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Research Article

On Sumudu Transform Method in Discrete Fractional Calculus

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In this paper, starting from the definition of the Sumudu transform on a general time scale, we define the generalized discrete Sumudu transform and present some of its basic properties. We obtain the discrete Sumudu transform of Taylor monomials, fractional sums, and fractional differences. We apply this transform to solve some fractional difference initial value problems.

1. Introduction

The fractional calculus, which is as old as the usual calculus, deals with the generalization of the integration and differentiation of integer order to arbitrary order. It has recently received a lot of attention because of its interesting applications in various fields of science, such as, viscoelasticity, diffusion, neurology, control theory, and statistics, see [1–6].

The analogous theory for discrete fractional calculus was initiated by Miller and Ross [7], where basic approaches, definitions, and properties of the theory of fractional sums and differences were reported. Recently, a series of papers continuing this research has appeared. We refer the reader to the papers [8–12] and the references cited therein.

In the early 1990's, Watugala [13, 14] introduced the Sumudu transform and applied it to solve ordinary differential equations. The fundamental properties of this transform, which are thought to be an alternative to the Laplace transform were then established in many articles [15–19].

The Sumudu transform is defined over the set of functions

$$A := \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, \left| f(t) \right| < M e^{|t|/\tau_j},\ \text{if } t \in (-1)^j \times [0, \infty) \right\} \tag{1.1}$$

by

$$F(u) := \mathbb{S}\{f\}(u) := \frac{1}{u}\int_0^{\infty} f(t)e^{-(t/u)}dt, \quad u \in (-\tau_1, \tau_2). \tag{1.2}$$

Although the Sumudu transform of a function has a deep connection to its Laplace transform, the main advantage of the Sumudu transform is the fact that it may be used to solve problems without resorting to a new frequency domain because it preserves scales and unit properties. By these properties, the Sumudu transform may be used to solve intricate problems in engineering and applied sciences that can hardly be solved when the Laplace transform is used. Moreover, some properties of the Sumudu transform make it more advantageous than the Laplace transform. Some of these properties are

(i) The Sumudu transform of a Heaviside step function is also a Heaviside step function in the transformed domain.

(ii) $\mathbb{S}\{t^n\}(u) = n!u^n$.

(iii) $\lim_{u\to-\tau_1} F(u) = \lim_{t\to-\infty} f(t)$.

(iv) $\lim_{u\to\tau_2} F(u) = \lim_{t\to\infty} f(t)$.

(v) $\lim_{t\to 0^{\mp}} f(t) = \lim_{u\to 0^{\mp}} F(u)$.

(vi) For any real or complex number $c, \mathbb{S}\{f(ct)\}(u) = F(cu)$.

In particular, since constants are fixed by the Sumudu transform, choosing $c = 0$, it gives $F(0) = f(0)$.

In dealing with physical applications, this aspect becomes a major advantage, especially in instances where keeping track of units, and dimensional factor groups of constants, is relevant. This means that in problem solving, u and $G(u)$ can be treated as replicas of t and $f(t)$, respectively [20].

Recently, an application of the Sumudu and Double Sumudu transforms to Caputo-fractional differential equations is given in [21]. In [22], the authors applied the Sumudu transform to fractional differential equations.

Starting with a general definition of the Laplace transform on an arbitrary time scale, the concepts of the h-Laplace and consequently the discrete Laplace transform were specified in [23]. The theory of time scales was initiated by Hilger [24]. This theory is a tool that unifies the theories of continuous and discrete time systems. It is a subject of recent studies in many different fields in which dynamic process can be described with discrete or continuous models.

In this paper, starting from the definition of the Sumudu transform on a general time scale, we define the discrete Sumudu transform and present some of its basic properties.

The paper is organized as follows: in Sections 2 and 3, we introduce some basic concepts concerning the calculus of time scales and discrete fractional calculus, respectively. In Section 4, we define the discrete Sumudu transform and present some of its basic properties. Section 5 is devoted to an application.

2. Preliminaries on Time Scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The most well-known examples are $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = \overline{q^{\mathbb{Z}}} := \{q^n : n \in \mathbb{Z}\} \bigcup \{0\}$, where $q > 1$. The forward and backward jump operators are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \tag{2.1}$$

respectively, where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is said to be left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. For details, see the monographs [25, 26].

The following two concepts are introduced in order to describe classes of functions that are integrable.

Definition 2.1 (see [25]). A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated if its right-sided limits exist at all right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

Definition 2.2 (see [25]). A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$.

The set $\mathbb{T}^\kappa$ is derived from the time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa := \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa := \mathbb{T}$.

Definition 2.3 (see [25]). A function $f : \mathbb{T} \to \mathbb{R}$ is said to be delta differentiable at a point $t \in \mathbb{T}^\kappa$ if there exists a number $f^\Delta(t)$ with the property that given any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left| [f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s] \right| \le \varepsilon |\sigma(t) - s| \quad \forall s \in U. \tag{2.2}$$

We will also need the following definition in order to define the exponential function on an arbitrary time scale.

Definition 2.4 (see [25]). A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$.

The set $\mathcal{R}$ of all regressive and rd-continuous functions forms an Abelian group under the "circle plus" addition $\oplus$ defined by

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t) \quad \forall t \in \mathbb{T}^\kappa. \tag{2.3}$$

The additive inverse $\ominus p$ of $p \in \mathcal{R}$ is defined by

$$(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)p(t)} \quad \forall t \in \mathbb{T}^\kappa. \tag{2.4}$$

Theorem 2.5 (see [25]). *Let $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$ be a fixed point. Then the exponential function $e_p(\cdot, t_0)$ is the unique solution of the initial value problem*

$$y^{\Delta} = p(t)y, \qquad y(t_0) = 1. \tag{2.5}$$

3. An Introduction to Discrete Fractional Calculus

In this section, we introduce some basic definitions and a theorem concerning the discrete fractional calculus.

Throughout, we consider the discrete set

$$\mathbb{N}_a := \{a, a+1, a+2, \ldots\}, \quad \text{where } a \in \mathbb{R} \text{ is fixed}. \tag{3.1}$$

Definition 3.1 (see [27]). Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given. Then the νth-order fractional sum of f is given by

$$\Delta_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - \sigma(s))^{\underline{\nu-1}} f(s) \quad \text{for } t \in \mathbb{N}_{a+\nu}. \tag{3.2}$$

Also, we define the trivial sum by

$$\Delta_a^{-0} f(t) := f(t) \quad \text{for } t \in N_a. \tag{3.3}$$

Note that the fractional sum operator $\Delta_a^{-\nu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu}$.

In the above equation the term $(t - \sigma(s))^{\underline{\nu-1}}$ is the generalized falling function defined by

$$t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)} \tag{3.4}$$

for any $t, \nu \in \mathbb{R}$ for which the right-hand side is well defined. As usual, we use the convention that division by a pole yields zero.

Definition 3.2 (see [27]). Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu \geq 0$ be given, and let $N \in \mathbb{N}$ be chosen such that $N - 1 < \nu \leq N$. Then the νth-order Riemann-Liouville fractional difference of f is given by

$$\Delta_a^{\nu} f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t) \quad \text{for } t \in \mathbb{N}_{a+N-\nu}. \tag{3.5}$$

It is clear that, the fractional difference operator Δ_a^{ν} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+N-\nu}$.

As stated in the following theorem, the composition of fractional operators behaves well if the inner operator is a fractional difference.

Theorem 3.3 (see [27]). *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and suppose $\nu, \mu > 0$ with $N-1 < \nu \leq N$. Then*

$$\Delta_{a+\mu}^{\nu} \Delta_a^{-\mu} f(t) = \Delta_a^{\nu-\mu} f(t) \quad \text{for } t \in N_{a+\mu+N-\nu}. \tag{3.6}$$

A disadvantage of the Riemann-Liouville fractional difference operator is that when applied to a constant c, it does not yield 0. For example, for $0 < v < 1$, we have

$$\Delta_a^{\nu} c = -\frac{c(t-a)^{\underline{-\nu}}}{\Gamma(1-\nu)}. \tag{3.7}$$

In order to overcome this and to make the fractional difference behave like the usual difference, the Caputo fractional difference was introduced in [12].

Definition 3.4 (see [12]). Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu \geq 0$ be given, and let $N \in \mathbb{N}$ be chosen such that $N-1 < \nu \leq N$. Then the νth-order Caputo fractional difference of f is given by

$$^{C}\Delta_a^{\nu} f(t) := \Delta_a^{-(N-\nu)} \Delta^N f(t) \quad \text{for } t \in \mathbb{N}_{a+N-\nu}. \tag{3.8}$$

It is clear that the Caputo fractional difference operator $^{C}\Delta_a^{\nu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+N-\nu}$ as well. And it follows from the definition of the Caputo fractional difference operator that

$$^{C}\Delta_a^{\nu}\, c = 0. \tag{3.9}$$

4. The Discrete Sumudu Transform

The following definition is a slight generalization of the one introduced by Jarad et al. [28].

Definition 4.1. The Sumudu transform of a regulated function $f : \mathbb{T}_a \to \mathbb{R}$ is given by

$$\mathbb{S}_a\{f\}(u) := \frac{1}{u} \int_a^{\infty} e_{\ominus(1/u)}(\sigma(t), a) f(t) \Delta t \quad \forall u \in \mathfrak{D}\{f\}, \tag{4.1}$$

where $a \in \mathbb{R}$ is fixed, $\mathbb{T}_a$ is an unbounded time scale with infimum a and $\mathfrak{D}\{f\}$ is the set of all nonzero complex constants u for which $1/u$ is regressive and the integral converges.

In the special case, when $\mathbb{T}_a = \mathbb{N}_a$, every function $f : \mathbb{N}_a \to \mathbb{R}$ is regulated and its discrete Sumudu transform can be written as

$$\mathbb{S}_a\{f\}(u) = \frac{1}{u} \sum_{k=0}^{\infty} \left(\frac{u}{u+1}\right)^{k+1} f(k+a) \tag{4.2}$$

for each $u \in \mathbb{C} \setminus \{-1, 0\}$ for which the series converges. For the convergence of the Sumudu transform, we need the following definition.

Definition 4.2 (see [27]). A function $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order r ($r > 0$) if there exists a constant $A > 0$ such that

$$|f(t)| \leq Ar^t \quad \text{for sufficiently large } t. \tag{4.3}$$

The following lemma can be proved similarly as in Lemma 12 in [27].

Lemma 4.3. *Suppose $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 0$. Then*

$$\mathbb{S}_a\{f\}(u) \text{ exists for all } u \in \mathbb{C} \setminus \{-1, 0\} \text{ such that } \left|\frac{u+1}{u}\right| > r. \tag{4.4}$$

The following lemma relates the shifted Sumudu transform to the original.

Lemma 4.4. *Let $m \in \mathbb{N}_0$ and $f : \mathbb{N}_{a-m} \to \mathbb{R}$ and $g : \mathbb{N}_a \to \mathbb{R}$ are of exponential order $r > 0$. Then for all $u \in \mathbb{C} \setminus \{-1, 0\}$ such that $|(u+1)/u| > r$,*

$$\mathbb{S}_{a-m}\{f\}(u) = \left(\frac{u}{u+1}\right)^m \mathbb{S}_a\{f\}(u) + \frac{1}{u}\sum_{k=0}^{m-1}\left(\frac{u}{u+1}\right)^{k+1} f(k+a-m), \tag{4.5}$$

$$\mathbb{S}_{a+m}\{g\}(u) = \left(\frac{u+1}{u}\right)^m \mathbb{S}_a\{g\}(u) - \frac{1}{u}\sum_{k=0}^{m-1}\left(\frac{u+1}{u}\right)^{m-1-k} g(k+a). \tag{4.6}$$

Proof. For all $u \in \mathbb{C} \setminus \{-1, 0\}$ such that $|(u+1)/u| > r$, we have

$$\begin{aligned}
\mathbb{S}_{a-m}\{f\}(u) &= \frac{1}{u}\sum_{k=0}^{\infty}\left(\frac{u}{u+1}\right)^{k+1} f(k+a-m) \\
&= \frac{1}{u}\sum_{k=m}^{\infty}\left(\frac{u}{u+1}\right)^{k+1} f(k+a-m) + \frac{1}{u}\sum_{k=0}^{m-1}\left(\frac{u}{u+1}\right)^{k+1} f(k+a-m) \\
&= \frac{1}{u}\sum_{k=0}^{\infty}\left(\frac{u}{u+1}\right)^{k+m+1} f(k+a) + \frac{1}{u}\sum_{k=0}^{m-1}\left(\frac{u}{u+1}\right)^{k+1} f(k+a-m) \\
&= \left(\frac{u}{u+1}\right)^m \mathbb{S}_a\{f\}(u) + \frac{1}{u}\sum_{k=0}^{m-1}\left(\frac{u}{u+1}\right)^{k+1} f(k+a-m),
\end{aligned}$$

$$\begin{aligned}
\mathbb{S}_{a+m}\{g\}(u) &= \frac{1}{u}\sum_{k=0}^{\infty}\left(\frac{u}{u+1}\right)^{k+1} g(k+a+m) \\
&= \frac{1}{u}\sum_{k=m}^{\infty}\left(\frac{u}{u+1}\right)^{k-m+1} g(k+a) \\
&= \frac{1}{u}\sum_{k=0}^{\infty}\left(\frac{u}{u+1}\right)^{k-m+1} g(k+a) - \frac{1}{u}\sum_{k=0}^{m-1}\left(\frac{u}{u+1}\right)^{k-m+1} g(k+a) \\
&= \left(\frac{u+1}{u}\right)^{m}\mathbb{S}_a\{g\}(u) - \frac{1}{u}\sum_{k=0}^{m-1}\left(\frac{u+1}{u}\right)^{m-1-k} g(k+a).
\end{aligned} \tag{4.7}$$

□

Taylor monomials are very useful for applying the Sumudu transform in discrete fractional calculus.

Definition 4.5 (see [27]). For each $\mu \in \mathbb{R} \setminus (-\mathbb{N})$, define the μth-Taylor monomial to be

$$h_\mu(t,a) := \frac{(t-a)^{\underline{\mu}}}{\Gamma(\mu+1)} \quad \text{for } t \in \mathbb{N}_a. \tag{4.8}$$

Lemma 4.6. *Let $\mu \in \mathbb{R} \setminus (-\mathbb{N})$ and $a, b \in \mathbb{R}$ such that $b - a = \mu$. Then for all $u \in \mathbb{C} \setminus \{-1, 0\}$ such that $|(u+1)/u| > 1$, one has*

$$\mathbb{S}_b\{h_\mu(\cdot,a)\}(u) = (u+1)^{\mu}. \tag{4.9}$$

Proof. By the general binomial formula

$$(x+y)^{\nu} = \sum_{k=0}^{\infty}\binom{v}{k} x^k y^{v-k} \tag{4.10}$$

for $\nu, x, y \in \mathbb{R}$ such that $|x| < |y|$, where

$$\binom{v}{k} := \frac{\nu^{\underline{k}}}{k!}, \tag{4.11}$$

as in [27], it follows from (4.10) and

$$\binom{-v}{k} = (-1)^k\binom{k+v-1}{v-1}, \tag{4.12}$$

where $k \in \mathbb{N}_0$ that

$$\frac{1}{(1-y)^{\nu}} = ((-y)+1)^{-\nu} = \sum_{k=0}^{\infty} \binom{k+v-1}{v-1} y^k \tag{4.13}$$

for $\nu \in \mathbb{R}$ and $|y| < 1$.

And since $b - a = \mu$, we have for all $u \in \mathbb{C} \setminus \{-1, 0\}$ such that $|(u+1)/u| > 1$,

$$\begin{aligned}
(u+1)^{\mu} &= \frac{1}{u+1} \frac{1}{(1-(u/(u+1)))^{\mu+1}} \\
&= \frac{1}{u+1} \sum_{k=0}^{\infty} \binom{k+\mu}{\mu} \left(\frac{u}{u+1}\right)^{k} \\
&= \frac{1}{u} \sum_{k=0}^{\infty} \binom{k+\mu}{\mu} \left(\frac{u}{u+1}\right)^{k+1} \\
&= \frac{1}{u} \sum_{k=0}^{\infty} \frac{(k+\mu)^{\underline{\mu}}}{\Gamma(\mu+1)} \left(\frac{u}{u+1}\right)^{k+1} \\
&= \frac{1}{u} \sum_{k=0}^{\infty} h_{\mu}(k+b, a) \left(\frac{u}{u+1}\right)^{k+1} \\
&= \mathbb{S}_b\{h_{\mu}(\cdot, a)\}(u).
\end{aligned} \tag{4.14}$$

□

Definition 4.7 (see [27]). Define the convolution of two functions $f, g : \mathbb{N}_a \to \mathbb{R}$ by

$$(f * g)(t) := \sum_{r=a}^{t} f(r) g(t-r+a) \quad \text{for } t \in \mathbb{N}_a. \tag{4.15}$$

Lemma 4.8. *Let $f, g : \mathbb{N}_a \to \mathbb{R}$ be of exponential order $r > 0$. Then for all $u \in \mathbb{C} \setminus \{-1, 0\}$ such that $|(u+1)/u| > r$,*

$$\mathbb{S}_a\{f * g\}(u) = (u+1)\mathbb{S}_a\{f\}(u)\mathbb{S}_a\{g\}(u). \tag{4.16}$$

Proof. Since

$$\begin{aligned}
\mathbb{S}_a\{f * g\}(u) &= \frac{1}{u}\sum_{k=0}^{\infty}\left(\frac{u}{u+1}\right)^{k+1}(f * g)(k+a) \\
&= \frac{1}{u}\sum_{k=0}^{\infty}\left(\frac{u}{u+1}\right)^{k+1}\sum_{r=a}^{k+a} f(r)g((k+a)-r+a) \\
&= \frac{1}{u}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\left(\frac{u}{u+1}\right)^{k+1} f(r+a)g(k-r+a),
\end{aligned} \tag{4.17}$$

the substitution $\tau = k - r$ yields

$$\begin{aligned}
\mathbb{S}_a\{f * g\}(u) &= \frac{1}{u}\sum_{\tau=0}^{\infty}\sum_{r=0}^{\infty}\left(\frac{u}{u+1}\right)^{\tau+r+1} f(r+a)g(\tau+a) \\
&= (u+1)\left(\frac{1}{u}\sum_{r=0}^{\infty}\left(\frac{u}{u+1}\right)^{r+1} f(r+a)\right)\left(\frac{1}{u}\sum_{\tau=0}^{\infty}\left(\frac{u}{u+1}\right)^{\tau+1} g(\tau+a)\right) \\
&= (u+1)\mathbb{S}_a\{f\}(u)\mathbb{S}_a\{g\}(u)
\end{aligned} \tag{4.18}$$

for all $u \in \mathbb{C} \setminus \{-1, 0\}$ such that $|(u+1)/u| > r$. □

Theorem 4.9. *Suppose $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$ and let $\nu > 0$ with $N - 1 < \nu \leq N$. Then for all $u \in \mathbb{C} \setminus \{-1, 0\}$ such that $|(u+1)/u| > r$,*

$$\mathbb{S}_{a+\nu}\{\Delta_a^{-\nu} f\}(u) = (u+1)^{\nu}\mathbb{S}_a\{f\}(u), \tag{4.19}$$

$$\mathbb{S}_{a+\nu-N}\{\Delta_a^{-\nu} f\}(u) = \frac{u^N}{(u+1)^{N-\nu}}\mathbb{S}_a\{f\}(u). \tag{4.20}$$

Proof. First note that the shift formula (4.5) implies that for all $u \in \mathbb{C} \setminus \{-1, 0\}$ such that $|(u+1)/u| > r$,

$$\begin{aligned}
\mathbb{S}_{a+\nu-N}\{\Delta_a^{-\nu} f\}(u) &= \frac{1}{u}\sum_{k=0}^{\infty}\left(\frac{u}{u+1}\right)^{k+1}\Delta_a^{-\nu} f(k+a+\nu-N) \\
&= \left(\frac{u}{u+1}\right)^{N}\mathbb{S}_{a+\nu}\{\Delta_a^{-\nu} f\}(u) + \frac{1}{u}\sum_{k=0}^{N-1}\left(\frac{u}{u+1}\right)^{k+1}\Delta_a^{-\nu} f(k+a+\nu-N) \\
&= \left(\frac{u}{u+1}\right)^{N}\mathbb{S}_{a+\nu}\{\Delta_a^{-\nu} f\}(u),
\end{aligned} \tag{4.21}$$

taking N zeros of $\Delta_a^{-\nu} f$ into account. Furthermore, by (4.9), (4.15), and (4.16),

$$\begin{aligned}
\mathbb{S}_{a+\nu}\{\Delta_a^{-\nu} f\}(u) &= \frac{1}{u}\sum_{k=0}^{\infty}\left(\frac{u}{u+1}\right)^{k+1}\Delta_a^{-\nu} f(k+a+\nu) \\
&= \frac{1}{u}\sum_{k=0}^{\infty}\left(\frac{u}{u+1}\right)^{k+1}\sum_{r=a}^{k+a}\frac{(k+a+\nu-\sigma(r))^{\underline{\nu-1}}}{\Gamma(\nu)} f(r) \\
&= \frac{1}{u}\sum_{k=0}^{\infty}\left(\frac{u}{u+1}\right)^{k+1}\sum_{r=a}^{k+a} f(r) h_{\nu-1}((k+a)-r+a, a-(\nu-1)) \\
&= \frac{1}{u}\sum_{k=0}^{\infty}\left(\frac{u}{u+1}\right)^{k+1}\left(f * h_{\nu-1}(\cdot, a-(\nu-1))\right)(k+a) \\
&= \mathbb{S}_a\{f * h_{\nu-1}(\cdot, a-(\nu-1))\}(u) \\
&= (u+1)\mathbb{S}_a\{f\}(u)\mathbb{S}_a\{h_{\nu-1}(\cdot, a-(\nu-1))\} \\
&= (u+1)(u+1)^{\nu-1}\mathbb{S}_a\{f\}(u) \\
&= (u+1)^{\nu}\mathbb{S}_a\{f\}(u).
\end{aligned} \tag{4.22}$$

Then we obtain

$$\begin{aligned}
\mathbb{S}_{a+\nu-N}\{\Delta_a^{-\nu} f\}(u) &= \left(\frac{u}{u+1}\right)^{N}\mathbb{S}_{a+\nu}\{\Delta_a^{-\nu} f\}(u) \\
&= \frac{u^N}{(u+1)^{N-\nu}}\mathbb{S}_a\{f\}(u).
\end{aligned} \tag{4.23}$$

□

Theorem 4.10. *Suppose $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$ and let $\nu > 0$ with $N-1 < \nu \leq N$. Then for all $u \in \mathbb{C} \setminus \{-1, 0\}$ such that $|(u+1)/u| > r$,*

$$\mathbb{S}_{a+N-\nu}\{\Delta_a^{\nu} f\}(u) = \frac{(u+1)^{N-\nu}}{u^N}\mathbb{S}_a\{f\}(u) - \sum_{k=0}^{N-1} u^{k-N}\Delta_a^{\nu-N+k} f(a+N-\nu). \tag{4.24}$$

Proof. Let f, r, ν, and N be as in the statement of the theorem. We already know from Theorem 3.8 in [28] that (4.24) holds when $\nu = N$, that is,

$$\mathbb{S}_a\left\{\Delta^N f\right\}(u) = \frac{1}{u^N}\mathbb{S}_a\{f\}(u) - \sum_{k=0}^{N-1} u^{k-N}\Delta^k f(a). \tag{4.25}$$

If $N-1<\nu<N$, then $0<N-\nu<1$ and hence it follows from (3.6), (4.19), and (4.25) that

$$\begin{aligned}
\mathbb{S}_{a+N-\nu}\{\Delta_a^{\nu}f\}(u) &= \mathbb{S}_{a+N-\nu}\left\{\Delta^N\Delta_a^{-(N-\nu)}f\right\}(u)\\
&= \frac{1}{u^N}\mathbb{S}_{a+N-\nu}\left\{\Delta_a^{-(N-\nu)}f\right\}(u)-\sum_{k=0}^{N-1}u^{k-N}\Delta^k\Delta_a^{-(N-\nu)}f(a+N-\nu)\\
&= \frac{(u+1)^{N-\nu}}{u^N}\mathbb{S}_a\{f\}(u)-\sum_{k=0}^{N-1}u^{k-N}\Delta_a^{\nu-N+k}f(a+N-\nu).
\end{aligned} \tag{4.26}$$

□

In the following theorem the Sumudu transform of the Caputo fractional difference operator is presented.

Theorem 4.11. *Suppose $f:\mathbb{N}_a\to\mathbb{R}$ is of exponential order $r\geq 1$ and let $\nu>0$ with $N-1<\nu\leq N$. Then for all $u\in\mathbb{C}\setminus\{-1,0\}$ such that $|(u+1)/u|>r$,*

$$\mathbb{S}_{a+N-\nu}\left\{{}^C\Delta_a^{\nu}f\right\}(u)=\frac{(u+1)^{N-\nu}}{u^N}\left[\mathbb{S}_a\{f\}(u)-\sum_{k=0}^{N-1}u^k\Delta^kf(a)\right]. \tag{4.27}$$

Proof. Let f,r,ν, and N be as in the statement of the theorem. We already know from (4.25) that $v=N$, (4.27) holds. If $N-1<\nu<N$, then $0<N-\nu<1$ and hence it follows from (4.19) and (4.25) that

$$\begin{aligned}
\mathbb{S}_{a+N-\nu}\left\{{}^C\Delta_a^{\nu}f\right\}(u) &= \mathbb{S}_{a+N-\nu}\left\{\Delta_a^{-(N-\nu)}\Delta^Nf\right\}(u)\\
&= (u+1)^{N-\nu}\mathbb{S}_a\left\{\Delta^Nf\right\}(u)\\
&= \frac{(u+1)^{N-\nu}}{u^N}\left[\mathbb{S}_a\{f\}(u)-\sum_{k=0}^{N-1}u^k\Delta^kf(a)\right].
\end{aligned} \tag{4.28}$$

□

Lemma 4.12. *Let $f:\mathbb{N}_a\to\mathbb{R}$ be given. For any $p\in\mathbb{N}_0$ and $\nu>0$ with $N-1<\nu\leq N$, one has*

$${}^C\Delta_a^{\nu+p}f(t)={}^C\Delta_a^{\nu}\Delta^pf(t)\qquad for\ t\in\mathbb{N}_{a+N-v}. \tag{4.29}$$

Proof. Let f, v, N, and p be given as in the statement of the lemma. Then

$$\begin{aligned}
{}^C\Delta_a^{\nu+p}f(t) &= \Delta_a^{-(N+p-\nu-p)}\Delta^{N+p}f(t)\\
&= \Delta_a^{-(N-\nu)}\Delta^N\Delta^pf(t)\\
&= {}^C\Delta_a^{\nu}\Delta^pf(t).
\end{aligned} \tag{4.30}$$

□

Corollary 4.13. *Suppose $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$, $\nu > 0$ with $N-1 < \nu \leq N$ and $p \in \mathbb{N}_0$. Then for all $u \in \mathbb{C} \setminus \{-1, 0\}$ such that $|(u+1)/u| > r$,*

$$\mathbb{S}_{a+N-\nu}\left\{{}^{C}\Delta_a^{\nu+p} f\right\}(u) = \frac{(u+1)^{N-\nu}}{u^{N+p}}\left[\mathbb{S}_a\{f\}(u) - \sum_{k=0}^{N+p-1} u^k \Delta^k f(a)\right]. \tag{4.31}$$

Proof. The proof follows from (4.25), (4.27), and (4.29). □

5. Applications

In this section, we will illustrate the possible use of the discrete Sumudu transform by applying it to solve some initial value problems. The following initial value problem was solved in Theorem 23 in [27] by using the Laplace transforms.

Example 5.1. Suppose $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$ and let $\nu > 0$ with $N-1 < \nu \leq N$. The unique solution to the fractional initial value problem

$$\begin{gathered} \Delta_{a+\nu-N}^{\nu} y(t) = f(t), \quad t \in \mathbb{N}_a \\ \Delta^k y(a+\nu-N) = A_k, \quad k \in \{0, 1, \ldots, N-1\}, \ A_k \in \mathbb{R} \end{gathered} \tag{5.1}$$

is given by

$$y(t) = \sum_{k=0}^{N-1} \alpha_k (t-a)^{\underline{\nu+k-N}} + \Delta_a^{-\nu} f(t), \quad t \in \mathbb{N}_{a+\nu-N}, \tag{5.2}$$

where

$$\alpha_k = \frac{\Delta_{a+\nu-N}^{\nu-N+k} y(a)}{\Gamma(\nu+k-N+1)} = \sum_{p=0}^{k}\sum_{j=0}^{k-p} \frac{(-1)^j}{k!}(k-j)^{\underline{N-\nu}}\binom{k}{p}\binom{k-p}{j} A_p \tag{5.3}$$

for $k \in \{0, 1, \ldots, N-1\}$.

Proof. Since f is of exponential order r, then $\mathbb{S}_a\{f\}(u)$ exists for all $u \in \mathbb{C} \setminus \{-1, 0\}$ such that $|(u+1)/u| > r$. So, applying the Sumudu transform to both sides of the fractional difference equation in (5.1), we have for all $u \in \mathbb{C} \setminus \{-1, 0\}$ such that $|(u+1)/u| > r$,

$$\mathbb{S}_a\left\{\Delta_{a+\nu-N}^{\nu} y\right\}(u) = \mathbb{S}_a\{f\}(u). \tag{5.4}$$

Then from (4.24), it follows

$$\frac{(u+1)^{N-\nu}}{u^N}\mathbb{S}_{a+\nu-N}\{y\}(u) - \sum_{k=0}^{N-1} u^{k-N} \Delta_{a+\nu-N}^{\nu-N+k} y(a) = \mathbb{S}_a\{f\}(u) \tag{5.5}$$

and hence

$$\mathbb{S}_{a+\nu-N}\{y\}(u)=\frac{u^N}{(u+1)^{N-\nu}}\mathbb{S}_a\{f\}(u)+\sum_{k=0}^{N-1}\frac{u^k}{(u+1)^{N-\nu}}\Delta_{a+\nu-N}^{\nu-N+k}y(a). \tag{5.6}$$

By (4.20), we have

$$\frac{u^N}{(u+1)^{N-\nu}}\mathbb{S}_a\{f\}(u)=\mathbb{S}_{a+\nu-N}\{\Delta_a^{-\nu}f\}(u). \tag{5.7}$$

Considering the terms in the summation, by using the shifting formula (4.5), we see that for each $k\in\{0,1,\ldots,N-1\}$,

$$\begin{aligned}
&\frac{u^k}{(u+1)^{N-\nu}}\\
&\quad=\left(\frac{u}{u+1}\right)^k(u+1)^{\nu+k-N}\\
&\quad=\left(\frac{u}{u+1}\right)^k\mathbb{S}_{a+\nu+k-N}\{h_{\nu+k-N}(\cdot,a)\}(u)\\
&\quad=\mathbb{S}_{a+\nu-N}\{h_{\nu+k-N}(\cdot,a)\}(u)-\frac{1}{u}\sum_{i=0}^{k-1}\left(\frac{u}{u+1}\right)^{i+1}h_{\nu+k-N}(i+a+\nu-N,a)\\
&\quad=\mathbb{S}_{a+\nu-N}\{h_{\nu+k-N}(\cdot,a)\}(u)
\end{aligned} \tag{5.8}$$

since

$$\begin{aligned}
h_{\nu+k-N}(i+a+\nu-N,a)&=\frac{(i+\nu-N)^{\underline{\nu+k-N}}}{\Gamma(\nu+k-N+1)}\\
&=\frac{\Gamma(i+\nu-N+1)}{\Gamma(i-k+1)\Gamma(\nu+k-N+1)}\\
&=0
\end{aligned} \tag{5.9}$$

for $i\in\{0,\ldots k-1\}$.

Consequently, we have

$$\begin{aligned}
\mathbb{S}_{a+\nu-N}\{y\}(u)&=\mathbb{S}_{a+\nu-N}\{\Delta_a^{-\nu}f\}(u)+\sum_{k=0}^{N-1}\Delta_{a+\nu-N}^{\nu-N+k}y(a)\mathbb{S}_{a+\nu-N}\{h_{\nu+k-N}(\cdot,a)\}(u)\\
&=\mathbb{S}_{a+\nu-N}\left\{\sum_{k=0}^{N-1}\Delta_{a+\nu-N}^{\nu-N+k}y(a)h_{\nu+k-N}(\cdot,a)+\Delta_a^{-\nu}f\right\}(u).
\end{aligned} \tag{5.10}$$

Since Sumudu transform is a one-to-one operator (see [28, Theorem 3.6]), we conclude that for $t \in \mathbb{N}_{a+\nu-N}$,

$$\begin{aligned} y(t) &= \sum_{k=0}^{N-1} \Delta_{a+\nu-N}^{\nu-N+k} y(a) h_{\nu+k-N}(t,a) + \Delta_a^{-\nu} f(t) \\ &= \sum_{k=0}^{N-1} \left(\frac{\Delta_{a+\nu-N}^{\nu-N+k} y(a)}{\Gamma(\nu+k-N+1)} \right) (t-a)^{\underline{\nu+k-N}} + \Delta_a^{-\nu} f(t), \end{aligned} \tag{5.11}$$

where

$$\frac{\Delta_{a+\nu-N}^{\nu-N+k} y(a)}{\Gamma(\nu+k-N+1)} = \sum_{p=0}^{k} \sum_{j=0}^{k-p} \frac{(-1)^j}{k!} (k-j)^{\underline{N-\nu}} \binom{k}{p} \binom{k-p}{j} \Delta^k y(a+\nu-N), \tag{5.12}$$

(see [27, Theorem 11]). □

Example 5.2. Consider the initial value problem (5.1) with the Riemann-Liouville fractional difference replaced by the Caputo fractional difference.

$$\begin{aligned} {}^{C}\Delta_{a+\nu-N}^{\nu} y(t) &= f(t), \quad t \in \mathbb{N}_a, \\ \Delta^k y(a+\nu-N) &= A_k, \quad k \in \{0, 1, \ldots, N-1\}, \ A_k \in \mathbb{R}. \end{aligned} \tag{5.13}$$

Applying the Sumudu transform to both sides of the difference equation, we get for all $u \in \mathbb{C} \setminus \{-1, 0\}$ such that $|(u+1)/u| > r$,

$$\mathbb{S}_a \left\{ {}^{C}\Delta_{a+\nu-N}^{\nu} y \right\}(u) = \mathbb{S}_a \{f\}(u). \tag{5.14}$$

Then from (4.27), it follows

$$\frac{(u+1)^{N-\nu}}{u^N} \left[\mathbb{S}_{a+\nu-N} \{y\}(u) - \sum_{k=0}^{N-1} u^k A_k \right] = \mathbb{S}_a \{f\}(u). \tag{5.15}$$

By (4.20), we have

$$\begin{aligned} \mathbb{S}_{a+\nu-N} \{y\}(u) &= \sum_{k=0}^{N-1} u^k A_k + \frac{u^N}{(u+1)^{N-\nu}} \mathbb{S}_a \{f\}(u) \\ &= \sum_{k=0}^{N-1} u^k A_k + \mathbb{S}_{a+\nu-N} \left\{ \Delta_a^{-\nu} f \right\}(u). \end{aligned} \tag{5.16}$$

Since from [28], we have

$$\mathbb{S}_0 \{t^{\underline{n}}\}(u) = n! u^n, \quad n \in \mathbb{N}_0, \tag{5.17}$$

hence

$$y(t) = \sum_{k=0}^{N-1} A_k \frac{(t-a-\nu+N)^{\underline{k}}}{k!} + \Delta_a^{-\nu} f(t). \tag{5.18}$$

Remark 5.3. The initial value problem (5.1) can also be solved by using Proposition 15 in [12].

Example 5.4. Consider the initial value problem

$$\begin{gathered} {}^C\Delta_{a+\nu-1}^{\nu+1} y(t) - {}^C\Delta_{a+\nu-1}^{\nu} y(t) = 0, \quad t \in \mathbb{N}_a, \\ \Delta^k y(a+v-N) = A_k, \quad k \in \{0.1\}, \ A_k \in \mathbb{R}, \end{gathered} \tag{5.19}$$

where $0 < \nu \leq 1$. Applying the Sumudu transform to both sides of the equation and using (4.31) and (4.27), respectively, we get

$$\frac{(u+1)^{1-\nu}}{u^2} \left[\mathbb{S}_{a+\nu-1}\{y\}(u) - A_0 - uA_1\right] - \frac{(u+1)^{1-\nu}}{u} \left[\mathbb{S}_{a+\nu-1}\{y\}(u) - A_0\right] = 0. \tag{5.20}$$

Hence we get

$$\mathbb{S}_{a+\nu-1}\{y\}(u) = (A_0 - A_1) + \frac{A_1}{1-u}. \tag{5.21}$$

Since from [28], we have

$$\mathbb{S}_0\left\{(1+\lambda)^t\right\}(u) = \frac{1}{1-\lambda u} \quad \text{for} \left|\frac{(1+\lambda)u}{u+1}\right| < 1, \tag{5.22}$$

then

$$y(t) = (A_0 - A_1) + A_1 2^{t-a-\nu+1}. \tag{5.23}$$

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