Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 370946, 18 pages
doi:10.1155/2012/370946

Research Article

A Note on Coincidence Degree Theory

Ali Sırma and Sebaheddin Ṣevgin

Department of Mathematics, Yüzüncü Yıl University, 65080 Van, Turkey

Correspondence should be addressed to Ali Sırma, alisirma01@gmail.com

Received 13 March 2012; Accepted 17 June 2012

Academic Editor: Yong Zhou

The background of definition of coincidence degree is explained, and some of its basic properties are given.

1. Introduction

Gaines and Mawhin introduced coincidence degree theory in 1970s in analyzing functional and differential equations [1, 2]. Mawhin has continued studies on this theory later on and has made so important contributions on this subject since then this theory is also known as Mahwin's coincidence degree theory. Coincidence theory is very powerful technique especially in existence of solutions problems in nonlinear equations. It has especially so broad applications in the existence of periodic solutions of nonlinear differential equations so that many researchers have used it for their investigations (see [3–32] and references therein). The main goal in the coincidence degree theory is to search the existence of a solutions of the operator equation

$$Lx = Nx \tag{1.1}$$

in some bounded and open set Ω in some Banach space for L being a linear operator and N nonlinear operator using Leray-Schauder degree theory. As it is known that, in finite dimensional case, for $\Omega \subset \mathbb{R}^n$, $f \in C(\overline{\Omega})$, and $p \in \mathbb{R}^n \setminus f(\partial\Omega)$, the degree of f on Ω with respect to p, $d(f,\Omega,p)$ is well defined. But unfortunately this is not the case in infinite dimension for $f \in C(\overline{\Omega})$ (see [33], page 172). Luckily, in an arbitrary Banach space X, Leray and Schauder proved that for $\Omega \in X$ open, bounded set, $M : \overline{\Omega} \to X$ compact operator and for $p \in X \setminus (I-M)(\partial\Omega)$ the degree of compact perturbation of identity $I-M$ in Ω with respect to p, $d(I-M,\Omega,p)$ is well defined [34]. One of the main useful properties of degree theory is that if $d(I-M,\Omega,p)\neq 0$ then $(I-M)x=p$ has at least one solution in Ω. In particular if we take $p=0$ and $d(I-M,\Omega,p)\neq 0$ then the compact operator M has at least one fixed point in Ω. In [1], Gaines and Mawhin studied existence of a solution of an operator equation (1.1) defined on a Banach space X in an open bounded set Ω using the Leray-Schauder degree theory. But since the operator $I-(L-N)$ is not compact in general the need to define a compact operator M such that its set of fixed points in Ω would be equal to a solution set of (1.1) in Ω aroused. In [1], the compact operator M is given and the coincidence degree for the couple (L,N) in Ω is defined by $d[(L,N),\Omega]=d(I-M,\Omega,0)$.

The aim of this paper is to make an effort to understand the theoretical background of the definition of coincidence degree which has similar properties with the Leray-Schauder degree for an operator couple (L,N) satisfying some special conditions, to analyze the dependence of coincidence degree to the components of the compact operator M and in this way to prepare good resource for one who wants to study and to improve the coincidence degree theory.

The paper is basically prepared using [1]. In this study, we tried to explain the theory that was given densely in [1]. Besides we give proofs of some results that their proofs not given in [1]. Namely, we give proofs of Lemmas 2.1, 2.2, and 3.19 and Theorems 3.3, 4.1, and 4.2. We state and prove Lemma 3.17 which is essential for Proposition 3.18. In Proposition 3.6 we show that the operator Π_Q is an isomorphism and explain important details, and, in Proposition 3.20, we show that A is an automorphism and make necessary explanations. Also in each proof we tried to make important contributions to make the proofs much more understandable and so that it can be improved by interested researchers.

In summary, in Section 2, some preliminaries which are used in the definition of coincidence degree are used. In Section 3, definition of coincidence degree for some linear perturbations of Fredholm mappings on normed spaces is given. In Section 4, some basic properties of coincidence degree are given.

2. Algebraic Preliminaries

In this section, we will give some facts that will be used throughout the paper.

Let X and Z be two vector spaces, the domain of operator L, Dom L is a linear subspace of X, and $L:\operatorname{Dom} L\to Z$ is a linear operator. Assume that the operators $P:X\to X$ and $Q:Z\to Z$ linear projection operators such that the chain

$$X \xrightarrow{P} \operatorname{Dom} L \xrightarrow{L} Z \xrightarrow{Q} Z \tag{2.1}$$

is exact, that is, $\operatorname{Im} P=\ker L$ and $\operatorname{Im} L=\ker Q$. Let us define the restriction of L to $\operatorname{Dom} L\cap\ker P$ as $L_P:\operatorname{Dom} L\cap\ker P\to\operatorname{Im} L$.

Now, let us give the following lemma about L_P.

Lemma 2.1. *L_P is an algebraic isomorphism.*

Proof. Firstly, let us show that L_P is one-to-one mapping. For this let us take $x\in\ker L_P\subset\ker L=\operatorname{Im} P$, so that there exists $y\in\operatorname{Dom} P$ such that $x=Py$. Since P is a projection operator we get $x=Py=P^2y=P(Py)=Px=0$. Therefore, $x=0$, so that we obtain that $\ker L_P=\{0\}$. This means that L_P is one-to-one.

Now let us show that L_P is onto. Since $P : X \to X$ is a projection operator, we can write the vector space X as direct sums $X = \ker P \oplus \operatorname{Im} P$. From the exactness of the chain above, we get $X = \ker P \oplus \ker L$. Take $z \in \operatorname{Im} L$, so that there exists $x \in \operatorname{Dom} L \subset X$ with $Lx = z$. Since $X = \ker P \oplus \ker L$, there exists unique elements $e \in \ker P$ and $f \in \ker L$ such that we can write $x = e + f$. From here, we can obtain $z = Lx = L(e + f) = Le + Lf = Le + 0 = Le$. This means that $e \in \operatorname{Dom} L$. So we get $e \in \operatorname{Dom} L$ and $e \in \ker P$ and $L_P e = z$. So the result follows. □

Now, let us define $K_P := L_P^{-1}$. It is clear that $K_P : \operatorname{Im} L \subset Z \to \operatorname{Dom} L \cap \ker P$ is one-to-one, onto, and $PK_P = 0$.

Lemma 2.2. *(1) On* $\operatorname{Im} L$*, we have* $LK_P = I$*. (2) On* $\operatorname{Dom} L$*, we have* $K_P L = I - P$*.*

Proof. (1) Take $x \in \operatorname{Im} L$. Therefore, $LK_P x = L(K_P(x)) = L_P(K_P(x)) = Ix$.

(2) Since $\operatorname{Im} P = \ker L$, then we have $LP = 0$, so we obtain $K_P L = K_P L(I - P)$. So that in order to prove (2), we need to show the equality $K_P L(I - P) = K_P L_P(I - P)$. If we can have $\operatorname{Im}(I - P) \subseteq \operatorname{Dom}(L_P) = \operatorname{Dom} L \cap \ker P$, then the result follows. Take $x \in \operatorname{Dom} L$. Since $P(x) \in \ker L \subset \operatorname{Dom} L$ and $\operatorname{Dom} L$ is a vector subspace of X, we have $(x - Px) \in \operatorname{Dom} L$.

Since $P(x - Px) = Px - P^2 x = Px - Px = 0$, then $(x - Px) \in \ker P$; therefore, we have $(x - Px) \in \operatorname{Dom} L \cap \ker P$. From here, we obtain $\operatorname{Im}(I - P) \subset \operatorname{Dom} L \cap \ker P$. So using (1), the result $K_P L(I - P) = K_P L_P(I - P) = I - P$ follows. □

Now, let us define the canonic surjection Π as

$$\begin{aligned} \Pi : Z &\longrightarrow \operatorname{Coker} L \\ z &\longmapsto z + \operatorname{Im} L. \end{aligned} \tag{2.2}$$

Here, $\operatorname{Coker} L = Z/\operatorname{Im} L$ is the quotient space of Z under the equivalence relation $z \sim z' \Leftrightarrow z - z' \in \operatorname{Im} L$. Thus, $\operatorname{Coker} L = \{\overline{z} = z + \operatorname{Im} L : z \in Z\}$. It is clear that the canonic surjection operator Π is linear and $\ker \Pi = \ker Q$.

Proposition 2.3. *If there exists an one-to-one operator* $\Lambda : \operatorname{Coker} L \to \ker L$*, then*

$$Lx = y, \quad y \in Z \tag{2.3}$$

will be equivalent to

$$(I - P)x = (\Lambda\Pi + K_{P,Q})y. \tag{2.4}$$

Here, the operator $K_{P,Q} : Z \to X$ *is defined as* $K_{P,Q} = K_P(I - Q)$*.*

Proof. Since $\operatorname{Im} L = \ker Q = \ker \Pi$, then for $y \in \operatorname{Im} L$ we have $Qy = 0$ and $\Lambda\Pi y = \Lambda \operatorname{Im} L = \Lambda\overline{0} = 0$. From here, it is seen that

$$\begin{aligned} Lx = y &\Longleftrightarrow Lx = y - Qy \Longleftrightarrow K_P Lx = K_P(y - Qy) \\ &\Longleftrightarrow (I - P)x = K_P(I - Q)y \Longleftrightarrow (I - P)x = (\Lambda\Pi + K_P(I - Q))y. \end{aligned} \tag{2.5}$$

□

Now, let us consider another projection operator couple (P', Q') that will make the chain

$$X \xrightarrow{P'} \text{Dom } L \xrightarrow{L} Z \xrightarrow{Q'} Z \tag{2.6}$$

exact, and let us search the relation of this operator couple with (P, Q).

From Lemma 2.2, since $LK_P = I$, and $LK_{P'} = I$ then we have $L(K_P - K_{P'}) = 0$. So for any $z \in \operatorname{Im} L$, we have $(K_P - K_{P'})z \in \ker L$. Therefore, we can write $K_P - K_{P'} : \operatorname{Im} L \to \ker L = \operatorname{Im} P = \operatorname{Im} P'$. Since the projection operator P behaves on $\operatorname{Im} P$ as an identity operator, we have $K_P - K_{P'} = P(K_P - K_{P'})$. As a result, the equality

$$K_P - K_{P'} = P(K_P - K_{P'}) = P'(K_P - K_{P'}) \tag{2.7}$$

follows.

Lemma 2.4. *The following relations hold.*

(i) $PK_{P'} + P'K_P = 0$, (ii) $K_{P'} = (I - P')K_P$.

Proof. (i) Using $PK_P = 0$, $P'K_{P'} = 0$, and (2.7), the result

$$\begin{gathered} P\left(K_p - K_{P'}\right) = P'(K_P - K_{P'}) \\ -PK_{P'} = P'K_P \end{gathered} \tag{2.8}$$

follows.

(ii) Again using (2.7) and (i), we obtain

$$\begin{gathered} K_P - K_{P'} = P'(K_P - K_{P'}) \\ \Longrightarrow K_P - K_{P'} = P'K_P - P'K_{P'} \\ \Longrightarrow K_P = P'K_P + (-PK_{P'}) \\ \Longrightarrow K_P = (I - PK_{P'}). \end{gathered} \tag{2.9}$$

In a similar manner, the equality $K_{P'} = (I - P'K_P)$ can be obtained. □

3. Definition of Coincidence Degree for Some Linear Perturbations of Fredholm Mappings on Normed Spaces

In this section, definition of coincidence degree for some linear perturbations of Fredholm mappings on normed spaces is given.

Let X and Z be two real norm spaces, $\Omega \subset X$ an open, bounded subset of X and $\overline{\Omega}$ an closure of Ω. Let us assume that the operators

$$L : \text{Dom } L \subset X \longrightarrow Z, \qquad N : \overline{\Omega} \subset X \longrightarrow Z \tag{3.1}$$

satisfy the following conditions:

(i) L is linear and $\operatorname{Im} L$ is an closed subset of Z,

(ii) $\ker L$ and $\operatorname{Coker} L = Z/\operatorname{Im} L$ are finite dimensional spaces and $\dim\ker L = \dim\operatorname{Coker} L$,

(iii) the operator $N : \overline{\Omega} \subset X \to Z$ is continuous and $\Pi N(\Omega)$ is bounded,

(iv) the operator $K_{P,Q}N : \overline{\Omega} \to Z$ is compact on $\overline{\Omega}$.

Definition 3.1. The operator L which satisfies the conditions (i) and (ii) will be called as Fredholm operator of index zero.

Definition 3.2. The operator $N : \overline{\Omega} \to Z$ which satisfies the conditions (iii) and (iv) will be called L-compact operator.

It is clear that if we take $X = Z$ and $L = I$ the operator Π reduced to zero operator and the operator $K_{P,Q}$ turns to an identity operator then L-compactness of N on $\overline{\Omega}$ reduced to usual compactness for operators.

Theorem 3.3. *Let Z be a Banach space. If the operator L is a Fredholm operator of index zero then there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that the chain*

$$X \xrightarrow{P} \operatorname{Dom} L \xrightarrow{L} Z \xrightarrow{Q} Z \tag{3.2}$$

will be exact.

Proof. Assume that $\ker L$ is finite dimensional, and the set $\{y_1, y_2, \ldots, y_n\}$ is a basis for $\ker L$. Define the vector subspaces as $X_k = \operatorname{span}\{y_1, y_2, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n\}$. Since X_k is finite dimensional so is a closed subspace of X (see [35, Theorem 2.4-3] and $y_k \notin X_k$. Let B be a basis of X such that $\{y_1, y_2, \ldots, y_n\} \subseteq B$. Now, let us define the linear operators which satisfy the conditions

$$\begin{gathered} F_k : X \longrightarrow \mathbb{R}, \\ F_k(y) = \begin{cases} 1, & \text{if } y = y_k \\ 0, & \text{if } y \in B - \{y_k\}. \end{cases} \end{gathered} \tag{3.3}$$

Therefore, the operator P defined by

$$Px = \sum_{k=1}^{n} F_k(x) y_k \tag{3.4}$$

is a continuous projection operator (see [36, Remark 2.1.19]).

Now, let us prove the existence of continuous projection operators Q on Z with $\ker Q = \operatorname{Im} L$. We know that there exists a subspace $\widetilde{Z}$ such that $Z = \operatorname{Im} L \oplus \widetilde{Z}$ (see [37, Proposition I]). The projection operator Q defined on Z with the rule

$$Q(z) = Q\left(z_{\ker Q} + z_{\widetilde{Z}}\right) = z_{\widetilde{Z}} \tag{3.5}$$

satisfies the relations $\ker Q = \operatorname{Im} L$ and $\operatorname{Im} Q = \widetilde{Z}$. Since dimCoker L is finite dimensional so is $\operatorname{Im} Q$, therefore it is closed in Z. Since Z is a Banach space, $\ker Q$ and $\operatorname{Im} Q$ are closed subsets of Z, therefore the projection operator Q is continuous (see [38, Theorem 6.12.6]). □

Moreover, the canonical surjection $\Pi : Z \to \operatorname{Coker} L$ is continuous with the quotient topology on $\operatorname{Coker} L$. Now, let us state two theorems that will be used in the proof of following proposition.

Theorem 3.4 (see [35]). *Assume X and Y are normed spaces and the operator $T : X \to Y$ is linear. Therefore,*

(a) *If T is bounded and $\dim(\operatorname{Im} T) < \infty$ then T is compact.*

(b) *If $\dim X < \infty$ then T is continuous and compact.*

Theorem 3.5 (see [35], Lemma 8.3-2). *Let X be normed space, $T : X \to X$ be a linear compact operator, and $S : X \to X$ a linear bounded (continuous) operator. So the operators TS and ST are also compact.*

The following proposition states that the condition (iv) does not depend on the choice of the projection operators P and Q.

Proposition 3.6. *Assume that the conditions (i), (ii), (iii) are all satisfied. If the condition (iv) is satisfied for the projection operator couple (P, Q) that makes the chain exact then for any projection operator couple (P', Q') that makes the chain exact is satisfied.*

Proof. Let us denote the restriction of Π to $\operatorname{Im} Q$ with Π_Q, and let us show that the linear operator $\Pi_Q : \operatorname{Im} Q \to Z/ImL$ is one-to-one and onto. Since $\ker(\Pi_Q) \in \operatorname{Im} L = \ker Q$ and $\ker Q \cap \operatorname{Im} Q = \{0\}$, then we have $\ker(\Pi_Q) = \{0\}$. Therefore Π_Q is one-to-one. To show surjection, let us take an arbitrary element $\overline{z} \in Z/\operatorname{Im} L$. So there exists $z \in Z$ such that $\Pi z = \overline{z}$ holds. Since the space Z can be written as $Z = \operatorname{Im} Q \oplus \ker Q = \operatorname{Im} Q \oplus \operatorname{Im} L$ there exist unique elements $z_{\operatorname{Im} Q} \in \operatorname{Im} Q$ and $z_{\operatorname{Im} L} \in \operatorname{Im} L$ such that the relation $z = z_{\operatorname{Im} Q} + z_{\operatorname{Im} L}$ is satisfied. Since we have

$$\overline{z} = \Pi\left(z_{\operatorname{Im} Q} + z_{\operatorname{Im} L}\right) = \Pi\left(z_{\operatorname{Im} Q}\right) + \Pi\left(z_{\operatorname{Im} L}\right) = \Pi\left(z_{\operatorname{Im} Q}\right), \tag{3.6}$$

then the surjectivity of Π_Q follows.

Since we have $\dim(\operatorname{Im} Q) \leq \dim(\operatorname{Im} \Pi_Q) \leq \dim(\operatorname{Coker} L) = n < \infty$, then $\operatorname{Im} Q$ is a finite dimensional linear subspace of Z. Similarly, $\operatorname{Im} Q'$ is also a finite dimensional subspace of Z. Therefore, since we have $\dim(\operatorname{Im}(Q - Q')) \leq \dim(\operatorname{Im} Q) + \dim(\operatorname{Im} Q')$, then $\operatorname{Im}(Q - Q')$ is also a finite dimensional subspace of Z.

Now, let us show that for an arbitrary $\alpha \in Z$ the relation $\Pi_Q^{-1}\Pi(\alpha) = Q(\alpha)$ holds. Since we can write $\alpha = Q\alpha + (I - Q)\alpha)$, then

$$\begin{aligned}\Pi_Q^{-1}\Pi(\alpha) &= \Pi_Q^{-1}\Pi((Q\alpha + (I - Q)\alpha)) = \Pi_Q^{-1}\Pi(Q\alpha) + \Pi_Q^{-1}\Pi((I - Q)\alpha)\\ &= \Pi_Q^{-1}\Pi(Q\alpha) = \Pi_Q^{-1}\Pi_Q(Q\alpha) = Q\alpha\end{aligned} \tag{3.7}$$

is obtained.

Let $\widetilde{K}_P$ denote the restriction of the operator K_P to the finite dimensional space $\operatorname{Im}(Q-Q')$. Using the results obtained until here in this proof and using the equality $K_{P'} = (I-P')K_P$,

$$\begin{aligned}K_{P',Q'}N &= K_{P'}(I - Q')N = (I - P')K_P(I - Q')N\\ &= (I - P')K_P(I - Q + Q + Q')N\\ &= (I - P')K_P(I - Q)N + (I - P')K_P(Q - Q')N\\ &= (I - P')K_{P,Q}N + (I - P')K_P(Q - Q')N\\ &= (I - P')K_{P,Q}N + (I - P')\widetilde{K}_P(Q - Q')N\\ &= (I - P')K_{P,Q}N + (I - P')\widetilde{K}_P\left(\Pi_Q^{-1}\Pi - \Pi_{Q'}^{-1}\Pi\right)N\\ &= (I - P')K_{P,Q}N + (I - P')\widetilde{K}_P\left(\Pi_Q^{-1} - \Pi_{Q'}^{-1}\right)\Pi N\end{aligned} \tag{3.8}$$

is achieved. Now, let us explain the operator $K_{P',Q'}N$ is compact. Since the operator $K_{P,Q}N$ is compact and $I - P'$ is continuous, then the operator $(I - P')K_{P,Q}N$ is compact. Since $\dim(\operatorname{Coker} L) = n < \infty$, then the operator $\Pi_Q^{-1} : \operatorname{Coker} L \to \operatorname{Im} Q$ is compact. From the same reason, the operator $\Pi_{Q'}^{-1}$ is also compact. Since the operators N, Π, and $I - P'$ are all continuous, the compactness of $K_{P',Q'}N$ follows. □

Proposition 3.7. *The element $x \in \operatorname{Dom} L \cap \overline{\Omega}$ is a solution of the operator equation* (1.1) *if and only if it satisfies*

$$(I - P)x = \left(\Lambda\Pi + K_{P,Q}\right)Nx. \tag{3.9}$$

In other words, the set of solutions of (1.1) *is equal to the set of fixed points of the operator $M : \overline{\Omega} \to X$ defined by*

$$M = P + \left(\Lambda\Pi + K_{P,Q}\right)N. \tag{3.10}$$

Here, $\Lambda : \operatorname{Coker} L \to \ker L$ is any isomorphism.

Proof. Clear from Proposition 2.3. □

Remark 3.8. Note that since $\operatorname{Im} P = \ker L \subset \operatorname{Dom} L$, $\operatorname{Im}\Lambda = \ker L \subset \operatorname{Dom} L$, and $\operatorname{Im} K_{P,Q} \subset \operatorname{Dom} L \cap \ker P \subset \operatorname{Dom} L$, then by definition $M : \overline{\Omega} \to \operatorname{Dom} L$. That is, any fixed points of M, if they exist, should be in the set $\overline{\Omega} \cap \operatorname{Dom} L$. Therefore, if (1.1) has a solution in $\overline{\Omega}$, then the solution should be in the set $\overline{\Omega} \cap \operatorname{Dom} L$.

Proposition 3.9. *Assume that the conditions (i)–(iv) hold. Then, the operator M is compact on $\overline{\Omega}$.*

Proof. The projection operator P is bounded and $\operatorname{Im} P = \ker L$ then $\operatorname{Im} P$ is a finite dimensional therefore, from Theorem 3.4 (a), P is compact. By assumption (iv), $K_{P,Q}N$ is compact. Beside these the operator $\Lambda : \operatorname{Coker} L \to \ker L$ is linear isomorphism and $\dim(\operatorname{Coker} L) = n < \infty$, therefore Λ is compact. Since $\Lambda\Pi$ is continuous then $\Lambda\Pi N$ is compact. As a result, we obtained the compactness of the operator M on a set $\overline{\Omega}$. □

Let $\partial\Omega$ denote the boundary of a set Ω.

(v) If $0 \notin (L - N)(\operatorname{Dom} L \cap \partial\Omega)$, then the Leray-Schauder degree $d(I - M, \Omega, 0)$ is well defined [34], since this condition by Proposition 3.7 gives us $0 \notin (I - M)(\partial\Omega)$.

Now, let us search how much the degree depends upon the choice of the operators P, Q, and Λ. To show this, we will need the following definition and results.

Let Λ_L will be the set of all linear isomorphism from $\operatorname{Coker} L$ to $\ker L$.

Definition 3.10. If there exists a continuous $\underline{\Lambda}$,

$$\begin{gathered} \underline{\Lambda} : \operatorname{Coker} L \times [0,1] \longrightarrow \ker L \\ \underline{\Lambda}(\cdot, 0) = \Lambda, \qquad \underline{\Lambda}(\cdot, 1) = \Lambda' \end{gathered} \tag{3.11}$$

such that for any $\lambda \in [0,1]$ the operator $\underline{\Lambda}(\cdot, \Lambda) \in \Lambda_L$ then the operator $\Lambda, \Lambda' \in \Lambda_L$ is called homotopic in Λ_L.

Being homotopic is an equivalence relation in the set Λ_L. Therefore, this equivalence relation divides the set Λ_L into homotopy classes.

Proposition 3.11. *The operators Λ and Λ' are homotopic in Λ_L if and only if* $\det(\Lambda\Lambda') > 0$.

Proof. Assume that Λ and Λ' are homotopic in Λ_L. From the condition (ii) we know that we have $\dim\ker L = \dim\operatorname{Coker} L = n$. Let $\underline{\Lambda}$ be the operator defined in Definition 3.10, $[a_1, a_2, \ldots, a_n]$ and $[b_1, b_2, \ldots, b_n]$ be bases of the spaces $\operatorname{Coker} L$ and $\ker L$, respectively. If for any $\lambda \in [0,1]$, $\Delta(\lambda)$ denotes the determinant of the matrix corresponding to $\underline{\Lambda}(\cdot, \lambda)$ with respect to these bases, then, for any $\lambda \in [0,1]$, $\Delta(\lambda) \neq 0$ since $\underline{\Lambda}(\cdot, \lambda)$ is an isomorphism for any $\lambda \in [0,1]$. Beside this, since $\underline{\Lambda}$ is continuous, then Δ is also continuous with respect to λ. Using continuity and the fact that for any $\lambda \in [0,1]$, $\Delta(\lambda) \neq 0$ we have the number $\Delta(\lambda)$ is always positive or negative, that is it has always same sign. In particular $\Delta(0)$ and $\Delta(1)$ have the same signs, therefore we have

$$\det\left(\Lambda'\Lambda^{-1}\right) = \det(\Lambda')\det\left(\Lambda^{-1}\right) = \frac{\det(\Lambda')}{\det(\Lambda)} = \frac{\Delta(1)}{\Delta(0)} > 0. \tag{3.12}$$

Conversely assume that $\det(\Lambda'\Lambda^{-1}) > 0$. With respect to bases of $\operatorname{Coker} L$ and $\ker L$, let $\widetilde{\Lambda}$ and $\widetilde{\Lambda'}$ denote the matrix representations of the operators Λ and Λ', respectively. By assumption $\det(\widetilde{\Lambda})$ and $\det(\widetilde{\Lambda'})$ have the same sign. Therefore, they belong to same connected component of the topological group $\mathrm{GL}(n,r)$. Since $\mathrm{GL}(n,r)$ is locally arcwise connected then the corresponding component is also path connected. Therefore, there exists a continuous operator

$$\begin{gathered}\underline{\widetilde{\Lambda}} : \operatorname{Coker} L \times [0,1] \rightarrow \ker L \\ \underline{\widetilde{\Lambda}}(\cdot,0) = \widetilde{\Lambda}, \quad \underline{\widetilde{\Lambda}}(\cdot,1) = \widetilde{\Lambda'}.\end{gathered} \tag{3.13}$$

Therefore, for any $\lambda \in [0,1]$, if we take $\underline{\Lambda}(\cdot,\lambda)$ as a family of isomorphisms corresponding to continuous matrices defined from $\operatorname{Coker} L$ to $\ker L$, then the proof will be completed. □

Corollary 3.12. Λ_L *is separated into two homotopy classes.*

Therefore, the set of all isomorphisms $\Lambda : \operatorname{Coker} L \rightarrow \ker L$ with the same sign of determinant will be in the same classes. So one class will be with positive determinant and the other one will be with negative determinant.

Note the following: let $\Lambda : \operatorname{Coker} L \rightarrow \ker L$ be any isomorphism from the set Λ_L. The sign of determinant of the matrix corresponding to Λ depends upon not only the basis chosen for $\operatorname{Coker} L$ and $\ker L$ but also the order of the elements in these basis. If the operators Λ and Λ' are homotopic with respect to chosen bases for $\operatorname{Coker} L$ and $\ker L$, then they are homotopic with respect to any basis chosen for these spaces.

Now, let us fix an orientation on $\operatorname{Coker} L$ and $\ker L$, and let $[a_1, a_2, \ldots, a_n]$ be a basis for $\operatorname{Coker} L$ for the chosen orientation.

Definition 3.13. Let the operator $\Lambda : \operatorname{Coker} L \rightarrow \ker L$ be given. If $[\Lambda a_1, \Lambda a_2, \ldots, \Lambda a_n]$ has the same orientation with basis chosen in $\ker L$, then the operator Λ is said to be an orientation preserving transformation. Otherwise, it is said to be an orientation reversing transformation.

Proposition 3.14. *If* $\operatorname{Coker} L$ *and* $\ker L$ *are oriented, then the operators* Λ *and* Λ' *are homotopic in* Λ_L *if and only if they are both orientation preserving or both orientation reversing transformations.*

Proof. Assume that $[a_1, a_2, \ldots, a_n]$ and $[b_1, b_2, \ldots, b_n]$ are, respectively, bases of $\operatorname{Coker} L$ and $\ker L$ with respect to chosen the orientation. The basis $[\Lambda a_1, \Lambda a_2, \ldots, \Lambda a_n]$ on $\ker L$ has the same orientation with $[b_1, b_2, \ldots, b_n]$ if and only if the determinant of the matrix $S = (s_{ij})$ defined by

$$\Lambda a_j = \sum_{j=1}^{n} s_{ij} b_j \tag{3.14}$$

will be positive. Namely, let M_1 be the transition matrix from the basis $[a_1, a_2, \ldots, a_n]$ to the basis $[b_1, b_2, \ldots, b_n]$ and M_2 be the transition matrix from the basis $[a_1, a_2, \ldots, a_n]$ to the basis $[\Lambda a_1, \Lambda a_2, \ldots, \Lambda a_n]$,

$$\begin{gathered}[a_1, a_2, \ldots, a_n] \xrightarrow{M_1} [b_1, b_2, \ldots, b_n], \\ [a_1, a_2, \ldots, a_n] \xrightarrow{M_2} [\Lambda a_1, \Lambda a_2, \ldots, \Lambda a_n], \\ [\Lambda a_1, \Lambda a_2, \ldots, \Lambda a_n] \xrightarrow{S} [b_1, b_2, \ldots, b_n],\end{gathered} \tag{3.15}$$

then we have $M_1 = SM_2$ and $\det(M_1) = \det(S)\det(M_2)$. $[\Lambda a_1, \Lambda a_2, \dots, \Lambda a_n]$ has the same orientation with $[b_1, b_2, \dots, b_n]$ if and only if the determinants of the matrixes M_1 and M_2 have the same sign. This is only possible in the case the determinant of S is positive. Therefore, since the determinant of the matrixes M_1 and M_2 have the same sign, using the relation $\det(M_1) = \det(S)\det(M_2)$, we obtain that $\det(S) > 0$.

Let us assume that S' is a matrix related to a basis $[\Lambda' a_1,\ \Lambda' a_2, \dots, \Lambda' a_n]$. In this case if the matrix $G = (g_{ij})$ is the matrix represent the operator $\Lambda'\Lambda^{-1}$ with respect to basis $[b_1, b_2, \dots, b_n]$, then we have

$$\begin{gathered}\sum_{j=1}^{n} s_{ij} b_j = \Lambda' a_i = \Lambda'\left(\Lambda\Lambda^{-1}\right) a_i = \left(\Lambda'\Lambda^{-1}\right)\Lambda a_i, \\ \left(\Lambda'\Lambda^{-1}\right) \sum_{k=1}^{n} s_{ki} b_k = \sum_{k=1}^{n} s_{ki} \left(\Lambda'\Lambda^{-1}\right)(b_k), \\ \sum_{k=1}^{n} s_{ki} \sum_{j=1}^{n} g_{jk} b_j.\end{gathered} \tag{3.16}$$

Therefore, $(S')^T = S^T G^T$ and $S' = GS$ are obtained. Since $\det(S) > 0$ and $\det(S') > 0$, then $\det(G) > 0$, that is $\det(\Lambda'\Lambda^{-1}) > 0$. This means that Λ' and Λ^{-1} have the same orientation.

Conversely, if the operators Λ' and Λ^{-1} have the same orientation, then $\det(\Lambda'\Lambda^{-1}) > 0$. Therefore, from the Proposition 3.11, Λ' and Λ^{-1} are homotopic. □

Lemma 3.15. *Let Y be a vector space and $S, S' : Y \to Y$ be two projection operators with* $\operatorname{Im} S = \operatorname{Im} S' \neq 0$. *Therefore the operator S'' defined by $S'' = aS + bS'$, $a, b \in \mathbb{R}$, is a projection operator with the property* $\operatorname{Im} S'' = \operatorname{Im} S$ *if and only if* $a + b = 1$.

Proof. Let a and b are real numbers and assume that the operator S'' defined $S'' = aS + bS'$ is a projection operator with its image is equal to $\operatorname{Im} S = \operatorname{Im} S'$. Since for any $x \in \operatorname{Im} S$ we have $Sx = x$ and for any $y \in Y$, $S'y \in \operatorname{Im} S' = \operatorname{Im} S$ then, for any $y \in Y$ we have $SS'y = S(S'y) = S'y$. Therefore, we get the relation $SS' = S'$. In a similar manner, the equality $S'S = S$ can be shown. So

$$\begin{aligned} aS + bS' = S'' = (S'')^2 &= (aS + bS')(aS + bS') \\ &= a^2S^2 + abSS' + abS'S + b^2(S')^2 = a^2S + abSS' + abS'S + b^2S' \\ &= a^2S + abS' + abS + b^2S' = a(aS + bS') + b(aS + bS') \\ &= (a + b)(aS + bS') \end{aligned} \tag{3.17}$$

is obtained. From here, we get the result $(a + b - 1)(aS + bS') = 0$, that is $(a + b - 1)S'' = 0$. The assumption $\operatorname{Im} S'' \neq 0$ forces the fact that $a + b = 1$.

Conversely, if $a+b=1$, then

$$\begin{aligned} S'' &= \left(S''\right)^2 = \left(aS+bS'\right)\left(aS+bS'\right) \\ &= a^2S^2 + abSS' + abS'S + b^2\left(S'\right)^2 = a^2S + abSS' + abS'S + b^2S' \\ &= a^2S + abS' + abS + b^2S' = (a+b)\left(aS+bS'\right) = aS+bS' = S'' \end{aligned} \tag{3.18}$$

is obtained. Therefore, S'' is a projection operator. Since $\operatorname{Im} S = \operatorname{Im} S'$ is a vector space and $S'' = aS + bS'$, then we have $\operatorname{Im} S'' \subseteq \operatorname{Im} S$. Now, let us take an arbitrary element $x \in \operatorname{Im} S = \operatorname{Im} S'' \neq \{0\}$. Therefore,

$$S''x = aSx + bS'x = ax + bx = (a+b)x = x \tag{3.19}$$

and from here we obtain $x \in \operatorname{Im} S''$ and $\operatorname{Im} S \subseteq \operatorname{Im} S''$. So the result $\operatorname{Im} S = \operatorname{Im} S''$ follows. □

Lemma 3.16. *If P and P' are projection operators onto* $\ker L$, *$a+b=1$ and $P'' = aP + bP'$, then $K_{P''} = aK_P + bK_{P'}$.*

Proof. In the case $\ker L = \{0\}$, the proof is clear. Assume that $\operatorname{Im} P = \operatorname{Im} P' = \ker L \neq \{0\}$. Since $a+b=1$ by Lemma 3.15, P'' is a projection operator and $\operatorname{Im} P'' = \operatorname{Im} P$. Since $K_{P'} = (I-P')K_P$ and $PK_P = 0$, then the relation

$$\begin{aligned} K_{P''} &= \left(I - P''\right)K_P = \left(I - aP - bP'\right)K_P \\ &= K_P - aPK_P - bP'K_P = 1 \cdot K_P - bP'K_P \\ &= (a+b)K_P - bP'K_P = aK_P + bK_P - bP'K_P \\ &= aK_P + b\left(I-P'\right)K_P = aK_P + bK_{P'} \end{aligned} \tag{3.20}$$

is obtained. □

Lemma 3.17. *Let Z be a vector space, $S, S' : Z \to Z$ two projection operators with* $\ker S = \ker S'$, *then for $a, b \in \mathbb{R}$, $a+b=1$ the operator S'' defined by $S'' = aS + bS'$ is a projection operator with* $\ker S'' = \ker S$.

Proof. First of all, let us show that S'' is a projection operator. Since

$$\begin{aligned} &S : Z \longrightarrow Z \qquad &&S' : Z \longrightarrow Z \\ &Z = \ker S \oplus \operatorname{Im} S, \qquad &&Z = \ker S' \oplus \operatorname{Im} S', \end{aligned} \tag{3.21}$$

then for any $z \in Z$ there exist unique elements $z_0 \in \ker S$, $z_1 \in \operatorname{Im} S$, $z_0' \in \ker S'$, and $z_1' \in \operatorname{Im} S'$ such that $z = z_0 + z_1$ and $z = z_0' + z_1'$ hold. Therefore,

$$\begin{aligned}
(S'')^2(z) &= a^2 S(z) + abSS'(z) + abS'S(z) + b^2 S'(z) \\
&= a^2 S(z_0' + z_1') + abSS'(z_0' + z_1') + abS'S(z_0 + z_1) + b^2 S'(z_0 + z_1) \\
&= a^2 S(z_1') + abSS'(z_1') + abS'S(z_1) + b^2 S'(z_1) \\
&= a^2 S(z_1') + abS(z_1') + abS'(z_1) + b^2 S'(z_1) = (a+b)(aS(z_1') + bS'(z_1)) \\
&= aS(z_1') + bS'(z_1) = aS(z_0') + aS(z_1') + bS'(z_0) + bS'(z_1) \\
&= aS(z) + bS'(z) = S''(z)
\end{aligned} \tag{3.22}$$

is obtained.

Now, let us show that $\ker S'' = \ker S = \ker S'$. For this, take an arbitrary element $x \in \ker S = \ker S'$. Therefore,

$$S''(x) = aS(x) + bS(x) = a.0 + b.0 = 0. \tag{3.23}$$

This means that $\ker S = \ker S' \subseteq \ker S''$. Now, take $x \in \ker S''$. Since $Z = \ker S \oplus \operatorname{Im} S$, then there exist unique elements $e \in \ker S$ and $f \in \operatorname{Im} S$ such that $x = e + f$ holds. Therefore, we obtain

$$\begin{aligned}
0 = S''(x) &= aS(x) + bS'(x) \\
&= aS(e+f) + bS'(e+f) = aS(f) + bS'(f) = af + bS'(f).
\end{aligned} \tag{3.24}$$

That is $af + bS'(f) = 0$. If $b = 0$, since $a + b = 1$ then $a = 1$ and then $f = 0$. So that $x \in \ker S = \ker S'$. If $b \neq 0$, then $S'(f) = -(\mathrm{a}/\mathrm{b})f$. Then,

$$S'(f) = S^2(f) = S'(S'(f)) = S'\left(-\frac{a}{b}f\right) = -\frac{a}{b}S'(f) \tag{3.25}$$

is obtained. In this case, we get $S'(f) = -(a/b)S'(f)$. Since $a + b = 1$, this gives us $S'(f) = 0$. From here, we get that $\ker S'' \subseteq \ker S = \ker S'$ is obtained. So in any case we showed that $\ker S'' = \ker S = \ker S'$. □

Proposition 3.18. *If the assumptions (i)–(v) hold, then Leray-Schauder degree $d[I - M, \Omega, 0]$ depends on only L, N, Ω and homotopy class of Λ in Λ_L.*

Proof. Let the operators P, P', Q, Q' be the projection operators with the properties $\operatorname{Im} P = \operatorname{Im} P' = \ker L$, $\ker Q = \ker Q' = \operatorname{Im} L$ and Λ, Λ' two isomorphisms from $\operatorname{Coker} L$ to $\ker L$ in the same homotopy class. From Lemma 3.15 and Lemma 3.16, it is clear that for each $\lambda \in [0,1]$ the operators

$$P(\lambda) = (1-\lambda)P + \lambda P', \qquad Q(\lambda) = (1-\lambda)Q + \lambda Q' \tag{3.26}$$

are the projection operators with the property of for each $\lambda \in [0,1]$, $\operatorname{Im} P(\lambda) = \ker L$ and $\ker Q(\lambda) = \operatorname{Im} L$. Beside this, from Lemma 3.16, we have $K_{P(\lambda)} = (1-\lambda)K_P + \lambda K_{P'}$. Let the operator $\underline{\Lambda} : \operatorname{Coker} L \times [0,1] \to \ker L$ be the operator given in Definition 3.1. Using Proposition 3.7, we see that for each $\lambda \in [0,1]$ the fixed points of the operator

$$\begin{gathered} \underline{M}(\cdot,\lambda) : \overline{\Omega} \longrightarrow X \\ \underline{M}(\cdot,\lambda) = P(\lambda) + \underline{\Lambda}\Pi N(\cdot,\lambda) + K_{P(\lambda),Q(\lambda)}N \end{gathered} \tag{3.27}$$

coincide with the solutions of the operator equation (1.1). From the condition (v), we have $0 \notin (L-N)(\operatorname{Dom} L \cap \partial\Omega)$ and

$$x \notin \underline{M}(x,\lambda), \quad \forall x \in \partial\Omega,\ \forall \lambda \in [0,1]. \tag{3.28}$$

Clearly, we have

$$\begin{gathered} \underline{M}(\cdot,0) = M = P + \left(\Lambda\Pi N + K_{P,Q}N\right), \\ \underline{M}(\cdot,1) = M' = P' + \left(\Lambda'\Pi N + K_{P',Q'}N\right). \end{gathered} \tag{3.29}$$

Now let us show that $\underline{M}$ is compact on $\overline{\Omega} \times [0,1]$. From the open form

$$\underline{M}(\cdot,\lambda) = (1-\lambda)Px + \lambda P'x + \underline{\Lambda}\Pi N(\cdot,\lambda) + ((1-\lambda)K_P + \lambda K_{P'})\left(I - (1-\lambda)Q - \lambda Q'\right)N, \tag{3.30}$$

it is clear that $\underline{M}$ is continuous. So, in order to show that the set $\underline{M}(\overline{\Omega} \times [0,1])$ is relatively compact the only delicate point is the last term. Using the fact that $K_{P'} = (I-P')K_P$, we obtain the last term as

$$\left(I - \lambda P'\right)K_P(I-Q)N + \lambda\left(I - \lambda P'\right)K_P\left(Q - Q'\right)N. \tag{3.31}$$

Therefore, compactness can be proven like in the proof of Proposition 3.9. Using the invariance of Leray-Schauder degree with respect to compact homotopy, we obtain that

$$d(I - M(\cdot,0), \Omega, 0) = d(I - M(\cdot,1), \Omega, 0), \tag{3.32}$$

that is,

$$d(I - M, \Omega, 0) = d\left(I - M', \Omega, 0\right). \tag{3.33}$$

□

Now, let us indicate how the degree $d(I - M, \Omega, 0)$ depends on homotopy class of Λ. For this, let us prove the following lemma.

Lemma 3.19. *If* $G : \ker L \to \ker L$ *is any automorphism and*

$$M' = P + \left(G\Lambda\Pi + K_{P,Q}\right)N \tag{3.34}$$

then the relation

$$I - M' = (I - P + GP)(I - M) \tag{3.35}$$

is satisfied.

Proof. Since $P^2 = P$, $PK_{P,Q} = 0$, and $P\Lambda = \Lambda$, then

$$\begin{aligned}
(I-P+GP)(I-M) &= I-M-P+PM+GP-GPM \\
&= I-P-\Lambda\Pi N-K_{P,Q}N-P+P^2+P\Lambda\Pi N+PK_{P,Q}N+GP-GPM \\
&= I-P-\Lambda\Pi N-K_{P,Q}N+P\Lambda\Pi N+GP-GPM \\
&= I-P-\Lambda\Pi N-K_{P,Q}N+P\Lambda\Pi N+GP-GP\left(P+\left(\Lambda\Pi N+K_{P,Q}N\right)\right) \\
&= I-P-\Lambda\Pi N-K_{P,Q}N+P\Lambda\Pi N+GP-GP^2-GP\Lambda\Pi N-GPK_{P,Q}N \\
&= I-P-\Lambda\Pi N-K_{P,Q}N+P\Lambda\Pi N-GP\Lambda\Pi N \\
&= I-\left(P+\left(GP\Lambda\Pi+K_{P,Q}\right)N\right) = I-M'.
\end{aligned} \tag{3.36}$$

□

Proposition 3.20. *If* $\Lambda, \Lambda' \in \Lambda_L$ *and*

$$M' = P + \left(\Lambda'\Pi + K_{P,Q}\right)N \tag{3.37}$$

then we have

$$d\left(I - M', \Omega, 0\right) = \operatorname{sgn}\left(\det\left(\Lambda'\Lambda^{-1}\right)\right) d(I - M, \Omega, 0). \tag{3.38}$$

Proof. In Lemma 3.19, if we take $G = \Lambda'\Lambda^{-1}$, then

$$I - M' = \left(I - P + \Lambda'\Lambda^{-1}P\right)(I - M) \tag{3.39}$$

is obtained. Now let us show that the operator $A = I - P + \Lambda'\Lambda^{-1}P$ is an automorphism on X. For this take, $x \in \ker L$, then we have

$$x - Px + \Lambda'\Lambda^{-1}Px = 0. \tag{3.40}$$

If we apply the operator P to both sides, we get $\Lambda'\Lambda^{-1}Px = 0$. Since $\Lambda'\Lambda^{-1}$ is one-to-one, this result gives $Px = 0$. If we substitute this result in (3.40) we obtain that $x = 0$, and therefore A

is one-to-one. For surjectivity, take $y \in X$. Therefore, there exists unique elements $k \in \ker P$, $j \in \operatorname{Im} P$ such that $y = k + j$. Now, we are looking for $x \in X$, $k' \in \ker P$, $j' \in \operatorname{Im} P$ such that $x = k' + j'$ and $Ax = y$. So

$$\begin{gathered} Ax = y \\ \left(I - P + \left(\Lambda'\Lambda^{-1}\right)P\right)x = y \\ \left(I - P + \left(\Lambda'\Lambda^{-1}\right)P\right)(k' + j') = k + j \\ k' + \Lambda'\Lambda^{-1}j' = k + j. \end{gathered} \tag{3.41}$$

Using uniqueness in direct sum and the fact that $\Lambda'\Lambda^{-1}$ is an automorphism on $\ker L = \operatorname{Im} P$, we get $k' = k$ and $\Lambda'\Lambda^{-1}j' = j$. Therefore, taking $x = k' + \Lambda'\Lambda^{-1}j'$ ontoness of the operator A is proved. Therefore, A is an automorphism on X. So using the identity $I - M' = (I - P + \Lambda'\Lambda^{-1}P)(I - M)$ and Leray Product Theorem, we have

$$d\left(I - M', \Omega, 0\right) = d(A, B_\epsilon(0), 0) \cdot d(I - M, \Omega, 0). \tag{3.42}$$

Therefore the result

$$\begin{aligned} d\left(I - M', \Omega, 0\right) &= d\left(I - P + \Lambda'\Lambda^{-1}P, B_\epsilon(0), 0\right) \cdot d(I - M, \Omega, 0) \\ &= d\left(\left(I - P + \Lambda'\Lambda^{-1}P\right)_{\ker L}, B_\epsilon(0) \cap \ker L, 0\right) \cdot d(I - M, \Omega, 0) \\ &= d\left(\Lambda'\Lambda^{-1}, B_\epsilon(0) \cap \ker L, 0\right) \cdot d(I - M, \Omega, 0) \\ &= \operatorname{sgn}\left(\det\left(\Lambda'\Lambda^{-1}\right)\right) \cdot d(I - M, \Omega, 0) \end{aligned} \tag{3.43}$$

is achieved. □

Corollary 3.21. *Under the assumptions of Proposition 3.18, Leray-Schauder degree* $|d(I - M, \Omega, 0)|$ *only depends upon* L*,* N*, and* Ω*.*

Now, if the orientation on the spaces $\ker L$ and $\operatorname{Coker} L$ is fixed, then we can give the following beautiful and fruitful definition.

Definition 3.22. If the operators L and N satisfy the conditions (i)–(v) then the coincidence degree of L and N in Ω defined by

$$d[(L, N), \Omega] = d(I - M, \Omega, 0). \tag{3.44}$$

Here, Λ in M is an orientation preserving isomorphism.

This definition is supported with all the arguments given in this paper.

4. Basic Properties of Coincidence Degree

In this section, we will see that the coincidence degree satisfies all the basic properties of the Leray-Schauder degree. First, let us consider the simplest case where $X = Z$ and $L = I$. In this situation, $\ker L = \{0\}$ and $\operatorname{Im} L = Z = X$, so that $\operatorname{Coker} L = Z/\operatorname{Im} L = \{\overline{0}\}$. Therefore $\dim\ker L = \dim\operatorname{Coker} L = 0$ and then the assumptions (i) and (ii) are clearly satisfied. Since $\operatorname{Im} P = \ker L = 0$ and $\ker Q = \operatorname{Im} L = Z$, then $P = 0$, and $Q = 0$. Thus $K_{P,Q} = K_P(I - Q) = I(I - 0) = I$, and $\Pi = 0$. Therefore, the conditions (iii) and (iv) reduced to the compactness of N on $\overline{\Omega}$. Since $L = I$ and $\operatorname{Dom} L = X$, then the condition (v) in this case means that N has no fixed point on $\partial\Omega$. Since $P = 0$, $\Pi = 0$, and $K_{P,Q} = 0$, then $M = P + (\Lambda\Pi + K_{P,Q})N = N$. Therefore,

$$d[(L,N),\Omega] = d[(I,N),\Omega] = d(I - N,\Omega,0). \tag{4.1}$$

That is the coincidence degree of L and N in this case is nothing but the Leray-Schauder degree of $I - N$.

Now, we will give the basic properties of coincidence degree.

Theorem 4.1. *Assume that the conditions (i) to (v) are satisfied. Then coincidence degree satisfies the following basic properties.*

(1) *Existence theorem: if* $d[(L,N),\Omega] \neq 0$, *then* $0 \in (L - N)(\operatorname{dom} L \cap \Omega)$.

(2) *Excision property: if* $\Omega_0 \in \Omega$ *is an open set such that* $(L - N)^{-1}(0) \in \Omega_0$, *then*

$$d[(L,N),\Omega] = d[(L,N),\Omega_0]. \tag{4.2}$$

(3) *Additivity property: if* $\Omega = \Omega_1 \cup \Omega_2$ *with* Ω_1 *and* Ω_2 *are open, bounded, disjoint subsets of* X, *then*

$$d[(L,N),\Omega] = d[(L,N),\Omega_1] + d[(L,N),\Omega_2]. \tag{4.3}$$

(4) *Invariance under homotopy property: if the operator*

$$\begin{aligned} \widetilde{N} : \overline{\Omega} \times [0,1] &\longrightarrow Z \\ (x,\lambda) &\longmapsto \widetilde{N}(x,\lambda) \end{aligned} \tag{4.4}$$

is L-compact in $\overline{\Omega}\times[0,1]$ *and such that for each* $\lambda \in [0,1]$, $0 \notin [L - \widetilde{N}(\cdot,\lambda)](\operatorname{dom} L \cap \partial\Omega)$, *then coincidence degree* $d[(L,N(\cdot,\lambda)),\Omega]$, *is independent of* λ *in* $[0,1]$. *In particular*

$$d[(L,N(\cdot,1)),\Omega] = d[(L,N(\cdot,0)),\Omega]. \tag{4.5}$$

Proof. (1) If $d(I - M,\Omega,0) = d[(L,N),\Omega] \neq 0$, then $\exists x \in \Omega$ such that $(I - M)x = 0$. But in fact, we know that $x \in \operatorname{Dom} L \cap \Omega$. Also, by Proposition 3.7, $Lx = Nx$. That is $0 \in (L - N)(\operatorname{dom} L \cap \Omega)$.

(2) Assume that $\Omega_0 \in \Omega$ is an open set such that $(L - N)^{-1}(0) \in \Omega_0$, then by Proposition 3.7, $(I - M)^{-1}(0) \in \Omega_0$. Therefore, by the excision property of the Leray-Schauder degree, $d(I - M, \Omega, 0) = d(I - M, \Omega_0, 0)$. So, by the definition of coincidence degree, we have $d[(L, N), \Omega] = d[(L, N), \Omega_0]$.

(3) If $\Omega = \Omega_1 \cup \Omega_2$ with Ω_1 and Ω_2 are open, bounded, disjoint subsets of X, then additive property of Leray-Schauder degree we have $d(I - M, \Omega, 0) = d(I - M, \Omega_1, 0) + d(I - M, \Omega_2, 0)$. So the result follows from the definition of coincidence degree.

(4) Since the operator $\widetilde{N}(\cdot, \lambda)$ is L-compact for each $\lambda \in [0,1]$ and for each $\lambda \in [0,1]$, $0 \notin [L - \widetilde{N}(\cdot, \lambda)](\operatorname{dom} L \cap \partial\Omega)$, then for each $\lambda \in [0,1]$ the coincidence degree $d[(L, N(\cdot, \lambda)), \Omega]$ is well defined. Since the operator $\widetilde{N}$ is L-compact on $\overline{\Omega} \times [0,1]$ then it is a homotopy of compact operators on $\overline{\Omega}$. Therefore, by invariance of the Leray-Schauder degree under homotopy property the result follows. □

The famous Borsuck theorem for degree theory is also valid for coincidence degree.

Theorem 4.2. *If Ω is symmetric with respect to 0 and contains it and if $N(-x) = -N(x)$ in Ω, then coincidence degree $d[(L, N), \Omega]$ is an odd integer.*

Proof. We proved that the operator M is compact on $\overline{\Omega}$. Since a projection operator P is linear then it is odd in Ω and N is odd in Ω then the operator $M = P + (\Lambda\Pi + K_{P,Q})N$ is odd in Ω. Therefore, the result follows from the validity of Borsuck theorem in the Leray-Schauder degree. □

References

[1] R. E. Gaines and J. L. Mawhin, *Coincidence Degree, and Nonlinear Differential Equations*, vol. 568 of *Lecture Notes in Mathematic*, Springer, 1977.

[2] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, vol. 40 of *Regional Conference Series in Mathematics*, American Mathematical Society, 1979.

[3] Y. Li and Y. Kuang, "Periodic solutions in periodic state-dependent delay equations and population models," *Proceedings of the American Mathematical Society*, vol. 130, no. 5, pp. 1345–1353, 2002.

[4] A. Sirma, C. Tunç, and S. Özlem, "Existence and uniqueness of periodic solutions for a kind of Rayleigh equation with finitely many deviating arguments," *Nonlinear Analysis*, vol. 73, no. 2, pp. 358–366, 2010.

[5] Z. Yang and J. Cao, "Existence of periodic solutions in neutral state-dependent delays equations and models," *Journal of Computational and Applied Mathematics*, vol. 174, no. 1, pp. 179–199, 2005.

[6] L. Wang and J. Shao, "New results of periodic solutions for a kind of forced Rayleigh-type equations," *Nonlinear Analysis*, vol. 11, no. 1, pp. 99–105, 2010.

[7] J. Shen and R. Liang, "Periodic solutions for a kind of second order neutral functional differential equations," *Applied Mathematics and Computation*, vol. 190, no. 2, pp. 1394–1401, 2007.

[8] Y. Chen and D. O'Regan, "Coincidence degree theory for mappings of class $L - (S_+)$," *Applicable Analysis*, vol. 85, no. 8, pp. 963–970, 2006.

[9] A. Buică, "Contributions to coincidence degree theory of asymptotically homogeneous operators," *Nonlinear Analysis*, vol. 68, no. 6, pp. 1603–1610, 2008.

[10] S. Lu and W. Ge, "Existence of periodic solutions for a kind of second-order neutral functional differential equation," *Applied Mathematics and Computation*, vol. 157, no. 2, pp. 433–448, 2004.

[11] Y. Li and L. Huang, "New results of periodic solutions for forced Rayleigh-type equations," *Journal of Computational and Applied Mathematics*, vol. 221, no. 1, pp. 98–105, 2008.

[12] W. Ge and J. Ren, "An extension of Mawhin's continuation theorem and its application to boundary value problems with a p-Laplacian," *Nonlinear Analysis*, vol. 58, no. 3-4, pp. 477–488, 2004.

[13] Z. Yang and J. Cao, "Existence of periodic solutions in neutral state-dependent delays equations and models," *Journal of Computational and Applied Mathematics*, vol. 174, no. 1, pp. 179–199, 2005.

[14] S. Lu and W. Ge, "Periodic solutions for a kind of second order differential equation with multiple deviating arguments," *Applied Mathematics and Computation*, vol. 146, no. 1, pp. 195–209, 2003.
[15] W. Xiong, Q. Zhou, B. Xiao, Y. Wang, and F. Long, "Periodic solutions for a kind of Liénard equation with two deviating arguments," *Nonlinear Analysis*, vol. 8, no. 3, pp. 787–796, 2007.
[16] S. Lu and W. Ge, "Periodic solutions for a kind of Liénard equation with a deviating argument," *Journal of Mathematical Analysis and Applications*, vol. 289, no. 1, pp. 231–243, 2004.
[17] S. Lu and W. Ge, "Some new results on the existence of periodic solutions to a kind of Rayleigh equation with a deviating argument," *Nonlinear Analysis*, vol. 56, no. 4, pp. 501–514, 2004.
[18] H. Xiankai and Z. G. Xian, "On existence of 2π-periodic solutions for delay Duffing equation $x'' + g(t, x(t-\tau(t))) = p(t)$," *Chinese Science Bulletin*, vol. 39, pp. 201–203, 1994.
[19] G.-Q. Wang and S. S. Cheng, "A priori bounds for periodic solutions of a delay Rayleigh equation," *Applied Mathematics Letters*, vol. 12, no. 3, pp. 41–44, 1999.
[20] B. Liu and L. Huang, "Periodic solutions for a class of forced Linard-type equations," *Acta Mathematicae Applicatae Sinica*, vol. 21, pp. 81–92, 2005.
[21] B. Liu, "Existence and uniqueness of periodic solutions for a kind of Rayleigh equation with two deviating arguments," *Computers & Mathematics with Applications*, vol. 55, no. 9, pp. 2108–2117, 2008.
[22] C. Huang, Y. He, L. Huang, and W. Tan, "New results on the periodic solutions for a kind of Rayleigh equation with two deviating arguments," *Mathematical and Computer Modelling*, vol. 46, no. 5-6, pp. 604–611, 2007.
[23] Q. Zhou and F. Long, "Existence and uniqueness of periodic solutions for a kind of Liénard equation with two deviating arguments," *Journal of Computational and Applied Mathematics*, vol. 206, no. 2, pp. 1127–1136, 2007.
[24] Y. Zhou and X. Tang, "On existence of periodic solutions of a kind of Rayleigh equation with a deviating argument," *Nonlinear Analysis*, vol. 69, no. 8, pp. 2355–2361, 2008.
[25] Y. Zhou and X. Tang, "Periodic solutions for a kind of Rayleigh equation with a deviating argument," *Computers & Mathematics with Applications*, vol. 53, no. 5, pp. 825–830, 2007.
[26] M.-L. Tang, X.-G. Liu, and X.-B. Liu, "New results on periodic solutions for a kind of Rayleigh equation," *Applications of Mathematics*, vol. 54, no. 1, pp. 79–85, 2009.
[27] Y. Li and L. Huang, "New results of periodic solutions for forced Rayleigh-type equations," *Journal of Computational and Applied Mathematics*, vol. 221, no. 1, pp. 98–105, 2008.
[28] L. Wang and J. Shao, "New results of periodic solutions for a kind of forced Rayleigh-type equations," *Nonlinear Analysis*, vol. 11, no. 1, pp. 99–105, 2010.
[29] L. Peng, B. Liu, Q. Zhou, and L. Huang, "Periodic solutions for a kind of Rayleigh equation with two deviating arguments," *Journal of the Franklin Institute*, vol. 343, no. 7, pp. 676–687, 2006.
[30] X.-G. Liu, M.-L. Tang, and R. R. Martin, "Periodic solutions for a kind of Liénard equation," *Journal of Computational and Applied Mathematics*, vol. 219, no. 1, pp. 263–275, 2008.
[31] Y. Q. Chen and Y. J. Cho, "Topological degree theory for multi-valued mappings of class $(S_+)_L$," *Archiv der Mathematik*, vol. 84, no. 4, pp. 325–333, 2005.
[32] D. O'Regan, Y. J. Cho, and Y.-Q. Chen, *Topological Degree Theory and Applications*, Taylor and Francis, Chapman and Hall/CRC, Florida, Fla, USA, 2006.
[33] I. Fonseca and W. Gangbo, *Degree Theory in Analysis and Applications*, vol. 2, Oxford University Press, 1995.
[34] A. G. Kartsatos, *Advanced Ordinary Differential Equations*, Mancorp, 1993.
[35] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, New-York, NY, USA, 1978.
[36] P. Drábek and J. Milota, *Methods of Nonlinear Analysis Applications to Differential Equations*, Birkhäuser, Basel, Switzerland, 2007.
[37] W. H. Greub, *Linear Algebra*, Springer, New York, NY, USA, 1967.
[38] E. S. Şuhubi, *Functional Analysis*, Kluwer Academic, 2003.