

Research Article

Global Attractor of Atmospheric Circulation Equations with Humidity Effect

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Received 1 June 2012; Accepted 15 July 2012

Academic Editor: Jinhu Lü

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Global attractor of atmospheric circulation equations is considered in this paper. Firstly, it is proved that this system possesses a unique global weak solution in $L^2(\Omega, R^4)$. Secondly, by using C-condition, it is obtained that atmospheric circulation equations have a global attractor in $L^2(\Omega, R^4)$.

1. Introduction

This paper is concerned with global attractor of the following initial-boundary problem of atmospheric circulation equations involving unknown functions (u, T, q, p) at $(x, t) = (x_1, x_2, t) \in \Omega \times (0, \infty)$ ($\Omega = (0, 2\pi) \times (0, 1)$ is a period of C^∞ field $(-\infty, +\infty) \times (0, 1)$):

$$\frac{\partial u}{\partial t} = P_r(\Delta u - \nabla p - \sigma u) + P_r(RT - \tilde{R}q)\vec{k} - (u \cdot \nabla)u, \quad (1.1)$$

$$\frac{\partial T}{\partial t} = \Delta T + u_2 - (u \cdot \nabla)T + Q, \quad (1.2)$$

$$\frac{\partial q}{\partial t} = L_e \Delta q + u_2 - (u \cdot \nabla)q + G, \quad (1.3)$$

$$\operatorname{div} u = 0, \quad (1.4)$$

where P_r , R , \tilde{R} , and L_e are constants, $u = (u_1, u_2)$, T , q , and p denote velocity field, temperature, humidity, and pressure, respectively; Q , G are known functions, and σ is constant matrix:

$$\sigma = \begin{pmatrix} \sigma_0 & \omega \\ \omega & \sigma_1 \end{pmatrix}. \quad (1.5)$$

The problems (1.1)–(1.4) are supplemented with the following Dirichlet boundary condition at $x_2 = 0, 1$ and periodic condition for x_1 :

$$\begin{aligned}(u, T, q) &= 0, & x_2 &= 0, 1, \\ (u, T, q)(0, x_2) &= (u, T, q)(2\pi, x_2),\end{aligned}\tag{1.6}$$

and initial value conditions

$$(u, T, q) = (u_0, T_0, q_0), \quad t = 0.\tag{1.7}$$

The partial differential equations (1.1)–(1.7) were firstly presented in atmospheric circulation with humidity effect [1]. Atmospheric circulation is one of the main factors affecting the global climate, so it is very necessary to understand and master its mysteries and laws. Atmospheric circulation is an important mechanism to complete the transports and balance of atmospheric heat and moisture and the conversion between various energies. On the contrary, it is also the important result of these physical transports, balance and conversion. Thus, it is of necessity to study the characteristics, formation, preservation, change and effects of the atmospheric circulation and master its evolution law, which is not only the essential part of human's understanding of nature, but also the helpful method of changing and improving the accuracy of weather forecasts, exploring global climate change, and making effective use of climate resources.

The atmosphere and ocean around the earth are rotating geophysical fluids, which are also two important components of the climate system. The phenomena of the atmosphere and ocean are extremely rich in their organization and complexity, and a lot of them cannot be produced by laboratory experiments. The atmosphere or the ocean or the couple atmosphere and ocean can be viewed as an initial and boundary value problem [2–5], or an infinite dimensional dynamical system [6–8]. We deduce the atmospheric circulation model (1.1)–(1.7) which is able to show features of atmospheric circulation and is easy to be studied from the very complex atmospheric circulation model based on the actual background and meteorological data, and we present global solutions of atmospheric circulation equations with the use of the T -weakly continuous operator [1]. In fact, there are numerous papers on this topic [9–13]. Compared with some similar papers, we add humidity function in this paper. We propose firstly the atmospheric circulation equation with humidity function which does not appear in the previous literature.

As far as the theory of infinite-dimensional dynamical system is concerned, we refer to [9–11, 14–18]. In the study of infinite dimensional dynamical system, the long-time behavior of the solution to equations is an important issue. The long-time behavior of the solution to equations can be shown by the global attractor with the finite-dimensional characteristics. Some authors have already studied the existence of the global attractor for some evolution equations [2, 3, 13, 19–21]. The global attractor strictly defined as ω -limit set of ball, which under additional assumptions is nonempty, compact, and invariant [13, 17]. Attractor theory has been intensively investigated within the science, mathematics, and engineering communities. Lü et al. [22–25] apply the current theoretical results or approaches to investigate the global attractor of complex multiscroll chaotic systems. We obtain existence of global attractor for the atmospheric circulation equations from the mathematical perspective in this paper.

The paper is organized as follows. In Section 2, we recall preliminary results. In Section 3, we present uniqueness of the solution to the atmospheric circulation equations. In Section 4, we obtain global attractor of the equations.

$\|\cdot\|_X$ denote norm of the space X ; C and C_i are variable constants. Let $H = \{\phi = (u, T, q) \in L^2(\Omega, R^4) \mid \phi \text{ satisfy (1.4), (1.6)}\}$, and $H_1 = \{\phi = (u, T, q) \in H^1(\Omega, R^4) \mid \phi \text{ satisfy (1.4), (1.6)}\}$.

2. Preliminaries

Let X and X_1 be two Banach spaces, $X_1 \subset X$ a compact and dense inclusion. Consider the abstract nonlinear evolution equation defined on X , given by

$$\begin{aligned} \frac{du}{dt} &= Lu + G(u), \\ u(x, 0) &= u_0, \end{aligned} \quad (2.1)$$

where $u(t)$ is an unknown function, $L : X_1 \rightarrow X$ a linear operator, and $G : X_1 \rightarrow X$ a nonlinear operator.

A family of operators $S(t) : X \rightarrow X$ ($t \geq 0$) is called a semigroup generated by (2.1) if it satisfies the following properties:

- (1) $S(t) : X \rightarrow X$ is a continuous map for any $t \geq 0$;
- (2) $S(0) = id : X \rightarrow X$ is the identity;
- (3) $S(t + s) = S(t) \cdot S(s)$, for all $t, s \geq 0$. Then, the solution of (2.1) can be expressed as

$$u(t, u_0) = S(t)u_0. \quad (2.2)$$

Next, we introduce the concepts and definitions of invariant sets, global attractors, and ω -limit sets for the semigroup $S(t)$.

Definition 2.1. Let $S(t)$ be a semigroup defined on X . A set $\Sigma \subset X$ is called an invariant set of $S(t)$ if $S(t)\Sigma = \Sigma$, for all $t \geq 0$. An invariant set Σ is an attractor of $S(t)$ if Σ is compact, and there exists a neighborhood $U \subset X$ of Σ such that for any $u_0 \in U$,

$$\inf_{v \in \Sigma} \|S(t)u_0 - v\|_X \rightarrow 0, \quad \text{as } t \rightarrow \infty. \quad (2.3)$$

In this case, we say that Σ attracts U . Particularly, if Σ attracts any bounded set of X , Σ is called a global attractor of $S(t)$ in X .

For a set $D \subset X$, we define the ω -limit set of D as follows:

$$\omega(D) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)D}, \quad (2.4)$$

where the closure is taken in the X -norm. Lemma 2.2 is the classical existence theorem of global attractor by Temam [13].

Lemma 2.2. Let $S(t) : X \rightarrow X$ be the semigroup generated by (2.1). Assume that the following conditions hold:

- (1) $S(t)$ has a bounded absorbing set $B \subset X$, that is, for any bounded set $A \subset X$ there exists a time $t_A \geq 0$ such that $S(t)u_0 \in B$, for all $u_0 \in A$ and $t > t_A$;
- (2) $S(t)$ is uniformly compact, that is, for any bounded set $U \subset X$ and some $T > 0$ sufficiently large, the set $\overline{\bigcup_{t \geq T} S(t)U}$ is compact in X .

Then the ω -limit set $\mathcal{A} = \omega(B)$ of B is a global attractor of (2.1), and \mathcal{A} is connected providing B is connected.

Definition 2.3 (see [19]). We say that $S(t) : X \rightarrow X$ satisfies C-condition, if for any bounded set $B \subset X$ and $\varepsilon > 0$, there exist $t_B > 0$ and a finite dimensional subspace $X_1 \subset X$ such that $\{PS(t)B\}$ is bounded, and

$$\|(I - P)S(t)u\|_X < \varepsilon, \quad \forall t \geq t_B, u \in B, \quad (2.5)$$

where $P : X \rightarrow X_1$ is a projection.

Lemma 2.4 (see [19]). Let $S(t) : X \rightarrow X$ ($t \geq 0$) be a dynamical systems. If the following conditions are satisfied:

- (1) there exists a bounded absorbing set $B \subset X$;
- (2) $S(t)$ satisfies C-condition,

then $S(t)$ has a global attractor in X .

From Linear elliptic equation theory, one has the following.

Lemma 2.5. The eigenvalue equation:

$$\begin{aligned} -\Delta T(x_1, x_2) &= \beta T(x_1, x_2), \quad (x_1, x_2) \in (0, 2\pi) \times (0, 1), \\ T &= 0, \quad x_2 = 0, 1, \\ T(0, x_2) &= T(2\pi, x_2) \end{aligned} \quad (2.6)$$

has eigenvalue $\{\beta_k\}_{k=1}^\infty$, and

$$0 < \beta_1 \leq \beta_2 \leq \dots, \quad \beta_k \longrightarrow \infty, \quad \text{as } k \longrightarrow \infty. \quad (2.7)$$

3. Uniqueness of Global Solution

Theorem 3.1. If $\tilde{\sigma}\beta_1 \geq \max\{(R+1)^2, ((\tilde{R}-1)^2/L_e)\}$, and β_1 is the first eigenvalue of elliptic equation (2.6), then the weak solution to (1.1)–(1.7) is unique.

Proof. From [1], $(u, T, q) \in L^\infty((0, T), H) \cap L^2((0, T), H_1)$, $0 < T < \infty$ is the weak solution to (1.1)–(1.7). Then for all $(v, S, z) \in H_1$, $0 \leq t \leq T$, we have

$$\begin{aligned} \frac{1}{P_r} \int_{\Omega} uv dx + \int_{\Omega} TS dx + \int_{\Omega} qz dx &= \int_0^t \int_{\Omega} \left[-\nabla u \nabla v - \sigma uv + (RT - \tilde{R}q)v_2 \right. \\ &\quad - \frac{1}{P_r} (u \cdot \nabla) uv - \nabla T \nabla S + u_2 S - (u \cdot \nabla) TS \\ &\quad \left. + QS - L_e \nabla q \nabla z + u_2 z - (u \cdot \nabla) qz + Gz \right] dx dt \\ &\quad + \frac{1}{P_r} \int_{\Omega} u_0 v dx + \int_{\Omega} T_0 S dx + \int_{\Omega} q_0 z dx. \end{aligned} \quad (3.1)$$

Set (u^1, T^1, q^1) and (u^2, T^2, q^2) are two weak solutions to (1.1)–(1.7), which satisfy (3.1). Let $(u, T, q) = (u^1, T^1, q^1) - (u^2, T^2, q^2)$. Then,

$$\begin{aligned} \frac{1}{P_r} \int_{\Omega} uv dx + \int_{\Omega} TS dx + \int_{\Omega} qz dx &= \int_0^t \int_{\Omega} \left[-\nabla u \nabla v - \sigma uv + (RT - \tilde{R}q)v_2 \right. \\ &\quad + \frac{1}{P_r} (u^2 \cdot \nabla) u^2 v - \frac{1}{P_r} (u^1 \cdot \nabla) u^1 v - \nabla T \nabla S \\ &\quad + u_2 S + (u^2 \cdot \nabla) T^2 S - (u^1 \cdot \nabla) T^1 S \\ &\quad - L_e \nabla q \nabla z + u_2 z + (u^2 \cdot \nabla) q^2 z \\ &\quad \left. - (u^1 \cdot \nabla) q^1 z \right] dx dt. \end{aligned} \quad (3.2)$$

Let $(v, S, z) = (u, T, q)$. We obtain from (3.2) the following:

$$\begin{aligned} &\frac{1}{P_r} \int_{\Omega} |u|^2 dx + \int_{\Omega} |T|^2 dx + \int_{\Omega} |q|^2 dx \\ &= \int_0^t \int_{\Omega} \left[-|\nabla u|^2 - \sigma u \cdot u + (RT - \tilde{R}q)u_2 + \frac{1}{P_r} (u^2 \cdot \nabla) u^2 u - \frac{1}{P_r} (u^1 \cdot \nabla) u^1 u \right. \\ &\quad \left. - |\nabla T|^2 + u_2 T + (u^2 \cdot \nabla) T^2 T - (u^1 \cdot \nabla) T^1 T - L_e |\nabla q|^2 + u_2 q \right. \\ &\quad \left. + (u^2 \cdot \nabla) q^2 q - (u^1 \cdot \nabla) q^1 q \right] dx dt \\ &\leq \int_0^t \int_{\Omega} \left[-|\nabla u|^2 - |\nabla T|^2 - L_e |\nabla q|^2 \right] dx dt \\ &\quad + \int_0^t \int_{\Omega} \left[-\tilde{\sigma} |u|^2 + (R+1)Tu_2 - (\tilde{R}-1)qu_2 \right] dx dt \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \int_{\Omega} \left[\frac{1}{P_r} (u \cdot \nabla) u^2 u + (u \cdot \nabla) T^2 T + (u \cdot \nabla) q^2 q \right] dx dt \\
& \leq \int_0^t \int_{\Omega} \left[-|\nabla u|^2 - |\nabla T|^2 - L_e |\nabla q|^2 \right] dx dt \\
& \quad + \int_0^t \int_{\Omega} \left[-\tilde{\sigma} |u|^2 + \tilde{\sigma} |u_2|^2 + \frac{(R+1)^2}{2\tilde{\sigma}} |T|^2 + \frac{(\tilde{R}-1)^2}{2\tilde{\sigma}} |q|^2 \right] dx dt \\
& \quad + \int_0^t \left[\frac{\sqrt{2}}{P_r} \|u\|_{L^2} \|\nabla u^2\|_{L^2} \|\nabla u\|_{L^2} + \sqrt{2} \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2} \|\nabla T^2\|_{L^2} \|T\|_{L^2}^{1/2} \|\nabla T\|_{L^2}^{1/2} \right. \\
& \quad \left. + \sqrt{2} \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2} \|\nabla q^2\|_{L^2} \|q\|_{L^2}^{1/2} \|\nabla q\|_{L^2}^{1/2} \right] dt \\
& \leq \int_0^t \int_{\Omega} \left[-|\nabla u|^2 - \frac{1}{2} |\nabla T|^2 - \frac{L_e}{2} |\nabla q|^2 \right] dx dt \\
& \quad + \int_0^t \left[\frac{\sqrt{2}}{P_r} \|u\|_{L^2} \|\nabla u^2\|_{L^2} \|\nabla u\|_{L^2} \right. \\
& \quad \left. + \sqrt{2} \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla T^2\|_{L^2} + \sqrt{2} \|\nabla T^2\|_{L^2} \|T\|_{L^2} \|\nabla T\|_{L^2} \right. \\
& \quad \left. + \sqrt{2} \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla q^2\|_{L^2} + \sqrt{2} \|q\|_{L^2} \|\nabla q\|_{L^2} \|\nabla q^2\|_{L^2} \right] dt \\
& \leq \int_0^t \int_{\Omega} \left[-|\nabla u|^2 - \frac{1}{2} |\nabla T|^2 - \frac{L_e}{2} |\nabla q|^2 \right] dx dt \\
& \quad + \int_0^t \left[\|\nabla u\|_{L^2}^2 + \frac{3}{P_r^2} \|u\|_{L^2}^2 \|\nabla u^2\|_{L^2}^2 + 3 \|u\|_{L^2}^2 \|\nabla T^2\|_{L^2}^2 + 3 \|u\|_{L^2}^2 \|\nabla q^2\|_{L^2}^2 \right. \\
& \quad \left. + \frac{1}{2} \|\nabla T\|_{L^2}^2 + \|\nabla T^2\|_{L^2}^2 \|T\|_{L^2}^2 + \frac{L_e}{2} \|\nabla q\|_{L^2}^2 + \frac{2}{L_e} \|q\|_{L^2}^2 \|\nabla q^2\|_{L^2}^2 \right] dt \\
& \leq \int_0^t \left[\frac{3}{P_r^2} \|u\|_{L^2}^2 \|\nabla u^2\|_{L^2}^2 + 3 \|u\|_{L^2}^2 \|\nabla T^2\|_{L^2}^2 + 3 \|u\|_{L^2}^2 \|\nabla q^2\|_{L^2}^2 \right. \\
& \quad \left. + \|\nabla T^2\|_{L^2}^2 \|T\|_{L^2}^2 + \frac{2}{L_e} \|q\|_{L^2}^2 \|\nabla q^2\|_{L^2}^2 \right] dt.
\end{aligned} \tag{3.3}$$

Then,

$$\begin{aligned}
& \|u\|_{L^2}^2 + \|T\|_{L^2}^2 + \|q\|_{L^2}^2 \\
& \leq C \int_0^t \left((\|u\|_{L^2}^2 + \|T\|_{L^2}^2 + \|q\|_{L^2}^2) \left(\|\nabla u^2\|_{L^2}^2 + \|\nabla T^2\|_{L^2}^2 + \|\nabla q^2\|_{L^2}^2 \right) \right) dt \\
& \leq C \int_0^t (\|u\|_{L^2}^2 + \|T\|_{L^2}^2 + \|q\|_{L^2}^2) dt.
\end{aligned} \tag{3.4}$$

By using the Gronwall inequality, it follows that

$$\|u\|_{L^2}^2 + \|T\|_{L^2}^2 + \|q\|_{L^2}^2 \leq 0, \quad (3.5)$$

which imply $(u, T, q) \equiv 0$. Thus, the weak solution to (1.1)–(1.7) is unique. \square

4. Existence of Global Attractor

Theorem 4.1. *If $\tilde{\sigma}\beta_1 \geq \max\{(R+1)^2, (\tilde{R}-1)^2/L_e\}$, and β_1 is the first eigenvalue of elliptic equation (2.6), then (1.1)–(1.7) have a global attractor in $L^2(\Omega, \mathbb{R}^4)$.*

Proof. According to Lemma 2.4, we prove Theorem 4.1 in the following two steps.

Step 1. Equations (1.1)–(1.7) have an absorbing set in H .

Multiply (1.1) by u and integrate the product in Ω :

$$\frac{1}{P_r} \int_{\Omega} \frac{du}{dt} u dx = \int_{\Omega} \left[\Delta u - \nabla p - \sigma u + (RT - \tilde{R}q)\tilde{\kappa} - \frac{1}{P_r}(u \cdot \nabla)u \right] u dx. \quad (4.1)$$

Then,

$$\frac{1}{2P_r} \frac{d}{dt} \int_{\Omega} u^2 dx = \int_{\Omega} \left[-|\nabla u|^2 - \sigma u \cdot u + (RT - \tilde{R}q)u_2 \right] dx. \quad (4.2)$$

Multiply (1.2) by T and integrate the product in Ω :

$$\int_{\Omega} \frac{dT}{dt} T dx = \int_{\Omega} [\Delta T + u_2 - (u \cdot \nabla)T + Q]T dx. \quad (4.3)$$

Then,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} T^2 dx = \int_{\Omega} \left(-|\nabla T|^2 + u_2 T dx + QT \right) dx. \quad (4.4)$$

Multiply (1.3) by q and integrate the product in Ω :

$$\int_{\Omega} \frac{dq}{dt} q dx = \int_{\Omega} [L_e \Delta q + u_2 - (u \cdot \nabla)q + Q]q dx. \quad (4.5)$$

Then,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} q^2 dx = \int_{\Omega} \left(-L_e |\nabla q|^2 + u_2 q dx + Gq \right) dx. \quad (4.6)$$

We deduce from (4.2)–(4.6) the following:

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left(\frac{1}{P_r} u^2 + T^2 + q^2 \right) dx &= \int_{\Omega} \left[-|\nabla u|^2 - |\nabla T|^2 - L_e |\nabla q|^2 - \sigma u \cdot u \right. \\
&\quad \left. + (R+1)Tu_2 - (\tilde{R}-1)qu_2 + QT + Gq \right] dx \\
&\leq \int_{\Omega} \left[-|\nabla u|^2 - |\nabla T|^2 - L_e |\nabla q|^2 - \tilde{\sigma} |u|^2 \right. \\
&\quad \left. + \tilde{\sigma} |u_2|^2 + \frac{(R+1)^2}{2\tilde{\sigma}} |T|^2 + \frac{(\tilde{R}-1)^2}{2\tilde{\sigma}} |q|^2 \right. \\
&\quad \left. + \varepsilon |T|^2 + \varepsilon |q|^2 + \frac{1}{\varepsilon} (|Q|^2 + |G|^2) \right] dx.
\end{aligned} \tag{4.7}$$

Let $\varepsilon > 0$ be appropriate small such that

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} (u^2 + T^2 + q^2) dx \\
\leq C_1 \int_{\Omega} [-|\nabla u|^2 - |\nabla T|^2 - |\nabla q|^2] dx + C_2 \int_{\Omega} (|Q|^2 + |G|^2) dx.
\end{aligned} \tag{4.8}$$

Then,

$$\frac{d}{dt} \int_{\Omega} (u^2 + T^2 + q^2) dx \leq -C_3 \int_{\Omega} (|u|^2 + |T|^2 + |q|^2) dx + C_4. \tag{4.9}$$

Applying the Gronwall inequality, it follows that

$$\|(u, T, q)(t)\|_{L^2}^2 \leq \|(u, T, q)(0)\|_{L^2}^2 e^{-C_3 t} + \frac{C_4}{C_3} (1 - e^{-C_3 t}). \tag{4.10}$$

Then, when $M^2 > C_4/C_3$, for any $(u_0, T_0, q_0) \in B$, here B is a bounded in H , there exists $t_* > 0$ such that

$$S(t)(u_0, T_0, q_0) = (u(t), T(t), q(t)) \in B_M, \quad t > t_*, \tag{4.11}$$

where B_M is a ball in H , at 0 of radius M . Thus, (1.1)–(1.7) have an absorbing B_M in H .

Step 2. C-condition is satisfied.

The eigenvalue equation:

$$\begin{aligned}\Delta u &= \lambda u, \\ u(x_1, 0) &= u(x_2, 0) = 0, \\ u(0, x_2) &= u(2\pi, x_2), \\ \operatorname{div} u &= 0\end{aligned}\tag{4.12}$$

has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k, \dots$ and eigenvector $\{e_k \mid k = 1, 2, 3, \dots\}$, and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq \dots$. If $k \rightarrow \infty$, then $\lambda_k \rightarrow -\infty$. $\{e_k \mid k = 1, 2, 3, \dots\}$ constitutes an orthogonal base of $L^2(\Omega)$.

For all $(u, T, q) \in H$, we have

$$\begin{aligned}u &= \sum_{k=1}^{\infty} u_k e_k, & \|u\|_{L^2}^2 &= \sum_{k=1}^{\infty} u_k^2, \\ T &= \sum_{k=1}^{\infty} T_k e_k, & \|T\|_{L^2}^2 &= \sum_{k=1}^{\infty} T_k^2, \\ q &= \sum_{k=1}^{\infty} q_k e_k, & \|q\|_{L^2}^2 &= \sum_{k=1}^{\infty} q_k^2.\end{aligned}\tag{4.13}$$

When $k \rightarrow \infty$, $\lambda_k \rightarrow -\infty$. Let δ be small positive constant, and $N = 1/\delta$. There exists positive integer k such that

$$-N \geq \lambda_j, \quad j \geq k+1.\tag{4.14}$$

Introduce subspace $E_1 = \operatorname{span}\{e_1, e_2, \dots, e_k\} \subset L^2(\Omega)$. Let E_2 be an orthogonal subspace of E_1 in $L^2(\Omega)$.

For all $(u, T, q) \in H$, we find that

$$\begin{aligned}u &= v_1 + v_2, & T &= T_1 + T_2, & q &= q_1 + q_2, \\ v_1 &= \sum_{i=1}^k x_i e_i \in E_1 \times E_1, & v_2 &= \sum_{j=k+1}^{\infty} x_j e_j \in E_2 \times E_2, \\ T_1 &= \sum_{i=1}^k T_i e_i \in E_1, & T_2 &= \sum_{j=k+1}^{\infty} T_j e_j \in E_2, \\ q_1 &= \sum_{i=1}^k q_i e_i \in E_1, & q_2 &= \sum_{j=k+1}^{\infty} q_j e_j \in E_2.\end{aligned}\tag{4.15}$$

Let $P_i : L^2(\Omega) \rightarrow E_i$ be the orthogonal projection. Thanks to Definition 2.3, we will prove that for any bounded set $B \subset H$ and $\varepsilon > 0$, there exists $t_0 > 0$ such that

$$\|P_1 S(t)B\|_H \leq M, \quad \forall t > t_0, \quad M \text{ is a constant},\tag{4.16}$$

$$\|P_2 S(t)B\|_H \leq \varepsilon, \quad \forall t > t_0, \quad (u_0, T_0, q_0) \in B.\tag{4.17}$$

From Step 1, $S(t)$ has an absorbing set B_M . Then for any bounded set $B \subset H$, there exists $t_* > 0$ such that $S(t)B \subset B_M$, for all $t > t_*$, which imply (4.16).

Multiply (1.1) by u and integrate over (Ω) . We obtain

$$\left(\frac{du}{dt}, u\right) = P_r(\Delta u, u) - P_r(\sigma u, u) + P_r\left(\left(RT + \tilde{R}q\right)\vec{\kappa}, u\right) - ((u \cdot \nabla)u, u). \quad (4.18)$$

Then,

$$\begin{aligned} \|u\|_{L^2}^2 &= P_r \int_0^t (\Delta u, u) dt - P_r \int_0^t (\sigma u, u) dt + P_r \int_0^t \left(\left(RT + \tilde{R}q\right)\vec{\kappa}, u\right) dt + \|u_0\|_{L^2}^2 \\ &= \varepsilon_1 P_r \int_0^t (\Delta u, u) dt + (1 - \varepsilon_1) P_r \int_0^t (\Delta u, u) dt - P_r \int_0^t (\sigma u, u) dt \\ &\quad + P_r \int_0^t \left(\left(RT + \tilde{R}q\right)\vec{\kappa}, u\right) dt + \|u_0\|_{L^2}^2, \end{aligned} \quad (4.19)$$

where ε_1 is a constant which needs to be determined.

From (4.14), we find that

$$\begin{aligned} (\Delta u, u) &= \sum_{i=1}^{\infty} \lambda_i u_i^2 = \sum_{i=1}^k \lambda_i u_i^2 + \sum_{j=k+1}^{\infty} \lambda_j u_j^2 \\ &\leq \lambda \sum_{i=1}^k u_i^2 - N \sum_{j=k+1}^{\infty} u_j^2 \\ &\leq \lambda \|u\|_{L^2}^2 - N \|v_2\|_{L^2}^2, \end{aligned} \quad (4.20)$$

where $\lambda = \max\{\lambda_1, \lambda_2, \dots, \lambda_k\}$.

Thanks to $(\Delta u, u) = -\int_{\Omega} |\nabla u|^2 dx = -\|\nabla u\|_{L^2}^2$ and $\|u\|_{L^2} \leq C \|\nabla u\|_{L^2}$, it follows that

$$(\Delta u, u) = -\|\nabla u\|_{L^2}^2 \leq -\frac{1}{C^2} \|u\|_{L^2}^2. \quad (4.21)$$

We deduce from (4.11) the following:

$$\|u\|_{L^2}^2 + \|T\|_{L^2}^2 + \|q\|_{L^2}^2 \leq M^2, \quad t \geq t_*. \quad (4.22)$$

Using (4.19)–(4.22), we find that

$$\begin{aligned} \|v_2\|_{L^2}^2 &\leq \|u\|_{L^2}^2 \\ &= \varepsilon_1 P_r \int_0^t (\Delta u, u) dt + (1 - \varepsilon_1) P_r \int_0^t (\Delta u, u) dt - P_r \int_0^t (\sigma u, u) dt \\ &\quad + P_r \int_0^t \left(\left(RT + \tilde{R}q\right)\vec{\kappa}, u\right) dt + \|u_0\|_{L^2}^2 \end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon_1 \lambda P_r \int_0^t \|u\|_{L^2}^2 dt - \varepsilon_1 N P_r \int_0^t \|v_2\|_{L^2}^2 dt - \frac{(1-\varepsilon_1)P_r}{C^2} \int_0^t \|u\|_{L^2}^2 dt \\
&\quad + \frac{P_r R}{2} \left(\int_0^t \|u\|_{L^2}^2 dt + \int_0^t \|T\|_{L^2}^2 dt \right) + \frac{P_r \tilde{R}}{2} \left(\int_0^t \|u\|_{L^2}^2 dt + \int_0^t \|q\|_{L^2}^2 dt \right) + \|u_0\|_{L^2}^2.
\end{aligned} \tag{4.23}$$

Let ε_1 satisfy $\varepsilon_1 \lambda \leq (1-\varepsilon_1)/C^2$ and $K_1 = P_r R M + P_r \tilde{R} M$. Then,

$$\|v_2\|_{L^2}^2 \leq -\varepsilon_1 N P_r \int_0^t \|v_2\|_{L^2}^2 dt + K_1 t + \|u_0\|_{L^2}^2, \quad t > t_*. \tag{4.24}$$

By the Gronwall inequality, we find that

$$\|v_2\|_{L^2}^2 \leq e^{-\varepsilon_1 N P_r t} \|u_0\|_{L^2}^2 + \frac{K_1}{\varepsilon_1 N P_r} \left(1 - e^{-\varepsilon_1 N P_r t} \right), \quad t > t_*. \tag{4.25}$$

Then, there exists $t_1 > t_*$ satisfying

$$e^{-\varepsilon_1 N P_r t_1} \|u_0\|_{L^2}^2 \leq \frac{K_1}{2\varepsilon_1 N P_r}, \quad \frac{K_1}{\varepsilon_1 N P_r} \left(1 - e^{-\varepsilon_1 N P_r t_1} \right) \leq \frac{K_1}{2\varepsilon_1 N P_r}. \tag{4.26}$$

Since $\delta = 1/N$, for $t > t_1$ it follows that

$$\|v_2\|_{L^2}^2 \leq \frac{K_1}{\varepsilon_1 N P_r} = \frac{K_1}{\varepsilon_1 P_r} \delta. \tag{4.27}$$

Multiply (1.2) by T and integrate over (Ω) . We obtain

$$\left(\frac{dT}{dt}, T \right) = (\Delta T, T) + (u_2, T) - ((u \cdot \nabla)T, T) + (Q, T). \tag{4.28}$$

Then,

$$\begin{aligned}
\|T\|_{L^2}^2 &= \int_0^t (\Delta T, T) dt + \int_0^t (u_2, T) dt + \|T_0\|_{L^2}^2 \\
&= \varepsilon_2 \int_0^t (\Delta T, T) dt + (1-\varepsilon_2) \int_0^t (\Delta T, T) dt + \int_0^t (u_2, T) dt + \|T_0\|_{L^2}^2,
\end{aligned} \tag{4.29}$$

where ε_2 is a constant which needs to be determined.

From (4.14), we find that

$$\begin{aligned}
 (\Delta T, T) &= \sum_{i=1}^{\infty} \lambda_i T_i^2 = \sum_{i=1}^k \lambda_i T_i^2 + \sum_{j=k+1}^{\infty} \lambda_j T_j^2 \\
 &\leq \lambda \sum_{i=1}^k T_i^2 - N \sum_{j=k+1}^{\infty} T_j^2 \\
 &\leq \lambda \|T\|_{L^2}^2 - N \|T_2\|_{L^2}^2,
 \end{aligned} \tag{4.30}$$

where $\lambda = \max\{\lambda_1, \lambda_2, \dots, \lambda_k\}$.

Since $(\Delta T, T) = -\int_{\Omega} |\nabla T|^2 dx = -\|\nabla T\|_{L^2}^2$ and $\|T\|_{L^2} \leq C\|\nabla T\|_{L^2}$, it follows that

$$(\Delta T, T) = -\|\nabla T\|_{L^2}^2 \leq -\frac{1}{C^2} \|T\|_{L^2}^2. \tag{4.31}$$

Using (4.22) and (4.29)–(4.31), we find that

$$\begin{aligned}
 \|T_2\|_{L^2}^2 &\leq \|T\|_{L^2}^2 \\
 &= \varepsilon_2 \int_0^t (\Delta T, T) dt + (1 - \varepsilon_2) \int_0^t (\Delta T, T) dt + \int_0^t (u_2, T) dt + \|T_0\|_{L^2}^2 \\
 &\leq \varepsilon_2 \lambda \int_0^t \|T\|_{L^2}^2 dt - \varepsilon_2 N \int_0^t \|T_2\|_{L^2}^2 dt - \frac{(1 - \varepsilon_2)}{C^2} \int_0^t \|T\|_{L^2}^2 dt \\
 &\quad + \frac{1}{2} \left(\int_0^t \|u\|_{L^2}^2 dt + \int_0^t \|T\|_{L^2}^2 dt \right) + \|T_0\|_{L^2}^2.
 \end{aligned} \tag{4.32}$$

Let ε_2 satisfy $\varepsilon_2 \lambda \leq (1 - \varepsilon_2)/C^2$. Then

$$\|T_2\|_{L^2}^2 \leq -\varepsilon_2 N \int_0^t \|T_2\|_{L^2}^2 dt + Mt + \|T_0\|_{L^2}^2, \quad t > t_*. \tag{4.33}$$

By the Gronwall inequality, we find that

$$\|T_2\|_{L^2}^2 \leq e^{-\varepsilon_2 N t} \|T_0\|_{L^2}^2 + \frac{M}{\varepsilon_2 N} (1 - e^{-\varepsilon_2 N t}), \quad t > t_*. \tag{4.34}$$

Then, there exists $t_2 > t_*$ satisfying

$$e^{-\varepsilon_2 N t_2} \|T_0\|_{L^2}^2 \leq \frac{M}{2\varepsilon_2 N}, \quad \frac{M}{\varepsilon_2 N} (1 - e^{-\varepsilon_2 N t_2}) \leq \frac{M}{2\varepsilon_2 N}. \tag{4.35}$$

Since $\delta = 1/N$, for $t > t_2$, it follows that

$$\|T_2\|_{L^2}^2 \leq \frac{M}{\varepsilon_2 N} = \frac{M}{\varepsilon_2} \delta. \quad (4.36)$$

Multiply (1.3) by q and integrate over (Ω) . We obtain

$$\left(\frac{dq}{dt}, q \right) = L_e(\Delta q, q) + (u_2, q) - ((u \cdot \nabla)q, q) + (G, q). \quad (4.37)$$

Then,

$$\begin{aligned} \|q\|_{L^2}^2 &= L_e \int_0^t (\Delta q, q) dt + \int_0^t (u_2, q) dt + \|q_0\|_{L^2}^2 \\ &= L_e \varepsilon_3 \int_0^t (\Delta q, q) dt + (1 - \varepsilon_3) \int_0^t (\Delta q, q) dt + \int_0^t (u_2, q) dt + \|q_0\|_{L^2}^2, \end{aligned} \quad (4.38)$$

where ε_3 is a constant which needs to be determined.

From (4.14), we find that

$$\begin{aligned} (\Delta q, q) &= \sum_{i=1}^{\infty} \lambda_i q_i^2 = \sum_{i=1}^k \lambda_i q_i^2 + \sum_{j=k+1}^{\infty} \lambda_j q_j^2 \\ &\leq \lambda \sum_{i=1}^k q_i^2 - N \sum_{j=k+1}^{\infty} q_j^2 \\ &\leq \lambda \|q\|_{L^2}^2 - N \|q_2\|_{L^2}^2, \end{aligned} \quad (4.39)$$

where $\lambda = \max\{\lambda_1, \lambda_2, \dots, \lambda_k\}$.

Since $(\Delta q, q) = -\int_{\Omega} |\nabla q|^2 dx = -\|\nabla q\|_{L^2}^2$ and $\|q\|_{L^2} \leq C \|\nabla q\|_{L^2}$, we see that

$$(\Delta q, q) = -\|\nabla q\|_{L^2}^2 \leq -\frac{1}{C^2} \|q\|_{L^2}^2. \quad (4.40)$$

Using (4.22) and (4.38)–(4.40), we obtain

$$\begin{aligned} \|q_2\|_{L^2}^2 &\leq \|q\|_{L^2}^2 \\ &= \varepsilon_3 L_e \int_0^t (\Delta q, q) dt + (1 - \varepsilon_3) L_e \int_0^t (\Delta q, q) dt + \int_0^t (u_2, q) dt + \|q_0\|_{L^2}^2 \\ &\leq \varepsilon_3 \lambda L_e \int_0^t \|q\|_{L^2}^2 dt - \varepsilon_3 N L_e \int_0^t \|q_2\|_{L^2}^2 dt - \frac{1 - \varepsilon_3}{C^2} L_e \int_0^t \|q\|_{L^2}^2 dt \\ &\quad + \frac{1}{2} \left(\int_0^t \|u\|_{L^2}^2 dt + \int_0^t \|q\|_{L^2}^2 dt \right) + \|q_0\|_{L^2}^2. \end{aligned} \quad (4.41)$$

Let ε_3 satisfy $\varepsilon_3\lambda \leq (1 - \varepsilon_3)/C^2$. Then,

$$\|q_2\|_{L^2}^2 \leq -\varepsilon_3 N L_e \int_0^t \|q_2\|_{L^2}^2 dt + Mt + \|q_0\|_{L^2}^2, \quad t > t_*. \quad (4.42)$$

By the Gronwall inequality, we find that

$$\|q_2\|_{L^2}^2 \leq e^{-\varepsilon_3 L_e N t} \|q_0\|_{L^2}^2 + \frac{M}{\varepsilon_3 L_e N} (1 - e^{-\varepsilon_3 L_e N t}), \quad t > t_*. \quad (4.43)$$

Then, there exists $t_3 > t_*$ satisfying

$$e^{-\varepsilon_3 L_e N t_3} \|q_0\|_{L^2}^2 \leq \frac{M}{2\varepsilon_3 L_e N}, \quad \frac{M}{\varepsilon_3 L_e N} (1 - e^{-\varepsilon_3 L_e N t_3}) \leq \frac{M}{2\varepsilon_3 L_e N}. \quad (4.44)$$

Since $\delta = 1/N$, for $t > t_3$, it follows that

$$\|q_2\|_{L^2}^2 \leq \frac{M}{\varepsilon_3 L_e N} = \frac{M}{\varepsilon_3 L_e} \delta. \quad (4.45)$$

From (4.27); (4.36) and (4.45) for all $\delta > 0$ there exists $t_0 = \max\{t_1, t_2, t_3\}$ such that when $t > t_0$, it follows that

$$\|P_2 S(t)(u_0, T_0, q_0)\|_H^2 = \|v_2\|_{L^2}^2 + \|T_2\|_{L^2}^2 + \|q_2\|_{L^2}^2 \leq \left(\frac{K_1}{\varepsilon_1 P_r} + \frac{M}{\varepsilon_2} + \frac{M}{\varepsilon_3 L_e} \right) \delta, \quad (4.46)$$

which imply (4.17). From Lemma 2.4, (1.1)–(1.7) have a global attractor in $L^2(\Omega, R^4)$. \square

Acknowledgments

This work was supported by national natural science foundation of China (no. 11271271) and the NSF of Sichuan Education Department of China (no. 11ZA102).

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