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Research Article

Variant Gradient Projection Methods for the Minimization Problems

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The gradient projection algorithm plays an important role in solving constrained convex minimization problems. In general, the gradient projection algorithm has only weak convergence in infinite-dimensional Hilbert spaces. Recently, H. K. Xu (2011) provided two modified gradient projection algorithms which have strong convergence. Motivated by Xu's work, in the present paper, we suggest three more simpler variant gradient projection methods so that strong convergence is guaranteed.

1. Introduction

Let H be a real Hilbert space and C a nonempty closed and convex subset of H. Let $f : H \to R$ be a real-valued convex function. Now we consider the following constrained convex minimization problem:

$$\min_{x\in C} f(x). \tag{1.1}$$

Assume that (1.1) is consistent; that is, it has a solution and we use S to denote its solution set. If f is Fréchet differentiable, then $x^* \in C$ solves (1.1) if and only if $x^* \in C$ satisfies the following optimality condition:

$$\left\langle \nabla f(x^*), x - x^* \right\rangle \geq 0, \quad \forall x \in C, \tag{1.2}$$

where ∇f denotes the gradient of f. Note that (1.2) can be rewritten as

$$\langle x^* - (x^* - \nabla f(x^*)), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.3}$$

This shows that the minimization (1.1) is equivalent to the fixed-point problem:

$$\mathrm{Proj}_C(x^* - \gamma \nabla f(x^*)) = x^*, \tag{1.4}$$

where $\gamma > 0$ is an any constant and Proj_C is the nearest point projection from H onto C. By using this relationship, the gradient-projection algorithm is usually applied to solve the minimization problem (1.1). This algorithm generates a sequence $\{x_n\}$ through the recursion:

$$x_{n+1} = \mathrm{Proj}_C(x_n - \gamma_n \nabla f(x_n)), \quad n \geq 0, \tag{1.5}$$

where the initial guess $x_0 \in C$ is chosen arbitrarily and $\{\gamma_n\}$ is a sequence of step sizes which may be chosen in different ways. The gradient-projection algorithm (1.5) is a powerful tool for solving constrained convex optimization problems and has well been studied in the case of constant stepsizes $\gamma_n = \gamma$ for all n. The reader can refer to [1–9]. It has recently been applied to solve split feasibility problems which find applications in image reconstructions and the intensity modulated radiation therapy (see [10–17]).

It is known [3] that if f has a Lipschitz continuous and strongly monotone gradient, then the sequence $\{x_n\}$ can be strongly convergent to a minimizer of f in C. If the gradient of f is only assumed to be Lipschitz continuous, then $\{x_n\}$ can only be weakly convergent if H is infinite dimensional. This gives naturally rise to a question.

Question 1. How to appropriately modify the gradient projection algorithm so as to have strong convergence?

For this purpose, recently, Xu [18] first introduced the following modification:

$$x_{n+1} = \theta_n h(x_n) + (1 - \theta_n)\mathrm{Proj}_C(x_n - \gamma_n \nabla f(x_n)), \quad n \geq 0, \tag{1.6}$$

where the sequences $\{\theta_n\} \subset (0, 1)$ and $\{\gamma_n\} \subset (0, \infty)$ satisfy the following conditions:

(i) $\lim_{n\to\infty}\theta_n = 0$, $\sum_{n=0}^{\infty} \theta_n = \infty$ and $\sum_{n=0}^{\infty} |\theta_{n+1} - \theta_n| < \infty$;

(ii) $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 2/L$ and $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Xu [18] proved that the sequence $\{x_n\}$ converges strongly to a minimizer of (1.1).

Remark 1.1. Xu's modification (1.6) is a convex combination of the gradient-projection algorithm (1.5) and a self-mapping $h(x)$ which is usually referred as a so-called viscosity item.

In [18], Xu presented another modification as follows:

$$
\begin{aligned}
y_n &= \mathrm{Proj}_C\left(x_n - \gamma_n \nabla f(x_n)\right), \quad n \geq 0, \\
C_n &= \left\{z \in C : \left\|y_n - z\right\| \leq \|x_n - z\|\right\}, \\
Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\
x_{n+1} &= \mathrm{Proj}_{C_n \cap Q_n} x_0.
\end{aligned}
\tag{1.7}
$$

Consequently, Xu [18] proved that Algorithm (1.7) also converges strongly to x^* which solves the minimization problem (1.1).

Remark 1.2. Equation (1.7) involved in additional projections which couple the gradient projection method (1.5) with the so-called CQ method.

It should be pointed out that Xu's modifications (1.6) and (1.7) are interesting and provide us with a direction for solving (1.1) in infinite-dimensional Hilbert spaces.

Motivated by Xu's work, in the present paper, we suggest three variant gradient projection methods so that strong convergence is guaranteed for solving (1.1) in infinite-dimensional Hilbert spaces. Our motivations are mainly in the two respects.

Reason 1. The solution of the minimization problem (1.1) is not always unique, so that there may be many solutions to the problem. In that case, a special solution (e.g., the minimum norm solution) must be found from among candidate solutions. The minimum norm problem is motivated by the following least squares solution to the constrained linear inverse problem:

$$
\begin{aligned}
Bx &= b, \\
x &\in \Omega,
\end{aligned}
\tag{1.8}
$$

where Ω is a nonempty closed convex subset of a real Hilbert space H, B is a bounded linear operator from H to another real Hilbert space H_1, B^* is the adjoint of B, and b is a given point in H_1. The least-squares solution to (1.8) is the least-norm minimizer of the minimization problem:

$$
\min_{x\in\Omega} \|Bx - b\|. \tag{1.9}
$$

For some related works, please see Solodov and Svaiter [19], Goebel and Kirk [20], and Martinez-Yanes and Xu [21].

Reason 2. Projection methods are used extensively in a variety of methods in optimization theory. Apart from theoretical interest, the main advantage of projection methods, which makes them successful in real-world applications, is computational (see [22–31]). In this respect, (1.7) is particularly useful. But we observe that (1.7) involves two half-spaces C_n and Q_n. If the sets C_n and Q_n are simple enough, then P_{C_n} and P_{Q_n} are easily executed. But $C_n \cap Q_n$ may be complicate, so that the projection $P_{C_n\cap Q_n}$ is not easily executed. This might seriously affect the efficiency of the method. Hence, it is interesting that one can relax C_n or Q_n from (1.7).

In the present paper, we suggest the following three methods:

$$x_{n+1} = \mathrm{Proj}_C\big(\theta_n h(x) + (1-\theta_n)x_n - \gamma_n \nabla f(x_n)\big), \quad n \geq 0,$$
$$x_{n+1} = \mathrm{Proj}_C\big(I - \gamma\nabla f\big)\mathrm{Proj}_C((1-\theta_n)x_n), \quad n \geq 0, \tag{1.10}$$

$$\begin{aligned} y_n &= \mathrm{Proj}_C\big(x_n - \gamma_n \nabla f(x_n)\big), \quad n \geq 0, \\ C_n &= \big\{z \in C_{n-1} : \|y_n - z\| \leq \|x_n - z\|\big\}, \\ x_{n+1} &= \mathrm{Proj}_{C_n} x_0. \end{aligned} \tag{1.11}$$

We will show that (1.10) can be used to find the minimum norm solution of the minimization problem (1.1), and (1.11) which is only involved in C_n also has strong convergence.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{2.1}$$

Recall that the (nearest point or metric) projection from H onto C, denoted Proj_C, assigns, to each $x \in H$, the unique point $\mathrm{Proj}_C(x) \in C$ with the property

$$\|x - \mathrm{Proj}_C(x)\| = \inf\{\|x - y\| : y \in C\}. \tag{2.2}$$

It is well known that the metric projection Proj_C of H onto C has the following basic properties:

(i) $\|\mathrm{Proj}_C(x) - \mathrm{Proj}_C(y)\| \leq \|x - y\|$, for all $x, y \in H$;

(ii) $\langle x - y, \mathrm{Proj}_C(x) - \mathrm{Proj}_C(y)\rangle \geq \|\mathrm{Proj}_C(x) - \mathrm{Proj}_C(y)\|^2$, for every $x, y \in H$;

(iii) $\langle x - \mathrm{Proj}_C(x), y - \mathrm{Proj}_C(x)\rangle \leq 0$, for all $x \in H$, $y \in C$.

Next we adopt the following notation:

(i) $x_n \to x$ means that x_n converges strongly to x;

(ii) $x_n \rightharpoonup x$ means that x_n converges weakly to x;

(iii) $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ is the weak ω-limit set of the sequence $\{x_n\}$.

Lemma 2.1 (see [32] (Demiclosedness Principle)). *Let C be a closed and convex subset of a Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}\,(T) \neq \emptyset$. *If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then*

$$(I - T)x = y. \tag{2.3}$$

In particular, if $y = 0$, then $x \in$ $\mathrm{Fix}\,(T)$.

Lemma 2.2 (see [33]). *Let C be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $x_0 \in H$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - x_0\| \leq \left\|x_0 - \mathrm{Proj}_C(x_0)\right\|, \quad \forall n \geq 0, \tag{2.4}$$

then $x_n \to \mathrm{Proj}_C(x_0)$.

Lemma 2.3 (see [34]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \tag{2.5}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 (see [35]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1. \tag{2.6}$$

Suppose

$$x_{n+1} = (1-\beta_n)y_n + \beta_n x_n, \tag{2.7}$$

for all $n \geq 0$, and

$$\limsup_{n\to\infty} \left(\left\|y_{n+1} - y_n\right\| - \|x_{n+1} - x_n\|\right) \leq 0. \tag{2.8}$$

Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

3. Main Results

In this section, we will state and prove our main results.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to R$ be a real-valued Fréchet differentiable convex function. Assume that the solution set S of* (1.1) *is nonempty. Assume that the gradient ∇f is L-Lipschitzian. Let $h : C \to H$ be a ρ-contraction with $\rho \in [0,1)$. Let $\{x_n\}$ be a sequence generated by the following hybrid gradient projection algorithm:*

$$x_{n+1} = \mathrm{Proj}_C\left(\theta_n h(x_n) + (1-\theta_n)x_n - \gamma_n \nabla f(x_n)\right), \quad n \geq 0, \tag{3.1}$$

where the sequences $\{\theta_n\} \subset (0,1)$ *and* $\{\gamma_n\} \subset (0,\infty)$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty}\theta_n = 0$, $\sum_{n=0}^{\infty}\theta_n = \infty$ *and* $\sum_{n=0}^{\infty}|\theta_{n+1} - \theta_n| < \infty$;

(ii) $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 2/L$ *and* $\sum_{n=0}^{\infty}|\gamma_{n+1} - \gamma_n| < \infty$.

Then the sequence $\{x_n\}$ *generated by* (3.1) *converges to a minimizer* $\widehat{x}$ *of* (1.1) *which is the unique solution of the following variational inequality:*

$$\widehat{x} \in S, \quad \langle (I-h)\widehat{x}, x - \widehat{x}\rangle \ge 0, \quad x \in S. \tag{3.2}$$

Proof. Take any $x^* \in S$. Since $x^* \in C$ solves the minimization problem (1.1) if and only if x^* solves the fixed-point equation, $x^* = \mathrm{Proj}_C(I - \gamma\nabla f)x^*$ for any fixed positive number γ. So, we have $x^* = \mathrm{Proj}_C(I - \gamma_n\nabla f)x^*$ for all $n \ge 0$. It can be rewritten as

$$x^* = \mathrm{Proj}_C\left(\theta_n x^* + (1-\theta_n)\left(x^* - \frac{\gamma_n}{1-\theta_n}\nabla f(x^*)\right)\right), \quad n \ge 0. \tag{3.3}$$

From condition (ii) $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 2/L$, there exist two constants a and b such that $0 < a \le \gamma_n \le b < 2/L$ for sufficiently large n; without loss of generality, we can assume $0 < a \le \gamma_n \le b < 2/L$ for all n. Since $\lim_{n\to\infty}\theta_n = 0$, without loss of generality, we can assume that $0 < \theta_n < 1 - bL/2$ for all $n \ge 0$. So, $0 < \liminf_{n\to\infty}(\gamma_n/(1-\theta_n)) \le \limsup_{n\to\infty}(\gamma_n/(1-\theta_n)) < 2/L$. Hence, $I - (\gamma_n/(1-\theta_n))\nabla f$ is nonexpansive.

From (3.1), we get

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\|\mathrm{Proj}_C(\theta_n h(x_n) + (1-\theta_n)x_n - \gamma_n\nabla f(x_n)) - \mathrm{Proj}_C(I - \gamma_n\nabla f)x^*\right\| \\
&= \Bigg\|\mathrm{Proj}_C\left(\theta_n h(x_n) + (1-\theta_n)\left(x_n - \frac{\gamma_n}{1-\theta_n}\nabla f(x_n)\right)\right) \\
&\qquad -\mathrm{Proj}_C\left(\theta_n x^* + (1-\theta_n)\left(x^* - \frac{\gamma_n}{1-\theta_n}\nabla f(x^*)\right)\right)\Bigg\| \\
&\le \theta_n\|h(x_n) - x^*\| + (1-\theta_n) \\
&\quad \times \left\|\left(I - \frac{\gamma_n}{1-\theta_n}\nabla f\right)x_n - \left(I - \frac{\gamma_n}{1-\theta_n}\nabla f\right)x^*\right\| \\
&\le \theta_n\|h(x_n) - h(x^*)\| + \theta_n\|h(x^*) - x^*\| + (1-\theta_n)\|x_n - x^*\| \\
&\le \theta_n\rho\|x_n - x^*\| + \theta_n\|h(x^*) - x^*\| + (1-\theta_n)\|x_n - x^*\| \\
&= (1 - (1-\rho)\theta_n)\|x_n - x^*\| + \theta_n\|h(x^*) - x^*\| \\
&\le \max\left\{\|x_n - x^*\|, \frac{1}{1-\rho}\|h(x^*) - x^*\|\right\}.
\end{aligned} \tag{3.4}$$

Thus, we deduce by induction that

$$\|x_n - x^*\| \le \max\left\{\|x_0 - x^*\|, \frac{1}{1-\rho}\|h(x^*) - x^*\|\right\}. \tag{3.5}$$

This indicates that the sequence $\{x_n\}$ is bounded and so are the sequences $\{h(x_n)\}$ and $\{\nabla f(x_n)\}$. Then, we can chose a constant $M > 0$ such that

$$\sup_{n\geq 0}\{\|h(x_n)\| + \|x_n\| + \|\nabla f(x_n)\|\} \leq M. \tag{3.6}$$

Next, we estimate $\|x_{n+1} - x_n\|$. By (3.1), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \big\|\mathrm{Proj}_C(\theta_n h(x_n) + (1-\theta_n)x_n - \gamma_n \nabla f(x_n)) \\
&\qquad -\mathrm{Proj}_C(\theta_{n-1} h(x_{n-1}) + (1-\theta_{n-1})x_{n-1} - \gamma_{n-1}\nabla f(x_{n-1}))\big\| \\
&\leq \big\|((1-\theta_n)x_n - \gamma_n\nabla f(x_n)) - ((1-\theta_{n-1})x_{n-1} - \gamma_{n-1}\nabla f(x_{n-1}))\big\| \\
&\quad + \|\theta_n h(x_n) - \theta_{n-1}h(x_{n-1})\| \\
&= \bigg\|(1-\theta_n)\left(I - \frac{\gamma_n}{1-\theta_n}\nabla f\right)x_n - (1-\theta_n)\left(I - \frac{\gamma_n}{1-\theta_n}\nabla f\right)x_{n-1} \\
&\qquad +(\theta_{n-1}-\theta_n)x_{n-1} + (\gamma_{n-1}-\gamma_n)\nabla f(x_{n-1})\bigg\| \\
&\quad + \|\theta_n(h(x_n) - h(x_{n-1})) + (\theta_n - \theta_{n-1})h(x_{n-1})\| \\
&\leq (1-\theta_n)\left\|\left(I - \frac{\gamma_n}{1-\theta_n}\nabla f\right)x_n - \left(I - \frac{\gamma_n}{1-\theta_n}\nabla f\right)x_{n-1}\right\| + |\theta_n - \theta_{n-1}|\|x_{n-1}\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\|\nabla f(x_{n-1})\| + \theta_n\|h(x_n) - h(x_{n-1})\| + |\theta_n - \theta_{n-1}|\|h(x_{n-1})\| \\
&\leq (1-(1-\rho)\theta_n)\|x_n - x_{n-1}\| + |\theta_n - \theta_{n-1}|(\|h(x_{n-1})\| + \|x_{n-1}\|) \\
&\quad + |\gamma_n - \gamma_{n-1}|\|\nabla f(x_{n-1})\| \\
&\leq (1-(1-\rho)\theta_n)\|x_n - x_{n-1}\| + (|\gamma_n - \gamma_{n-1}| + |\theta_n - \theta_{n-1}|)M.
\end{aligned} \tag{3.7}$$

Then, we can combine the last inequality and Lemma 2.3 to conclude that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.8}$$

Now we show that the weak limit set $\omega_w(x_n) \subset S$. Choose any $\tilde{x} \in \omega_w(x_n)$. Then there must exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \tilde{x}$. At the same time, the real number sequence $\{\gamma_{n_j}\}$ is bounded. Thus, there exists a subsequence $\{\gamma_{n_{j_i}}\}$ of $\{\gamma_{n_j}\}$ which converges to γ. Without loss of generality, we may assume that $\gamma_{n_j} \to \gamma$. Note that $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 2/L$. So, $\gamma \in (0, 2/L)$. That is, $\gamma_{n_j} \to \gamma \in (0, 2/L)$ as $j \to \infty$. Next, we only need to show that $\tilde{x} \in S$. First, from (3.8) we have that $\|x_{n_j+1} - x_{n_j}\| \to 0$. Then, we have

$$\begin{aligned}
&\left\|x_{n_j} - \mathrm{Proj}_C(I-\gamma\nabla f)x_{n_j}\right\| \\
&\quad\le \left\|x_{n_j} - x_{n_j+1}\right\| + \left\|x_{n_j+1} - \mathrm{Proj}_C\left(I-\gamma_{n_j}\nabla f\right)x_{n_j}\right\| \\
&\qquad + \left\|\mathrm{Proj}_C\left(I-\gamma_{n_j}\nabla f\right)x_{n_j} - \mathrm{Proj}_C(I-\gamma\nabla f)x_{n_j}\right\| \\
&\quad= \Big\|\mathrm{Proj}_C\left(\theta_{n_j}h\left(x_{n_j}\right) + \left(1-\theta_{n_j}\right)x_{n_j} - \gamma_{n_j}\nabla f\left(x_{n_j}\right)\right) \\
&\qquad\quad - \mathrm{Proj}_C\left(I-\gamma_{n_j}\nabla f\right)x_{n_j}\Big\| \\
&\qquad + \left\|\mathrm{Proj}_C\left(I-\gamma_{n_j}\nabla f\right)x_{n_j} - \mathrm{Proj}_C(I-\gamma\nabla f)x_{n_j}\right\| + \left\|x_{n_j} - x_{n_j+1}\right\| \\
&\quad\le \theta_{n_j}\left(\left\|h\left(x_{n_j}\right)\right\| + \left\|x_{n_j}\right\|\right) + \left|\gamma_{n_j} - \gamma\right|\left\|\nabla f\left(x_{n_j}\right)\right\| + \left\|x_{n_j} - x_{n_j+1}\right\| \\
&\quad\longrightarrow 0.
\end{aligned} \tag{3.9}$$

Since $\gamma \in (0, 2/L)$, $\mathrm{Proj}_C(I-\gamma\nabla f)$ is nonexpansive. It then follows from Lemma 2.1 (demiclosedness principle) that $\tilde{x} \in \mathrm{Fix}\,(\mathrm{Proj}_C(I-\gamma\nabla f))$. Hence, $\tilde{x} \in S$ because of $S = \mathrm{Fix}\,(\mathrm{Proj}_C(I-\gamma\nabla f))$. So, $\omega_w(x_n) \subset S$.

Finally, we prove that $x_n \to \hat{x}$, where $\hat{x}$ is the unique solution of the VI (3.2). First, we show that $\limsup_{n\to\infty}\langle (I-h)\hat{x}, x_n - \hat{x}\rangle \ge 0$. Observe that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfying

$$\limsup_{n\to\infty}\langle (I-h)\hat{x}, x_n - \hat{x}\rangle = \lim_{j\to\infty}\langle (I-h)\hat{x}, x_{n_j} - \hat{x}\rangle. \tag{3.10}$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$ such that $x_{n_{j_i}} \rightharpoonup \tilde{x}$. Without loss of generality, we assume that $x_{n_j} \rightharpoonup \tilde{x}$. Then, we obtain

$$\limsup_{n\to\infty}\langle (I-h)\hat{x}, x_n - \hat{x}\rangle = \lim_{j\to\infty}\langle (I-h)\hat{x}, x_{n_j} - \hat{x}\rangle = \langle (I-h)\hat{x}, \tilde{x} - \hat{x}\rangle \ge 0. \tag{3.11}$$

By using the property (ii) of Proj_C, we have

$$\begin{aligned}
&\|x_{n+1} - \hat{x}\|^2 \\
&\quad= \left\|\mathrm{Proj}_C(\theta_n h(x_n) + (1-\theta_n)x_n - \gamma_n\nabla f(x_n)) - \mathrm{Proj}_C(\theta_n\hat{x} + (1-\theta_n)\hat{x} - \gamma_n\nabla f(\hat{x}))\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\langle \theta_n(h(x_n)-\widehat{x}) + (1-\theta_n)\left(\left(I-\frac{\gamma_n}{1-\theta_n}\nabla f\right)x_n - \left(I-\frac{\gamma_n}{1-\theta_n}\nabla f\right)\widehat{x}\right), x_{n+1}-\widehat{x}\right\rangle \\
&\leq \theta_n\langle h(x_n)-h(\widehat{x}), x_{n+1}-\widehat{x}\rangle + \theta_n\langle h(\widehat{x})-\widehat{x}, x_{n+1}-\widehat{x}\rangle \\
&\quad + (1-\theta_n)\left\|\left(I-\frac{\gamma_n}{1-\theta_n}\nabla f\right)x_n - \left(I-\frac{\gamma_n}{1-\theta_n}\nabla f\right)\widehat{x}\right\|\|x_{n+1}-\widehat{x}\| \\
&\leq \theta_n\rho\|x_n-\widehat{x}\|\|x_{n+1}-\widehat{x}\| + \theta_n\langle h(\widehat{x})-\widehat{x}, x_{n+1}-\widehat{x}\rangle + (1-\theta_n)\|x_n-\widehat{x}\|\|x_{n+1}-\widehat{x}\| \\
&= (1-(1-\rho)\theta_n)\|x_n-\widehat{x}\|\|x_{n+1}-\widehat{x}\| + \theta_n\langle h(x^*)-\widehat{x}, x_{n+1}-\widehat{x}\rangle \\
&\leq \frac{1-(1-\rho)\theta_n}{2}\|x_n-\widehat{x}\|^2 + \frac{1}{2}\|x_{n+1}-\widehat{x}\|^2 + \theta_n\langle h(\widehat{x})-\widehat{x}, x_{n+1}-\widehat{x}\rangle.
\end{aligned}
\tag{3.12}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1}-\widehat{x}\|^2 \leq{}& (1-(1-\rho)\theta_n)\|x_n-\widehat{x}\|^2 \\
&+ (1-\rho)\theta_n\left\{\frac{2}{1-\rho}\langle h(\widehat{x})-\widehat{x}, x_{n+1}-\widehat{x}\rangle\right\}.
\end{aligned}
\tag{3.13}
$$

From Lemma 2.3, (3.11), and (3.13), we deduce that $x_n \to \widehat{x}$. This completes the proof. □

From Theorem 3.1, we obtain immediately the following theorem.

Theorem 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to R$ be a real-valued Fréchet differentiable convex function. Assume $S \neq \emptyset$. Assume that the gradient ∇f is L-Lipschitzian. Let $\{x_n\}$ be a sequence generated by the following hybrid gradient projection algorithm:*

$$
x_{n+1} = \mathrm{Proj}_C((1-\theta_n)x_n - \gamma_n\nabla f(x_n)), \quad n \geq 0, \tag{3.14}
$$

where the sequences $\{\theta_n\} \subset (0,1)$ and $\{\gamma_n\} \subset (0,\infty)$ satisfy the following conditions:

(i) $\lim_{n\to\infty}\theta_n = 0$, $\sum_{n=0}^{\infty}\theta_n = \infty$ *and* $\sum_{n=0}^{\infty}|\theta_{n+1}-\theta_n| < \infty$;

(ii) $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 2/L$ *and* $\sum_{n=0}^{\infty}|\gamma_{n+1}-\gamma_n| < \infty$.

Then the sequence $\{x_n\}$ generated by (3.14) *converges to a minimizer $\widehat{x}$ of* (1.1) *which is the minimum norm element in S.*

Proof. In Theorem 3.1, we note that h is a non-self mapping from C to the whole space H. Hence, if we chose $h(x) \equiv 0$ for all $x \in C$, then Algorithm (3.1) reduces to (3.14). And sequence x_n converges strongly to $\widehat{x} = \mathrm{Proj}_S(0)$ which is obviously the minimum norm element in S. The proof is completed. □

Next, we suggest another simple algorithm for dropping the assumption $\sum_{n=0}^{\infty}|\theta_{n+1}-\theta_n| < \infty$.

Theorem 3.3. *Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to R$ be a real-valued Fréchet differentiable convex function. Assume $S \neq \emptyset$. Assume that the gradient ∇f is L-Lipschitzian. Let $\{x_n\}$ be a sequence generated by the following hybrid gradient projection algorithm:*

$$x_{n+1} = \operatorname{Proj}_C(I - \gamma \nabla f)\operatorname{Proj}_C((1-\theta_n)x_n), \quad n \geq 0, \tag{3.15}$$

where $\gamma \in (0, 2/L)$ is a constant and the sequences $\{\theta_n\} \subset (0,1)$ satisfy the following conditions:

(1) $\lim_{n\to\infty}\theta_n = 0$;

(2) $\sum_{n=0}^{\infty}\theta_n = \infty$.

Then the sequence $\{x_n\}$ generated by (3.15) *converges to a minimizer $\widehat{x}$ of* (1.1) *which is the minimum norm element in S.*

Proof. Claim 1. The sequence $\{x_n\}$ is bounded.

Take $x^* \in S$. Then we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \left\|\operatorname{Proj}_C(I-\gamma\nabla f)\operatorname{Proj}_C((1-\theta_n)x_n) - \operatorname{Proj}_C(I-\gamma\nabla f)x^*\right\| \\ &\leq \|(1-\theta_n)x_n - x^*\| \\ &\leq (1-\theta_n)\|x_n - x^*\| + \theta_n\|x^*\| \\ &\leq \max\{\|x_n - x^*\|, \|x^*\|\}.\end{aligned} \tag{3.16}$$

By induction,

$$\|x_n - x^*\| \leq \max\{\|x_0 - x^*\|, \|x^*\|\}. \tag{3.17}$$

Claim 2. $\|x_n - \operatorname{Proj}_C(I-\gamma\nabla f)x_n\| \to 0$ and $\omega_w(x_n) \subset S$.

By the similar argument as that in [18, page 366], we can write

$$\operatorname{Proj}_C(I-\gamma\nabla f) = (1-\beta)I + \beta T, \tag{3.18}$$

where T is nonexpansive and $\beta = (2+\gamma L)/4 \subset (0,1)$. Then we can rewrite (3.15) as

$$\begin{aligned}x_{n+1} &= \left[(1-\beta)I + \beta T\right]\operatorname{Proj}_C((1-\theta_n)x_n) \\ &= (1-\beta)x_n + \beta T\operatorname{Proj}_C((1-\theta_n)x_n) + (1-\beta)\left(\operatorname{Proj}_C((1-\theta_n)x_n) - x_n\right) \\ &= (1-\beta)x_n + \beta y_n,\end{aligned} \tag{3.19}$$

where

$$y_n = T\operatorname{Proj}_C((1-\theta_n)x_n) + \frac{1-\beta}{\beta}\left(\operatorname{Proj}_C((1-\theta_n)x_n) - x_n\right). \tag{3.20}$$

It follows that

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\quad = \left\| T\,\mathrm{Proj}_C((1-\theta_{n+1})x_{n+1}) + \frac{1-\beta}{\beta}(\mathrm{Proj}_C((1-\theta_{n+1})x_{n+1}) - x_{n+1}) \right. \\
&\qquad \left. - T\,\mathrm{Proj}_C((1-\theta_n)x_n) - \frac{1-\beta}{\beta}(\mathrm{Proj}_C((1-\theta_n)x_n) - x_n) \right\| \\
&\quad \leq \|T\,\mathrm{Proj}_C((1-\theta_{n+1})x_{n+1}) - T\,\mathrm{Proj}_C((1-\theta_n)x_n)\| \\
&\qquad + \frac{1-\beta}{\beta}\|\mathrm{Proj}_C((1-\theta_{n+1})x_{n+1}) - x_{n+1}\| \\
&\qquad + \frac{1-\beta}{\beta}\|\mathrm{Proj}_C((1-\theta_n)x_n) - x_n\| \\
&\quad \leq \|x_{n+1} - x_n\| + \frac{\theta_{n+1}}{\beta}\|x_{n+1}\| + \frac{\theta_n}{\beta}\|x_n\|.
\end{aligned} \tag{3.21}$$

So,

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{3.22}$$

This together with Lemma 2.4 implies that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.23}$$

Thus,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta)\|y_n - x_n\| = 0. \tag{3.24}$$

Note that

$$\begin{aligned}
&\|x_n - \mathrm{Proj}_C(I - \gamma\nabla f)x_n\| \\
&\quad \leq \|x_n - x_{n+1}\| + \|x_{n+1} - \mathrm{Proj}_C(I - \gamma\nabla f)x_n\| \\
&\quad = \|x_n - x_{n+1}\| + \|\mathrm{Proj}_C(I - \gamma\nabla f)\mathrm{Proj}_C((1-\theta_n)x_n) - \mathrm{Proj}_C(I - \gamma\nabla f)x_n\| \\
&\quad \leq \|x_n - x_{n+1}\| + \theta_n\|x_n\|.
\end{aligned} \tag{3.25}$$

Therefore,

$$\lim_{n\to\infty}\|x_n - \mathrm{Proj}_C(I - \gamma\nabla f)x_n\| = 0. \tag{3.26}$$

Now repeating the proof of Theorem 3.1, we conclude that $\omega_w(x_n) \subset S$.

Claim 3. $\limsup_{n\to\infty}\langle \hat{x}, x_n - \hat{x}\rangle \geq 0$ where $\hat{x}$ is the minimum norm element in S.

Observe that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfying

$$\limsup_{n\to\infty}\langle \hat{x}, x_n - \hat{x}\rangle = \lim_{j\to\infty}\left\langle \hat{x}, x_{n_j} - \hat{x}\right\rangle. \tag{3.27}$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$ such that $x_{n_{j_i}} \rightharpoonup \tilde{x} \in S$. Without loss of generality, we assume that $x_{n_j} \rightharpoonup \tilde{x} \in S$. Then, we obtain

$$\limsup_{n\to\infty}\langle \hat{x}, x_n - \hat{x}\rangle = \lim_{j\to\infty}\left\langle \hat{x}, x_{n_j} - \hat{x}\right\rangle = \langle \hat{x}, \tilde{x} - \hat{x}\rangle \geq 0. \tag{3.28}$$

Claim 4. $x_n \to \hat{x}$. From (3.15), we have

$$\begin{aligned}
\|x_{n+1} - \hat{x}\|^2 &= \left\|\mathrm{Proj}_C(I - \gamma\nabla f)\mathrm{Proj}_C((1-\theta_n)x_n) - \mathrm{Proj}_C(I - \gamma\nabla f)\hat{x}\right\|^2 \\
&\leq \|(1-\theta_n)x_n - \hat{x}\|^2 \\
&= \|(1-\theta_n)(x_n - \hat{x}) - \theta_n\hat{x}\|^2 \\
&= (1-\theta_n)^2\|x_n - \hat{x}\|^2 - 2\theta_n(1-\theta_n)\langle \hat{x}, x_n - \hat{x}\rangle + \theta_n^2\|\hat{x}\|^2 \\
&\leq (1-\theta_n)\|x_n - \hat{x}\|^2 + \theta_n\left\{-2(1-\theta_n)\langle \hat{x}, x_n - \hat{x}\rangle + \theta_n\|\hat{x}\|^2\right\}.
\end{aligned} \tag{3.29}$$

It is obvious that $\limsup_{n\to\infty}(-2(1-\theta_n)\langle \hat{x}, x_n - \hat{x}\rangle + \theta_n\|\hat{x}\|^2) \leq 0$. Then we can apply Lemma 2.3 to the last inequality to conclude that $x_n \to \hat{x}$. The proof is completed. □

Next, we suggest another algorithm with the additional projections applied to the gradient projection algorithm. We show that this algorithm has strong convergence.

Theorem 3.4. *Let C be a closed convex subset of a real Hilbert space H. Let $f : C \to R$ be a real-valued Fréchet differentiable convex function. Assume $S \neq \emptyset$. Assume that the gradient ∇f is L-Lipschitzian. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = \mathrm{Proj}_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{gathered}
y_n = \mathrm{Proj}_C(x_n - \gamma_n\nabla f(x_n)), \quad n \geq 0, \\
C_n = \{z \in C_{n-1} : \|y_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} = \mathrm{Proj}_{C_n}x_0.
\end{gathered} \tag{3.30}$$

where the sequence $\{\gamma_n\} \subset (0, \infty)$ satisfies the condition $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 2/L$. Then the sequence $\{x_n\}$ generated by (3.30) converges to $\hat{x} = \mathrm{Proj}_S(x_0)$.

Proof. It is obvious that C_n is convex. For any $x^* \in S$, we have

$$\begin{aligned} \left\| y_n - x^* \right\| &= \left\| \mathrm{Proj}_C(x_n - \gamma_n \nabla f(x_n)) - \mathrm{Proj}_C(x^* - \gamma_n \nabla f(x^*)) \right\| \\ &\leq \| x_n - x^* \|. \end{aligned} \tag{3.31}$$

This implies that $x^* \in C_n$. Hence, $S \subset C_n$. From $x_{n+1} = \mathrm{Proj}_{C_n} x_0$, we have

$$\langle x_0 - x_{n+1}, x_{n+1} - y \rangle \geq 0, \quad \forall y \in C_n. \tag{3.32}$$

Since $S \subset C_n$, we have

$$\langle x_0 - x_{n+1}, x_{n+1} - u \rangle \geq 0, \quad \forall u \in S. \tag{3.33}$$

So, for $u \in S$, we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_{n+1}, x_{n+1} - u \rangle \\ &= \langle x_0 - x_{n+1}, x_{n+1} - x_0 + x_0 - u \rangle \\ &= -\| x_0 - x_{n+1} \|^2 + \langle x_0 - x_{n+1}, x_0 - u \rangle \\ &\leq -\| x_0 - x_{n+1} \|^2 + \| x_0 - x_{n+1} \| \| x_0 - u \|. \end{aligned} \tag{3.34}$$

Hence,

$$\| x_0 - x_{n+1} \| \leq \| x_0 - u \|, \quad \forall u \in S. \tag{3.35}$$

This implies that $\{x_n\}$ is bounded.

From $x_n = \mathrm{Proj}_{C_{n-1}} x_0$ and $x_{n+1} = \mathrm{Proj}_{C_n} x_0 \in C_n \subset C_{n-1}$, we have

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0. \tag{3.36}$$

Hence,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\| x_0 - x_n \|^2 + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leq -\| x_0 - x_n \|^2 + \| x_0 - x_n \| \| x_0 - x_{n+1} \|, \end{aligned} \tag{3.37}$$

and therefore

$$\| x_0 - x_n \| \leq \| x_0 - x_{n+1} \|. \tag{3.38}$$

This implies that

$$\lim_{n\to\infty} \|x_n - x_0\| \text{ exists.} \tag{3.39}$$

From (3.36) and (3.39), we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\
&\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \\
&\longrightarrow 0.
\end{aligned} \tag{3.40}$$

By the fact $x_{n+1} \in C_n$, we get

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\| \longrightarrow 0. \tag{3.41}$$

Therefore, from (3.40) and (3.41), we deduce

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x_n - \operatorname{Proj}_C(I - \gamma_n \nabla f)x_n\| = 0. \tag{3.42}$$

Now (3.42) and Lemma 2.1 guarantee that every weak limit point of $\{x_n\}$ is a fixed point of $\operatorname{Proj}_C(I - \gamma_n \nabla f)$. That is, $\omega_w(x_n) \subset \operatorname{Fix}(\operatorname{Proj}_C(I - \gamma_n \nabla f)) = S$. At the same time, if we choose $u = \operatorname{Proj}_S(x_0)$ in (3.35), we have

$$\|x_0 - x_{n+1}\| \le \|x_0 - \operatorname{Proj}_S(x_0)\|. \tag{3.43}$$

This fact and Lemma 2.2 ensure the strong convergence of $\{x_n\}$ to $\operatorname{Proj}_S(x_0)$. This completes the proof. □

Now we give some remarks on our variant gradient projection methods.

Remark 3.5. Under the same control parameters, the gradient projection methods (3.1) and (1.6) are all strong convergent. However, (3.1) seems to have more advantage than (1.6) as h is a non-self-mapping.

Remark 3.6. The gradient projection method (3.14) is similar to (1.5) by using $(1 - \theta_n)x_n$ instead of x_n. But (3.1) has strong convergence, and especially (3.1) converges strongly to the minimum norm element of S.

Remark 3.7. The advantage of the gradient projection method (3.15) is that it has strong convergence under some weaker assumptions on parameter θ_n.

Remark 3.8. The gradient projection method (3.30) is simpler than (1.7).

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