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Research Article

Asymptotic Behavior of Approximated Solutions to Parabolic Equations with Irregular Data

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Let Ω be a smooth bounded domain in $\mathbb{R}^N, (N \geq 3)$. We consider the asymptotic behavior of solutions to the following problem $u_t - \operatorname{div}(a(x)\nabla u) + \lambda f(u) = \mu$ in $\Omega \times \mathbb{R}^+, u = 0$ on $\partial\Omega \times \mathbb{R}^+, u(x,0) = u_0(x)$ in Ω, where $u_0 \in L^1(\Omega)$, μ is a finite Radon measure independent of time. We provide the existence and uniqueness results on the approximated solutions. Then we establish some regularity results on the solutions and consider the long-time behavior.

1. Introduction

We consider the asymptotic behavior of solutions to the following equations

$$\begin{aligned} u_t - \operatorname{div}(a(x)\nabla u) + \lambda f(u) = \mu, &\quad \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, &\quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), &\quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, $u_0 \in L^1(\Omega)$, $\lambda \geq 0,\mu$ is a finite Radon measure independent of time, $a(x)$ is a matrix with bounded, measurable entries, and satisfying the ellipticity assumption

$$a(x)\xi \cdot \xi \geq c|\xi|^2, \quad \text{for any } \xi \in \mathbb{R}^{\mathrm{N}}, \text{with } c > 0. \tag{1.2}$$

Concerning the nonlinear term, we assume that f is a C^1 function satisfying,

$$f'(s) \geq -l, \tag{1.3}$$

$$\left|f'(s)\right| \leq C\left(1 + |s|^{p-2}\right), \quad p \geq 2, \tag{1.4}$$

$$C|s|^p - k \leq f(s)s \leq C|s|^p + k, \quad p \geq 2, \tag{1.5}$$

both for all $s \in R$, where l, C, k are positive constants.

Parabolic equations with L^1 or measure data arise in many physical models, control problems, and in models of turbulent flows in oceanography and climatology [1–4]. Existence and regularity results for parabolic equations with L^1 and measure data have been studied widely by many authors in the past decades, see [5–10]. The usual approach to study problems with these kinds of data is approximation. The basic reference for these arguments is [7], where the authors obtained weak solutions (in distribution sense) to nonlinear parabolic equations. In our setting, such a solution is a function $u \in L^1((0,T); W_0^{1,1}(\Omega))$ such that $f(u) \in L^1(Q_T)$ for any $T > 0$, and

$$-\int_{Q_T} u\varphi_t dx\, dt + \int_{Q_T} a(x)\nabla u \nabla \varphi dx\, dt + \lambda \int_{Q_T} f(u)\varphi dx\, dt = \int_{Q_T} \varphi d\mu + \int_\Omega u_0(x)\varphi(0,x)dx, \tag{1.6}$$

for any $\varphi \in C_c^\infty([0,T) \times \Omega)$.

Generally, the regularity of weak solutions (in distribution sense) is not strong enough to ensure uniqueness [8]. But one may select a weak solution which is "better" than the others. Since one may prove that the weak solution obtained from approximation does not depend on the approximation chosen for the irregular data. In such a sense, it is the only weak solution which is found by means of approximations; we may call it approximated solutions. Such a concept was first introduced by [9]. Here in the present paper, we will focus ourselves to the scope of approximated solutions, that is, weak solutions obtained as limits of approximations.

The long-time behavior of parabolic problems with irregular data (such as L^1 data, measure data) have been considered by many authors [11–16]. In [11, 12], existence of global attractors for porous media equations and m-Laplacian equations with irregular initial data were deeply studied, while in [13, 14] the convergence to the equilibrium for the solutions of parabolic problems with measrued data were thoroughly investigated. In [15, 16], we considered the existence of global attractors for the parabolic equations with L^1 data.

In this paper, we intend to consider the asymptotic behavior of approximated solutions to problem (1.1) with measure data. Precisely speaking, we assume that the forcing term in the equations is just a finite Radon measure. For the case $\lambda > 0$, to ensure the existence result for large p in (1.5) [17], we restrict ourselves to diffuse measures, that is, μ does not charge the sets of zero parabolic 2-capacity (see details for parabolic p-capacity in [18]). We first provide the existence result for problem (1.1) and prove the uniqueness of the approximated solution. Then using some decomposition techniques, we establish some new regularity results and show the existence of a global attractor $\mathcal{A}$ in $L^{p-1}(\Omega) \cap W_0^{1,q}(\Omega)$ with $q < \max\{N/(N-1), (2p-2)/p\}$, which attracts every bounded subset of $L^1(\Omega)$ in the norm of $L^r(\Omega) \cap H_0^1(\Omega)$, for any $r \in [1, \infty)$.

For the case $\lambda = 0$, we consider general bounded Radon measure μ which is independent of time. We provide the uniqueness of approximated solutions for the parabolic problem and its corresponding elliptic problem. Then we prove that the approximated solution of the parabolic equations converges to the unique approximated solution of the corresponding elliptic equations in the norm topology of $L^r(\Omega) \cap H_0^1(\Omega)$, for any $r \in [1, \infty)$, though they all lie in some less regular spaces.

Our main results can be stated as follows.

Theorem 1.1. *Assume that $u_0 \in L^1(\Omega)$, $\lambda > 0$, μ is a bounded Radon measure, which does not charge the sets of zero parabolic 2-capacity and is independent of time, f is a C^1 function satisfying assumptions* (1.3)–(1.5). *Then the semigroup $\{S(t)\}_{t\geq 0}$, generated by approximated solutions of problem* (1.1), *possesses a global attractor $\mathcal{A}$ in $L^1(\Omega)$. Moreover, $\mathcal{A}$ is compact and invariant in $L^{p-1}(\Omega) \cap W_0^{1,q}(\Omega)$ with $q < \max\{N/(N-1), (2p-2)/p\}$, and attracts every bounded subset of $L^1(\Omega)$ in the norm topology of $L^r(\Omega) \cap H_0^1(\Omega)$, $1 \leq r < \infty$.*

Theorem 1.2. *Assume that $u_0 \in L^1(\Omega)$, $\lambda = 0$, μ is a bounded Radon measure independent of time. Then the approximated solution $u(t)$ of problem* (1.1) *is unique and converges to the unique approximated solution of the corresponding elliptic equations in the norm topology of $L^r(\Omega) \cap H_0^1(\Omega)$, for any $1 \leq r < \infty$.*

Remark 1.3. Though $u(t)$ and v all lie in some less-regular spaces, $u(t)$ converges to v in stronger norm, that is, $u(t) - v$ converges to 0 in $L^r(\Omega) \cap H_0^1(\Omega)$, $1 \leq r < \infty$. Such a result, in some sense, sharpens the result of [13], where the author showed that $u(t)$ converges to v in $L^1(\Omega)$.

We organize the paper as follows: in Section 2, we provide the existence of approximated solutions, prove the uniqueness result and some useful lemmas; in Section 3, we establish some improved regularity results on the approximated solutions. At last, in Section 4, we prove the main theorems.

For convenience, for any $T > 0$ we use Q_T to denote $\Omega \times (0, T)$ hereafter. Also, we denote by $|E|$ the Lebesgue measure of the set E, and denote by C any positive constant which may be different from each other even in the same line.

2. Existence Results and Useful Lemmas

We begin this section by providing some existence results on the approximated solutions.

Definition 2.1. A function u is called an approximated solution of problem (1.1), if $u \in L^1((0, T); W_0^{1,1}(\Omega))$, $f(u) \in L^1(Q_T)$ for any $T > 0$, and

$$-\int_{Q_T} u\varphi_t dx\, dt + \int_{Q_T} a(x)\nabla u \nabla \varphi dx\, dt + \lambda \int_{Q_T} f(u)\varphi dx\, dt = \int_{Q_T} \varphi d\mu + \int_{\Omega} u_0(x)\varphi(0, x) dx, \tag{2.1}$$

for any $\varphi \in C_c^{\infty}([0,T) \times \Omega)$, and moreover, u is obtained as limit of solutions to the following approximated problem

$$\begin{gathered} u_t^n - \operatorname{div}(a(x)\nabla u^n) + \lambda f(u^n) = \mu^n, \quad \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0^n, \quad \text{in } \Omega, \end{gathered} \tag{2.2}$$

where $\{\mu^n\}, \{u_0^n\}$ is a smooth approximation of data μ, u_0.

Theorem 2.2. *Under the assumptions of Theorem 1.1, problem* (1.1) *has a unique approximated solution $u \in C([0,T]; L^1(\Omega))$, for any $T > 0$, satisfying*

(i) *if $p \geq (2N+2)/N$, then $u \in L^q((0,T); W_0^{1,q}(\Omega))$ with $q < (2p-2)/p$;*

(ii) *if $2 \leq p < (2N+2)/N$, then $u \in L^q((0,T); W_0^{1,q}(\Omega))$ with $q < (N+2)/(N+1)$.*

Proof. According to [18, Theorem 2.12], if a Radon measure μ on Q_T does not charge the sets of zero parabolic 2-capacity and is independent of time, μ can actually be identified as a Radon measure which is absolutely continuous with respect to the elliptic 2-capacity. Using Theorem 2.1 of [19], μ can be decomposed as $\mu = g + \operatorname{div} G$, where $g \in L^1(\Omega), G \in (L^2(\Omega))^N$. Hence, we need only to consider the following problem

$$\begin{gathered} u_t - \operatorname{div}(a(x)\nabla u) + \lambda f(u) = g + \operatorname{div} G, \quad \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0, \quad \text{in } \Omega. \end{gathered} \tag{2.3}$$

The proof of existence part of the theorem is similar to [9]. Besides, one can prove $u \in C([0,T]; L^1(\Omega))$ using arguments similar to CLAIM 2 in [8]. So we omit the details of them and only prove the uniqueness result.

Let $\{g^n\}_{n\in\mathbb{N}}, \{u_0^n\}_{n\in\mathbb{N}}$ be a smooth approximation of data g and u_0 with

$$\|u_0^n\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}, \qquad \|\tilde{g}^n\|_{L^1(\Omega)} \leq \|g\|_{L^1(\Omega)}, \tag{2.4}$$

and let $\{\tilde{g}^n\}_{n\in\mathbb{N}}, \{\tilde{u}_0^n\}_{n\in\mathbb{N}}$ be another smooth approximation of the data with

$$\|\tilde{u}_0^n\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}, \qquad \|\tilde{g}^n\|_{L^1(\Omega)} \leq \|g\|_{L^1(\Omega)}. \tag{2.5}$$

Assume that $u, \tilde{u}$ are two approximated solutions to problem (1.1), obtained as limit of the solutions to the following two approximated problems, respectively,

$$\begin{gathered} u_t^n - \operatorname{div}(a(x)\nabla u^n) + \lambda f(u^n) = g^n + \operatorname{div} G, \quad \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0^n, \quad \text{in } \Omega, \end{gathered} \tag{2.6}$$

$$\begin{gathered} \tilde{u}_t^n - \operatorname{div}(a(x)\nabla \tilde{u}^n) + \lambda f(\tilde{u}^n) = \tilde{g}^n + \operatorname{div} G, \quad \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = \tilde{u}_0^n, \quad \text{in } \Omega. \end{gathered} \tag{2.7}$$

Now we prove that $u = \tilde{u}$. For any $k > 0$, define $\varphi_k(s)$ as

$$\varphi_k(s) = \begin{cases} k, & s > k \\ s, & |s| \le k \\ -k, & s < -k. \end{cases} \tag{2.8}$$

Let $\Psi_k(\sigma) = \int_0^\sigma \varphi_k(s)ds$ be its primitive function. Taking $\varphi_k(u^n - \tilde{u}^n)$ as a test function in (2.6) and (2.7), we deduce that

$$\begin{aligned} &\int_\Omega \Psi_k(u^n - \tilde{u}^n)(t)dx - \int_\Omega \Psi_k(u_0^n - \tilde{u}_0^n)dx + \int_{Q_T} a(x)\left(\nabla\varphi_k(u^n - \tilde{u}^n)\right)^2 dx\,dt \\ &\quad = \lambda \int_{Q_T} \left(f(\tilde{u}^n) - f(u^n)\right)\varphi_k(u^n - \tilde{u}^n)dx\,dt + \int_{Q_T} \left(g^n - \tilde{g}^n\right)\varphi_k(u^n - \tilde{u}^n)dx\,dt \\ &\quad \le \int_{Q_T} \lambda l(u^n - \tilde{u}^n)\varphi_k(u^n - \tilde{u}^n)dx\,dt + \int_{Q_T} \left|g^n - \tilde{g}^n\right| dx\,dt. \end{aligned} \tag{2.9}$$

Hence, from the assumptions on f, we get

$$\int_\Omega \Psi_k(u^n - \tilde{u}^n)(t)dx \le 2l\lambda \int_0^T \int_\Omega \Psi_k(u^n - \tilde{u}^n)dx\,dt + \int_{Q_T} \left|g^n - \tilde{g}^n\right| dx\,dt + \int_\Omega \Psi_k\left(u_0^n - \tilde{u}_0^n\right)dx. \tag{2.10}$$

Let $n \to \infty$, we have

$$\int_\Omega \Psi_k(u - \tilde{u})(t)dx \le 2l\lambda \int_0^T \int_\Omega \Psi_k(u - \tilde{u})(t)dx\,dt. \tag{2.11}$$

Thus for all $k > 0$, we have

$$\sup_{[0,T]} \int_\Omega \Psi_k(\tilde{u} - u)(t)dx \le 2lT \sup_{[0,T]} \int_\Omega \Psi_k(\tilde{u} - u)dx. \tag{2.12}$$

Taking T' small enough such that $2l\lambda T' < 1$, we deduce that $\Psi_k(\overline{u} - u) = 0$ for all $k > 0$ in $Q_{T'}$, thus $u \equiv \overline{u}$ in $Q_{T'}$. Dividing $[0, T]$ into several intervals to carry out the same arguments, we obtain the uniqueness of the approximated solution.

Similar to [20], we can prove the following. □

Theorem 2.3. *Under the assumptions of Theorem 1.1, there exists at least one approximated solution v to the stationary problem of corresponding to problem* (1.1)*, with*

(i) *if $p \ge (2N-2)/(N-2)$, we have $v \in W_0^{1,q}(\Omega)$ for $q < (2p-2)/p$;*

(ii) *if $2 \le p < (2N-2)/(N-2)$, we have $v \in W_0^{1,q}(\Omega)$ for $q < N/(N-1)$.*

Remark 2.4. Note that if v is an approximated solution to following problem

$$\begin{gathered} -\operatorname{div}(a(x)\nabla v) + \lambda f(v) = g + \operatorname{div} G, \quad \text{in } \Omega, \\ v = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.13}$$

then there is a sequence $\{v^n\}$ converges to v, where v^n is the solution of the corresponding approximated problem

$$\begin{gathered} -\operatorname{div}(a(x)\nabla v^n) + \lambda f(v^n) = g^n + \operatorname{div} G, \quad \text{in } \Omega, \\ v^n = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{2.14}$$

And hence v^n is a solution of parabolic equations

$$\begin{gathered} v_t^n - \operatorname{div}(a(x)\nabla v^n) + \lambda f(v^n) = g^n + \operatorname{div} G, \quad \text{in } \Omega \times \mathbb{R}^+, \\ v^n = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ v^n(0) = v^n(x), \quad \text{in } \Omega. \end{gathered} \tag{2.15}$$

Thus, v is an approximated solution of problem (1.1) with initial data $u_0 = v(x)$.

Under the assumptions of Theorem 1.2, the problem turns out to be

$$\begin{gathered} u_t - \operatorname{div}(a(x)\nabla u) = \mu, \quad \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x, 0) = u_0, \quad \text{in } \Omega. \end{gathered} \tag{2.16}$$

The existence of approximated solutions to problem (1.1) and the elliptic equations corresponding to it follows directly from Sections IV and II of [7]. Form Section 2.3 of [21], we know that the approximated solution to the stationary equations is actually a duality solution, and hence unique. Furthermore, it is not difficult to prove that an approximated solution for linear parabolic equations turns out to be a duality solution, and hence unique too.

Lemma 2.5. *Under the assumptions of Theorem 1.2, an approximated solution to the parabolic problem* (1.1), (2.16) *turns out to be a duality solution, and conversely.*

Proof. The proof is mainly similar to that of Theorem 6 in [22]. We just sketch it. Let u be an approximated solution, then there exist a smooth approximation $\{\mu^n\}_{n\in\mathbb{N}}$, $\{u_0^n\}_{n\in\mathbb{N}}$ of data μ and u_0, such that the solution of the approximated problem of (2.16) with data μ^n and u_0^n converges to u. Let $h \in C_c^\infty(Q_T)$ and ω be the solution of the the following parabolic problem

$$\begin{gathered}
-\omega_t - \operatorname{div}(a^*(x)\nabla\omega) = h, \quad \text{in } \Omega\times(0,T), \\
\omega = 0, \quad \text{on } \partial\Omega\times(0,T), \\
\omega(x,T) = 0, \quad \text{in } \Omega,
\end{gathered} \tag{2.17}$$

where $a^*(x)$ is the transposed matrix of $a(x)$. Taking ω as a test function in the approximated problem and taking u^n as a test function in the problem above, then let n go to infinity we obtain that the approximated solution is a duality solution. Form the uniqueness of duality solutions [13], we get the conclusion. □

Now we provide two lemmas which are useful in analyzing the regularity and asymptotic behavior of the solutions to problem (1.1).

Lemma 2.6 (see [15]). *Let X, Y be two Banach spaces, let X be separable, reflexive, and let $X \subset Y$ with dual X^*. Suppose that $\{u^n\}$ is uniformly bounded in $L^\infty((0,T);X)$ with*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|u^n(t)\|_X \le C, \tag{2.18}$$

and that $u^n \to u$ weakly in $L^r((0,T);X)$ for some $r \in (1,\infty)$. Then

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|u(t)\|_X \le C. \tag{2.19}$$

Moreover, if $u \in C([0,T];Y)$, then in fact

$$\sup_{t\in[0,T]} \|u(t)\|_X \le C. \tag{2.20}$$

Lemma 2.7 (see [16]). *Let X, Y be two Banach spaces with imbedding $X \hookrightarrow Y$, let $\{S(t)\}_{t\ge0}$ be a continuous semigroup on Y. Assume that $\{S(t)\}_{t\ge0}$ is asymptotically compact in X and has an absorbing set $B_0 \subset X$, that is, for any bounded set $B \subset Y$, there exists a $T = T(B)$ such that*

$$S(t)B \subset B_0, \quad \forall t \ge T. \tag{2.21}$$

Then $\{S(t)\}_{t\ge0}$ has a global attractor $\mathcal{A}$ in X, which is compact, invariant in X and attracts every bounded sets of Y in the topology of X.

Remark 2.8. To obtain global attractors, one usually needs the semigroup to be norm-to-norm continuous, weak-to-weak, or norm-to-weak continuous [23–27]. Here, to obtain global attractors in the space X, we need neither of them. We only need the semigroup to be continuous in a less-regular space Y.

3. Improved Regularity Results on the Approximated Solutions

In this section, we prove the following regularity results on the approximated solution u to problem (1.1).

Theorem 3.1. *Under the assumptions of Theorem 2.2, let $u(t)$ be the approximated solution to problem* (1.1)*. Then u admits the decomposition $u(x,t) = w(x,t) + v(x)$, with v being an approximated solution to problem* (2.13)*, and w being an approximated solution of the following problem*

$$\begin{gathered} w_t - \operatorname{div}(a(x)\nabla w) + \lambda f(v+w) - \lambda f(v) = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \\ w = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ w(x,0) = u_0 - v, \quad \text{in } \Omega. \end{gathered} \tag{3.1}$$

Moreover, we have

(i) *$w \in L^\infty((\delta,T);L^q(\Omega))$ for any $0 < \delta < T, 1 \le q < \infty$. Moreover, there exists a constant M_q and a time $t_q(u_0,g,G)$ such that $\|w(t)\|_{L^q(\Omega)} \le M_q$ for all $t \ge t_q(u_0,g,G)$.*

(ii) *$w \in L^\infty((\delta,T);H_0^1(\Omega))$ for any $0 < \delta < T$. Moreover, there exists a constant ρ and a time $T_0(u_0,g,G)$ such that $\|w(t)\|_{H_0^1(\Omega)} \le \rho$ for all $t \ge T_0(u_0,g,G)$.*

Proof. We follow the lines of [15, 28]. Let $\{g^n\}$ be a sequence of smooth data which converges to g in $L^1(\Omega)$ and $\|g^n\|_{L^1(\Omega)} \le \|g\|_{L^1(\Omega)}$. Let v^n be a solution of the following approximated problem for each n,

$$\begin{gathered} -\operatorname{div}(a(x)\nabla v^n) + \lambda f(v^n) = g^n + \operatorname{div} G, \quad \text{in } \Omega, \\ v^n = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.2}$$

Then v^n converges (up to subsequences) to an approximated solution v strongly in $L^1(\Omega)$, and weakly in $W_0^{1,q}(\Omega), 1 \le q < N/(N-1)$. Let $\{u^n\}$ be a sequence of solutions to the following approximated problem

$$\begin{gathered} u_t^n - \operatorname{div}(a(x)\nabla u^n) + \lambda f(u^n) = g^n(x) + \operatorname{div} G, \quad \text{in } \Omega \times \mathbb{R}^+, \\ u^n(x) = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u^n(x,0) = u_0^n, \quad \text{in } \Omega, \end{gathered} \tag{3.3}$$

where u_0^n converges to u_0 with $\|u_0^n\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)}$. Similar to [8, 29], we know that

$$\begin{gathered} u^n \longrightarrow u \quad \text{weakly in } L^q\Big((0,T);W_0^{1,q}(\Omega)\Big),\ q < \frac{N+2}{N+1}, \\ u^n \longrightarrow u \quad \text{in } C\Big([0,T];L^1(\Omega)\Big). \end{gathered} \tag{3.4}$$

Now let $w^n(t) = u^n(t) - v^n$. Then w^n satisfies

$$\begin{gathered} w_t^n - \operatorname{div}(a(x)\nabla w^n) + \lambda f(v^n + w^n) - \lambda f(v^n) = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \\ w^n = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ w^n(x,0) = u_0^n - v^n, \quad \text{in } \Omega. \end{gathered} \tag{3.5}$$

Similarly, we have w^n (up to subsequences) converges to the approximated solution w of problem (3.1) in $C([0,T];L^1(\Omega))$ and weakly in $L^q((0,T);W_0^{1,q}(\Omega)), q < (N+2)/(N+1)$.

Now we prove (i). Taking $\varphi_1(u^n)$ as test function in (3.3) (for simplicity we take $\lambda = 1$), we deduce that

$$\frac{d}{dt}\int_\Omega \Psi_1(u^n)dx + \int_\Omega f(u^n)\varphi_1(u^n)dx \le \|g\|_{L^1(\Omega)} + C\|G\|_{L^2(\Omega)}. \tag{3.6}$$

Since

$$f(u^n)\varphi_1(u^n) \ge \Big(C|u^n|^{p-1} - C\Big)|\varphi_1(u^n)|, \tag{3.7}$$

we have

$$\frac{d}{dt}\int_\Omega \Psi_1(u^n)dx + C\int_\Omega |\Psi_1(u^n)|dx \le \|g\|_{L^1(\Omega)} + C\|G\|_{L^2(\Omega)} + C|\Omega|. \tag{3.8}$$

The Gronwall's inequality implies that

$$\int_\Omega \Psi_1(u^n(t))dx \le \|u_0\|_{L^1(\Omega)}e^{-Ct} + C|\Omega| + C\|g\|_{L^1(\Omega)} + C\|G\|_{L^2(\Omega)}. \tag{3.9}$$

Noticing that

$$\int_\Omega |u^n(t)|dx \le \int_\Omega \Psi_1(u^n(t))dx + |\Omega|, \tag{3.10}$$

we obtain that

$$\int_\Omega |u^n(t)|dx \le \|u_0\|_{L^1(\Omega)}e^{-Ct} + C|\Omega| + C\|g\|_{L^1(\Omega)} + C\|G\|_{L^2(\Omega)}, \quad t \ge 0. \tag{3.11}$$

Moreover, integrating (3.6) between t and $t+1$ and using (3.7) we have

$$\int_t^{t+1}\int_\Omega |u^n|^{p-1}dx\,d\xi \le C\Big(\|u_0\|_{L^1(\Omega)}e^{-Ct}+|\Omega|+\|g\|_{L^1(\Omega)}+\|G\|_{L^2(\Omega)}\Big). \tag{3.12}$$

Similarly, taking $\varphi_1(v^n)$ as test function in (3.2), we can deduce that

$$\int_\Omega |f(v^n)|dx \le C\int_\Omega\Big(|v^n|^{p-1}+1\Big)dx \le C\Big(\|g\|_{L^1(\Omega)}+\|G\|_{L^2(\Omega)}+|\Omega|\Big). \tag{3.13}$$

Hence,

$$\int_t^{t+1}\int_\Omega |w^n|^{p-1}dx\,d\xi \le C\Big(\|g\|_{L^1(\Omega)}+\|G\|_{L^2(\Omega)}+\|u_0\|_{L^1(\Omega)}+|\Omega|\Big), \tag{3.14}$$

with C independent of n, for $t\ge 0$.

Now we use bootstrap method in the case $p\ge 3$. The case $2\le p<3$ can be treated similarly with minor modifications. Multiplying (3.5) by $|w^n|^{q_0-2}w^n, q_0 = p-1\ge 2$, and integrating on Ω, we obtain

$$\frac{1}{q_0}\frac{d}{dt}\int_\Omega |w^n|^{q_0}dx + (q_0-1)c\int_\Omega |\nabla w^n|^2|w^n|^{q_0-2}dx \le l\int_\Omega |w^n|^{q_0}dx. \tag{3.15}$$

Since $|\nabla w^n|^2|w^n|^{q_0-2} = (2/q_0)^2|\nabla(|w^n|^{(q_0-2)/2}w^n)|^2$, we deduce that

$$\frac{d}{dt}\int_\Omega |w^n|^{q_0}dx + \int_\Omega\Big|\nabla\Big(|w^n|^{(q_0-2)/2}w^n\Big)\Big|^2dx \le C\int_\Omega |w^n|^{q_0}dx. \tag{3.16}$$

Integrating (3.16) between s and $t+1$ $(t\le s<t+1)$, it yields

$$\int_\Omega |w^n(t+1)|^{q_0}dx \le C\int_s^{t+1}\int_\Omega |w^n|^{q_0}dx\,d\xi + \int_\Omega |w^n(s)|^{q_0}dx. \tag{3.17}$$

Integrating the above inequality with respect to s between t and $t+1$, we get

$$\int_\Omega |w^n(t+1)|^{q_0}dx \le C\int_t^{t+1}\int_\Omega |w^n|^{q_0}dx\,d\xi. \tag{3.18}$$

Therefore,

$$\int_\Omega |w^n(t)|^{q_0}dx \le C,\quad \forall t\ge 1. \tag{3.19}$$

Integrating (3.16) on $[t, t+1]$ for $t \geq 1$, we deduce that

$$\int_t^{t+1}\int_\Omega \left|\nabla\left(|w^n|^{(q_0-2)/2}w^n\right)\right|^2 dx\,d\xi \leq C\int_t^{t+1}\int_\Omega |w^n(\xi)|^{q_0}dx\,d\xi + C\int_\Omega |w^n(t)|^{q_0}dx \leq C. \quad (3.20)$$

Note that (3.20) insures that, for any $t \geq 1$, there exists at least a $t_0 \in [t, t+1]$ such that

$$\int_\Omega \left|\nabla\left(|w^n(t_0)|^{(q_0-2)/2}w^n(t_0)\right)\right|^2 dx \leq C. \quad (3.21)$$

Standard Sobolev imbedding implies that

$$\int_\Omega |w^n(t_0)|^{q_0(N/(N-2))}dx \leq C. \quad (3.22)$$

Now multiplying (3.5) by $|w^n|^{q_1-2}w^n$, $q_1 = (N/(N-2))q_0$, we have

$$\begin{aligned}\frac{N-2}{Nq_0}\frac{d}{dt}\int_\Omega |w^n|^{(N/(N-2))q_0}dx + C(c, q_0, N)\int_\Omega \left|\nabla\left(|w^n|^{(Nq_0-2N+4)/(2N-4)}w^n\right)\right|^2 dx \\ \leq l\int_\Omega |w^n|^{(N/(N-2))q_0}dx.\end{aligned} \quad (3.23)$$

Using Hölder inequality, and Young inequality we deduce that

$$\int_\Omega |w^n|^{(N/(N-2))q_0}dx \leq \varepsilon'\int_\Omega \left|\nabla\left(|w^n|^{(Nq_0-2N+4)/(2N-4)}w^n\right)\right|^2 dx + C_\varepsilon\left(\int_\Omega |w^n|^{q_0}dx\right)^{N/(N-2)}. \quad (3.24)$$

Taking (3.24) into (3.23), it yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega |w^n|^{(N/(N-2))q_0}\mathrm{d}x + C'(c, q_0, N)\int_\Omega \left|\nabla\left(|w^n|^{(Nq_0-2N+4)/(2N-4)}w^n\right)\right|^2 dx \\ \leq C'_\varepsilon\left(\int_\Omega |w^n|^{q_0}dx\right)^{N/(N-2)}.\end{aligned} \quad (3.25)$$

Integrating (3.25) between t_0 and $t_0 + s$, $0 < s \leq 1$, we have

$$\int_\Omega |w^n(t_0+s)|^{(N/(N-2))q_0}dx \leq \int_\Omega |w^n(t_0)|^{(N/(N-2))q_0}dx + C'_\varepsilon\left(\int_\Omega |w^n|^{q_0}dx\right)^{N/(N-2)}. \quad (3.26)$$

Therefore, from (3.19) and (3.22) we get

$$\int_\Omega |w^n(t)|^{(N/(N-2))q_0}dx = \int_\Omega |w^n(t)|^{q_1}dx \leq C, \quad \forall t \geq 2, \quad (3.27)$$

with C independent of n. Integrating (3.25) between t and $t+1$ for $t \geq 2$, we obtain

$$\int_t^{t+1}\int_\Omega \left|\nabla\left(|w^n|^{(q_1-2)/2}w^n\right)\right|^2 dx\,d\xi = \int_t^{t+1}\int_\Omega \left|\nabla\left(|w^n|^{(Nq_0-2N+4)/(2N-4)}w^n\right)\right|^2 dx\,d\xi \leq C. \quad (3.28)$$

Similar to (3.22), for any $t \geq 2$, there exists at least a $t_0 \in [t, t+1]$ such that

$$\int_\Omega |w^n(t_0)|^{q_0(N^2/(N-2)^2)}dx \leq C. \quad (3.29)$$

Bootstrap the above processes, we can deduce that

$$\int_\Omega |w^n(t)|^{q_k}dx \leq C, \quad \forall t \geq T_k, \quad (3.30)$$

with $q_k = (N/(N-2))^k q_0$, and C independent of n. Note that $w^n \to w$ in $C([0,T];L^1(\Omega))$ and $w \in C([0,T];L^1(\Omega))$. From Lemma 2.6, we have

$$\|w(t)\|_{L^{q_k}(\Omega)}^{q_k} = \int_\Omega |w(t)|^{q_k}dx \leq C, \quad \forall t \geq T_k. \quad (3.31)$$

Taking k large enough, we get the second part of (i) proved. If the integration are taken over $[t, t+\delta_0]$ instead of $[t, t+1]$, we get the first part of (i).

Now we are in the position to prove (ii). We multiply (3.5) with w^n and deduce that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |w^n|^2dx + c\int_\Omega |\nabla w^n|^2dx \leq l\int_\Omega |w^n|^2dx, \quad (3.32)$$

integrating over $[t, t+1]$, $t \geq T'$, we get

$$\int_t^{t+1}\int_\Omega |\nabla w^n|^2dx\,dt \leq C, \quad (3.33)$$

with C independent of n. Now, multiplying (3.5) with w_t^n, we obtain

$$\int_\Omega |w_t^n|^2dx + \frac{d}{dt}\int_\Omega a(x)|\nabla w^n|^2dx + \frac{d}{dt}\int_\Omega \left(F(w^n+v^n) - F(v^n) - f(v^n)w^n\right)dx = 0, \quad (3.34)$$

where $F(v+\sigma) = \int_0^\sigma f(v+s)ds$. Integrating (3.34) between s and $t+1$ $(t \leq s < t+1)$ gives

$$\begin{aligned}&\int_\Omega |\nabla w^n(t+1)|^2dx + \int_\Omega \left(F(w^n(t+1)+v^n) - F(v^n) - f(v^n)w^n(t+1)\right)dx \\ &\quad\leq C\int_\Omega |\nabla w^n(s)|^2dx + C\int_\Omega \left(F(w^n(s)+v^n) - F(v^n) - f(v^n)w^n(s)\right)dx.\end{aligned} \quad (3.35)$$

Now, integrating the above inequality with respect to s between t and $t+1$ we have

$$\begin{aligned}
&\int_\Omega |\nabla w^n(t+1)|^2 dx + \int_\Omega \left(F(w^n(t+1)+v^n) - F(v^n) - f(v^n)w^n(t+1)\right)dx \\
&\quad \le C\int_t^{t+1}\int_\Omega |\nabla w^n|^2 dx\, d\xi + C\int_t^{t+1}\left|\int_\Omega \left(F(w^n+v^n) - F(v^n) - f(v^n)w^n\right)dx\right| d\xi.
\end{aligned} \tag{3.36}$$

Since

$$\begin{gathered}
\int_\Omega \left(F(w^n+v^n) - F(v^n) - f(v^n)w^n\right)dx = \int_\Omega\int_0^1 \left(f(v^n+sw^n) - f(v^n)\right)w^n ds\, dx, \\
f(v^n+\tau w^n) - f(v^n) = \int_0^1 f'(v^n+\theta\tau w^n)\tau w^n d\theta, \quad 0 \le \tau \le 1.
\end{gathered} \tag{3.37}$$

We deduce that

$$\left|\int_\Omega \left(F(w^n+v^n) - F(v^n) - f(v^n)w^n\right)dx\right| \le \int_\Omega \left|f'\left(v^n+\tau' w^n\right)\right| |w^n|^2 dx, \quad 0 \le \tau' \le 1. \tag{3.38}$$

From the assumption (1.4) on f, we have

$$\begin{aligned}
&\left|\int_\Omega \left(F(w^n+v^n) - F(v^n) - f(v^n)w^n\right)dx\right| \\
&\quad \le \int_\Omega C\left(|v^n|^{p-2} + |w^n|^{p-2} + 1\right)|w^n|^2 dx \\
&\quad \le C\left(\|v^n\|^{p-2}_{L^{p-1}(\Omega)} + \|w^n\|^2_{L^{2p-2}(\Omega)} + \|w^n\|^p_{L^p(\Omega)} + 1\right).
\end{aligned} \tag{3.39}$$

Using results in (3.13) and (3.30), we know that

$$\left|\int_\Omega \left(F(w^n+v^n) - F(v^n) - f(v^n)w^n\right)dx\right| \le C, \quad \forall t \ge T''. \tag{3.40}$$

Set $T_0 = \max\{T', T''\}$. Combining (3.33), (3.34), (3.36), and (3.40) we have

$$\int_\Omega |\nabla w^n(t)|^2 dx \le C, \quad \forall t \ge T_0 + 1, \tag{3.41}$$

$$\int_t^{t+1}\int_\Omega \left|w_t^n\right|^2 dx\, d\xi \le C, \quad \forall t \ge T_0 + 1. \tag{3.42}$$

From Lemma 2.6, we obtain

$$\|w\|_{H_0^1(\Omega)} \leq C, \quad \forall t \geq T_0 + 1. \tag{3.43}$$

Thus we get the second part of (ii) proved. Taking integration over $[t, t+\delta_0]$ instead of $[t, t+1]$, the first part of (ii) follows. The proof is completed now. □

4. Proof of the Main Theorems

Let $\{S(t)\}_{t\geq 0}$ be the semigroup generated by problem (1.1) and let $v(x)$ be an approximated solution to problem (2.3). Define

$$S_1(t)(u_0 - v(x)) = S(t)u_0 - v(x). \tag{4.1}$$

Then it is easy to verify that $\{S_1(t)\}_{t\geq 0}$ is a continuous semigroup in $(L^1(\Omega) - v)$ and hence in $L^1(\Omega)$. From the results in Section 3, we know that the semigroup $\{S(t)\}_{t\geq 0}$ possesses a global attractor $\mathcal{A}$ in $L^1(\Omega)$. To verify the second part of Theorem 1.1, we prove the following theorem.

Theorem 4.1. *Under the assumptions of Theorem 1.1, the semigroup $\{S_1(t)\}_{t\geq 0}$ possesses a global attractor $\mathcal{A}_v$ in $L^r(\Omega) \cap H_0^1(\Omega), 1 \leq r < \infty$, that is, $\mathcal{A}_v$ is compact, invariant in $L^r(\Omega) \cap H_0^1(\Omega), 1 \leq r < \infty$, and attracts every bounded initial set of $L^1(\Omega)$ in the norm topology of $L^r(\Omega) \cap H_0^1(\Omega),\ 1 \leq r < \infty$.*

Proof. From Theorem 3.1, we know that $\{S(t)\}_{t\geq 0}$ possesses an absorbing set B_0 in $L^1(\Omega)$. Also $\{S_1(t)\}_{t\geq 0}$ possesses absorbing sets $B_1(= B_0 - v)$, B_2, respectively, in $L^1(\Omega)$ and $L^r(\Omega) \cap H_0^1(\Omega)$ for any $1 \leq r < \infty$.

In the next, we prove the asymptotic compactness of $\{S_1(t)\}_{t\geq 0}$. Before that we establish the following estimate

$$\int_\Omega \left|w_t^n\right|^2 dx \leq C, \quad \text{for } t \text{ large enough}. \tag{4.2}$$

Actually, differentiating (3.5) in time and denoting $\widetilde{w^n} = w_t^n$, we have

$$\widetilde{w^n}_t - \operatorname{div}\left(a(x)\nabla \widetilde{w^n}\right) + f'(v^n + w^n)\widetilde{w^n} = 0. \tag{4.3}$$

Multiplying (4.3) by $\widetilde{w^n}$ and using (1.3), we deduce that

$$\frac{d}{dt}\int_\Omega \left|\widetilde{w^n}\right|^2 dx \leq l\int_\Omega \left|\widetilde{w^n}\right|^2 dx. \tag{4.4}$$

Integrating the above inequality between s and $t+1$ $(t \le s < t+1)$ gives

$$\int_{\Omega} \left|\widetilde{w^n}(t+1)\right|^2 dx \le 2l \int_s^{t+1} \int_{\Omega} \left|\widetilde{w^n}\right|^2 dx\, d\xi + \int_{\Omega} \left|\widetilde{w^n}(s)\right|^2 dx. \tag{4.5}$$

Integrating the above inequality with respect to s between t and $t+1$, using (3.42) we obtain

$$\int_{\Omega} \left|\widetilde{w^n}(t)\right|^2 dx \le (2l+1) \int_t^{t+1} \int_{\Omega} \left|\widetilde{w^n}\right|^2 dx\, d\xi \le (2l+1)C, \quad \forall t \ge T_0 + 1. \tag{4.6}$$

Now we prove the asymptotic compactness of the semigroup $\{S_1(t)\}_{t\ge 0}$, that is, for any sequences $\{w_{0,k}\} \subset B_1, t_k \to \infty$, sequence $\{w_k(t_k)\}$ has convergent subsequences, where $w_k(t) = S_1(t)w_{0,k}$. Since $\{S_1(t)\}_{t\ge 0}$ is compact in $L^r(\Omega), 1 \le r < \infty$, there is a subsequence $\{w_{k_i}(t_{k_i})\}$, which is a Cauchy sequence in $L^r(\Omega), 1 \le r < \infty$. Denoting $u_{k_i}(t_{k_i}) = w_{k_i}(t_{k_i}) + v, u_{k_j}(t_{k_j}) = w_{k_j}(t_{k_j}) + v$, we deduce that

$$\begin{aligned}
c&\left\| w_{k_i}(t_{k_i}) - w_{k_j}\left(t_{k_j}\right)\right\|^2_{H_0^1(\Omega)} \\
&\le \left\langle \operatorname{div}(a(x)\nabla(w_{k_i}(t_{k_i}) + v)) - \operatorname{div}\left(a(x)\nabla\left(w_{k_j}\left(t_{k_j}\right) + v\right)\right), w_{k_i}(t_{k_i}) - w_{k_j}\left(t_{k_j}\right)\right\rangle \\
&= \left\langle \partial_t w_{k_i}(t_{k_i}) - \partial_t w_{k_j}\left(t_{k_j}\right) + f(u_{k_i}(t_{k_i})) - f\left(u_{k_j}\left(t_{k_j}\right)\right), w_{k_i}(t_{k_i}) - w_{k_j}\left(t_{k_j}\right)\right\rangle \\
&\le \left\|\partial_t w_{k_i}(t_{k_i}) - \partial_t w_{k_j}(t_{k_j})\right\|_{L^2(\Omega)} \left\| w_{k_i}(t_{k_i}) - w_{k_j}\left(t_{k_j}\right)\right\|_{L^2(\Omega)} \\
&\quad + \left\| f(w_{k_i}(t_{k_i}) + v) - f\left(w_{k_j}\left(t_{k_j}\right) + v\right)\right\|_{L^\sigma(\Omega)} \left\| w_{k_i}(t_{k_i}) - w_{k_j}\left(t_{k_j}\right)\right\|_{L^{\sigma'}(\Omega)}.
\end{aligned} \tag{4.7}$$

We then conclude form (4.6) that $\{w_{k_i}(t_{k_i})\}$ is a Cauchy sequence in $H_0^1(\Omega)$, and thus $\{S_1(t)\}_{t\ge 0}$ is asymptotically compact in $L^r(\Omega) \cap H_0^1(\Omega), 1 \le r < \infty$.

Using Lemma 2.7, we conclude that $\{S_1(t)\}_{t\ge 0}$ possesses a global attractor $\mathcal{A}_v$, which is compact, invariant in $L^r(\Omega) \cap H_0^1(\Omega)$, and attracts every bounded initial sets of $L^1(\Omega)$ in the topology of $L^r(\Omega) \cap H_0^1(\Omega)$. □

Completion of the Proof of Theorem 1.1

Note that

$$\begin{gathered}
\mathcal{A}_v = \cap_{s\ge 0}\overline{\cup_{t\ge s} S_1(t)B_2}^{L^1(\Omega)} = \cap_{s\ge 0}\overline{\cup_{t\ge s} S_1(t)B_1}^{L^1(\Omega)} = \cap_{s\ge 0}\overline{\cup_{t\ge s} (S(t)B_0 - v)}^{L^1(\Omega)}, \\
\mathcal{A} = \cap_{s\ge 0}\overline{\cup_{t\ge s} S(t)B_0}^{L^1(\Omega)}.
\end{gathered} \tag{4.8}$$

Thus we have

$$\mathcal{A} = \mathcal{A}_v + v. \tag{4.9}$$

The above relation between $\mathcal{A}$ and $\mathcal{A}_v$ implies the conclusion of Theorem 1.1 directly.

Proof of Theorem 1.2. Let $u(t), v$ be the approximated solution to the parabolic and its corresponding elliptic problem respectively. Since the approximated solution is a duality solution and conversely, we conclude that $u(t)$ converges to v in $L^1(\Omega)$ as $t \to \infty$. Using arguments similar to Section 3, we can prove similar regularity results for $w(= u - v)$ and then prove the asymptotic compactness of the semigroup $S_1(t)$ as in Theorem 4.1. Thus, we obtain that $w(t)$ converges to 0 in $L^r(\Omega)$, $1 \le r < \infty$, as $t \to \infty$. Moreover from the asymptotic compactness of the semigroup $S_1(t)$, we know that $w(t)$ converges to 0 in $H_0^1(\Omega)$ as $t \to \infty$. Else, we have a sequence $t_n \to \infty$, such that $C > \|w(t_n)\|_{H_0^1(\Omega)} \ge \epsilon > 0$. Since the semigroup $S_1(t)$ is asymptotically compact, there is a subsequence $t_{n_j} \to \infty$, such that $w(t_{n_j})$ converges to a function χ in $H_0^1(\Omega)$ and hence in $L^1(\Omega)$. Thus $\chi = 0$. A contradiction! □

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