials defined on the interval  $[0, 1]$ , that is, for  $q \in \mathcal{L}^\alpha$ .

#### 3.2. Relationship between the First- and Higher-Order Eigenvalues

Let us identify  $\mathcal{L}^\alpha = L^\alpha([0, 1], \mathbb{R})$  with  $L^\alpha(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ , where  $\alpha \in [1, \infty]$ . For  $n \geq 2$ , define the mapping  $\mathcal{T}_n : \mathcal{L}^\alpha \rightarrow \mathcal{L}^\alpha$  by

$$\mathcal{T}_n(q) = q_n, \quad q_n(t) := n^p q(nt), \quad t \in \mathbb{R}. \tag{3.6}$$

**Lemma 3.2** (see [17, Lemma 2.2]). *For any integer  $n \geq 2$ , let  $q$  and  $q_n$  be as in (3.6). Then there hold*

$$\bar{q}_n = n^p \bar{q}, \quad \|q_n\|_\alpha = n^p \|q\|_\alpha, \quad \lambda_n^\sigma(q_n) = n^p \lambda_1^\sigma(q), \quad (3.7)$$

where the superscript  $\sigma$  can be chosen to be  $D$  or  $N$ .

When  $n \geq 2$ ,  $\mathcal{T}_n$  of (3.6) is only injective but not surjective from  $B_\alpha[r/n^p]$  to  $B_\alpha[r]$ . It followed from the last two equalities in (3.7) that

$$\begin{aligned} \sup_{q \in B_\alpha[r]} \lambda_n^\sigma(q) &\geq n^p \cdot \sup_{q \in B_\alpha[r/n^p]} \lambda_1^\sigma(q), \\ \inf_{q \in B_\alpha[r]} \lambda_n^\sigma(q) &\leq n^p \cdot \inf_{q \in B_\alpha[r/n^p]} \lambda_1^\sigma(q), \end{aligned} \quad (3.8)$$

for any  $\alpha \in [1, \infty]$ ,  $n \geq 2$ , and  $\sigma$  chosen as  $D$  or  $N$ .

In fact, (3.8) can be proved to be equalities. Therefore, we need only to consider extremal values of the first eigenvalues. For cases  $\alpha > 1$ , the converse inequalities can be proved by using critical equations (4.16). See Remark 4.1. Moreover, results for case  $\alpha = 1$  can be obtained by limiting approaches  $\alpha \downarrow 1$ . Case  $\alpha = \infty$  is trivial.

### 3.3. Continuous Differentiability in Strong Topology

**Theorem 3.3** (see [14, Theorem 1.2]). *Given  $\alpha \in [1, \infty]$  and an admissible  $n$ . The functional*

$$(\mathcal{L}^\alpha, \|\cdot\|_\alpha) \longrightarrow \mathbb{R}, \quad q \longmapsto \lambda_n^\sigma(q) \quad (3.9)$$

is continuously Fréchet differentiable. The Fréchet derivative  $\partial_q \lambda_n^\sigma(q) \in (\mathcal{L}^\alpha, \|\cdot\|_\alpha)^*$  is the following bounded linear functional:

$$\mathcal{L}^\alpha \ni h \longmapsto - \int_0^1 |E_n^\sigma(t)|^p h(t) dt \in \mathbb{R}, \quad (3.10)$$

or written simply as

$$\partial_q \lambda_n^\sigma(q) = -|E_n^\sigma|^p. \quad (3.11)$$

Here  $E_n^\sigma(t) = E_n^\sigma(t; q)$  is a normalized eigenfunction associated with  $\lambda_n^\sigma(q)$  so that the  $L^p$  norm  $\|E_n^\sigma\|_p = 1$ .

*Remark 3.4.* Since (3.11) is always a negative functional of  $\mathcal{L}^\alpha$ ,  $\alpha \in [1, \infty]$ , eigenvalues possess the following monotonicity:

$$q_1, q_2 \in \mathcal{L}^1, \quad q_1 \leq q_2 \implies \lambda_n^\sigma(q_1) \geq \lambda_n^\sigma(q_2). \quad (3.12)$$

Moreover, if in addition,  $q_1(t) < q_2(t)$  holds on a subset of  $[0, 1]$  of positive measure, the conclusion inequality in (3.12) is strict.

### 3.4. Strong Continuity in Weak Topology

**Theorem 3.5** (see [14, Theorem 1.1]). *Given  $\alpha \in [1, \infty]$  and an admissible  $n$ , the following functional is continuous:*

$$(\mathcal{L}^\alpha, w_\alpha) \longrightarrow \mathbb{R}, \quad q \longmapsto \lambda_n^\sigma(q). \quad (3.13)$$

This theorem shows that eigenvalues  $\lambda_n^\sigma(q)$  have very strong continuous dependence on potentials. For other differential operators, similar results can be found in [9–11, 13, 16, 26, 30].

## 4. Extremal Problems in $\mathcal{L}^\alpha$ Balls, $1 < \alpha < \infty$

In this section, we always assume that  $1 < \alpha < \infty$ ,  $r > 0$ ,  $|h| < r$  and  $\sigma = D$  or  $N$ .

### 4.1. Preliminary Results for Minimization/Maximization Problems

For any admissible integer  $n$ , let us take some notations as follows

$$\begin{aligned} \mathbf{I}_{n,\alpha}(r) &:= \inf_{q \in B_\alpha[r]} \lambda_n^\sigma(q), & \mathbf{M}_{n,\alpha}(r) &:= \sup_{q \in B_\alpha[r]} \lambda_n^\sigma(q), \\ \mathfrak{S}_{n,\alpha}(r, h) &:= \inf_{q \in B_\alpha[r, h]} \lambda_n^\sigma(q), & \mathfrak{M}_{n,\alpha}(r, h) &:= \sup_{q \in B_\alpha[r, h]} \lambda_n^\sigma(q). \end{aligned} \quad (4.1)$$

If  $n = 0$ , then  $\sigma$  can only be chosen as  $N$ . For any positive integers  $n$ , these notations are still reasonable, and the extremal values are independent of the choice of  $\sigma = N$  or  $\sigma = D$  due to the relationship between Dirichlet and Neumann eigenvalues [29, Theorem 4.3]. Furthermore, these extremal values on balls with radius  $r$  are exactly the same as those on spheres with radius  $r$ , that is,

$$\mathbf{I}_{n,\alpha}(r) = \inf_{q \in S_\alpha[r]} \lambda_n^\sigma(q), \quad \mathbf{M}_{n,\alpha}(r) = \sup_{q \in S_\alpha[r]} \lambda_n^\sigma(q), \quad (4.2)$$

$$\mathfrak{S}_{n,\alpha}(r, h) = \inf_{q \in S_\alpha[r, h]} \lambda_n^\sigma(q), \quad \mathfrak{M}_{n,\alpha}(r, h) = \sup_{q \in S_\alpha[r, h]} \lambda_n^\sigma(q). \quad (4.3)$$

Equation (4.2) follows immediately from (3.12), the monotonicity of eigenvalues. Equation (4.3) follows from the facts that the closure of  $S_\alpha[r, h]$  in the space  $(\mathcal{L}^\alpha, w_\alpha)$  is  $B_\alpha[r, h]$  (See Lemma 2.6(i)), and  $\lambda_n(q)$  is strongly continuous in  $q \in (\mathcal{L}^\alpha, w_\alpha)$ .

Note that  $B_\alpha[r]$  and  $B_\alpha[r, h]$  are compact and sequentially compact in the space  $(\mathcal{L}^\alpha, w_\alpha)$  (see Lemma 2.5). Then all these extremal values in balls are actually minima or maxima. Among all these extremal values, the supremum of the principal Neumann

eigenvalues with potentials on  $S_\alpha[r, h]$  is a case apart, unlike any other. It is well known that [31, Lemma 3.3]

$$\lambda_0^N(q) \leq -\bar{q} \quad \forall q, \quad \lambda_0^N(q) = -\bar{q} \iff q \text{ is constant.} \quad (4.4)$$

Therefore one has

$$\mathfrak{M}_{0,\alpha}(r, h) = \max_{q \in B_\alpha[r, h]} \lambda_0^N(q) = \sup_{q \in S_\alpha[r, h]} \lambda_0^N(q) = \lambda_0^N(h) = -h. \quad (4.5)$$

Moreover,  $\mathfrak{M}_{0,\alpha}(r, h)$  cannot be attained by any potential on  $S_\alpha[r, h]$ . Consequently,

$$\mathbf{M}_{0,\alpha}(r) = \max_{q \in B_\alpha[r]} \lambda_0^N(q) = \lambda_0^N(-r) = r = \max_{q \in S_\alpha[r]} \lambda_0^N(q). \quad (4.6)$$

All other minimizers and maximizers are located exactly on the boundaries of the balls, that is,

$$\mathbf{I}_{n,\alpha}(r) = \min_{q \in S_\alpha[r]} \lambda_n^\sigma(q), \quad n = 0, 1, 2, \dots \quad (4.7)$$

$$\mathbf{M}_{n,\alpha}(r) = \max_{q \in S_\alpha[r]} \lambda_n^\sigma(q), \quad n = 1, 2, 3, \dots \quad (4.8)$$

$$\mathfrak{S}_{n,\alpha}(r, h) = \min_{q \in S_\alpha[r, h]} \lambda_n^\sigma(q), \quad n = 0, 1, 2, \dots \quad (4.9)$$

$$\mathfrak{M}_{n,\alpha}(r, h) = \max_{q \in S_\alpha[r, h]} \lambda_n^\sigma(q), \quad n = 1, 2, 3, \dots \quad (4.10)$$

Again, (4.7) and (4.8) are immediate consequences from monotonicity of eigenvalues. Especially, the minimizers/maximizers on the sphere  $S_\alpha[r]$  are nonnegative/nonpositive. It is natural to use the Lagrangian Multiplier Method (LMM for short) to deal with extremal value problems (4.7) and (4.8). Due to the restriction  $\bar{q} = h$ , it is not easy to obtain monotonicity of eigenvalues on  $q \in B_\alpha[r, h]$ . We refer the proof of (4.9) and (4.10) to [19, Lemma 3.1], where LMM is applied to exclude any minimizer or maximizer in the interior of  $B_\alpha[r, h]$ .

## 4.2. Variational Construction for Minimizer/Maximizer in $B_\alpha[r]$

The arguments in this subsection follow in part those of [17, 18, 20, 21].



The extremal value problems (4.7) and (4.8) can be considered uniformly. The only constraint wherein is  $\|q\|_\alpha = r$ . As  $\alpha \in (1, \infty)$ , the norm  $\|q\|_\alpha$  is continuously Fréchet differentiable in  $q \in (\mathcal{L}^\alpha, \|\cdot\|_\alpha)$ , with the Fréchet derivative as follows:

$$\partial_q \|q\|_\alpha = \|q\|_\alpha^{1-\alpha} \phi_\alpha(q), \tag{4.11}$$

where  $\phi_\alpha : \mathbb{R} \rightarrow \mathbb{R}$  is  $\phi_\alpha(x) := |x|^{\alpha-2}x$ . The eigenvalues  $\lambda_n^\sigma(q)$  are also Fréchet differentiable in  $q$ , and the Fréchet derivative is as in (3.11). By the Lagrangian Multiplier Method, the critical potential  $q_\alpha = q_{n,r,\alpha}$  satisfies the following equation:

$$-|E_\alpha(t)|^p = c_\alpha \phi_\alpha(q_\alpha(t)) \quad \text{a.e. } t \in [0, 1] \tag{4.12}$$

for some constant  $c_\alpha = c_{n,r,\alpha}$ . Here  $E_\alpha(t) = E_{n,r,\alpha}^\sigma(t)$  is a normalized eigen-function associated with  $\lambda_n^\sigma(q_\alpha) = \ell_\alpha$ . Since the minimizer is nonnegative and the maximizer is nonpositive,  $c_\alpha < 0$  for the minimization problem (4.7) and  $c_\alpha > 0$  for the maximization problem (4.8). Let  $s_\alpha = \text{sign } c_\alpha = \pm 1$  and

$$y_\alpha(t) = \frac{s_\alpha E_\alpha(t)}{\sqrt[p]{|c_\alpha|}}. \tag{4.13}$$

Then  $y_\alpha = y_{n,r,\alpha}^\sigma$  is also an eigenfunction associated with  $\ell_\alpha$ . Specifically, we identify it by  $y_\alpha'(0) > 0$  for Dirichlet case and  $y_\alpha(0) > 0$  for Neumann case. This identified  $y_\alpha$  is called the critical eigenfunction, and it satisfies the original ODE (3.1), that is,

$$(\phi_p(y_\alpha'))' + \ell_\alpha \phi_p(y_\alpha) + q_\alpha(t) \phi_p(y_\alpha) = 0, \tag{4.14}$$

and corresponding boundary conditions. By (4.12), the critical potential can also be written as

$$q_\alpha(t) = -s_\alpha \phi_{\alpha^*}(|y_\alpha(t)|^p) = -s_\alpha |y_\alpha(t)|^{p(\alpha^*-1)}. \tag{4.15}$$

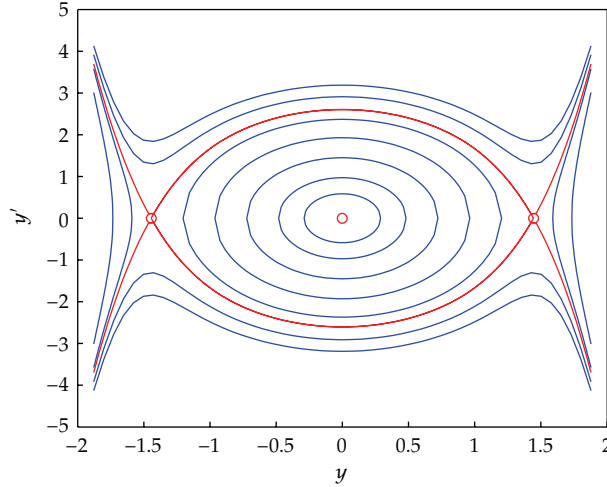
Substituting (4.15) into (4.14), one has

$$(\phi_p(y_\alpha'))' + \ell_\alpha \phi_p(y_\alpha) - s_\alpha \phi_{p\alpha^*}(y_\alpha) = 0, \tag{4.16}$$

which is the critical equation to problem (4.7) and problem (4.8). Note that this is a stationary Schrödinger equation, and it is independent of the orders  $n$  of eigenvalues  $\lambda_n(q)$ . By (4.15), the constraint  $\|q_\alpha\|_\alpha = r$  is transformed to

$$\int_0^1 |y_\alpha(t)|^{p\alpha^*} dt = r^\alpha. \tag{4.17}$$

*Remark 4.1.* Due to the autonomy and the symmetry of critical equation (4.16),  $y_{n,r,\alpha}(t)$  is a periodic solution of (4.16) with minimal period  $2/n$ . In fact, one has  $y_{n,r,\alpha}(t+1/n) \equiv -y_{n,r,\alpha}(t)$



**Figure 1:** Phase portrait of critical equation (4.16), where  $s_\alpha = 1$ .

(we refer the proof of this to Lemma 3.2 in [20]). By (4.15),  $q_{n,r,\alpha}(t)$  is periodic with minimal period  $1/n$ . Therefore, the maximizer  $q_{n,r,\alpha}$  is in the range of  $\mathcal{T}_n$ . See (3.6). This tells us that the converse inequalities of (3.8) are also true. More precisely, there hold

$$I_{n,\alpha}(r) = n^p I_{1,\alpha}\left(\frac{r}{n^p}\right), \quad M_{n,\alpha}(r) = n^p M_{1,\alpha}\left(\frac{r}{n^p}\right) \tag{4.18}$$

for any integer  $n \geq 2$  and any  $r > 0$ . This is why we need only to consider extremal values of  $\lambda_0^N(q)$  and  $\lambda_1^\sigma(q)$ .

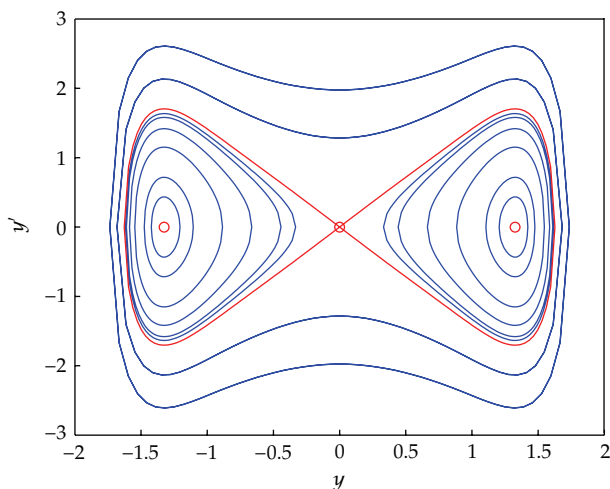
The phase portraits of (4.16) are distinguished in three cases. See Figures 1–3. In Figure 1,  $s_\alpha = 1$ . This figure corresponds to the maximization problem  $M_{1,\alpha}$ , and the eigenfunction  $y_\alpha$  is certain nonconstant periodic orbit surrounding the equilibrium  $(0, 0)$ . In Figure 2,  $s_\alpha = -1$  and  $\ell_\alpha < 0$ . The minimization problem  $I_{0,\alpha}$  is illustrated in this figure, and the positive eigenfunction  $y_\alpha$  corresponds to some non-constant periodic orbit surrounding the rightmost equilibrium. In Figure 3,  $s_\alpha = -1$  and  $\ell_\alpha \geq 0$ . When the minimization problem  $I_{1,\alpha}$  is considered, both Figures 2 and 3 should be taken into account. In fact, the bigger  $r$  is, the smaller  $I_{1,\alpha}(r)$  is. For  $r$  large enough,  $I_{1,\alpha}(r)$  is negative and  $y_\alpha$  should be some sign-changing periodic orbit outside the homoclinic orbits in Figure 2. For  $r > 0$  small enough,  $I_{1,\alpha}(r)$  is nonnegative, and  $y_\alpha$  should be some non-constant periodic orbit surrounding the equilibrium  $(0, 0)$  in Figure 3.

To study the parameter  $\ell_\alpha$  in (4.16), introduce an additional parameter as follows:

$$b_\alpha := \max_{t \in [0,1]} y_\alpha(t) = \|y_\alpha\|_\infty, \tag{4.19}$$

and besides, for case  $n = 0$  one more parameter as follows:

$$a_\alpha := \min_{t \in [0,1]} y_\alpha(t). \tag{4.20}$$



**Figure 2:** Phase portrait of critical equation (4.16), where  $\varsigma_\alpha = -1$  and  $\ell_\alpha < 0$ .

A first integral of (4.16) is

$$|y'_\alpha(t)|^p + F_\alpha(y_\alpha(t)) = F_\alpha(b_\alpha), \quad t \in [0, 1], \tag{4.21}$$

where

$$F_\alpha(x) = F_\alpha(x, \ell_\alpha) := (p - 1)^{-1} \left( \ell_\alpha |x|^p - \frac{\varsigma_\alpha |x|^{p\alpha^*}}{\alpha^*} \right), \quad x \in \mathbb{R}. \tag{4.22}$$

Note that the minimal period of  $y_\alpha$  is 2, and  $y_\alpha$  satisfies the constraint (4.17). Theoretically,  $\ell_\alpha = \mathbf{M}_{1,\alpha}$  (or  $\ell_\alpha = \mathbf{I}_{1,\alpha}$ ) and  $b_\alpha$  are implicitly determined by singular integrals as follows:

$$\begin{aligned} \int_0^{b_\alpha} \frac{dx}{(F_\alpha(b_\alpha) - F_\alpha(x))^{1/p}} &= \frac{1}{2}, \\ \int_0^{b_\alpha} \frac{x^{p\alpha^*} dx}{(F_\alpha(b_\alpha) - F_\alpha(x))^{1/p}} &= \frac{r^\alpha}{2}, \end{aligned} \tag{4.23}$$

where  $\varsigma_\alpha = 1$  in (4.22) for  $\mathbf{M}_{1,\alpha}$ , and  $\varsigma_\alpha = -1$  for  $\mathbf{I}_{1,\alpha}$ .

Choose  $\varsigma_\alpha = -1$  in (4.22). Then  $\ell_\alpha = \mathbf{I}_{0,\alpha}$ ,  $a_\alpha$  and  $b_\alpha$  are implicitly determined by

$$\begin{aligned} \int_{a_\alpha}^{b_\alpha} \frac{dx}{(F_\alpha(b_\alpha) - F_\alpha(x))^{1/p}} &= 1, \\ \int_{a_\alpha}^{b_\alpha} \frac{x^{p\alpha^*} dx}{(F_\alpha(b_\alpha) - F_\alpha(x))^{1/p}} &= r^\alpha, \\ F_\alpha(a_\alpha) &= F_\alpha(b_\alpha). \end{aligned} \tag{4.24}$$

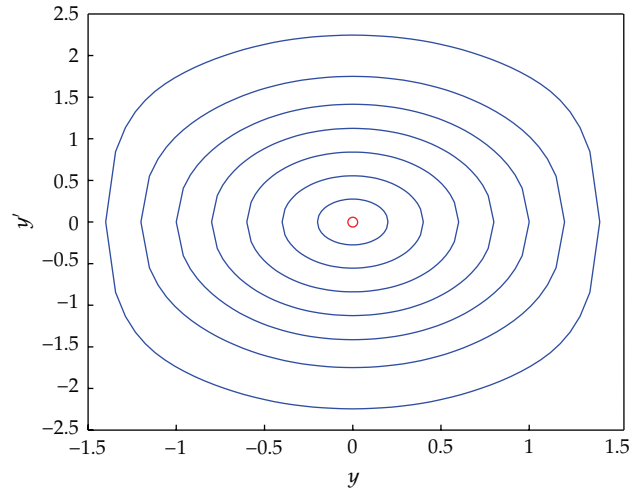


Figure 3: Phase portrait of critical equation (4.16), where  $\varsigma_\alpha = -1$  and  $\ell_\alpha \geq 0$ .

### 4.3. Variational Construction for Minimizer/Maximizer in $B_\alpha[r, h]$

The extremal value problems (4.9) and (4.10) can be considered uniformly by LMM as in the previous subsection. The critical equation is

$$(\phi_p(y'_\alpha))' + \ell_\alpha \phi_p(y_\alpha) - \varsigma_\alpha \phi_{\alpha^*}(y_\alpha^p - m_\alpha) y_\alpha = 0, \quad (4.25)$$

where  $\varsigma_\alpha = 1$  for maximization problems and  $\varsigma_\alpha = -1$  for minimization problems. The existence of the new constant  $m_\alpha$  is caused by the constraint  $\bar{q} = h$ . The critical potential is

$$q_\alpha(t) \equiv -\varsigma_\alpha \phi_{\alpha^*}(y_\alpha^p(t) - m_\alpha). \quad (4.26)$$

The constraint  $\|q_\alpha\|_\alpha = r$  is

$$\int_0^1 |y_\alpha^p(t) - m_\alpha|^{\alpha^*} dt = r^\alpha. \quad (4.27)$$

The constraint  $\bar{q}_\alpha = h$  is

$$\int_0^1 \phi_{\alpha^*}(y_\alpha^p(t) - m_\alpha) dt = -\varsigma_\alpha h. \quad (4.28)$$

A first integral of the critical equation is

$$|y'_\alpha|^p + \frac{\ell_\alpha}{p-1} |y_\alpha|^p - \frac{\varsigma_\alpha |y_\alpha^p - m_\alpha|^{\alpha^*}}{(p-1)\alpha^*} = \text{const.} \quad (4.29)$$

The existence of  $m_\alpha$  brings more complexity and difficulties. For the case of the Sturm-Liouville operator, we refer the readers to [19, 23] for more details.

### 5. Extremal Problems in $\mathcal{L}^1$ Balls

In this section we will solve the following extremal problems:

$$\mathbf{I}_n(r) = \inf_{q \in B_1[r]} \lambda_n^\sigma(q), \quad \mathbf{M}_n(r) = \sup_{q \in B_1[r]} \lambda_n^\sigma(q), \quad (5.1)$$

$$\mathfrak{S}_n(r, h) := \inf_{q \in B_1[r, h]} \lambda_n^\sigma(q), \quad \mathfrak{M}_n(r, h) := \sup_{q \in B_1[r, h]} \lambda_n^\sigma(q). \quad (5.2)$$

By the topological facts in Section 2 and the strong continuity of eigenvalues in weak topology (see Theorem 3.5), the extremal values in  $\mathcal{L}^1$  balls are limits of those extremal values in  $\mathcal{L}^\alpha$  balls, that is,

$$\begin{aligned} \mathbf{I}_n(r) &= \lim_{\alpha \downarrow 1} \mathbf{I}_{n,\alpha}(r), & \mathbf{M}_n(r) &= \lim_{\alpha \downarrow 1} \mathbf{M}_{n,\alpha}(r), \\ \mathfrak{S}_n(r, h) &= \lim_{\alpha \downarrow 1} \mathfrak{S}_{n,\alpha}(r, h), & \mathfrak{M}_n(r, h) &= \lim_{\alpha \downarrow 1} \mathfrak{M}_{n,\alpha}(r, h), \end{aligned} \quad (5.3)$$

for any  $r > 0$ ,  $|h| < r$  and any admissible integer  $n$ . By relationship (4.18), we need only consider for cases  $n = 0$  and  $n = 1$  in (5.1). Similar arguments hold for (5.2).

For simplicity, we only illustrate the extremal value problems (5.1) with potentials varied in  $B_1[r]$  and refer to [19, 23] for (5.2) with potentials varied in  $B_1[r, h]$ .

Let us use a uniform notation  $\ell_0$  to denote the extremal values  $\mathbf{I}_0(r)$ ,  $\mathbf{I}_1(r)$ , or  $\mathbf{M}_1(r)$ , respectively, in different extremal values problems, that is,

$$\ell_0 = \lim_{\alpha \downarrow 1} \ell_\alpha. \quad (5.4)$$

Note that  $\mathbf{I}_0(r) > -\infty$  (see [18, Lemma 2.3]) and  $\mathbf{M}_1(r) < \infty$  due to the asymptotical distribution of large eigenvalues [29, 32]. Thus in each case,  $\ell_0$  is finite and  $\{\ell_\alpha\}_{\alpha>1}$  is bounded.

Two different limiting approaches to this limit  $\ell_0$  are reviewed in the following two subsections. One is from the viewpoint of singular integrals as in [20, 21], where singular integrals involving extremal values for the Sturm-Liouville operator are analyzed directly and delicately. To overcome the difficulties caused by the presence of the  $p$ -Laplace operator, the other limiting approach is from the viewpoint of conservation laws (involving eigenfunctions and extremal values) as in [10, 17, 18, 23]. Compared with the singular integral method, the conservation law method can simplify and refine the limiting process because more information from eigen-functions can be used.

#### 5.1. Limiting Approach from the Viewpoint of Singular Integrals ( $p = 2$ )

By (5.4), to obtain the extremal values  $\ell_0$  in  $L^1$  balls, it is natural to compute firstly the extremal values  $\ell_\alpha$  in  $L^\alpha$  balls and then let  $\alpha \downarrow 1$ . Since  $\ell_\alpha$  will finally be formulated by singular

integrals, we say that such a limiting approach is from the viewpoint of singular integrals. We have only applied such a method to the Sturm-Liouville operator because of the complexity in analyzing singular integrals. More difficulties will be added to such a process by the presence of  $p$ -Laplacian.

Extremal values  $\mathbf{I}_1(r)$  and  $\mathbf{M}_1(r)$  for the Sturm-Liouville operator are studied in [20]. As stated in the previous section,  $\ell_\alpha = \mathbf{M}_{1,\alpha}$  (or  $\ell_\alpha = \mathbf{I}_{1,\alpha}$ ) and  $b_\alpha$  are implicitly determined by (4.23), where the exponent  $p = 2$ . After some transformation, (4.23) is analyzed in [20], and these extremal values in  $\mathcal{L}^\alpha$  balls are finally expressed by some singular integrals. More precisely, for any  $d \in (0, \infty)$ , let

$$\begin{aligned} A &= A_\alpha(d) = \frac{d^{2(\alpha^*-1)}}{\alpha^*}, \\ \mathbf{T}_\alpha(d) &= 4 \int_0^1 \frac{dx}{\sqrt{1-x^2 + A(1-x^{2\alpha^*})}}, \\ \mathbf{U}_\alpha(d) &= 4(\alpha^*A)^\alpha \int_0^1 \frac{x^{2\alpha^*} dx}{\sqrt{1-x^2 + A(1-x^{2\alpha^*})}}, \\ \widehat{\mathbf{T}}_\alpha(d) &= 4 \int_0^1 \frac{dx}{\sqrt{x^2-1 + A(1-x^{2\alpha^*})}}, \\ \widehat{\mathbf{U}}_\alpha(d) &= 4(\alpha^*A)^\alpha \int_0^1 \frac{x^{2\alpha^*} dx}{\sqrt{x^2-1 + A(1-x^{2\alpha^*})}}, \end{aligned} \quad (5.5)$$

and for any  $d \in (0, 1)$ , let

$$\begin{aligned} B &= B_\alpha(d) = d^{2(\alpha^*-1)} = d^{2/(\alpha-1)}, \\ \check{\mathbf{T}}_\alpha(d) &= 4 \int_0^1 \frac{dx}{\sqrt{1-x^2 - B(1-x^{2\alpha^*})/\alpha^*}}, \\ \check{\mathbf{U}}_\alpha(d) &= 4B^\alpha \int_0^1 \frac{x^{2\alpha^*} dx}{\sqrt{1-x^2 - B(1-x^{2\alpha^*})/\alpha^*}}, \end{aligned} \quad (5.6)$$

and we define

$$\begin{aligned} \mathbf{Z}_{1,\alpha}(x) &= \begin{cases} 0 & \text{for } x = \pi^2, \\ \frac{1}{4}(2\sqrt{x})^{2-1/\alpha} (\mathbf{U}_\alpha(\mathbf{T}_\alpha^{-1}(2\sqrt{x})))^{1/\alpha} & \text{for } x \in (0, \pi^2), \\ \mathbf{R}_{1,\alpha} = \mathbf{K}(2\alpha^*) & \text{for } x = 0, \\ \frac{1}{4}(2\sqrt{-x})^{2-1/\alpha} (\widehat{\mathbf{U}}_\alpha(\widehat{\mathbf{T}}_\alpha^{-1}(2\sqrt{-x})))^{1/\alpha} & \text{for } x \in (-\infty, 0), \end{cases} \\ \check{\mathbf{Y}}_{1,\alpha} &= \frac{1}{4}(2\sqrt{x})^{2-1/\alpha} (\check{\mathbf{U}}_\alpha(\check{\mathbf{T}}_\alpha^{-1}(2\sqrt{x})))^{1/\alpha} \quad \text{for } x \in [\pi^2, \infty). \end{aligned} \quad (5.7)$$

Then there hold

$$\mathbf{I}_{1,\alpha}(r) = \mathbf{Z}_{1,\alpha}^{-1}(r), \quad \mathbf{M}_{1,\alpha}(r) = \check{\mathbf{Y}}_{1,\alpha}^{-1}(r). \tag{5.8}$$

Note that  $\mathbf{I}_{1,\alpha}(r)$  and  $\mathbf{M}_{1,\alpha}(r)$  cannot be written as elementary functions of the radius  $r$ . However, after some delicate analysis on the singular integrals  $\mathbf{T}_\alpha, \mathbf{U}_\alpha, \hat{\mathbf{T}}_\alpha, \hat{\mathbf{U}}_\alpha, \check{\mathbf{T}}_\alpha$  and  $\check{\mathbf{U}}_\alpha$ , the limits of  $\mathbf{I}_{1,\alpha}(r)$  and  $\mathbf{M}_{1,\alpha}(r)$  as  $\alpha \downarrow 1$  are proved to be elementary functions of  $r$ , that is,

$$\mathbf{M}_1(r) = \frac{1}{4} \left( \pi + \sqrt{\pi^2 + 4r} \right)^2, \tag{5.9}$$

$$\mathbf{I}_1(r) = \mathbf{Z}_1^{-1}(r), \tag{5.10}$$

where

$$\mathbf{Z}_1(x) = \begin{cases} 2\sqrt{-x} \coth\left(\frac{\sqrt{-x}}{2}\right) & \text{for } x \in (-\infty, 0), \\ 4 & \text{for } x = 0, \\ 2\sqrt{x} \cot\left(\frac{\sqrt{x}}{2}\right) & \text{for } x \in (0, \pi^2] \end{cases} \tag{5.11}$$

is a decreasing diffeomorphism mapping  $(-\infty, \pi^2]$  onto  $[0, \infty)$ .

Extremal values  $\mathbf{I}_0(r)$  for the Sturm-Liouville operator are studied in [21], again by singular integral method in the limiting approach. Now  $\ell_\alpha = \mathbf{I}_{0,\alpha}$ ,  $a_\alpha$ , and  $b_\alpha$  are implicitly determined by (4.24) with  $p = 2$ . One can imagine that difficulty increases due to the presence of the additional new parameter  $a_\alpha$ . We only state the final results here and refer the readers to [21] for details. Let

$$\mathbf{Z}_0(x) = 2\sqrt{-x} \tanh\left(\frac{\sqrt{-x}}{2}\right), \quad x \leq 0, \tag{5.12}$$

which is a decreasing diffeomorphism from  $(-\infty, 0]$  onto  $[0, \infty)$ . Then there holds

$$\mathbf{I}_0(r) = \mathbf{Z}_0^{-1}(r). \tag{5.13}$$

### 5.2. Limiting Approach from the Viewpoint of Conservation Laws ( $p > 1$ )

The first integral (4.21) of the critical equation (4.16) can also be written as

$$|y'_\alpha(t)|^p + \frac{\ell_\alpha}{p-1} |y_\alpha(t)|^p - \frac{s_\alpha}{(p-1)\alpha^*} |y_\alpha(t)|^{p\alpha^*} = \frac{e_\alpha}{p-1}, \quad t \in [0, 1], \tag{5.14}$$

where

$$e_\alpha := (p-1)F_\alpha(b_\alpha) = \ell_\alpha |b_\alpha|^p - \frac{s_\alpha}{\alpha^*} |b_\alpha|^{p\alpha^*} \quad (5.15)$$

is a constant. We call (5.14) a conservation law of (4.16).

Define

$$G_\alpha(t) := \frac{|y_\alpha(t)|^{p\alpha^*}}{\alpha^*}, \quad t \in [0, 1]. \quad (5.16)$$

It follows from (4.17) that  $\|G_\alpha\|_1 = r^\alpha / \alpha^* \rightarrow 0$  as  $\alpha \downarrow 1$ . Passing to a subsequence if necessary,

$$\lim_{\alpha \downarrow 1} G_\alpha(t) = 0 \quad \text{a.e. } t \in [0, 1]. \quad (5.17)$$

Motivated by (5.17), one attempt to compute  $\ell_0$  in (5.4) is to consider the limit equation of the conservation law (5.14). Intuitively, if we assume that  $e_\alpha \rightarrow e_0$  and  $y_\alpha \rightarrow y_0$  in appropriate sense, then it followed from (5.4) and (5.17) that the limit equation of (5.14) should be of the form (5.29), which is a first-order ODE simpler than (5.14). Boundary conditions on the critical eigen-functions  $y_\alpha$  and the restriction that the critical potentials  $q_\alpha \in S_\alpha[r]$  will give more information on  $y_0$ . All these conditions on  $y_0$  and the solution to (5.29) will finally lead to the answer to the extremal value  $\ell_0$ . Compared with the previous singular integral method, such a limiting approach to  $\ell_0$  from the viewpoint of conservation laws cannot only greatly simplify the analysis but also deal with the  $p$ -Laplace operator.

In the limiting approach from the viewpoint of conservation laws, it is natural to study the convergence of critical eigen-functions  $y_\alpha$  as  $\alpha \downarrow 1$ . To this end, the boundedness of  $\|y_\alpha\|_p$  and  $\|y'_\alpha\|_p$  as  $\alpha \downarrow 1$  should be taken into consideration.

For both Neumann and Dirichlet eigenvalues, multiplying (4.16) by  $y_\alpha(t)$ , integrating over  $[0, 1]$  and taking use of the restriction (4.17), that is,  $\|y_\alpha\|_{p\alpha^*}^{p\alpha^*} = r^\alpha$ , one has

$$-\|y'_\alpha\|_p^p + \ell_\alpha \|y_\alpha\|_p^p - s_\alpha r^\alpha = 0. \quad (5.18)$$

Integrating (5.14) over  $[0, 1]$ , one has

$$\|y'_\alpha\|_p^p + \frac{\ell_\alpha}{p-1} \|y_\alpha\|_p^p - \frac{s_\alpha}{(p-1)\alpha^*} r^\alpha = \frac{e_\alpha}{p-1}. \quad (5.19)$$

Eliminating  $\|y'_\alpha\|_p^p$  from the above two equalities, one has

$$p\ell_\alpha \|y_\alpha\|_p^p = e_\alpha + s_\alpha \left(p - \frac{1}{\alpha}\right) r^\alpha. \quad (5.20)$$



For the principal Neumann eigenvalues, there hold an additional equality as follows:

$$e_\alpha \left\| \frac{1}{y_\alpha} \right\|_p^p = - \frac{\|q_\alpha\|_1}{\alpha}. \tag{5.21}$$

This equality can be deduced from (4.14), (5.14), and (4.17).

Note that  $\|\ell_\alpha + q_\alpha\|_1 \leq |\ell_\alpha| + \|q_\alpha\|_1 \leq |\ell_\alpha| + \|q_\alpha\|_\alpha = |\ell_\alpha| + r$  is bounded as  $\alpha \downarrow 1$ . The boundedness of  $\|\ell_\alpha + q_\alpha\|_1$  and the restriction (4.17) guarantees (passing to a subsequence if necessary) the convergence of  $\lim_{\alpha \downarrow 1} b_\alpha \in (0, \infty)$  and

$$y_\alpha \longrightarrow y_0 \neq 0 \quad \text{in } (\mathcal{C}, \|\cdot\|_\infty) \text{ as } \alpha \downarrow 1. \tag{5.22}$$

Here  $\mathcal{C} = C([0, 1], \mathbb{R})$ . For detailed proof, see Lemmas 2.1 and 4.1 in [18]. Consequently,  $\|y_\alpha\|_p$  is bounded as  $\alpha \downarrow 1$ , and by (5.20),

$$e_\alpha = (p - 1)F_\alpha(b_\alpha) = \ell_\alpha |b_\alpha|^p - \mathfrak{s}_\alpha \frac{|b_\alpha|^{p\alpha^*}}{\alpha^*} \quad (\longrightarrow e_0 \text{ as } \alpha \downarrow 1). \tag{5.23}$$

By (4.17), there holds  $r^\alpha \leq b_\alpha^{p\alpha^*}$ , and hence

$$\lim_{\alpha \downarrow 1} b_\alpha^{p\alpha^*} \geq 1. \tag{5.24}$$

It followed from (5.23) and (5.24) that

$$\lim_{\alpha \downarrow 1} b_\alpha = 1, \quad e_0 = \ell_0 - \lim_{\alpha \downarrow 1} \frac{\mathfrak{s}_\alpha |b_\alpha|^{p\alpha^*}}{\alpha^*}. \tag{5.25}$$

Note that  $\|q_\alpha\|_1 \leq \|q_\alpha\|_\alpha = r$ . On the other hand, it follows from (4.15) and (4.17) that

$$\|q_\alpha\|_1 = \int_0^1 \frac{|y_\alpha(t)|^{p\alpha^*}}{|y_\alpha(t)|^p} dt \geq \frac{1}{\|y_\alpha\|_\infty^p} \int_0^1 |y_\alpha(t)|^{p\alpha^*} dt = \frac{r^\alpha}{b_\alpha^p} \longrightarrow r \tag{5.26}$$

as  $\alpha \downarrow 1$ . Thus one has

$$s\text{-}\lim_{\alpha \downarrow 1} \|q_\alpha\|_1 = r. \tag{5.27}$$

By the monotonicity and the symmetry of the eigen-function  $y_\alpha$ , (4.21) can be written as

$$y'_\alpha(t) = \kappa_p^{-1} \left( e_\alpha - \ell_\alpha y_\alpha^p(t) - \mathfrak{s}_\alpha G_\alpha(t) \right)^{1/p}, \quad t \in I_\sigma, \tag{5.28}$$

where  $\kappa_p := (p-1)^{1/p}$ . For the Dirichlet eigen-function,  $y_\alpha(0) = 0$ ,  $y_\alpha(1/2) = b_\alpha$  and  $I_\sigma = I_D = [0, 1/2]$ . For the Neumann eigen-function,  $y_\alpha(0) = a_\alpha = \min_{t \in [0,1]} y_\alpha(t)$ ,  $y_\alpha(1) = b_\alpha$  and  $I_\sigma = I_N = [0, 1]$ . Since there holds (5.17), the Lebesgue-dominated convergence theorem can be applied to the integral equation equivalent to (5.28) with corresponding boundary conditions. Then it can be proved that  $y_0 = \lim_{\alpha \downarrow 1} y_\alpha \in C^1(I_\sigma)$  and

$$y_0' = \kappa_p^{-1} (e_0 - \ell_0 y_0^p)^{1/p}, \quad t \in I_\sigma. \quad (5.29)$$

As the limit of Dirichlet eigen-functions  $y_\alpha^D$ ,  $y_0$  satisfies the boundary conditions as follows:

$$y_0(0) = 0, \quad y_0\left(\frac{1}{2}\right) = 1. \quad (5.30)$$

As the limit of Neumann eigen-functions  $y_\alpha^N$ ,  $y_0$  satisfies the boundary conditions as follows:

$$y_0(0) = a_0 = \min_{t \in [0,1]} y_0(t) \in (0, 1), \quad y_0(1) = 1. \quad (5.31)$$

The limiting equality of (5.20) is

$$p\ell_0 \|y_0\|_p^p = e_0 + (p-1)r. \quad (5.32)$$

By (5.27), the limiting equality of (5.21) is

$$e_0 \left\| \frac{1}{y_0} \right\|_p^p = -r. \quad (5.33)$$

Either  $\ell_0 = \mathbf{M}_1(r)$  or  $\ell_0 = \mathbf{I}_1(r)$  is determined by ODE (5.29), the boundary conditions (5.30) and the restriction (5.32). Meanwhile,  $\ell_0 = \mathbf{I}_0(r)$  is determined by (5.29), (5.31), (5.32) and (5.33). To solve (5.29) and compute  $\ell_0$ , we need more information about  $e_0$ ,  $\ell_0$ , and  $a_0$ .

*Case 1* (maximization problem  $\mathbf{M}_1(r)$ ). In this case, there holds  $e_0 = \ell_0 > 0$ . In fact, we have

$$\ell_0 = \mathbf{M}_1(r) \geq \lambda_1(-r) = r > 0. \quad (5.34)$$

The rightmost equilibrium in the phase portrait Figure 1 is  $(\ell_\alpha^{(\alpha-1)/p}, 0)$ . Thus we have

$$0 < b_\alpha < \ell_\alpha^{(\alpha-1)/p}, \quad \lim_{\alpha \downarrow 1} \frac{b_\alpha^{p\alpha^*}}{\alpha^*} \leq \lim_{\alpha \downarrow 1} \frac{\ell_\alpha^\alpha}{\alpha^*} = 0. \quad (5.35)$$

Consequently,  $e_0 = \ell_0$  by (5.25).

Finally,  $\ell_0 = \mathbf{M}_1(r)$  can be proved to be the unique root of

$$x - \pi_p x^{1/p^*} = r, \quad x \in [0, \infty), \quad (5.36)$$

where

$$\pi_p = \frac{2\pi(p-1)^{1/p}}{p \sin(\pi/p)}. \tag{5.37}$$

We refer to [17] for details. Note that if  $p = 2$ , result (5.36) is consistent with (5.9).

*Case 2* (Minimization problem  $I_1(r)$ ). In this case, there hold  $e_0 > 0$  and  $e_0 > \ell_0$ . In fact,  $s_\alpha = -1$  for the minimization problems. It follows from (5.14) that

$$e_\alpha = (p-1)F_\alpha(b_\alpha) \geq \ell_\alpha |y_\alpha(t)|^p, \quad \forall t \in [0, 1]. \tag{5.38}$$

Set  $t = 0$  and  $1/2$  in (5.38), respectively, and let  $\alpha \downarrow 1$ . We see that  $e_0 \geq 0$  and  $e_0 \geq \ell_0$ . Furthermore, cases  $e_0 = 0$  and  $e_0 = \ell_0$  can be excluded by checking (5.32) after solving (5.29)-(5.30) and taking into account that

$$\ell_0 = I_1(r) \leq \lambda_1(r) = \pi_p^p - r. \tag{5.39}$$

In this case, till now there is no such restrictions as positivity or negativity on  $\ell_0$ . This is consistent to the phase portrait analysis in Section 4.2.

Finally,  $(e_0, \ell_0)$  ( $e_0 > 0, e_0 > \ell_0$ ) can be uniquely determined by

$$\int_0^1 \frac{\kappa_p du}{(e_0 - \ell_0 u^p)^{1/p}} = \frac{1}{2}, \tag{5.40}$$

$$r = 2(p-1)^{-1/p^*} (e_0 - \ell_0)^{1/p^*}.$$

For details we refer to [18]. Note that if  $p = 2$ , the integral in (5.40) can be evaluated explicitly, and one can get (5.10).

*Case 3* (minimization problem  $I_0(r)$ ). In this case, there hold  $\ell_0 < 0$  and  $e_0 = \ell_0 a_0^p < 0$ . In fact, it follows from (4.4) that

$$\ell_0 = I_0(r) < -r < 0. \tag{5.41}$$

By (5.33),  $e_0 < 0$ . Note that (5.38) still holds for this minimization problem. Let  $t = 0$  in (5.38) and  $\alpha \downarrow 1$ . One has  $e_0 \geq \ell_0 a_0^p$ . One can exclude the case  $e_0 > \ell_0 a_0^p$  by checking (5.33) after solving (5.29)-(5.31).

Finally,  $(a_0, \ell_0)$  ( $0 < a_0 < 1, \ell_0 < 0$ ) can be uniquely determined by

$$\kappa_p \int_{a_0}^1 \frac{du}{(\ell_0 a_0^p - \ell_0 u^p)^{1/p}} = 1, \tag{5.42}$$

$$(p-1)^{-1/p^*} (\ell_0 a_0^p - \ell_0)^{1/p^*} = r.$$

For details we refer to [18]. Note that if  $p = 2$ , the integral in (5.42) can be evaluated explicitly, and one can get (5.13).

## 6. Extremal Problems for Eigenvalues of Measure Differential Equations

Generalized ordinary differential equations (GODEs for short) [33, 34] can describe the jumps of solutions caused by impulses, and so forth. In this paper we will only consider the so-called measure differential equations (MDEs for short) [26, 27], a special class of GODE.

By a real measure  $\mu$  on  $[0, 1]$ , it means that it is an element of the dual space  $\mathcal{M}_0 := (\mathcal{C}, \|\cdot\|_\infty)^*$  of the Banach space  $(\mathcal{C}, \|\cdot\|_\infty)$ . Then, for any  $u \in \mathcal{C}$  and any subinterval  $J$  of  $[0, 1]$ , the Riemann-Stieltjes integral and the Lebesgue-Stieltjes integral [35]

$$\int_{[0,1]} u \, d\mu \quad \int_J u \, d\mu \quad (6.1)$$

are defined. Given a measure  $\mu \in \mathcal{M}_0$ , following the notations in [27], the second-order linear MDE with the measure  $\mu$  is written as

$$dx^\bullet + x \, d\mu(t) = 0, \quad t \in [0, 1]. \quad (6.2)$$

The solution  $x(t)$  of MDE (6.2) with the initial value  $(x(0), x^\bullet(0)) = (x_0, v_0) \in \mathbb{R}^2$  is explained using the system of integral equations as follows:

$$\begin{aligned} x(t) &= x_0 + \int_{[0,t]} v(s) \, ds, \quad t \in [0, 1], \\ v(t) &= \begin{cases} v_0, & t = 0, \\ v_0 - \int_{[0,t]} x(s) \, d\mu(s), & t \in (0, 1]. \end{cases} \end{aligned} \quad (6.3)$$

Here  $v(t) := x^\bullet(t)$  is the generalized right-hand derivative of  $y(t)$  or the velocity of  $y(t)$ . By (6.3), it is well known that solutions of initial value problems of (6.2) are uniquely defined on  $[0, 1]$  [33, 34]. See also [26, 27]. From (6.3), one sees that the solutions themselves are continuous, that is,  $x \in \mathcal{C}$ . However,  $v(t) = x^\bullet(t)$  may have discontinuity at those  $t$  such that the density  $d\mu(t)/dt$  does not exist. In case  $\mu \in \mathcal{M}_0$  is absolutely continuous with respect to the Lebesgue measure  $\ell$  on  $[0, 1]$ , that is,  $d\mu(t)/dt = q(t) \in \mathcal{L}^1$ , solutions of MDE (6.2) reduce to that for ODE

$$x'' + q(t)x = 0. \quad (6.4)$$

Let  $\mu \in \mathcal{M}_0$  be a (real) measure on  $[0, 1]$ . The corresponding eigenvalue problem

$$dx^\bullet + \lambda x \, dt + x \, d\mu(t) = 0, \quad t \in [0, 1] \quad (6.5)$$

has been studied in [26, 27]. Like problem (1.1), eigenvalues of (6.5) with the Dirichlet boundary condition  $x(0) = x(1) = 0$  are a real, increasing sequence  $\{\lambda_n^D(\mu)\}_{n \in \mathbb{N}}$ , accumulating at  $+\infty$ . With the Neumann boundary condition  $x'(0) = x'(1) = 0$ , eigenvalues of (6.5) are a real, increasing sequence  $\{\lambda_n^N(\mu)\}_{n \in \mathbb{Z}^+}$ , again accumulating at  $+\infty$ .

By the Riesz representation theorem [25], the measure space  $\mathcal{M}_0$  can be characterized using real functions of  $[0, 1]$  of bounded variations. For  $\mu \in \mathcal{M}_0$ , we use  $\|\mu\|_V$  to denote the total variation of  $\mu$  on  $[0, 1]$ . Then  $(\mathcal{M}_0, \|\cdot\|_V)$  is a Banach space. Since  $\mathcal{M}_0$  is the dual space of  $(C, \|\cdot\|_\infty)$ , one has also in  $\mathcal{M}_0$  the weak\* topology  $w_*$  defined as follows:  $\mu_n \rightarrow \mu$  in  $(\mathcal{M}_0, w_*)$  if and only if

$$\lim_{n \rightarrow \infty} \int_{[0,1]} u \, d\mu_n = \int_{[0,1]} u \, d\mu \quad \forall u \in C. \tag{6.6}$$

It is well known that bounded subsets of  $(\mathcal{M}_0, \|\cdot\|_V)$  are relatively compact and relatively sequentially compact in  $(\mathcal{M}_0, w_*)$  [24, 25].

Some deep results on the dependence of eigenvalues  $\lambda_n^\sigma(\mu)$  of MDE on measures  $\mu$  are as follows.

**Theorem 6.1** (see [26, 27]). *(i) In the norm topology  $\|\cdot\|_V$ ,  $\lambda_n^\sigma(\mu)$  is continuously Fréchet differentiable in  $\mu \in (\mathcal{M}_0, \|\cdot\|_V)$ . Moreover, the formula for the Fréchet derivative  $\partial_\mu \lambda_n^\sigma(\mu)$  is similar to (3.11).*

*(ii) In the weak\* topology  $w_*$ ,  $\lambda_n^\sigma(\mu)$  are continuous in  $\mu \in (\mathcal{M}_0, w_*)$ .*

Let us introduce spheres and balls of measures as follows:

$$S_0[r] := \{\mu \in \mathcal{M}_0 : \|\mu\|_V = r\}, \quad B_0[r] := \{\mu \in \mathcal{M}_0 : \|\mu\|_V \leq r\}, \tag{6.7}$$

where  $r \geq 0$ . Because of Theorem 6.1 and the compactness of  $B_0[r]$  in  $(\mathcal{M}_0, w_*)$ , the following extremal problems

$$I_n(r) := \min_{\mu \in B_0[r]} \lambda_n^\sigma(\mu), \quad M_n(r) := \max_{\mu \in B_0[r]} \lambda_n^\sigma(\mu), \tag{6.8}$$

where  $n \in \mathbb{Z}^+$  are well posed. Moreover, both the minimum and the maximum can be realized by some measures from  $B_0[r]$ . Here, when  $n \in \mathbb{N}$ , the values  $I_n(r)$  and  $M_n(r)$  are the same for the Dirichlet and the Neumann eigenvalues. Since the Fréchet derivatives  $\partial_\mu \lambda_n^\sigma(\mu)$  are nonzero, one sees that problems (6.8) are the same as

$$I_n(r) \equiv \min_{\mu \in S_0[r]} \lambda_n^\sigma(\mu), \quad M_n(r) \equiv \max_{\mu \in S_0[r]} \lambda_n^\sigma(\mu). \tag{6.9}$$

That is, the minimizing and maximizing measures of (6.8) must be on the sphere  $S_0[r]$ .

Recall from Theorem 6.1 that the functionals  $\lambda_n^\sigma(\mu)$  we are minimizing/maximizing are continuously differentiable. However,  $S_0[r]$  is not differentiable in the space  $(\mathcal{M}_0, \|\cdot\|_V)$ . The solution of problems (6.9) appeals for the LMM using the sub-differentials [36, 37].

For the zeroth Neumann eigenvalues  $\lambda_0^N(\mu)$ , problems (6.9) have been solved using this idea in [22]. The results are as follows.

**Theorem 6.2** (see [22]). *For any  $r \geq 0$ , one has*

$$\begin{aligned} I_0(r) &= \min_{\mu \in S_0[r]} \lambda_0^N(\mu) = \lambda_0^N(r\delta_0) = \lambda_0^N(r\delta_1) = \mathbf{Z}_0^{-1}(r), \\ M_0(r) &= \max_{\mu \in S_0[r]} \lambda_0^N(\mu) = \lambda_0^N(-r\ell) = r, \end{aligned} \quad (6.10)$$

where  $\mathbf{Z}_0$  is defined by (5.12), and for  $a \in [0, 1]$ ,  $\delta_a$  denotes the unit Dirac measure located at  $a$ , and  $\ell$  is the Lebesgue measure of  $[0, 1]$ .

*Remark 6.3.* Result (6.10) can give another explanation to result (5.13) in Section 5. To see this, one can notice that any integrable potential  $q \in \mathcal{L}^1$  induces an absolutely continuous measure  $\mu_q$  defined by

$$\mu_q(t) := \int_{[0,t]} q(s) ds \quad \text{for } t \in [0, 1]. \quad (6.11)$$

Since  $\|\mu_q\|_{\mathbf{V}} = \|q\|_1$ , one sees that  $B_1[r] \subset (\mathcal{L}^1, \|\cdot\|_1)$  can be isometrically embedded into  $B_0[r] \subset (\mathcal{M}_0, \|\cdot\|_{\mathbf{V}})$ . Hence one has

$$\inf_{q \in B_1[r]} \lambda_0^N(q) \geq \min_{\mu \in S_0[r]} \lambda_0^N(\mu). \quad (6.12)$$

On the other hand, the minimizing (singular) measures  $r\delta_0$  and  $r\delta_1$  in (6.9) can be approximated by sequences  $\{\mu_{q_n}\}$  in the weak\* topology  $w_*$  where  $q_n$  are smooth potentials from  $B_1[r]$ . Hence we have

$$\min_{\mu \in S_0[r]} \lambda_0^N(\mu) = \lim_{n \rightarrow \infty} \lambda_0^N(\mu_{q_n}) \geq \inf_{q \in B_1[r]} \lambda_0^N(q). \quad (6.13)$$

Thus one has (5.13).

Let us return to the minimizing/maximizing problems of eigenvalues of (1.1) and (3.1) for potentials  $q$  in  $B_1[r]$ . Though  $B_1[r] \subset \mathcal{L}^1$  has no compactness in  $\|\cdot\|_1$  or  $w_1$ , it has been shown that all maximizing problems

$$\sup_{q \in B_1[r]} \lambda_n^\sigma(q) \quad (6.14)$$

can be realized by potentials from  $S_1[r] \subset B_1[r]$ . In fact, these maximizing potentials are step potentials. For detailed construction, we refer to [17] for general  $p \in (1, \infty)$ .

On the other hand, because of the noncompactness of  $B_1[r]$ ,  $r > 0$ , the corresponding minimizing problems

$$\inf_{q \in B_1[r]} \lambda_n^\sigma(q) \quad (6.15)$$

cannot be realized by any potential from  $B_1[r]$ , but by some singular measures from  $S_0[r] \subset B_0[r]$ . The following results hold for eigenvalues of (1.1), that is,  $p = 2$  in (3.1). For example, from (6.10) and (5.13), one has

$$\inf_{q \in B_1[r]} \lambda_0^N(q) = \lambda_0^N(r\delta_0) = \lambda_0^N(r\delta_1). \tag{6.16}$$

For the first Dirichlet eigenvalues  $\lambda_1^D(q)$ , one has from [20, 26] that

$$\inf_{q \in B_1[r]} \lambda_1^D(q) = \lambda_1^D(r\delta_{1/2}). \tag{6.17}$$

The main result of [23] states that

$$\inf_{q \in \tilde{B}_1[r]} \lambda_0^N(q) = \lambda_0^N\left(\pm \frac{r}{2}(\delta_0 - \delta_1)\right) = -\frac{r^2}{4}, \tag{6.18}$$

where  $\tilde{B}_1[r] = \{q \in \mathcal{L}^1 : \|q\|_1 = r, \bar{q} = 0\}$ . Results (6.16)–(6.18) can yield a natural explanation to what kinds of integrable potentials in  $B_1[r]$  will decrease the eigenvalues.

### 7. Some Open Problems

We end this paper with two open problems.

*Problem 1* (eigenvalue gaps). Let us consider, for example, the Dirichlet eigenvalues  $\lambda_n(q)$ ,  $n \in \mathbb{N}$ , of problem (1.1) with integrable potentials  $q \in \mathcal{L}^1$ . In applied sciences, it is important to study eigenvalue gaps like  $\lambda_{n+1}(q) - \lambda_n(q)$ ,  $n \in \mathbb{N}$ . See [38–42]. In most literature, only single-well and symmetric potentials are considered, and lower and upper bounds for these gaps are obtained. Because of the boundedness of  $\lambda_n(q)$  for  $q$  in bounded subsets of  $(\mathcal{L}^1, \|\cdot\|_1)$ , the following extremal problems for eigenvalue gaps:

$$\inf_{q \in S_1[r]} (\lambda_{n+1}(q) - \lambda_n(q)) \quad \sup_{q \in S_1[r]} (\lambda_{n+1}(q) - \lambda_n(q)), \quad n \in \mathbb{N}, \tag{7.1}$$

are well posed. The problem is how to solve these explicitly, including the extremal values and the corresponding minimizers/maximizers.

*Problem 2* (eigenvalues of the beam equation with integrable potentials). Given an integrable potential  $q \in \mathcal{L}^1$ , consider the eigenvalue problem of the beam equation, or the Euler-Bernoulli equation [43–45],

$$x^{(4)} + q(t)x = \lambda x, \quad t \in [0, 1], \tag{7.2}$$

with the Lidstone boundary condition

$$x(0) = x(1) = 0, \quad x''(0) = x''(1) = 0. \tag{7.3}$$

Eigenvalues of problem (7.2)-(7.3) are still denoted by  $\{\lambda_n(q)\}_{n \in \mathbb{N}}$ . Like the second-order Sturm-Liouville operator (1.1), it is desirable to develop analogous ideas so that the following extremal problems for eigenvalues:

$$\inf_{q \in S_1[r]} \lambda_n(q), \quad \sup_{q \in S_1[r]} \lambda_n(q), \quad (7.4)$$

and the extremal problems for eigenvalue gaps like

$$\inf_{q \in S_1[r]} (\lambda_2(q) - \lambda_1(q)), \quad \sup_{q \in S_1[r]} (\lambda_2(q) - \lambda_1(q)), \quad (7.5)$$

can be solved in a complete way.

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## References

- [1] A. Derlet, J. P. Gossez, and P. Takáč, "Minimization of eigenvalues for a quasilinear elliptic Neumann problem with indefinite weight," *Journal of Mathematical Analysis and Applications*, vol. 371, no. 1, pp. 69–79, 2010.
- [2] C.-Y. Kao, Y. Lou, and E. Yanagida, "Principal eigenvalue for an elliptic problem with indefinite weight on cylindrical domains," *Mathematical Biosciences and Engineering*, vol. 5, no. 2, pp. 315–335, 2008.
- [3] Y. Lou and E. Yanagida, "Minimization of the principal eigenvalue for an elliptic boundary value problem with indefinite weight, and applications to population dynamics," *Japan Journal of Industrial and Applied Mathematics*, vol. 23, no. 3, pp. 275–292, 2006.
- [4] X. Liang, X. Lin, and H. Matano, "A variational problem associated with the minimal speed of travelling waves for spatially periodic reaction-diffusion equations," *Transactions of the American Mathematical Society*, vol. 362, no. 11, pp. 5605–5633, 2010.
- [5] X. Liang, Y. Yi, and X. Q. Zhao, "Spreading speeds and traveling waves for periodic evolution systems," *Journal of Differential Equations*, vol. 231, no. 1, pp. 57–77, 2006.
- [6] S. Karaa, "Sharp estimates for the eigenvalues of some differential equations," *SIAM Journal on Mathematical Analysis*, vol. 29, no. 5, pp. 1279–1300, 1998.
- [7] M. G. Krein, "On certain problems on the maximum and minimum of characteristic values and on the Lyapunov zones of stability," *American Mathematical Society Translations*, vol. 2, no. 1, pp. 163–187, 1955.
- [8] W. Li and P. Yan, "Various half-eigenvalues of scalar  $p$ -Laplacian with indefinite integrable weights," *Abstract and Applied Analysis*, vol. 2009, Article ID 109757, 27 pages, 2009.
- [9] W. Li and P. Yan, "Continuity and continuous differentiability of half-eigenvalues in potentials," *Communications in Contemporary Mathematics*, vol. 12, no. 6, pp. 977–996, 2010.
- [10] G. Meng, P. Yan, and M. Zhang, "Spectrum of one-dimensional  $p$ -Laplacian with an indefinite integrable Weight," *Mediterranean Journal of Mathematics*, vol. 7, no. 2, pp. 225–248, 2010.
- [11] G. Meng and M. R. Zhang, "Continuity in weak topology: first order linear systems of ODE," *Acta Mathematica Sinica, English Series*, vol. 26, no. 7, pp. 1287–1298, 2010.
- [12] M. M. Möller and A. Zettl, "Differentiable dependence of eigenvalues of operators in Banach spaces," *Journal of Operator Theory*, vol. 36, pp. 335–355, 1996.
- [13] J. P. Pöschel and E. Trubowitz, *The Inverse Spectral Theory*, Academic Press, New York, NY, USA, 1987.



- [14] P. Yan and M. Zhang, "Continuity in weak topology and extremal problems of eigenvalues of the  $p$ -laplacian," *Transactions of the American Mathematical Society*, vol. 363, no. 4, pp. 2003–2028, 2011.
- [15] A. Zettl, *Sturm-Liouville Theory, Mathematical Surveys and Monographs*, vol. 121, American Mathematical Society, Providence, RI, USA, 2005.
- [16] M. Zhang, "Continuity in weak topology: higher order linear systems of ODE," *Science in China, Series A*, vol. 51, no. 6, pp. 1036–1058, 2008.
- [17] G. Meng, P. Yan, and M. Zhang, "Maximization of eigenvalues of one-dimensional  $p$ -Laplacian with integrable potentials," *Communications in Contemporary Mathematics*. In press.
- [18] G. Meng, P. Yan, and M. Zhang, "Minimization of eigenvalues of one-dimensional  $p$ -Laplacian with integrable potentials," *Journal of Optimization Theory and Applications*. In press.
- [19] G. Meng, P. Yan, and M. Zhang, "Minimizing and maximizing eigenvalues of Sturm-Liouville operators with integrable potentials," preprint, <http://faculty.math.tsinghua.edu.cn/~mzhang/pubs/publ.htm>.
- [20] Q. Wei, G. Meng, and M. Zhang, "Extremal values of eigenvalues of Sturm-Liouville operators with potentials in  $L^1$  balls," *Journal of Differential Equations*, vol. 247, no. 2, pp. 364–400, 2009.
- [21] M. Zhang, "Extremal values of smallest eigenvalues of Hill's operators with potentials in  $L^1$  balls," *Journal of Differential Equations*, vol. 246, no. 11, pp. 4188–4220, 2009.
- [22] M. R. Zhang, "Extremal eigenvalues of measure differential equations with fixed variation," *Science China Mathematics*, vol. 53, no. 10, pp. 2573–2588, 2010.
- [23] M. Zhang, "Minimization of the zeroth Neumann eigenvalues with integrable potentials," *Annales de l'Institut Henri Poincaré Analyse Non Linéaire*, vol. 29, pp. 501–523, 2012.
- [24] N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Interscience Publishers, New York, NY, USA, 1958.
- [25] R. E. Megginson, *An Introduction to Banach Space Theory, Graduate Texts in Mathematics*, vol. 183, Springer, New York, NY, USA, 1998.
- [26] G. Meng, *Continuity of Solutions and Eigenvalues in Measures with Weak\* Topology, Doctoral Dissertation*, Tsinghua University, Beijing, China, 2009.
- [27] G. Meng and M. Zhang, "Dependence of solutions and eigenvalues of measure differential equations on measures," preprint, <http://faculty.math.tsinghua.edu.cn/~mzhang/pubs/publ.htm>.
- [28] P. L. Binding and P. Drábek, "Sturm-Liouville theory for the  $p$ -Laplacian," *Studia Scientiarum Mathematicarum Hungarica*, vol. 40, no. 4, pp. 375–396, 2003.
- [29] M. Zhang, "The rotation number approach to eigenvalues of the one-dimensional  $p$ -laplacian with periodic potentials," *Journal of the London Mathematical Society*, vol. 64, no. 1, pp. 125–143, 2001.
- [30] M. Cuesta and H. R. Quoirin, "A weighted eigenvalue problem for the  $p$ -Laplacian plus a potential," *Nonlinear Differential Equations and Applications*, vol. 16, no. 4, pp. 469–491, 2009.
- [31] M. Zhang, "Certain classes of potentials for  $p$ -Laplacian to be non-degenerate," *Mathematische Nachrichten*, vol. 278, no. 15, pp. 1823–1836, 2005.
- [32] B. M. Brown and M. S. P. Eastham, "Titchmarsh's asymptotic formula for periodic eigenvalues and an extension to the  $p$ -Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 2, pp. 1255–1266, 2008.
- [33] A. B. Mingarelli, *Volterra-Stieltjes Integral Equations and Generalized Ordinary Differential Expressions*, vol. 989 of *Lecture Notes in Mathematics*, Springer, New York, NY, USA, 1983.
- [34] S. Schwabik, *Generalized Ordinary Differential Equations*, World Scientific, Singapore, 1992.
- [35] M. Carter and B. van Brunt, *The Lebesgue-Stieltjes Integral: A Practical Introduction*, Springer, New York, NY, USA, 2000.
- [36] F. H. Clark, *Optimization and Nonsmooth Analysis*, John Wiley and Sons, New York, NY, USA, 1983.
- [37] E. Zeidler, *Nonlinear Functional Analysis and its Applications, III, Variational Methods and Optimization*, Springer, New York, NY, USA, 1985.
- [38] M. S. Ashbaugh and R. Benguria, "Optimal lower bounds for eigenvalue gaps for Schrödinger operators with symmetric single-well potentials and related results," in *Maximum Principles and Eigenvalue Problems in Partial Differential Equations (Knoxville, TN, 1987)*, vol. 175 of *Pitman Research Notes Mathematical Series*, pp. 134–145, Longman Science and Technology, Harlow, Harlow, 1988.
- [39] M. S. Ashbaugh, E. M. Harrell, and R. Svirsky, "On minimal and maximal eigenvalue gaps and their causes," *Pacific Journal of Mathematics*, vol. 147, no. 1, pp. 1–24, 1991.

- [40] M.-J. Huang and T.-M. Tsai, "The eigenvalue gap for one-dimensional Schrödinger operators with symmetric potentials," *Proceedings of the Royal Society of Edinburgh Section A*, vol. 139, no. 2, pp. 359–366, 2009.
- [41] S. Karaa, "Extremal eigenvalue gaps for the Schrödinger operator with Dirichlet boundary conditions," *Journal of Mathematical Physics*, vol. 39, no. 4, pp. 2325–2332, 1998.
- [42] R. Lavine, "The eigenvalue gap for one-dimensional convex potentials," *Proceedings of the American Mathematical Society*, vol. 121, pp. 815–821, 1994.
- [43] V. G. Papanicolaou, "The periodic Euler-Bernoulli equation," *Transactions of the American Mathematical Society*, vol. 355, no. 9, pp. 3727–3759, 2003.
- [44] V. G. Papanicolaou and D. Kravvaritis, "The floquet theory of the periodic Euler-Bernoulli equation," *Journal of Differential Equations*, vol. 150, no. 1, pp. 24–41, 1998.
- [45] L. Pipes, *Applied Mathematics for Engineers and Physicists*, McGraw-Hill, New York, NY, USA, 1958.