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Research Article

On Some Extensions of Szasz Operators Including Boas-Buck-Type Polynomials

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This paper is concerned with a new sequence of linear positive operators which generalize Szasz operators including Boas-Buck-type polynomials. We establish a convergence theorem for these operators and give the quantitative estimation of the approximation process by using a classical approach and the second modulus of continuity. Some explicit examples of our operators involving Laguerre polynomials, Charlier polynomials, and Gould-Hopper polynomials are given. Moreover, a Voronovskaya-type result is obtained for the operators containing Gould-Hopper polynomials.

1. Introduction

The approximation theory, which is concerned with the approximation of functions by simpler calculated functions, is a branch of mathematical analysis. In 1885, Weierstrass identified the set of continuous functions on a closed and bounded interval through uniform approximation by polynomials. Later, Bernstein gave the first impressive example for these polynomials.

In 1953, Korovkin [1] published his celebrated theorem on the approximation of sequences of linear positive operators. This theorem contains a simple and easily applicable criterion to check if a sequence of linear positive operators converges uniformly to the function. One of the well-known examples of linear positive operators is Szasz operators [2]

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \tag{1.1}$$

where $n \in \mathbb{N}$, $x \geq 0$, and $f \in C[0,\infty)$ whenever the above sum converges. Many researchers have dealt with the generalization of Szasz operators in a natural way.

Later, Jakimovski and Leviatan [3] presented a generalization of Szasz operators with Appell polynomials. Let $g(z) = \sum_{k=0}^{\infty} a_k z^k (a_0 \neq 0)$ be an analytic function in the disc$|z| < R$ $(R > 1)$ and assume that $g(1) \neq 0$. The Appell polynomials $p_k(x)$ have generating functions of the form

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k. \tag{1.2}$$

Under the assumption $p_k(x) \geq 0$ for $x \in [0,\infty)$, Jakimovski and Leviatan introduced the linear positive operators $P_n(f;x)$ via

$$P_n(f;x) := \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad \text{for } n \in \mathbb{N}, \tag{1.3}$$

and gave the approximation properties of these operators.

Remark 1.1. For $g(z) = 1$, in view of the generating functions (1.2), we easily find $p_k(x) = x^k/k!$ and from (1.3) we meet again the Szasz operators given by (1.1).

Then, Ismail [4] obtained another generalization of the Szasz operators (1.1) and also Jakimovski and Leviatan operators (1.3) by means of Sheffer polynomials. Let $A(z) = \sum_{k=0}^{\infty} a_k z^k$ $(a_0 \neq 0)$ and $H(z) = \sum_{k=1}^{\infty} h_k z^k$ $(h_1 \neq 0)$ be analytic functions in the disc$|z| < R$ $(R > 1)$ where a_k and h_k are real. The Sheffer polynomials $p_k(x)$ have generating functions of the type

$$A(t)e^{xH(t)} = \sum_{k=0}^{\infty} p_k(x)t^k, \quad |t| < R. \tag{1.4}$$

Using the following assumptions:

$$\begin{aligned} &\text{(i) for } x \in [0,\infty), \quad p_k(x) \geq 0, \\ &\text{(ii) } A(1) \neq 0, \qquad H'(1) = 1. \end{aligned} \tag{1.5}$$

Ismail investigated the approximation properties of linear positive operators given by

$$T_n(f;x) := \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \quad \text{for } n \in \mathbb{N}. \tag{1.6}$$

Remark 1.2. For $H(t) = t$, one can observe that the generating functions (1.4) reduce to (1.2) and from this fact, the operators (1.6) return to the operators (1.3).

Remark 1.3. For $H(t) = t$ and $A(t) = 1$, it is easy to get the Szasz operators from the operators (1.6).

Recently, Varma et al. [5] constructed linear positive operators including Brenke-type polynomials. Brenke-type polynomials [6] have generating functions of the form

$$A(t)B(xt) = \sum_{k=0}^{\infty} p_k(x)t^k, \tag{1.7}$$

where A and B are analytic functions

$$\begin{aligned} A(t) &= \sum_{r=0}^{\infty} a_r t^r, \quad a_0 \neq 0, \\ B(t) &= \sum_{r=0}^{\infty} b_r t^r, \quad b_r \neq 0 \ (r \geq 0), \end{aligned} \tag{1.8}$$

and have the following explicit expression:

$$p_k(x) = \sum_{r=0}^{k} a_{k-r} b_r x^r, \quad k = 0, 1, 2, \ldots. \tag{1.9}$$

Using the following assumptions

$$\begin{aligned} &\text{(i)}\ A(1) \neq 0, \quad \frac{a_{k-r} b_r}{A(1)} \geq 0, \quad 0 \leq r \leq k,\ k = 0, 1, 2, \ldots, \\ &\text{(ii)}\ B : [0, \infty) \longrightarrow (0, \infty), \\ &\text{(iii)}\ (1.7) \text{ and the power series } (1.8) \text{ converge for } |t| < R \quad (R > 1), \end{aligned} \tag{1.10}$$

Varma et al. introduced the following linear positive operators involving the Brenke-type polynomials

$$L_n(f; x) := \frac{1}{A(1)B(nx)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{1.11}$$

where $x \geq 0$ and $n \in \mathbb{N}$.

Remark 1.4. Let $B(t) = e^t$. In this case, the operators (1.11) (resp., (1.7)) reduce to the operators given by (1.3) (resp., (1.2)).

Remark 1.5. Let $B(t) = e^t$ and $A(t) = 1$. We meet again the Szasz operators (1.1).

In this paper, our aim is to construct linear positive operators by using Boas-Buck-type polynomials including the Brenke-type polynomials, Sheffer polynomials, and Appell polynomials with special cases. Boas-Buck-type polynomials [7] have generating functions of the type

$$A(t)B(xH(t)) = \sum_{k=0}^{\infty} p_k(x)t^k, \tag{1.12}$$

where A, B, and H are analytic functions

$$A(t) = \sum_{k=0}^{\infty} a_k t^k \quad (a_0 \neq 0), \qquad B(t) = \sum_{k=0}^{\infty} b_k t^k \quad (b_k \neq 0),$$
$$H(t) = \sum_{k=1}^{\infty} h_k t^k \quad (h_1 \neq 0). \tag{1.13}$$

We will restrict ourselves to the Boas-Buck-type polynomials satisfying

$$\begin{aligned}
&\text{(i) } A(1) \neq 0, \qquad H'(1) = 1, \qquad p_k(x) \geq 0, \quad k = 0, 1, 2, \ldots, \\
&\text{(ii) } B : \mathbb{R} \longrightarrow (0, \infty), \\
&\text{(iii) (1.12) and the power series (1.13) converge for } |t| < R (R > 1).
\end{aligned} \tag{1.14}$$

Now, given the above restrictions, we present a new form of linear positive operators with Boas-Buck-type polynomials as follows:

$$\mathcal{B}_n(f; x) := \frac{1}{A(1)B(nxH(1))} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{1.15}$$

where $x \geq 0$ and $n \in \mathbb{N}$.

Remark 1.6. Let $H(t) = t$. The operators (1.15) (resp., (1.12)) reduce to the operators given by (1.11) (resp., (1.7)).

Remark 1.7. Let $B(t) = e^t$. The operators (1.15) (resp., (1.12)) return to the operators given by (1.6) (resp., (1.4)).

Remark 1.8. Let $H(t) = t$ and $B(t) = e^t$. It is obvious that one can get the operators (1.3) from the operators (1.15). In addition, if we choose $A(t) = 1$, we meet again the well-known Szasz operators (1.1).

The paper is divided into three sections. Following the introduction, Section 2 is devoted to obtain qualitative and quantitative results for the operators (1.15). In the last section, we give some significant illustrations with the help of Laguerre, Charlier, and Gould-Hopper polynomials for the operators (1.15). Moreover, we give a Voronovskaya-type theorem for the operators including Gould-Hopper polynomials.

2. Approximation Properties of $\mathcal{B}_n$ Operators

In this section, with the help of well-known Korovkin's theorem, we get approximation results by means of $\mathcal{B}_n$ linear positive operators. Next, we present quantitative results for estimating the error of approximation using the classical approach and the second modulus of continuity.

Lemma 2.1. *For the operators $\mathcal{B}_n$, one has*

$$\begin{aligned}
&\mathcal{B}_n(1;x) = 1,\\
&\mathcal{B}_n(s;x) = \frac{B'(nxH(1))}{B(nxH(1))}x + \frac{A'(1)}{nA(1)},\\
&\mathcal{B}_n\left(s^2;x\right) = \frac{B''(nxH(1))}{B(nxH(1))}x^2 + \frac{[2A'(1) + (1 + H''(1))A(1)]B'(nxH(1))}{nA(1)B(nxH(1))}x\\
&\qquad + \frac{A''(1) + A'(1)}{n^2A(1)},
\end{aligned} \tag{2.1}$$

for any $x \in [0,\infty)$.

Proof. From the generating functions of the Boas-Buck-type polynomials given by (1.12), we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty} p_k(nx) = A(1)B(nxH(1)),\\
&\sum_{k=0}^{\infty} kp_k(nx) = A'(1)B(nxH(1)) + nxA(1)B'(nxH(1)),\\
&\sum_{k=0}^{\infty} k^2p_k(nx) = \left(A''(1) + A'(1)\right)B(nxH(1)) + \left(2A'(1) + A(1) + A(1)H''(1)\right)\\
&\qquad \times B'(nxH(1))nx + A(1)B''(nxH(1))(nx)^2.
\end{aligned} \tag{2.2}$$

With regard to these equalities, we get the assertions of the lemma. □

Let us define the class of E as follows:

$$E := \left\{f : x \in [0,\infty), \frac{f(x)}{1 + x^2} \text{ is convergent as } x \longrightarrow \infty\right\}. \tag{2.3}$$

Theorem 2.2. *Let $f \in C[0,\infty) \cap E$ and assume that*

$$\lim_{y\to\infty}\frac{B'(y)}{B(y)} = 1, \qquad \lim_{y\to\infty}\frac{B''(y)}{B(y)} = 1. \tag{2.4}$$

Then,

$$\lim_{n\to\infty} \mathcal{B}_n(f;x) = f(x) \tag{2.5}$$

uniformly on each compact subset of $[0,\infty)$.

Proof. According to Lemma 2.1 and taking into account the assumptions (2.4), we find

$$\lim_{n\to\infty} \mathcal{B}_n\left(s^i;x\right) = x^i, \quad i = 0,1,2. \tag{2.6}$$

The above-mentioned convergences are satisfied uniformly in each compact subset of $[0,\infty)$. Applying the universal Korovkin-type property (vi) of Theorem 4.1.4 from [8], we lead to the desired result. □

In order to estimate the rate of convergence, we will give some definitions and lemmas.

Definition 2.3. Let $f \in \widetilde{C}[0,\infty)$ and $\delta > 0$. The modulus of continuity $\omega(f;\delta)$ of the function f is defined by

$$\omega(f;\delta) := \sup_{\substack{x,y\in[0,\infty)\\ |x-y|\le\delta}} \left|f(x) - f(y)\right|, \tag{2.7}$$

where $\widetilde{C}[0,\infty)$ is the space of uniformly continuous functions on $[0,\infty)$.

Definition 2.4. The second modulus of continuity of the function $f \in C[a,b]$ is defined by

$$\omega_2(f;\delta) := \sup_{0<t\le\delta} \|f(\cdot+2t) - 2f(\cdot+t) + f(\cdot)\|, \tag{2.8}$$

where $\|f\| = \max_{x\in[a,b]}|f(x)|$.

Lemma 2.5 (Gavrea and Raşa [9]). *Let $g \in C^2[0,a]$ and $(K_n)_{n\ge0}$ be a sequence of linear positive operators with the property $K_n(1;x) = 1$. Then,*

$$\left|K_n(g;x) - g(x)\right| \le \|g'\|\sqrt{K_n\left((s-x)^2;x\right)} + \frac{1}{2}\|g''\|K_n\left((s-x)^2;x\right). \tag{2.9}$$

Lemma 2.6 (Zhuk [10]). *Let $f \in C[a,b]$ and $h \in (0,(b-a)/2)$. Let f_h be the second-order Steklov function attached to the function f. Then, the following inequalities are satisfied:*

$$\begin{aligned} &\text{(i)}\ \|f_h - f\| \le \frac{3}{4}\omega_2(f;h),\\ &\text{(ii)}\ \|f_h''\| \le \frac{3}{2h^2}\omega_2(f;h). \end{aligned} \tag{2.10}$$

Lemma 2.7. *For $x \in [0, \infty)$, one has*

$$\mathcal{B}_n\left((s-x)^2; x\right) = \frac{B''(nxH(1)) - 2B'(nxH(1)) + B(nxH(1))}{B(nxH(1))}x^2 + \frac{A(1)(H''(1)+1)B'(nxH(1)) + 2A'(1)[B'(nxH(1)) - B(nxH(1))]}{nA(1)B(nxH(1))}x + \frac{A''(1) + A'(1)}{n^2 A(1)}. \tag{2.11}$$

Proof. Using the linearity property of $\mathcal{B}_n$ operators, one can write

$$\mathcal{B}_n\left((s-x)^2; x\right) = \mathcal{B}_n\left(s^2; x\right) - 2x\mathcal{B}_n(s; x) + x^2\mathcal{B}_n(1; x). \tag{2.12}$$

Applying Lemma 2.1, we obtain the equality stated in the lemma. □

Generally, we use the modulus of continuity and second modulus of continuity to obtain quantitative error estimation for convergence by linear positive operators. Now, we will calculate the rate of convergence in the following two theorems.

Theorem 2.8. *Let $f \in \widetilde{C}[0, \infty) \cap E$. $\mathcal{B}_n$ operators verify the following inequality:*

$$\left|\mathcal{B}_n(f; x) - f(x)\right| \le 2\omega\left(f; \sqrt{\vartheta_n(x)}\right), \tag{2.13}$$

where

$$\begin{aligned}\vartheta := \vartheta_n(x) &= \mathcal{B}_n\left((s-x)^2; x\right) \\ &= \frac{B''(nxH(1)) - 2B'(nxH(1)) + B(nxH(1))}{B(nxH(1))}x^2 \\ &\quad + \frac{A(1)(H''(1)+1)B'(nxH(1)) + 2A'(1)[B'(nxH(1)) - B(nxH(1))]}{nA(1)B(nxH(1))}x \\ &\quad + \frac{A''(1) + A'(1)}{n^2 A(1)}.\end{aligned} \tag{2.14}$$

Proof. Making use of Lemma 2.1 and the property of modulus of continuity, we deduce

$$\begin{aligned}\left|\mathcal{B}_n(f; x) - f(x)\right| &\le \frac{1}{A(1)B(nxH(1))}\sum_{k=0}^{\infty} p_k(nx)\left|f\left(\frac{k}{n}\right) - f(x)\right| \\ &\le \left\{1 + \frac{1}{A(1)B(nxH(1))}\frac{1}{\delta}\sum_{k=0}^{\infty} p_k(nx)\left|\frac{k}{n} - x\right|\right\}\omega(f; \delta).\end{aligned} \tag{2.15}$$

Taking into account the Cauchy-Schwarz inequality and then by using Lemma 2.7, we get

$$\begin{aligned}\sum_{k=0}^{\infty} p_k(nx)\left|\frac{k}{n}-x\right| &\le \left\{\sum_{k=0}^{\infty} p_k(nx)\right\}^{1/2}\left\{\sum_{k=0}^{\infty} p_k(nx)\left|\frac{k}{n}-x\right|^2\right\}^{1/2} \\ &= A(1)B(nxH(1))\sqrt{\mathcal{B}_n\left((s-x)^2;x\right)}.\end{aligned} \tag{2.16}$$

Considering the last inequality in (2.15), we obtain

$$\left|\mathcal{B}_n(f;x)-f(x)\right| \le \left\{1+\frac{1}{\delta}\sqrt{\vartheta_n(x)}\right\}\omega(f;\delta), \tag{2.17}$$

where $\vartheta_n(x)$ is given by (2.14). In inequality (2.17), by choosing $\delta = \sqrt{\vartheta_n(x)}$, we get the desired result. □

Theorem 2.9. *For $f \in C[0,a]$, the following estimate*

$$\left|\mathcal{B}_n(f;x)-f(x)\right| \le \frac{2}{a}\|f\|h^2+\frac{3}{4}\left(a+2+h^2\right)\omega_2(f;h) \tag{2.18}$$

holds, where

$$h := h_n(x) = \sqrt[4]{\mathcal{B}_n\left((s-x)^2;x\right)}. \tag{2.19}$$

Proof. Let f_h be the second-order Steklov function attached to the function f. With regard to the identity $\mathcal{B}_n(1;x)=1$, we have

$$\begin{aligned}\left|\mathcal{B}_n(f;x)-f(x)\right| &\le \left|\mathcal{B}_n(f-f_h;x)\right|+\left|\mathcal{B}_n(f_h;x)-f_h(x)\right|+\left|f_h(x)-f(x)\right| \\ &\le 2\left\|f_h-f\right\|+\left|\mathcal{B}_n(f_h;x)-f_h(x)\right|.\end{aligned} \tag{2.20}$$

Taking account of the fact that $f_h \in C^2[0,a]$, it follows from Lemma 2.5

$$\left|\mathcal{B}_n(f_h;x)-f_h(x)\right| \le \|f_h'\|\sqrt{\mathcal{B}_n\left((s-x)^2;x\right)}+\frac{1}{2}\|f_h''\|\mathcal{B}_n\left((s-x)^2;x\right). \tag{2.21}$$

If one combines Landau inequality with Lemma 2.6, we can write

$$\begin{aligned}\|f_h'\| &\le \frac{2}{a}\|f_h\|+\frac{a}{2}\|f_h''\| \\ &\le \frac{2}{a}\|f\|+\frac{3a}{4}\frac{1}{h^2}\omega_2(f;h).\end{aligned} \tag{2.22}$$

From the last inequality and Lemma 2.6, (2.21) becomes by taking $h = \sqrt[4]{\mathcal{B}_n((s-x)^2;x)}$

$$\left|\mathcal{B}_n(f_h;x) - f_h(x)\right| \le \frac{2}{a}\|f\|h^2 + \frac{3a}{4}\omega_2(f;h) + \frac{3}{4}h^2\omega_2(f;h). \tag{2.23}$$

Substituting (2.23) in (2.20), hence Lemma 2.6 gives the proof of the theorem. □

Remark 2.10. In Theorem 2.9, we present the proof for $h \in (0, a/2)$. For the special case $B(t) = e^t, H(t) = t, A(t) = 1$ and $x = 0$, one can get $h = 0$ from the equality $h := h_n(x) = \sqrt[4]{\mathcal{B}_n((s-x)^2;x)}$. The inequality obtained in Theorem 2.9 still holds true when $h = 0$.

Remark 2.11. Note that in Theorems 2.8–2.9 when $n \to \infty$, respectively, ϑ and h tend to zero under the assumptions (2.4).

3. Examples

Example 3.1. Laguerre polynomials are one of the most important classical orthogonal polynomials in the literature. Such polynomials are used in every area of mathematics. In addition, these polynomials have served with several interesting properties for physicists. For example, Laguerre polynomials arise as solutions of the Coulomb potential in quantum mechanics.

Laguerre polynomials have generating functions of the form

$$\frac{1}{(1-t)^{\alpha+1}}\exp\left(-\frac{xt}{1-t}\right) = \sum_{k=0}^{\infty} L_k^{(\alpha)}(x)t^k, \quad |t| < 1, \tag{3.1}$$

and explicit expressions

$$L_k^{(\alpha)}(x) = \sum_{m=0}^{k} \frac{(\alpha+k)!}{(k-m)!(\alpha+m)!m!}(-x)^m, \quad \alpha > -1. \tag{3.2}$$

It is clear that Laguerre polynomials are Boas-Buck-type polynomials. Note that when $x \in (-\infty, 0]$, $L_k^{(\alpha)}(x)$ are positive. For ensuring the restrictions (1.14) and the assumptions (2.4), we have to modify the generating functions (3.1) as follows:

$$\frac{1}{(1-(t/2))^{\alpha+1}}\exp\left(\frac{xt}{2(2-t)}\right) = \sum_{k=0}^{\infty} \frac{L_k^{(\alpha)}(-x/2)}{2^k}t^k, \quad |t| < 2. \tag{3.3}$$

With the help of generating functions (3.3), we find the following linear positive operators including Laguerre polynomials from the operators (1.15)

$$\widetilde{L}_n(f;x) = e^{-(nx/2)}\sum_{k=0}^{\infty}\frac{L_k^{(\alpha)}(-(nx/2))}{2^{\alpha+k+1}}f\left(\frac{k}{n}\right), \tag{3.4}$$

where $\alpha > -1$ and $x \in [0,\infty)$.

Remark 3.2. It is worthy to note that we obtain new linear positive operators different from the one given in [4].

Example 3.3. Varma and Taşdelen [11] gave the following linear positive operators involving Charlier polynomials as a generalization of the Szasz operators:

$$L_n(f;x,a) = e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}f\left(\frac{k}{n}\right), \tag{3.5}$$

where $a > 1$, $x \in [0,\infty)$ and $C_k^{(a)}(x)$ Charlier polynomials have the generating functions of the type

$$e^t\left(1-\frac{t}{a}\right)^x = \sum_{k=0}^{\infty}\frac{C_k^{(a)}(x)}{k!}t^k, \quad |t| < a. \tag{3.6}$$

Note that one can get the operators (3.5) as an example of the operators given by the equality (1.6) defined in [4].

On the other hand, Charlier polynomials are also the Boas-Buck-type polynomials by choosing

$$A(t) = e^t, \qquad B(t) = e^t, \qquad H(t) = \ln\left(1-\frac{t}{a}\right). \tag{3.7}$$

For ensuring the restrictions (1.14) and the assumptions (2.4), we have to change the generating functions (3.6) by

$$e^t e^{-(a-1)x\,\ln(1-(t/a))} = \sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)x)}{k!}t^k, \quad |t| < a. \tag{3.8}$$

In view of the generating functions (3.8), we have the linear positive operators (3.5) investigated in [11] from the operators (1.15).

Example 3.4. Gould-Hopper polynomials [12] have the generating functions of the form

$$e^{ht^{d+1}}\exp(xt) = \sum_{k=0}^{\infty}g_k^{d+1}(x,h)\frac{t^k}{k!} \tag{3.9}$$

and the explicit representations

$$g_k^{d+1}(x,h) = \sum_{s=0}^{[k/(d+1)]} \frac{k!}{s!(k-(d+1)s)!} h^s x^{k-(d+1)s}, \tag{3.10}$$

where, as usual, $[\cdot]$ denotes the integer part. Gould-Hopper polynomials $g_k^{d+1}(x,h)$ are d-orthogonal polynomial set of Hermite type [13]. Van Iseghem [14] and Maroni [15] discovered the notion of d-orthogonality. Gould-Hopper polynomials are Boas-Buck-type polynomials with

$$A(t) = e^{ht^{d+1}}, \qquad B(t) = e^t, \qquad H(t) = t. \tag{3.11}$$

Under the assumption $h \geq 0$; the restrictions (1.14) and condition (2.4) for the operators $\mathcal{B}_n$ given by (1.15) are satisfied. With the help of the generating functions (3.9), we obtain the explicit form of $\mathcal{B}_n$ operators including Gould-Hopper polynomials by

$$L_n^*(f;x) = e^{-nx-h} \sum_{k=0}^{\infty} \frac{g_k^{d+1}(nx,h)}{k!} f\left(\frac{k}{n}\right), \tag{3.12}$$

where $x \in [0,\infty)$.

Remark 3.5. First time, the operators L_n^* are given in [5] as an explicit example of the operators (1.11). For $h = 0$, we obtain that

$$g_k^{d+1}(nx,0) = (nx)^k. \tag{3.13}$$

Substituting $h = 0$ and $g_k^{d+1}(nx,0) = (nx)^k$ in the operators (3.12), we get the well-known Szasz operators. By the help of L_n^* operators, we present an attractive generalization of the Szasz operators with Gould-Hopper polynomials.

Next, we give a Voronovskaya-type theorem for the operators L_n^*. In order to prove this theorem, we need some auxiliary results.

Lemma 3.6. *For the operators L_n^*, one has*

$$\begin{aligned}
L_n^*(1;x) &= 1, \\
L_n^*(s;x) &= x + \frac{h(d+1)}{n}, \\
L_n^*\left(s^2;x\right) &= x^2 + \frac{2h(d+1)+1}{n}x + \frac{h(h+1)(d+1)^2}{n^2}, \\
L_n^*\left(s^3;x\right) &= x^3 + \frac{3h(d+1)+3}{n}x^2 \\
&\quad + \frac{3h^2(d+1)^2 + 3h(d+1)(d+2)+1}{n^2}x + \frac{h\left(h^2+3h+1\right)(d+1)^3}{n^3},
\end{aligned}$$

$$L_n^*\left(s^4;x\right) = x^4 + \frac{4h(d+1)+6}{n}x^3 + \frac{6h^2(d+1)^2+6h(d+1)(d+3)+7}{n^2}x^2 + \frac{6h^2(d+1)^2(2d+3)+2h(d+1)\left(2d^2+7d+7\right)+4h^3(d+1)^3+1}{n^3}x + \frac{h\left(h^3+6h^2+7h+1\right)(d+1)^4}{n^4}. \tag{3.14}$$

Proof. By virtue of the generating functions (3.9) for Gould-Hopper polynomials, we obtain the above equalities. □

Lemma 3.7. *For $x \geq 0$, the following equalities hold:*

$$\begin{aligned} L_n^*\left((s-x)^2;x\right) &= \frac{x}{n} + \frac{h(h+1)(d+1)^2}{n^2}, \\ L_n^*\left((s-x)^4;x\right) &= \frac{3}{n^2}x^2 + \frac{6h^2(d+1)^2+2h(d+1)(3d+5)+1}{n^3}x \\ &\quad + \frac{h\left(h^3+6h^2+7h+1\right)(d+1)^4}{n^4}. \end{aligned} \tag{3.15}$$

Proof. According to Lemma 3.6, it is easy to get the above equalities. □

Theorem 3.8. *Let $f \in C^2[0,a]$. Then, one has*

$$\lim_{n\to\infty} n\left[L_n^*(f;x) - f(x)\right] = h(d+1)f'(x) + \frac{xf''(x)}{2}. \tag{3.16}$$

Proof. In view of Taylor formula for the function f, we find

$$f(s) = f(x) + (s-x)f'(x) + \frac{(s-x)^2}{2!}f''(x) + (s-x)^2\eta(s;x), \tag{3.17}$$

where $\eta(s;x) \in C[0,a]$ and $\lim_{s\to x}\eta(s;x) = 0$. Applying L_n^* to the both sides of (3.17), we get

$$\begin{aligned} L_n^*(f;x) &= f(x) + f'(x)L_n^*(s-x;x) + \frac{f''(x)}{2}L_n^*\left((s-x)^2;x\right) \\ &\quad + L_n^*\left((s-x)^2\eta(s;x);x\right). \end{aligned} \tag{3.18}$$

According to Lemmas 3.6–3.7, (3.18) becomes

$$L_n^*(f;x) = f(x) + f'(x)\frac{h(d+1)}{n} + \frac{f''(x)}{2}\left[\frac{x}{n} + \frac{h(h+1)(d+1)^2}{n^2}\right] + I, \tag{3.19}$$

where

$$I := e^{-nx-h} \sum_{k=0}^{\infty} \frac{g_k^{d+1}(nx,h)}{k!} \left(\frac{k}{n} - x\right)^2 \eta\left(\frac{k}{n}; x\right). \tag{3.20}$$

Now, we consider the sum I as follows:

$$\begin{aligned} I = {} & e^{-nx-h} \sum_{|(k/n)-x| \le \delta} \frac{g_k^{d+1}(nx,h)}{k!} \left(\frac{k}{n} - x\right)^2 \eta\left(\frac{k}{n}; x\right) \\ & + e^{-nx-h} \sum_{|(k/n)-x| > \delta} \frac{g_k^{d+1}(nx,h)}{k!} \left(\frac{k}{n} - x\right)^2 \eta\left(\frac{k}{n}; x\right). \end{aligned} \tag{3.21}$$

From the continuity of function η, it results that for all $\varepsilon > 0$, there exists a positive δ such that if $|(k/n) - x| \le \delta$, then $|\eta(k/n; x)| < \varepsilon$. Furthermore, since the function η is bounded, we can write $|\eta(k/n; x)| < M$ for $|(k/n) - x| > \delta$. In view of these facts, (3.21) leads to

$$I \le \varepsilon L_n^*\left((s-x)^2; x\right) + M e^{-nx-h} \sum_{|(k/n)-x| > \delta} \frac{g_k^{d+1}(nx,h)}{k!} \left(\frac{k}{n} - x\right)^2. \tag{3.22}$$

Taking into account the fact

$$e^{-nx-h} \sum_{|(k/n)-x| > \delta} \frac{g_k^{d+1}(nx,h)}{k!} \left(\frac{k}{n} - x\right)^2 \le \frac{1}{\delta^2} L_n^*\left((s-x)^4; x\right) \tag{3.23}$$

in the last inequality, we have

$$I \le \varepsilon L_n^*\left((s-x)^2; x\right) + \frac{M}{\delta^2} L_n^*\left((s-x)^4; x\right). \tag{3.24}$$

Substituting the inequality (3.24) in the equality (3.19), then from Lemma 3.7, we obtain

$$\begin{aligned} L_n^*(f; x) - f(x) \le {} & f'(x) \frac{h(d+1)}{n} + \left(\varepsilon + \frac{f''(x)}{2}\right) \left[\frac{x}{n} + \frac{h(h+1)(d+1)^2}{n^2}\right] \\ & + \frac{M}{\delta^2} \left[\frac{3}{n^2} x^2 + \frac{6h^2(d+1)^2 + 2h(d+1)(3d+5) + 1}{n^3} x \right. \\ & \left. + \frac{h\left(h^3 + 6h^2 + 7h + 1\right)(d+1)^4}{n^4}\right]. \end{aligned} \tag{3.25}$$

Equivalently, we can write

$$L_n^*(f;x) - f(x) = \mathcal{O}\left(\frac{1}{n}\right)\left\{f'(x)h(d+1) + \left(\varepsilon + \frac{f''(x)}{2}\right)\left[x + \frac{h(h+1)(d+1)^2}{n}\right]\right.$$
$$+ \frac{M}{\delta^2}\left[\frac{3}{n}x^2 + \frac{6h^2(d+1)^2 + 2h(d+1)(3d+5)+1}{n^2}x\right.$$
$$\left.\left.+\frac{h\left(h^3+6h^2+7h+1\right)(d+1)^4}{n^3}\right]\right\}. \tag{3.26}$$

Taking limits for $n \to \infty$, (3.26) becomes

$$\lim_{n\to\infty} n\left[L_n^*(f;x) - f(x)\right] = h(d+1)f'(x) + \frac{xf''(x)}{2}, \tag{3.27}$$

which completes the proof. □

Remark 3.9. Theorem 3.8 is an explicit example of Gonska's result given in [16]. It is worthy to note that this Voronovskaya-type result is given for the L_n^* operators which contain Gould-Hopper polynomials.

Remark 3.10. Taking $h = 0$ in Theorem 3.8, we get a Voronovskaya-type result for the Szasz operators.

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