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Research Article

Ground-State Solutions for a Class of N-Laplacian Equation with Critical Growth

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We investigate the existence of ground-state solutions for a class of N-Laplacian equation with critical growth in $\mathbb{R}^N$. Our proof is based on a suitable Trudinger-Moser inequality, Pohozaev-Pucci-Serrin identity manifold, and mountain pass lemma.

1. Introduction

Consider the following N-Laplacian equation:

$$\begin{gathered} -\Delta_N u + |u|^{N-2}u = f(u), \quad \text{in } \mathbb{R}^N, \\ u > 0, \quad \text{in } \mathbb{R}^N,\ u \in W^{1,N}\left(\mathbb{R}^N\right), \end{gathered} \tag{1.1}$$

where $N \geq 2$. $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian, the nonlinear term $f(u)$ has critical growth.

The interest in these problems lies in that fact that the order of the Laplacian is the same as the dimension N of the underlying space. The classical case of this problem that $N = 2$, and the the problem (1.1) reduces to

$$-\Delta u + u = f(u), \quad \text{in } \mathbb{R}^2, \tag{1.2}$$

has been treated by Atkinson and Peletier [1] and Berestycki and Lions [2]. They obtained the existence of ground-state solution which the nonlinear term $f(u)$ is subcritical growth.

Alves et al. [3] extend their results to the critical growth. As $N \neq 2$, do Ó and Medeiros [4] consider the following N-Laplacian equation problem:

$$-\Delta_N u = g(u), \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $g : \mathbb{R} \mapsto \mathbb{R}$ has a subcritical growth and obtain a mountain pass characterization of the ground-state solution for the problem (1.3). In the present paper, we will improve and complement some of the results cited above.

Assume the function $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous and satisfies the following conditions:

(g_1) $\lim_{s \to 0^+} (f(s)/s|s|^{N-2}) = 0$;

(g_2) There exist constants $\alpha_0, b_1, b_2 > 0$ such that $|f(s)| \leq b_1|s|^{N-1} + b_2[\exp(\alpha_0|s|^{N/(N-1)}) - S_{N-2}(\alpha_0, s)]$, where $S_{N-2}(\alpha_0, s) = \sum_{k=0}^{N-2}(\alpha_0^k/k!)|s|^{Nk/(N-1)}$;

(g_3) There exist $\lambda > 0$ and $q > N$ such that $f(s) \geq \lambda s^{q-1}$, for every $s \geq 0$.

Remark 1.1. Condition (g_2) implies that f has a critical growth with critical exponent α_0.

Consider the energy functional $I : W^{1,N}(\mathbb{R}^N) \mapsto \mathbb{R}$

$$I(u) = \frac{1}{N}\int_{\mathbb{R}^N}\left(|\nabla u|^N + |u|^N\right)dx - \int_{\mathbb{R}^N} F(u)dx, \tag{1.4}$$

where $F(s) = \int_0^s f(t)dt$. By a ground-state solution, we mean a solution such $\omega \in W^{1,N}(\mathbb{R}^N)$ such that $I(\omega) \leq I(u)$ for every nontrivial solution u of the problem (1.1). Let $C_q > 0$ denote the best constant of Sobolev embeddings:

$$W^{1,N}\left(\mathbb{R}^N\right) \hookrightarrow L^q\left(\mathbb{R}^N\right), \tag{1.5}$$

for $q \in (N, +\infty)$, that is,

$$C_q\left[\int_{\mathbb{R}^N} |u|^q dx\right]^{N/q} \leq \int_{\mathbb{R}^N}\left(|\nabla u|^N + |u|^N\right)dx, \tag{1.6}$$

for all $u \in W^{1,N}(\mathbb{R}^N)$.

Now we state our main theorem in this paper.

Theorem 1.2. *If f satisfies (g_1), (g_2), and (g_3), with*

$$\lambda > \left(\frac{q-N}{q}\right)^{(q-N)/N} C_q^{q/N}, \tag{1.7}$$

then the problem (1.1) *possesses a nontrivial ground-state solution.*

In this paper, we complement some results [4] from subcritical case to the critical case. Furthermore, the ground-state solution to the problem (1.1) is obtained without assuming that the function

$$s \mapsto \frac{f(s)}{|s|^{N-1}} \tag{1.8}$$

is increasing for $s > 0$ (see [5]), and the so-called Ambrosetti-Rabinowitz condition: there exists $\theta > N$, such that for all $x \in \mathbb{R}^N$,

$$0 < \theta F(x,u) \leq u f(x,u). \tag{1.9}$$

The paper is organized as follows. Section 2 contains some technical results which allows us to give a variational approach for our results. In Section 3, we prove our main results.

2. The Variational Framework

For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^N)$ denotes the Lebesgue spaces with the norm $\|u\|_{L^p(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |u|^p dx)^{1/p}$, $W^{1,p}(\mathbb{R}^N)$ denotes the Sobolev spaces with the norm $\|u\|_{W^{1,p}(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) dx)^{1/p}$. As $p = N$, we have the following version of Trudinger-Moser inequality.

Lemma 2.1 (see [6]). *If $N \geq 2$, $\alpha > 0$ and $u \in W^{1,N}(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} \left[\exp\left(\alpha |u|^{N/(N-1)}\right) - S_{N-2}(\alpha, u)\right] dx < \infty. \tag{2.1}$$

Moreover, if $\|\nabla u\|^N_{L^N(\mathbb{R}^N)} \leq 1$, $\|u\|_{L^N(\mathbb{R}^N)} \leq M < \infty$ and $\alpha < \alpha_N$, then there exists a constant C, which depends only on N, M, and α, such that

$$\int_{\mathbb{R}^N} \left[\exp\left(\alpha |u|^{N/(N-1)}\right) - S_{N-2}(\alpha, u)\right] dx \leq C(N, M, \alpha), \tag{2.2}$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ and $\omega^{1/(N-1)}$ is the measure of the unit sphere in $\mathbb{R}^N$.

In the sequel, since we seek positive solutions, and assume that $f(s) = 0$ for $s \leq 0$. Consider the following minimization problem:

$$\min\left\{ \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^N dx : \int_{\mathbb{R}^N} G(u) dx = 0 \right\}, \tag{2.3}$$

where $g(s) = f(s) - s|s|^{N-2}$, $G(s) = \int_0^s g(t) dt = F(s) - (1/N)s^N$. Since the problem (1.1) is an autonomous problem, under the Schwarz symmetric process, we can minimize the problem

(2.3) on the space $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$, the subspace of $W^{1,N}(\mathbb{R}^N)$ formed by radially symmetric functions. Indeed, let u^* be the Schwarz symmetrization of u, we have

$$\int_{\mathbb{R}^N} G(u^*)dx = \int_{\mathbb{R}^N} G(u)dx, \qquad \int_{\mathbb{R}^N} |\nabla u^*|^N dx \leq \int_{\mathbb{R}^N} |\nabla u|^N dx. \tag{2.4}$$

Hence, we can minimize the problem (2.3) on the space $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$ (see [7]). Now, we defined the following notations

$$\begin{gathered} m = \inf\{I(u) : u \text{ is nontrivial solution of the problem (1.1)}\} \\ A = \inf\left\{\frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N dx : \int_{\mathbb{R}^N} G(u)dx = 0\right\} \\ b = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)), \end{gathered} \tag{2.5}$$

where $\Gamma = \{\gamma \in C([0,1], W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)) : \gamma(0) = 0, I(\gamma(1)) < 0\}$.

We recall that Pohozaev-Pucci-Serrin identity shows that any solutions u of the problem (1.1) should satisfies the Pohozaev-Pucci-Serrin identity:

$$(N-p)\int_{\mathbb{R}^N} |\nabla u|^p dx = Np\int_{\mathbb{R}^N} G(u)dx. \tag{2.6}$$

Then, as $p = N$, we have $\int_{\mathbb{R}^N} G(u)dx = 0$.

Hence, we have the Pohozaev identity manifold:

$$\begin{aligned} \mathcal{P} &= \left\{u \in W^{1,N}\left(\mathbb{R}^N\right) \setminus \{0\} : (N-p)\int_{\mathbb{R}^N} |\nabla u|^p dx = Np\int_{\mathbb{R}^N} G(u)dx\right\} \\ &= \left\{u \in W^{1,N}\left(\mathbb{R}^N\right) \setminus \{0\} : \int_{\mathbb{R}^N} G(u)dx = 0\right\}. \end{aligned} \tag{2.7}$$

So, we have

$$A = \inf_{u\in\mathcal{P}} \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N dx, \qquad m = \inf_{u\in\tau} I(u), \tag{2.8}$$

where $\tau = \{u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\} : I'(u) = 0\}$.

In what follows, we will show that A is attained, and afterwards we prove that

$$m = A = b, \tag{2.9}$$

thereby proving that the problem (1.1) has a ground-state solution.

3. The Proof of Theorem 1.2

In this section, we prove that A is attained, the equality (2.9) is satisfied. Hence the proof of Theorem 1.2 is obtained.

In the following, we consider the following minimax value:

$$c = \inf_{0 \neq v \in W^{1,N}\left(\mathbb{R}^N\right)} \max_{t \geq 0} I(tv). \tag{3.1}$$

Now, we show a sufficient condition, on a sequence $\{v_n\}$ to get a convergence like $F(v_n) \to F(v)$ in $L^1(\mathbb{R}^N)$.

Lemma 3.1. *Assume that f satisfies (g_1) and (g_2), and let $\{v_n\}$ be a sequence in $W^{1,N}_{rad}(\mathbb{R}^N)$ such that $\|\nabla v_n\|^N_{L^N(\mathbb{R}^N)} \leq 1, \|v_n\|_{L^N(\mathbb{R}^N)} \leq M < \infty$, then we have*

$$\int_{\mathbb{R}^N} F(v_n) dx \longrightarrow \int_{\mathbb{R}^N} F(v) dx, \tag{3.2}$$

where $v_n \rightharpoonup v$ in $W^{1,N}(\mathbb{R}^N)$.

Proof. Without loss of generality, we assume that there exist $v \in W^{1,N}_{\text{rad}}(\mathbb{R}^N)$ such that

$$\begin{aligned} v_n &\rightharpoonup v, \quad \text{in } W^{1,N}_{\text{rad}}\left(\mathbb{R}^N\right), \\ v_n &\rightharpoonup v, \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{3.3}$$

Let v^* is the Schwarz symmetrization of v, then we have

$$\begin{gathered} \int_{\mathbb{R}^N} v\left[\exp\left(\alpha_0 |v|^{N/(N-1)}\right) - S_{N-2}(\alpha_0, v)\right] dx = \int_{\mathbb{R}^N} v^*\left[\exp\left(\alpha_0 |v^*|^{N/(N-1)}\right) - S_{N-2}(\alpha_0, v^*)\right] dx, \\ \int_{\mathbb{R}^N} |v|^N dx = \int_{\mathbb{R}^N} |v^*|^N dx. \end{gathered} \tag{3.4}$$

From (g_1), we obtain that for $\epsilon > 0$, there exists $\delta > 0$, such that

$$f(s) \leq \epsilon |s|^{N-1}, \quad \text{for } |s| < \delta, \tag{3.5}$$

so, we have

$$F(s) \leq \frac{\epsilon}{N} |s|^N, \quad \text{for } |s| < \delta. \tag{3.6}$$

From (g_2), we obtain

$$F(s) \leq C_1 |s|^N + C_2 |s| \left[\exp\left(\alpha_0 |s|^{N/(N-1)}\right) - S_{N-2}(\alpha_0, u)\right], \quad \text{for } |s| \geq \delta. \tag{3.7}$$

There two estimates yield

$$F(s) \le C_3|s|^N + C_2|s|\Big[\exp\Big(\alpha_0|s|^{N/(N-1)}\Big) - S_{N-2}(\alpha_0, u)\Big], \quad \text{for } s > 0. \tag{3.8}$$

On one hand, from Lemma 2.1, we obtain that there exists a constant C, which depends only on N, M, and α such that

$$\int_{\mathbb{R}^N} \exp\Big(\alpha|v_n|^{N/(N-1)}\Big) \le C. \tag{3.9}$$

When $\|\nabla v_n\|^N_{L^N(\mathbb{R}^N)} \le 1,\ \|v_n\|_{L^N(\mathbb{R}^N)} \le M < \infty$ and $\alpha < \alpha_N$. Hence, we have

$$\begin{aligned}
\int_{|x|\le r} F(v_n) &\le C_3\int_{\mathbb{R}^N}|v_n^*|^N dx + C_2\int_{|x|\le r}|v_n^*|\Big[\exp\Big(\alpha_0|v_n^*|^{N/(N-1)}\Big) - S_{N-2}(\alpha_0, v_n^*)\Big]dx \\
&\le C_3M^N + C_2\int_{|x|\le r}|v_n^*|\exp\Big(\alpha_0|v_n^*|^{N/(N-1)}\Big)dx \\
&\le C_3M^N + C_3\left(\int_{|x|\le r}|v_n^*|^{\mu}\right)^{1/\mu}\left(\int_{|x|\le r}\exp\Big(\beta\alpha_0|v_n^*|^{N/(N-1)}\Big)dx\right)^{1/\beta} \\
&\le C_3M^N + C_4M\left(\int_{|x|\le r}\exp\Big(\beta\alpha_0|v_n^*|^{N/(N-1)}\Big)dx\right)^{1/\beta} \\
&\le C_5,
\end{aligned} \tag{3.10}$$

where C_i $(i = 2,3,4,5)$ are positive constants, the continuous imbedding $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^{\mu}(\mathbb{R}^N)$, $1/\mu + 1/\beta = 1$ and $\beta\alpha_0 < \alpha_N$.

Then, by Dominated convergence theorem, we obtain

$$\int_{|x|\le r} F(v_n)dx \longrightarrow \int_{|x|\le r} F(v)dx. \tag{3.11}$$

On the other hand,

$$\begin{aligned}
\int_{|x|>r} F(v_n)dx &\le C_3\int_{|x|>r}|v_n|^N dx + C_2\int_{|x|>r}|v_n|\Big[\exp\Big(\alpha_0|v_n|^{N/(N-1)}\Big) - S_{N-2}(\alpha_0, v_n)\Big]dx \\
&\int_{|x|>r}|v_n|\Big[\exp\Big(\alpha_0|v_n|^{N/(N-1)}\Big) - S_{N-2}(\alpha_0, v_n)\Big]dx \\
&= \sum_{j=N-1}^{\infty}\frac{\alpha_0^j}{j!}\int_{|x|>r}|v_n|\cdot|v_n|^{Nj/(N-1)}dx \\
&= \sum_{j=N-1}^{\infty}\frac{\alpha_0^j}{j!}\int_{|x|>r}|v_n^*||v_n^*|^{Nj/(N-1)}dx,
\end{aligned} \tag{3.12}$$

where v_n^* is the Schwarz symmetrization of v_n. Notice that the estimate

$$\int_{|x|>r} \frac{1}{|x|^{1+Nj/(N-1)}} dx = \omega_{N-1} \int_r^\infty \frac{t^{N-1}}{t^{1+Nj/(N-1)}} dt = \left(\frac{\omega_{N-1}}{Nj/(N-1) - N + 1} \right) r^{N-1-Nj/(N-1)} \leq \frac{\omega_{N-1}}{r}, \tag{3.13}$$

for all $j \geq N-1$, together with the Radial Lemma [4] leads to

$$\begin{aligned} &\sum_{j=N-1}^{\infty} \frac{\alpha_0^j}{j!} \int_{|x|>r} |v_n^*||v_n^*|^{Nj/(N-1)} dx \\ &\leq M\left(\frac{N}{\omega_{N-1}}\right)^{1/N} \sum_{j=N-1}^{\infty} \frac{\alpha_0^j}{j!} \left(\frac{N}{\omega_{N-1}}\right)^{j/(N-1)} M^{Nj/(N-1)} \int_{|x|>r} |x|^{-1-Nj/(N-1)} dx \\ &\leq \frac{C(N)}{r}. \end{aligned} \tag{3.14}$$

Thus, given $\delta > 0$, there exists $r > 0$ such that

$$\int_{|x|>r} |v_n|^N dx < \delta, \qquad \int_{|x|>r} \left[\exp\left(\alpha_0 |v_n|^{N/(N-1)}\right) - S_{N-2}(\alpha_0, v_n)\right] dx < \delta. \tag{3.15}$$

Which implies that

$$\int_{|x|>r} F(v_n) dx \leq C\delta, \qquad \int_{|x|>r} F(v) dx \leq C\delta. \tag{3.16}$$

Using the estimate

$$\left| \int_{\mathbb{R}^N} (F(v_n) - F(v)) dx \right| \leq \left| \int_{|x|\leq r} (F(v_n) - F(v)) dx \right| + \left| \int_{|x|>r} (F(v_n) - F(v)) dx \right|, \tag{3.17}$$

we get

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}^N} F(v_n) - \int_{\mathbb{R}^N} F(v) dx \right| \leq C\delta, \tag{3.18}$$

Hence, we obtain that

$$\int_{\mathbb{R}^N} F(v_n) dx \longrightarrow \int_{\mathbb{R}^N} F(v) dx. \tag{3.19}$$

□

Lemma 3.2. *The numbers A and c satisfy the inequality $A \le c$.*

Proof. For each $v \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}$, since we only consider the nontrivial solutions of the problem (1.1), we divide them into two cases to consider.

Case 1. Let $v^+ = \max\{v, 0\} \ne 0$, we define the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \int_{\mathbb{R}^N} G(tv)dx = \int_{\mathbb{R}^N} \left[F(tv) - \frac{t^N v^N}{N}\right] dx. \tag{3.20}$$

By (g_1), we obtain that there exists $\delta > 0$, $0 < c_0 < 1$ such that $|s| < \delta$, and

$$\left|f(s)\right| < c_0 |s|^{N-1}. \tag{3.21}$$

Hence

$$\begin{aligned} h(t) &\le \int_{\mathbb{R}^N} \int_0^{tv} c_0 |s|^{N-1} dx - \frac{1}{N} \int_{\mathbb{R}^N} t^N v^N dx \\ &= \frac{c_0}{N} \int_{\mathbb{R}^N} |tv|^N dx - \frac{1}{N} \int_{\mathbb{R}^N} |tv|^N dx, \end{aligned} \tag{3.22}$$

we obtain that $h(t) < 0$ for t small enough. On the other hand, by (g_2), we obtain that $h(t) > 0$ for t large enough. In this way, there exists $t_0 > 0$ such that $h(t_0) = 0$, That is, $t_0 v \in \mathcal{P}$. Hence

$$A \le \frac{1}{N} \int_{R^N} |\nabla(t_0 v)|^N dx = I(t_0 v) \le \max_{t \ge 0} I(tv). \tag{3.23}$$

Case 2. Let $v^+ = \max\{v, 0\} = 0$, since $f(s) = 0$ for all $s < 0$, we obtain

$$\max_{t \ge 0} I(tv) = +\infty. \tag{3.24}$$

As a consequence,

$$A \le c. \tag{3.25}$$

Combining Cases 1 and 2, we obtain that $A \le c$. □

Lemma 3.3. *The number A defined by* (2.8) *is positive, that is, $A > 0$.*

Proof. Clearly, $A \ge 0$. Assume by contradiction that $A = 0$ and let $\{u_n\}$ be a minimizing sequence in $W^{1,N}(\mathbb{R}^{\mathbb{N}})$ to A, that is,

$$\frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_n|^N dx \longrightarrow A = 0 \quad \text{with} \quad \int_{\mathbb{R}^N} G(u_n) dx = 0. \tag{3.26}$$

For each $\lambda_n > 0$, set $v_n(x) = u_n(x/\lambda_n)$ satisfying

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_n|^N dx = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_n|^N dx. \tag{3.27}$$

Similarly, we have

$$\begin{gathered}\int_{\mathbb{R}^N} G(v_n)dx = \lambda_n^N \int_{\mathbb{R}^N} G(u_n)dx = 0,\\ \int_{\mathbb{R}^N} |v_n|^N \, dx = \lambda_n^N \int_{\mathbb{R}^N} |u_n|^N dx.\end{gathered} \tag{3.28}$$

We choose $\lambda_n^N = 1/\int_{\mathbb{R}^N} |u_n|^N dx$, so $\int_{\mathbb{R}^N} |v_n|^N dx = 1$. Then we get

$$\begin{gathered}\frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_n|^N dx \longrightarrow A = 0,\\ \int_{\mathbb{R}^N} |v_n|^N dx = 1, \qquad \int_{\mathbb{R}^N} G(v_n)dx = 0.\end{gathered} \tag{3.29}$$

In what follows, we study in the space $W^{1,N}_{\text{rad}}(\mathbb{R}^N)$. Firstly, we assume that there exists $v \in W^{1,N}_{\text{rad}}(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ in $W^{1,N}_{\text{rad}}(\mathbb{R}^N)$.

On one hand, since $(1/N)\int_{\mathbb{R}^N} |\nabla v_n|^N \to A = 0$, then $\exists N_0 > 0$, for all $0 < \epsilon < 1$, when $n > N_0$, we have $\int_{\mathbb{R}^N} |\nabla v_n|^N dx < \epsilon < 1$ and we also know that $\int_{\mathbb{R}^N} |v_n|^N dx = 1$, so $\|v_n\|_{L^N(\mathbb{R}^N)} \le M < \infty$. From Lemma 3.1, we have

$$\int_{\mathbb{R}^N} F(v_n)dx \longrightarrow \int_{\mathbb{R}^N} F(v)dx. \tag{3.30}$$

Note that

$$\int_{\mathbb{R}^N} G(v_n)dx = \int_{\mathbb{R}^N} \left(F(v_n) - \frac{1}{N}|v_n|^N\right)dx = 0, \tag{3.31}$$

so we have

$$\int_{\mathbb{R}^N} F(v_n)dx = \frac{1}{N}\int_{\mathbb{R}^N} |v_n|^N dx = \frac{1}{N}. \tag{3.32}$$

Hence, we have

$$\int_{\mathbb{R}^N} F(v)dx = \frac{1}{N}. \tag{3.33}$$

It implies that $v \neq 0$. On the other hand, since $v_n \rightharpoonup v$ in $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$, and the space $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$ is a reflexible Banach space, we have

$$\lim_{n\to\infty}\inf \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_n|^N dx \geq \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^N dx \geq 0. \tag{3.34}$$

Since

$$\lim_{n\to\infty}\inf \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_n|^N dx = A = 0, \tag{3.35}$$

we get

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^N dx = 0. \tag{3.36}$$

From which it follows that $v = 0$, we have an absurd. Hence, we have

$$A > 0. \tag{3.37}$$

□

Lemma 3.4. *If $\lambda > (q - N/q)^{(q-N)/N} C_q^{q/N}$, then $c < 1/N$.*

Proof. From (g_3), we have $f(s) > \lambda s^{q-1}$, for all $s \geq 0$. Now we choose $\psi \in W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$ such that

$$\psi \geq 0, \qquad \|\psi\|_q^N = C_q^{-1}, \qquad \|\psi\|_{W^{1,N}(\mathbb{R}^N)} = 1. \tag{3.38}$$

Hence, we have

$$\begin{aligned} c \leq \max_{t\geq 0} I(t\psi) &= \max_{t\geq 0}\left\{\frac{1}{N}\int_{\mathbb{R}^N}\left(|\nabla(t\psi)|^N + |t\psi|^N\right)dx - \int_{\mathbb{R}^N} F(t\psi)dx\right\} \\ &= \max_{t\geq 0}\left\{\frac{t^N}{N} - \int_{\mathbb{R}^N}\int_0^{t\psi} f(s)ds\,dx\right\} \\ &\leq \max_{t\geq 0}\left\{\frac{t^N}{N} - \lambda\int_{\mathbb{R}^N}\int_0^{t\psi} s^{q-1}ds\,dx\right\} \\ &= \max_{t\geq 0}\left\{\frac{t^N}{N} - \frac{\lambda t^q}{q}\int_{\mathbb{R}^N}\psi^q dx\right\}. \end{aligned} \tag{3.39}$$

Let $K(t) = t^N/N - (\lambda t^q/q)\int_{\mathbb{R}^N}\psi^q dx$, then $K(t)$ is continuous function, we have

$$K'(t) = t^{N-1} - \lambda t^{q-1}\int_{\mathbb{R}^N}\psi^q dx = 0. \tag{3.40}$$

By a simple calculation, when $t_0 = (1/\lambda \int_{\mathbb{R}^N} \psi^q dx)^{1/(q-N)} > 0$, we have

$$\begin{aligned}
\max_{t>0} K(t) &= K(t_0) = \frac{1}{N}\left(\frac{1}{\lambda \int_{\mathbb{R}^N} \psi^q dx}\right)^{N/(q-N)} - \left(\frac{\lambda}{q}\int_{\mathbb{R}^N} \psi^q dx\right)\left(\frac{1}{\lambda \int_{\mathbb{R}^N} \psi^q dx}\right)^{q/(q-N)} \\
&= \frac{q-N}{Nq}\lambda^{-N/(q-N)} C_q^{(q/N)\cdot(N/(q-N))} \\
&< \frac{q-N}{Nq}\left(\frac{q-N}{q}\right)^{((q-N)/N)\cdot(-N/(q-N))} C_q^{(q/N)(-N/(q-N))} C_q^{q/(q-N)} \\
&= \frac{1}{N}.
\end{aligned} \tag{3.41}$$

Hence, we have

$$c < \frac{1}{N}. \tag{3.42}$$

□

Lemma 3.5. *The number A is attained, that is, there exists $u \in W_{rad}^{1,N}(\mathbb{R}^N)$ such that $A = \int_{\mathbb{R}^N} |\nabla u|^N dx$ and $\int_{\mathbb{R}^N} G(u) dx = 0$.*

Proof. Let $\{u_n\}$ be a minimizing sequence in $W_{\text{rad}}^{1,N}(\mathbb{R}^N)$ for A, that is,

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla u_n|^N dx \longrightarrow A \quad (n \longrightarrow \infty), \qquad \int_{\mathbb{R}^N} G(u_n) dx = 0. \tag{3.43}$$

Arguing as in Lemma 3.3, we assume that $\int_{\mathbb{R}^N} |u_n|^N dx = 1$. From (3.43), Lemmas 3.3 and 3.4, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^N dx = NA \le Nc < 1. \tag{3.44}$$

From Lemma 3.1,

$$\int_{\mathbb{R}^N} F(u_n) dx \longrightarrow \int_{\mathbb{R}^N} F(u) dx, \tag{3.45}$$

where $u_n \rightharpoonup u$ in $W^{1,N}(\mathbb{R}^N)$, as $n \to \infty$.

By (3.43) and (3.45), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} F(u_n) dx &= \frac{1}{N}\int_{\mathbb{R}^N} |u_n|^N dx = \frac{1}{N}, \\
\int_{\mathbb{R}^N} F(u) dx &= \frac{1}{N}.
\end{aligned} \tag{3.46}$$

It implies that

$$u \neq 0, \tag{3.47}$$

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N dx \le \liminf_{n\to\infty} \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u_n|^N dx = A, \tag{3.48}$$

$$\int_{\mathbb{R}^N} |u|^N dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^N dx = 1. \tag{3.49}$$

From (3.48) and (3.49), we have

$$\int_{\mathbb{R}^N} G(u)dx = \int_{\mathbb{R}^N} F(u)dx - \frac{1}{N}\int_{\mathbb{R}^N} |u|^N dx = \frac{1}{N} - \frac{1}{N}\int_{\mathbb{R}^N} |u|^N dx \ge 0. \tag{3.50}$$

If $\int_{\mathbb{R}^N} G(u)dx \neq 0$, from (3.50), we have $\int_{\mathbb{R}^N} G(u)dx > 0$. Consider the function h defined in Lemma 3.2 relative to the function:

$$h(t) = \int_{\mathbb{R}^N} G(tu)dx. \tag{3.51}$$

We concludes that $h(t) < 0$ for t small enough. On the other hand, $h(1) = \int_{\mathbb{R}^N} G(u)dx > 0$. In this way, we obtain that there is $t_0 \in (0,1)$ such that $h(t_0) = 0$, that is,

$$\int_{\mathbb{R}^N} G(t_0 u)dx = 0. \tag{3.52}$$

Hence, from (3.48),

$$0 < \frac{1}{N}\int_{\mathbb{R}^N} |\nabla (t_0 u)|^N dx = \frac{1}{N} t_0^N \int_{\mathbb{R}^N} |\nabla u|^N dx \le t_0^N A < A. \tag{3.53}$$

However, from (3.52), we have $t_0 u \in \mathcal{P}$. Hence, we obtain

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla (t_0 u)|^N dx \ge A. \tag{3.54}$$

Which is contradictory with (3.53).

Thus, we obtain

$$\int_{\mathbb{R}^N} G(u)dx = 0. \tag{3.55}$$

It implies $u \in \mathcal{P}$ and

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N dx \ge A. \tag{3.56}$$

From (3.48) and (3.56), we obtain that

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N dx = A, \tag{3.57}$$

with $\int_{\mathbb{R}^N} G(u)dx = 0,\ u \neq 0.$

We obtain that A is attained. □

Proof of Theorem 1.2. From Lemma 3.5, there is $u \in W^{1,N}_{\text{rad}}(\mathbb{R}^N) \setminus \{0\}$ such that

$$\frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N dx = A, \qquad \int_{\mathbb{R}^N} G(u)dx = 0. \tag{3.58}$$

we will prove that $m = b = A$.

By Lagrange multipliers, there exists $\rho \in \mathbb{R}$, such that

$$\int_{\mathbb{R}^N} |\nabla u|^{N-2}\nabla u \nabla v dx = \rho \int_{\mathbb{R}^N} g(u)v\, dx, \tag{3.59}$$

for every $v \in W^{1,N}(\mathbb{R}^N)$.

Define the rescaled function $u_{\rho^{1/N}} = u(\rho^{-1/N}x)$, which is a nontrivial solution of (1.1) with

$$\begin{gathered}\int_{\mathbb{R}^N} \left|\nabla u_{\rho^{1/N}}\right|^N dx = \int_{\mathbb{R}^N} |\nabla u|^N dx,\\ \int_{\mathbb{R}^N} G\left(u_{\rho^{1/N}}\right)dx = \rho\int_{\mathbb{R}^N} G(u)dx = 0.\end{gathered} \tag{3.60}$$

Thus, we have

$$m \le I\left(u_{\rho^{1/N}}\right) = \frac{1}{N}\int_{\mathbb{R}^N} \left|\nabla u_{\rho^{1/N}}\right|^N dx - \int_{\mathbb{R}^N} G\left(u_{\rho^{1/N}}\right)dx = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N dx = A. \tag{3.61}$$

So, we have

$$m \le A. \tag{3.62}$$

For each $\gamma \in \Gamma$, one has $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$ from [4]. We obtain that there exists $t_0 \in [0,1]$ such that $\gamma(t_0) \in \mathcal{P}$, that is, $\gamma(t_0)$ satisfied that $\int_{\mathbb{R}^N} G(\gamma(t_0))dx = 0$ and then

$$A \le \frac{1}{N}\int_{\mathbb{R}^N} \left|\nabla\gamma(t_0)\right|^N dx - \frac{1}{N}\int_{\mathbb{R}^N} G(\gamma(t_0))dx = I(\gamma(t_0)). \tag{3.63}$$

Hence $A \le I(\gamma(t_0)) \le \max_{t\in[0,1]} I(\gamma(t))$ for every $\gamma \in \Gamma$, we obtain that

$$A \le b. \tag{3.64}$$

From (3.62) and (3.64), we obtain that $m \le A \le b$.

On the other hand, for every nontrivial solution $\omega \in W^{1,N}(\mathbb{R}^N)$ of the problem (1.1), there exists a path $\gamma_\omega \in \Gamma$ such that $\omega \in \gamma_\omega([0,1])$ and $\max_{t\in[0,1]} I(\gamma_\omega(t)) = I(\omega)$. Consequently, $b \le I(\omega),\ b \le m$.

In conclusion, we obtain

$$m = A = b. \tag{3.65}$$

Hence, the function $u_{\rho^{1/N}}$ is a ground-state solution of the problem (1.1). □

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