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Research Article

Duality Fixed Point and Zero Point Theorems and Applications

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The following main results have been given. (1) Let E be a p-uniformly convex Banach space and let $T : E \to E^*$ be a $(p-1)$-L-Lipschitz mapping with condition $0 < (pL/c^2)^{1/(p-1)} < 1$. Then T has a unique generalized duality fixed point $x^* \in E$ and (2) let E be a p-uniformly convex Banach space and let $T : E \to E^*$ be a q-α-inverse strongly monotone mapping with conditions $1/p + 1/q = 1$, $0 < (q/(q-1)c^2)^{q-1} < \alpha$. Then T has a unique generalized duality fixed point $x^* \in E$. (3) Let E be a 2-uniformly smooth and uniformly convex Banach space with uniformly convex constant c and uniformly smooth constant b and let $T : E \to E^*$ be a L-lipschitz mapping with condition $0 < 2b/c^2 < 1$. Then T has a unique zero point x^*. These main results can be used for solving the relative variational inequalities and optimal problems and operator equations.

1. Introduction and Preliminaries

Let E be a real Banach space with the dual E^* and let T be an operator from E into E^*. Firstly, for $p \geq 2$, we consider the variational inequality problem of finding an element $x^* \in E$ such that

$$\langle Tx^*, x^* - x \rangle \geq 0, \quad \forall \|x\| \leq \|x^*\|^{(p-1)^2}. \tag{1.1}$$

Taking $p = 2$, the problem (1.1) becomes the following variational inequality problem of finding an element $x^* \in E$ such that

$$\langle Tx^*, x^* - x \rangle \geq 0, \quad \forall \|x\| \leq \|x^*\|. \tag{1.2}$$

Secondly, for $p \geq 2$, we consider the optimal problem of finding an element $x^* \in E$ such that

$$\left(\|x^*\|^{(p-1)} - \|Tx^*\|\right)^2 = \min_{x\in E}\left(\|x\|^{(p-1)} - \|Tx\|\right)^2. \tag{1.3}$$

Taking $p = 2$, the problem (1.3) becomes the following optimal problem of finding an element $x^* \in E$ such that

$$(\|x^*\| - \|Tx^*\|)^2 = \min_{x\in E}(\|x\| - \|Tx\|)^2. \tag{1.4}$$

Thirdly, for $p \geq 2$, we consider the operator equation problem of finding an element $x^* \in E$ such that

$$\langle Tx^*, x^*\rangle = \|Tx^*\|^p = \|x^*\|^{p(p-1)}. \tag{1.5}$$

Taking $p = 2$, the problem (1.5) becomes the following operator equation problem of finding an element $x^* \in E$ such that

$$\langle Tx^*, x^*\rangle = \|Tx^*\|^2 = \|x^*\|^2. \tag{1.6}$$

Finally, we consider the operator equation problem of finding an element $x^* \in E$ such that

$$Tx^* = 0. \tag{1.7}$$

Let E be a real Banach space with the dual E^*. Let p be a given real number with $p > 1$. The generalized duality mapping J_p from E into 2^{E^*} is defined by

$$J_p(x) = \left\{f \in E^* : \langle x, f\rangle = \|f\|^p,\ \|f\| = \|x\|^{p-1}\right\}, \quad \forall x \in E, \tag{1.8}$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. In particular, $J = J_2$ is called the normalized duality mapping and $J_p(x) = \|x\|^{p-2}J(x)$ for all $x \neq 0$. If E is a Hilbert space, then $J = I$, where I is the identity mapping. The duality mapping J has the following properties:

(i) if E is smooth, then J is single-valued;

(ii) if E is strictly convex, then J is one-to-one;

(iii) if E is reflexive, then J is surjective;

(iv) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;

(v) if E^* is uniformly convex, then J is uniformly continuous on each bounded subsets of E and J is single-valued and also one-to-one.

For more details, see [1].

In this paper, we firstly present the definition of duality fixed point for a mapping T from E into its dual E^* as follows.

Let E be a Banach space with a single-valued generalized duality mapping $J_p : E \to E^*$. Let $T : E \to E^*$. An element $x^* \in E$ is said to be a generalized duality fixed point of T if $Tx^* = J_p x^*$. An element $x^* \in E$ is said to be a duality fixed point of T if $Tx^* = Jx^*$.

Example 1.1. Let E be a smooth Banach space with the dual E^*, and let $A : E \to E^*$ be an operator, then an element $x^* \in E$ is a zero point of A if and only if x^* is a duality fixed point of $J + \lambda A$ for any $\lambda > 0$. Namely, the x^* is a duality fixed point of $J + \lambda A$ for any $\lambda > 0$ if and only if x^* is a fixed point of $J_\lambda = (J + \lambda A)^{-1} J : E \to E$ (if A is maximal monotone, then J_λ is, namely, the resolvent of A).

Example 1.2. In Hilbert space, the fixed point of an operator is always duality fixed point.

Example 1.3. Let E be a p-uniformly convex Banach space with the dual E^*, then any element of E must be the generalized duality fixed point of the generalized normalized duality mapping J_p.

Conclusion 1. If x^* is a generalized duality fixed point of T, then x^* must be a solution of variational inequality problem (1.1).

Proof. Suppose x^* is a generalized duality fixed point of T, then

$$\langle Tx^*, x^* \rangle = \langle J_p x^*, x^* \rangle = \left\| J_p x^* \right\|^p = \|Tx^*\|^p = \|x^*\|^{p(p-1)}. \tag{1.9}$$

Obverse that

$$\begin{aligned} \langle Tx^*, x^* - x \rangle &= \langle Tx^*, x^* \rangle - \langle Tx^*, x \rangle \\ &\geq \|Tx^*\|^p - \|Tx^*\| \|x\| \\ &= \|Tx^*\| \left(\|Tx^*\|^{p-1} - \|x\| \right) \\ &= \|Tx^*\| \left(\|x^*\|^{(p-1)^2} - \|x\| \right) \geq 0 \end{aligned} \tag{1.10}$$

for all $\|x\| \leq \|x^*\|^{(p-1)^2}$. □

Taking $p = 2$, we have the following result.

Conclusion 2. If x^* is a duality fixed point of T, then x^* must be a solution of variational inequality problem (1.2).

Conclusion 3. If x^* is a generalized duality fixed point of T, then x^* must be a solution of the optimal problem (1.3). Therefore, x^* is also a solution of operator equation problem (1.5).

Proof. If x^* is a generalized duality fixed point of T, then $Tx^* = J_p x^*$, so that

$$\langle Tx^*, x^* \rangle = \langle J_p x^*, x^* \rangle = \left\| J_p x^* \right\|^p = \|Tx^*\|^p = \|x^*\|^{p(p-1)}. \tag{1.11}$$

All conclusions are obvious. □

Take $p = 2$, we have the following result.

Conclusion 4. If x^* is a duality fixed point of T, then x^* must be a solution of the optimal problem (1.4). Therefore, x^* is also a solution of operator equation problem (1.6).

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be strictly convex if for any $x, y \in U$, $x \neq y$ implies $\|(x+y)/2\| < 1$. It is also said to be uniformly convex if for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that for any $x, y \in U$, $\|x-y\| \geq \varepsilon$ implies $\|(x+y)/2\| < 1-\delta$. It is well known that a uniformly convex Banach space is reflexive and strictly convex. And we define a function $\delta : [0,2] \to [0,1]$ called the modulus of convexity of E as follows:

$$\delta(\varepsilon) = \left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\right\}. \tag{1.12}$$

It is well known that E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let p be a fixed real number with $p \geq 2$. Then E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0,2]$. For example, see [2, 3] for more details. The constant $1/c$ is said to be uniformly convexity constant of E.

A Banach space E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1.13}$$

exists for all $x, y \in U$. It is also said to be uniformly smooth if the above limit is attained uniformly for $x, y \in U$. One should note that no Banach space is p-uniformly convex for $1 < p < 2$; see [4] for more details. It is well known that the Hilbert and the Lebesgue $L^q(1 < q \leq 2)$ spaces are 2-uniformly convex and uniformly smooth. Let X be a Banach space and let $L^q(X) = \{\Omega, \Sigma, \mu; X\}, 1 < q \leq \infty$ be the Lebesgue-Bochner space on an arbitrary measure space (Ω, Σ, μ). Let $2 \leq p < \infty$ and let $1 < q \leq p$. Then $L^q(X)$ is p-uniformly convex if and only if X is p-uniformly convex; see [3].

Let $\rho_E : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in U,\ \|y\| \leq t\right\}. \tag{1.14}$$

A Banach space E is said to be uniformly smooth if $\rho_E(t)/t \to 0$ as $t \to 0$. Let $q > 1$. A Banach space E is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. It is well known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable, in particular, the norm of E is Fréchet differentiable. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p,2\}$-uniformly smooth for every $p > 1$.

Lemma 1.4 (see [5, 6]). *Let E be a p-uniformly convex Banach space with $p \geq 2$. Then, for all $x, y \in E$, $j(x) \in J_p(x)$ and $j(y) \in J_p(y)$,*

$$\langle x-y, j(x)-j(y) \rangle \geq \frac{c^p}{c^{p-2}p} \|x-y\|^p, \tag{1.15}$$

where J_p is the generalized duality mapping from E into E^* and $1/c$ is the p-uniformly convexity constant of E.

Lemma 1.5. *Let E be a p-uniformly convex Banach space with $p \geq 2$. Then J_p is one-to-one from E onto $J_p(E) \subset E^*$ and for all $x, y \in E$,*

$$\|x-y\| \leq \left(\frac{p}{c^2}\right)^{1/(p-1)} \|J_p(x)-J_p(y)\|^{1/(p-1)}, \tag{1.16}$$

where J_p is the generalized duality mapping from E into E^* with range $J_p(E)$, and $1/c$ is the p-uniformly convexity constant of E.

Proof. Let E be a p-uniformly convex Banach space with $p \geq 2$, then $J = J_2$ is one-to-one from E onto E^*. Since $J_p(x) = \|x\|^{p-2}J(x)$, then $J_p(x)$ is single valued. From (1.5) we have

$$\langle x-y, J_p(x)-J_p(y) \rangle \geq \frac{c^p}{c^{p-2}p} \|x-y\|^p, \tag{1.17}$$

which implies that

$$\|x-y\| \|J_p(x)-J_p(y)\| \geq \frac{c^p}{c^{p-2}p} \|x-y\|^p. \tag{1.18}$$

That is

$$\|J_p(x)-J_p(y)\| \geq \frac{c^p}{c^{p-2}p} \|x-y\|^{p-1}. \tag{1.19}$$

Hence

$$\|x-y\| \leq \left(\frac{p}{c^2}\right)^{1/(p-1)} \|J_p(x)-J_p(y)\|^{1/(p-1)}. \tag{1.20}$$

Then (1.6) has been proved. Therefore, from (1.6) we can see, for any $x, y \in E$, that $J_p(x) = J_p(y)$ implies that $x = y$. □

2. Duality Contraction Mapping Principle and Applications

Let E be a Banach space with the dual E^*. An operator $T : E \to E^*$ is said to be p-L-Lipschitz, if

$$\|Tx - Ty\| \le L\|x - y\|^p, \quad \forall x, y \in E, \tag{2.1}$$

where $L \in (0, +\infty)$, $p \in [1, +\infty)$ are two constants. If $p = 1$, the operator T is said to be L-Lipschitz.

Theorem 2.1 (generalized duality contraction mapping principle). *Let E be a p-uniformly convex Banach space and let $T : E \to E^*$ be a $(p-1)$-L-Lipschitz mapping with condition $0 < (pL/c^2)^{1/(p-1)} < 1$. Then T has a unique generalized duality fixed point $x^* \in E$ and for any given guess $x_0 \in E$, the iterative sequence $x_{n+1} = J_p^{-1}Tx_n$ converges strongly to this generalized duality fixed point x^*.*

Proof. Let $A = J_p^{-1}T$, then A is a mapping from E into itself. By using Lemma 1.5, we have

$$\begin{aligned}
\|Ax - Ay\| &= \left\|J_p^{-1}Tx - J_p^{-1}Ty\right\| \\
&\le \left(\frac{p}{c^2}\right)^{1/(p-1)} \|Tx - Ty\|^{1/(p-1)} \\
&\le \left(\frac{p}{c^2}\right)^{1/(p-1)} \left(L\|x - y\|^{p-1}\right)^{1/(p-1)} \\
&\le \left(\frac{pL}{c^2}\right)^{1/(p-1)} \|x - y\|
\end{aligned} \tag{2.2}$$

for all $x, y \in E$, where $0 < (pL/c^2)^{1/(p-1)} < 1$. By using Banach's contraction mapping principle, there exists a unique element $x^* \in E$ such that $Ax^* = x^*$. That is, $Tx^* = J_px^*$, so x^* is a generalized unique duality fixed point of T. Further, the Picard iterative sequence $x_{n+1} = Ax_n = J_p^{-1}Tx_n$ $(n = 0, 1, 2, \ldots)$ converges strongly to this generalized duality fixed point x^*. □

Taking $p = 2$, we have the following result.

Theorem 2.2 (duality contraction mapping principle). *Let E be a 2-uniformly convex Banach space and let $T : E \to E^*$ be a L-Lipschitz mapping with condition $0 < (2L/c^2) < 1$. Then T has a unique duality fixed point $x^* \in E$ and for any given guess $x_0 \in E$, the iterative sequence $x_{n+1} = J^{-1}Tx_n$ converges strongly to this duality fixed point x^*.*

From Conclusions 1–4 and Theorem 2.1, we have the following result for solving the variational inequality problems (1.1) and (1.2), the optimal problems (1.3) and (1.4), and the operator equation problems (1.5) and (1.6).

Theorem 2.3. *Let E be a p-uniformly convex Banach space and let $T : E \to E^*$ be a $(p-1)$-L-Lipschitz mapping with condition $0 < (pL/c^2)^{1/(p-1)} < 1$. Then the variational inequality problem*

(1.1) *(the optimal problem* (1.2) *and operator equation problem* (1.3)*) has solutions and for any given guess* $x_0 \in E$*, the iterative sequence* $x_{n+1} = J_p^{-1}Tx_n$ *converges strongly to a solution of the variational inequality problem* (1.1) *(the optimal problem* (1.3) *and the operator equation problem* (1.5)*).*

Taking $p = 2$, we have the following result.

Theorem 2.4. *Let E be a 2-uniformly convex Banach space, let $T : E \longrightarrow E^*$ be a $(p-1)$-L-Lipschitz mapping with condition $0 < (2L/c^2) < 1$. Then the variational inequality problem* (1.2) *(the optimal problem* (1.4) *and operator equation problem* (1.6)*) has solutions and for any given guess $x_0 \in E$, the iterative sequence $x_{n+1} = J^{-1}Tx_n$ converges strongly to a solution of the variational inequality problem* (1.2) *(the optimal problem* (1.4) *and the operator equation problem* (1.6)*).*

Theorem 2.5 (generalized duality Mann weak convergence theorem). *Let E be a p-uniformly convex Banach space which satisfying Opial's condition, let $T : E \longrightarrow E^*$ be a $(p-1)$-L-Lipschitz mapping with nonempty generalized duality fixed point set. Assume $0 < (pL/c^2)^{1/(p-1)} \leq 1$, and the real sequence $\{\alpha_n\} \subset [0,1]$ satisfies the condition $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = +\infty$. Then for any given guess $x_0 \in E$, the generalized Mann iterative sequence*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n J_p^{-1}Tx_n \tag{2.3}$$

converges weakly to a generalized duality fixed point of T.

Proof. Letting $A = J^{-1}T$, by using Lemma 1.4, we have

$$\|Ax - Ay\| = \left\|J^{-1}Tx - J^{-1}Ty\right\| \leq \frac{2}{c^2}\|Tx - Ty\| \leq \frac{2L}{c^2}\|x - y\| \leq \|x - y\|, \tag{2.4}$$

for all $x, y \in E$. Hence A is a nonexpansive mapping from E into itself. In addition, we have

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n J^{-1}Tx_n = (1-\alpha_n)x_n + \alpha_n Ax_n. \tag{2.5}$$

By using the well-known result, we know that the sequence $\{x_n\}$ converges weakly to a fixed point x^* of A $(Ax^* = x^*)$. This point x^* is also a duality fixed point of T $(Tx^* = Jx^*)$. □

Take $p = 2$, we have the following result.

Theorem 2.6 (duality Mann weak convergence theorem). *Let E be a 2-uniformly convex Banach space which satisfy Opial's condition and let $T : E \longrightarrow E^*$ be a L-Lipschitz mapping with nonempty duality fixed point set. Assume $0 < 2L/c^2 \leq 1$, and the real sequence $\{\alpha_n\} \subset [0,1]$ satisfies the condition $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = +\infty$. Then for any given guess $x_0 \in E$, the generalized Mann iterative sequence*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n J^{-1}Tx_n \tag{2.6}$$

converges weakly to a duality fixed point of T.

Theorem 2.7 (duality Halpern strong convergence theorem). *Let E be a p-uniformly convex Banach space which satisfying Opial's condition, let $T : E \longrightarrow E^*$ be a $(p-1)$-L-Lipschitz mapping with* *nonempty generalized duality fixed point set. Assume* $0 < (pL/c^2)^{1/(p-1)} \leq 1$, *and the real sequence* $\{\alpha_n\} \subset [0,1]$ *satisfies the condition:*

(C_1): $\lim_{n\to\infty}\alpha_n = 0$;

(C_2): $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C_3): $\lim_{n\to\infty}(\alpha_{n+1} - \alpha_n)/\alpha_{n+1} = 0$ *or* $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$.

Let u, x_0 *be given, then iterative sequence*

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_p^{-1} T x_n, \tag{2.7}$$

converges strongly to a generalized duality fixed point of T.

Proof. Let $A = J_p^{-1}T$, then A is a mapping from E into itself. By using Lemma 1.5, we have

$$\begin{aligned} \|Ax - Ay\| &= \left\| J_p^{-1}Tx - J_p^{-1}Ty \right\| \\ &\leq \left(\frac{p}{c^2}\right)^{1/(p-1)} \|Tx - Ty\|^{1/(p-1)} \\ &\leq \left(\frac{p}{c^2}\right)^{1/(p-1)} \left(L\|x-y\|^{p-1}\right)^{1/(p-1)} \\ &\leq \left(\frac{pL}{c^2}\right)^{1/(p-1)} \|x-y\| \end{aligned} \tag{2.8}$$

for all $x, y \in E$, where $0 < (pL/c^2)^{1/(p-1)} \leq 1$. Hence A is a nonexpansive mapping from E into itself. In addition, we have

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_p^{-1} T x_n = \alpha_n u + (1-\alpha_n) A x_n. \tag{2.9}$$

By using the well-known result of Xu [7, Theorem 2.3], we know that the iterative sequence $\{x_n\}$ converges strongly to a fixed point of nonexpansive mapping A. Hence the sequence $\{x_n\}$ converges strongly to a generalized duality fixed point of T. □

Theorem 2.8. *Letting* H *be a Hilbert space, then one has its uniformly convexity constant* $1/c \geq \sqrt{2}/2$, *that is* $c \leq \sqrt{2}$.

Proof. If $c > \sqrt{2}$. For any $x \neq y$, by using Lemma 1.4, we have

$$\|x-y\| = \left\| J^{-1}x - J^{-1}y \right\| \leq \frac{2}{c^2}\|x-y\| < \|x-y\|. \tag{2.10}$$

This is a contradiction. □

3. Fixed Point Theorem of Inverse Strongly Monotone Mappings

Definition 3.1. Letting E be a Banach space, the mapping $T : E \to E^*$ is called q-α-inverse strongly monotone, if

$$\langle Tx - Ty,\ x - y\rangle \geq \alpha \|Tx - Ty\|^q, \quad \forall x, y \in E. \tag{3.1}$$

Lemma 3.2. *Let E be a Banach space and let $T : E \to E^*$ be a q-α-inverse strongly monotone mapping. Then T is $1/(q-1) - (1/\alpha)^{(1/(q-1))}$-Lipschitz.*

Proof. Let $T : E \to E^*$ be a q-α-inverse strongly monotone mapping, that is,

$$\langle Tx - Ty,\ x - y\rangle \geq \alpha \|Tx - Ty\|^q, \quad \forall x, y \in E. \tag{3.2}$$

It follows from the above inequality that

$$\alpha \|Tx - Ty\|^q \leq \langle Tx - Ty,\ x - y\rangle \leq \|Tx - Ty\| \|x - y\|, \quad \forall x, y \in E, \tag{3.3}$$

which leads to

$$\|Tx - Ty\|^q \leq \frac{1}{\alpha} \|Tx - Ty\| \|x - y\|, \quad \forall x, y \in E. \tag{3.4}$$

Further

$$\|Tx - Ty\|^{q-1} \leq \frac{1}{\alpha} \|x - y\|, \quad \forall x, y \in E, \tag{3.5}$$

and hence

$$\|Tx - Ty\| \leq \left(\frac{1}{\alpha}\right)^{1/(q-1)} \|x - y\|^{1/(q-1)}, \quad \forall x, y \in E. \tag{3.6}$$

Then T is $1/(q-1) - (1/\alpha)^{1/(q-1)}$-Lipschitz. $\square$

Theorem 3.3 (fixed point theorem of inverse strongly monotone mappings). *Let E be a p-uniformly convex Banach space and let $T : E \to E^*$ be a q-α-inverse strongly monotone mapping with conditions $1/p + (1/q) = 1$, $0 < (q/(q-1)c^2)^{q-1} < \alpha$. Then T has a unique generalized duality fixed point $x^* \in E$ and for any given guess $x_0 \in E$, the iterative sequence $x_{n+1} = J_p^{-1} T x_n$ converges strongly to this generalized duality fixed point x^*.*

Proof. Letting $A = J_p^{-1}T$, then A is a mapping from E into itself. By using Lemma 1.5 and Lemma 3.2, we have

$$\begin{aligned}
\|Ax - Ay\| &= \left\|J_p^{-1}Tx - J_p^{-1}Ty\right\| \\
&\leq \left(\frac{p}{c^2}\right)^{1/(p-1)} \|Tx - Ty\|^{1/(p-1)} \\
&\leq \left(\frac{p}{c^2}\right)^{1/(p-1)} \left(\left(\frac{1}{\alpha}\right)^{1/(q-1)} \|x - y\|^{1/(q-1)}\right)^{1/(p-1)} \\
&= \left(\frac{p}{c^2}\right)^{1/(p-1)} \frac{1}{\alpha}\|x - y\| \\
&= \left(\frac{q}{(q-1)c^2}\right)^{q-1} \frac{1}{\alpha}\|x - y\|
\end{aligned} \tag{3.7}$$

for all $x, y \in E$. It follows from the condition $0 < (q/(q-1)c^2)^{q-1} < \alpha$ that $0 < (q/(q-1)c^2)^{q-1}(1/\alpha) < 1$. By using Banach's contraction mapping principle, there exists a unique element $x^* \in E$ such that $Ax^* = x^*$. That is, $Tx^* = J_p x^*$, so x^* is a generalized unique duality fixed point of T. Further, the Picard iterative sequence $x_{n+1} = Ax_n = J_p^{-1}Tx_n$ $(n = 0, 1, 2, \ldots)$ converges strongly to this generalized duality fixed point x^*. □

Taking $p = 2$, we have the following results.

Lemma 3.4. *Let E be a Banach space and let $T : E \to E^*$ be a 2-α-inverse strongly monotone mapping. Then T is $(1/\alpha)$-Lipschitz.*

Theorem 3.5 (fixed point theorem of inverse strongly monotone mappings). *Let E be a 2-uniformly convex Banach space, let $T : E \to E^*$ be a 2-α-inverse strongly monotone mapping with condition $0 < 2/c^2 < \alpha$. Then T has a unique duality fixed point $x^* \in E$ and for any given guess $x_0 \in E$, the iterative sequence $x_{n+1} = J^{-1}Tx_n$ converges strongly to this generalized duality fixed point x^*.*

4. Application for Zero Point of Operators

Lemma 4.1 (see [8]). *Let E be a p-uniformly smooth Banach space with uniformly smooth constant b. Then*

$$\|Jx - Jy\| \leq b\|x - y\|^{p-1}, \quad \forall x, y \in E. \tag{4.1}$$

Theorem 4.2. *Let E be a 2-uniformly smooth and uniformly convex Banach space with uniformly convex constant c and uniformly smooth constant b and let $T : E \to E^*$ be a L-lipschitz mapping with condition $0 < 2b/c^2 < 1$. Then T has a unique zero point x^* and for any given guess $x_0 \in E$, the iterative sequence $x_{n+1} = J^{-1}(J + rT)x_n$ converges strongly to this zero point x^*.*

Proof. Let $A = J^{-1}(J + rT)$, then A is a mapping from E into itself. By using Lemma 1.5 and Lemma 4.1, we have

$$\begin{aligned}
\|Ax - Ay\| &= \left\| J^{-1}(J + rT)x - J^{-1}(J + rT)y \right\| \\
&\le \frac{2}{c^2}\|(J + rT)x - (J + rT)y\| \\
&= \frac{2}{c^2}\|(Jx - Jy) - r(Tx - Ty)\| \\
&\le \frac{2}{c^2}(\|Jx - Jy\| + r\|Tx - Ty\|) \\
&\le \frac{2}{c^2}(b + rL)\|x - y\|
\end{aligned} \tag{4.2}$$

for all $x, y \in E$. Observing the condition $0 < 2b/c^2 < 1$, it follows that, there exists a positive number $r > 0$ such that $0 < 2/c^2(b+rL) < 1$. By using Banach's contraction mapping principle, there exists a unique element $x^* \in E$ such that $Ax^* = x^*$. That is, $(J + rT)x^* = Jx^*$ which implies $rTx^* = 0$, so x^* is a zero point of T. Further, the Picard iterative sequence $x_{n+1} = Ax_n = J^{-1}(J + rT)x_n$ $(n = 0, 1, 2, \ldots)$ converges strongly to this zero point x^*. □

Remark 4.3. Under the conditions of Theorem 4.2, we know that the operator equation $Tx = 0$ has a unique solution which can be computed by the iterative scheme $x_{n+1} = Ax_n = J^{-1}(J + rT)x_n$ $(n = 0, 1, 2, \ldots)$ starting any given guess $x_0 \in E$.

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References

[1] T. Ibaraki and W. Takahashi, "A new projection and convergence theorems for the projections in Banach spaces," *Journal of Approximation Theory*, vol. 149, no. 1, pp. 1–14, 2007.
[2] D. Butnariu, S. Reich, and A. J. Zaslavski, "Asymptotic behavior of relatively nonexpansive operators in Banach spaces," *Journal of Applied Analysis*, vol. 7, no. 2, pp. 151–174, 2001.
[3] D. Butnariu, S. Reich, and A. J. Zaslavski, "Weak convergence of orbits of nonlinear operators in reflexive Banach spaces," *Numerical Functional Analysis and Optimization*, vol. 24, no. 5-6, pp. 489–508, 2003.
[4] Y. Su and X. Qin, "General iteration algorithm and convergence rate optimal model for common fixed points of nonexpansive mappings," *Applied Mathematics and Computation*, vol. 186, no. 1, pp. 271–278, 2007.
[5] C. Martinez-Yanes and H.-K. Xu, "Strong convergence of the CQ method for fixed point iteration processes," *Nonlinear Analysis*, vol. 64, no. 11, pp. 2400–2411, 2006.
[6] S.-Y. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 134, no. 2, pp. 257–266, 2005.
[7] H.-K. Xu, "Another control condition in an iterative method for nonexpansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 65, no. 1, pp. 109–113, 2002.
[8] S.-S. Chang, Y. J. Cho, and H. Zhou, *Iterative Methods for Nonlinear Operator Equations in Banach Spaces*, Nova Science, Huntington, NY, USA, 2002.