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Research Article

Multiplicity Result of Positive Solutions for Nonlinear Differential Equation of Fractional Order

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We investigate the existence of multiple positive solutions for a class of boundary value problems of nonlinear differential equation with Caputo's fractional order derivative. The existence results are obtained by means of the Avery-Peterson fixed point theorem. It should be point out that this is the first time that this fixed point theorem is used to deal with the boundary value problem of differential equations with fractional order derivative.

1. Introduction

In this paper, we consider the existence and multiple existence of positive solutions for following boundary value problem of differential equation involving the Caputo's fractional order derivative

$$\begin{gathered} D_{0+}^{\alpha}u(t) = f\left(t, u(t), D_{0+}^{\beta}u(t)\right), \quad t \in (0,1), \\ u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0, \end{gathered} \tag{1.1}$$

where $1 < \alpha < 2, 0 < \beta < \alpha - 1$ and $f : C([0,1] \times R^{+} \times R, R^{+})$. Here D_{0+}^{α} is the Caputo's derivative of fractional order.

Due to the development of the theory of fractional calculus and its applications, such as in the fields of control theory, blood flow phenomena, Bode's analysis of feedback amplifiers, aerodynamics, and polymer rheology and many work on fractional calculus, fractional order differential equations has appeared [1–7]. Recently, there have been many results concerning the solutions or positive solutions of boundary value problems for nonlinear fractional differential equations, see [8–28] and references along this line.

For example, Bai and Lü [12] considered the following Dirichlet boundary value problem of fractional differential equation:

$$D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad t \in (0,1),\ u(0) = 0 = u(1),\ 1 < \alpha \le 2. \tag{1.2}$$

By means of different fixed-point theorems on cone, some existence and multiplicity results of positive solutions were obtained. Jiang and Yuan [20] improved the results in [12] by discussing some new positive properties of the Green function for problem (1.2). By using the fixed point theorem on a cone due to Krasnoselskii, the authors established the existence results of positive solution for problem (1.2). Recently, Caballero et al. [21] obtained the existence and uniqueness of positive solution for singular boundary value problem (1.2). The existence results were established in the case that the nonlinear term f may be singular at $t = 0$. As to positive solutions of problem (1.1), under the case that the nonlinear term was not involved with the derivative of the function $u(t)$, Zhang [13] obtained the existence and multiplicity results of positive solutions by means of a fixed-point theorem on cones.

There are also some results concerning multipoint boundary value problems for differential equations of fractional order. Bai [23] investigated the existence and uniqueness of positive solution for three-point boundary value problem

$$D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad t \in (0,1),\ u(0) = 0,\ u(1) = \beta u(\eta), \tag{1.3}$$

where $1 < \alpha \le 2$, $\eta \in (0,1)$, $0 < \beta\eta^{\alpha-1} < 1$. In [23], the uniqueness of positive solution was obtained by the use of contraction map principle and some existence results of positive solutions were established by means of the fixed point index theory. Very recently, Wang et al. [26] considered the boundary value problem of fractional differential equation with integral condition

$$\begin{gathered} D_{0+}^{\alpha}u(t) + q(t)f(t,u(t)) = 0, \quad t \in (0,1),\ n-1 < \alpha < n, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \int_0^1 u(s)dA(s), \end{gathered} \tag{1.4}$$

where $\alpha > 2$, $\int_0^1 u(s)dA(s)$ was given by Riemann-Stieltjes integral with a signed measure. By using the fixed point theorem, the existence of positive solution for this problem was established.

However, in this work, the derivative of the unknown function $u(t)$ was not involved in the nonlinear term explicitly. To our best knowledge, there are few papers considering the positive solution of boundary value problem of nonlinear fractional differential equations which the derivative of the unknown function $u(t)$ is involved in the nonlinear term. In [29], Guo and Ge proved a new fixed point theorem, which can be regarded as an extension of Krasnoselskii's fixed point theorem in a cone. By applying this new theorem, Guo and

Ge obtained the existence of positive solutions for second-order three-point boundary value problem

$$\begin{aligned} &u'' + f(t, u(t), u'(t)) = 0, \quad t \in [0,1], \\ &u(0) = 0, \quad u(1) = \alpha u(\eta), \quad \eta \in (0,1), \end{aligned} \tag{1.5}$$

where f depended on the first order derivative of u. Very recently, Yang et al. [30] considered following boundary value problem

$$\begin{aligned} &D_{0+}^{\alpha} u(t) = f(t, u(t), u'(t)), \quad t \in (0,1), \\ &u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0. \end{aligned} \tag{1.6}$$

By means of Schauder's fixed point theorem and the fixed point theorem duo to Guo and Ge, some results on the existence of positive solutions were obtained.

In [31], Avery and Peterson gave an new triple fixed point theorem, which can be regarded as an extension of Leggett-Williams fixed point theorem. By using this method, many results concerning the existence of at least three positive solutions of boundary value problems of differential equation with integer order were established, see [32–37]. For example, by using the Avery-Peterson fixed point theorem, Yang et al. [32] established the existence of at least three positive solutions of second-order multipoint boundary value problem

$$\begin{aligned} &u'' + f(t, u(t), u'(t)) = 0, \quad t \in [0,1], \\ &u'(0) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i). \end{aligned} \tag{1.7}$$

But by using the Avery-Peterson fixed point theorem, the nonlinear terms are often assumed to be nonnegative to ensure the concavity or convexity of the unknown function. When the differential equations of fractional order are considered, we cannot derive the concavity or convexity of function $u(t)$ by the sign of its fractional order derivative. Thus the Avery-Peterson fixed point theorem cannot directly be used to consider the boundary value problem of nonlinear differential equation with fractional order where the derivative of the unknown function $u(t)$ is involved in the nonlinear term explicitly.

In this paper, by obtaining some new inequalities of the unknown function and defining a special cone, we overcome the difficulties brought by the lack of the concavity or convexity of unknown function $u(t)$. By an application of Avery-Peterson fixed point theorem, the existence of at least three positive solutions of problem (1.1) is established. It should be pointed out that it is the first time that the Avery-Peterson fixed point theorem is used to deal with the positive solutions of boundary value problem of differential equations with fractional order derivative.

2. Preliminary Results

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u(t) : (0, \infty) \to R$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds \tag{2.1}$$

provided the right side is point-wise defined on $(0, \infty)$.

Definition 2.2. The Caputo's fractional derivative of order $\alpha > 0$ of a continuous function $u(t) : (0, \infty) \to R$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.2}$$

where $n-1 < \alpha \le n$, provided that the right side is point-wise defined on $(0, \infty)$.

Lemma 2.3. *Let $\alpha > 0$. Then the fractional differential equation $D_{0+}^{\alpha} u(t) = 0$ has solutions*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad c_i \in R, \ i = 0, 1, \ldots, n-1. \tag{2.3}$$

Lemma 2.4. *Let $\alpha > 0$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad c_i \in R, \ i = 1, 2, \ldots, n-1. \tag{2.4}$$

Definition 2.5. Let E be a real Banach space over R. A nonempty convex closed set $P \subset E$ is said to be a cone provided that

(1) $au \in P$, for all $u \in P$, $a \ge 0$,

(2) $u, -u \in P$ implies $u = 0$.

Definition 2.6. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 2.7. The map α is said to be a continuous nonnegative convex functional on cone P of a real Banach space E provided that $\alpha : P \to [0, +\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \le t\alpha(x) + (1-t)\alpha(y), \quad x, y \in P, \ t \in [0, 1]. \tag{2.5}$$

Definition 2.8. The map β is said to be a continuous nonnegative concave functional on cone P of a real Banach space E provided that $\beta : P \to [0, +\infty)$ is continuous and

$$\beta(tx + (1-t)y) \ge t\beta(x) + (1-t)\beta(y), \quad x, y \in P, \ t \in [0, 1]. \tag{2.6}$$

Let γ and θ be nonnegative continuous convex functionals on P, α a nonnegative continuous concave functional on P, and ψ a nonnegative continuous functional on P. Then for positive numbers a, b, c, and d, we define the following convex sets:

$$\begin{gathered} P(\gamma, d) = \{x \in P \mid \gamma(x) < d\}, \\ P(\gamma, \alpha, b, d) = \{x \in P \mid b \leq \alpha(x), \gamma(x) \leq d\}, \\ P(\gamma, \theta, \alpha, b, c, d) = \{x \in P \mid b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\}, \end{gathered} \tag{2.7}$$

and a closed set

$$R(\gamma, \psi, a, d) = \{x \in P \mid a \leq \psi(x), \gamma(x) \leq d\}. \tag{2.8}$$

Lemma 2.9 (see [31]). *Let P be a cone in Banach space E. Let γ, θ be nonnegative continuous convex functionals on P, α a nonnegative continuous concave functional on P, and ψ a nonnegative continuous functional on P satisfying*

$$\psi(\lambda x) \leq \lambda \psi(x), \quad \text{for } 0 \leq \lambda \leq 1, \tag{2.9}$$

such that for some positive numbers l and d,

$$\alpha(x) \leq \psi(x), \qquad \|x\| \leq l\gamma(x), \tag{2.10}$$

for all $x \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, c with $a < b$ such that

- (S_1) *$\{x \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma, \theta, \alpha, b, c, d)$;*
- (S_2) *$\alpha(Tx) > b$ for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > c$;*
- (S_3) *$0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.*

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that

$$\gamma(x_i) \leq d, \quad i = 1, 2, 3; \qquad b < \alpha(x_1); \qquad a < \psi(x_2), \qquad \alpha(x_2) < b; \qquad \psi(x_3) < a. \tag{2.11}$$

3. Main Results

Lemma 3.1 (see [30]). *Given $y(t) \in C[0, 1]$, then boundary value problem*

$$D_{0+}^{\alpha} u(t) = y(t), \qquad u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0 \tag{3.1}$$

is equivalent to

$$\begin{aligned} u(t) &= \int_0^t \left(\frac{(1-s)^{\alpha-1}(1-t) + (1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)} \right) y(s)ds \\ &\quad + \int_t^1 \left(\frac{(1-s)^{\alpha-1}(1-t)}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)} \right) y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds, \end{aligned} \tag{3.2}$$

where

$$G(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha-1}(1-t) + (1-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}, & 0 \le s \le t \le 1, \\ \dfrac{(1-s)^{\alpha-1}(1-t)}{\Gamma(\alpha)} + \dfrac{(1-s)^{\alpha-2}(1-t)}{\Gamma(\alpha-1)}, & 0 \le t \le s \le 1. \end{cases} \tag{3.3}$$

Lemma 3.2. *Given $y(t) \in C[0,1]$, assume that $u(t)$ is a solution of boundary value problem*

$$D_{0+}^{\alpha} u(t) = y(t), \qquad u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0. \tag{3.4}$$

Then

$$\begin{aligned} D^{\beta} u(t) &= \int_0^t \left(\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{t^{1-\beta}(1-s)^{\alpha-2}(\alpha-s)}{\Gamma(\alpha)\Gamma(2-\beta)} \right) y(s)ds \\ &\quad + \int_t^1 -\frac{t^{1-\beta}(1-s)^{\alpha-2}(\alpha-s)}{\Gamma(\alpha)\Gamma(2-\beta)} y(s)ds. \end{aligned} \tag{3.5}$$

Proof. From Lemmas 2.3 and 2.4, we get that

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - C_1 - C_2 t, \qquad u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds - C_2. \tag{3.6}$$

The boundary condition $u(0) + u'(0) = 0$ implies that $C_1 + C_2 = 0$. Considering the boundary condition $u(1) + u'(1) = 0$, we have

$$C_2 = -C_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)ds + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s)ds. \tag{3.7}$$

From the definition of the Caputo derivative of fractional order, we see

$$\begin{aligned}
D^{\beta}u(t) &= D^{\beta}\left(\int_0^t \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds - C_1 - C_2 t\right) \\
&= \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}y(s)ds - \frac{C_2}{\Gamma(2-\beta)}t^{1-\beta} \\
&= \int_0^t \left(\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{t^{1-\beta}(1-s)^{\alpha-2}(\alpha-s)}{\Gamma(\alpha)\Gamma(2-\beta)}\right)y(s)ds \\
&\quad + \int_t^1 -\frac{t^{1-\beta}(1-s)^{\alpha-2}(\alpha-s)}{\Gamma(\alpha)\Gamma(2-\beta)}y(s)ds.
\end{aligned} \tag{3.8}$$

□

Lemma 3.3 (see [30]). *The function $G(t,s)$ satisfies the following conditions:*

(1) *$G(t,s) \in C([0,1]\times[0,1))$, $G(t,s) > 0$, for $t,s \in (0,1)$;*

(2) *there exist a positive function $\gamma(s), K(s) \in C(0,1)$ such that*

$$\max_{0\le t\le 1} \le K(s), \quad s\in(0,1), \qquad \min_{s\in[1/3,2/3]} G(t,s) \ge \gamma(s)K(s), \quad 0<s<1, \tag{3.9}$$

where

$$K(s) = \frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, \tag{3.10}$$

$$\gamma(s) = \frac{1}{3}\frac{(1-s)^{\alpha-1} + (\alpha-1)(1-s)^{\alpha-2}}{2(1-s)^{\alpha-1} + (\alpha-1)(1-s)^{\alpha-2}} > \frac{1}{6}, \quad s\in[0,\ 1). \tag{3.11}$$

Lemma 3.4. *Assume that $y(t) > 0$ and $u(t)$ is a solution of boundary value problem* (3.1)*. There exists a positive constant γ_0 such that*

$$\max_{0\le t\le 1} |u(t)| \le \gamma_0 \max_{0\le t\le 1} D^{\beta}|u(t)|. \tag{3.12}$$

Proof. From Lemma 3.2,

$$\begin{aligned}
\max_{0\le t\le 1}\left|D^{\beta}u(t)\right| = \max_{0\le t\le 1}\Bigg|&\int_0^t\left(\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{t^{1-\beta}(1-s)^{\alpha-2}(\alpha-s)}{\Gamma(\alpha)\Gamma(2-\beta)}\right)y(s)ds \\
&-\int_t^1 \frac{t^{1-\beta}(1-s)^{\alpha-2}(\alpha-s)}{\Gamma(\alpha)\Gamma(2-\beta)}y(s)ds\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\geq \max_{0\leq t\leq 1}\left|D^{\beta}u(1)\right| = \int_0^1\left(\frac{(1-s)^{\alpha-2}(\alpha-s)}{\Gamma(\alpha)\Gamma(2-\beta)} - \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right)y(s)ds \\
&= \int_0^1 k(s)y(s)ds.
\end{aligned} \tag{3.13}$$

Denote

$$\begin{aligned}
h(s) = \frac{K(s)}{k(s)} &= \frac{2(1-s)^{\alpha-1}/\Gamma(\alpha) + (1-s)^{\alpha-2}/\Gamma(\alpha-1)}{(1-s)^{\alpha-2}(\alpha-s)/\Gamma(\alpha)\Gamma(2-\beta) - (1-s)^{\alpha-\beta-1}/\Gamma(\alpha-\beta)} \\
&= \frac{2(1-s)/\Gamma(\alpha) + 1/\Gamma(\alpha-1)}{(\alpha-s)/\Gamma(\alpha)\Gamma(2-\beta) - (1-s)^{1-\beta}/\Gamma(\alpha-\beta)}.
\end{aligned} \tag{3.14}$$

By a simple computation, we have

$$\begin{aligned}
h(s) &\leq \frac{2/\Gamma(\alpha) + 1/\Gamma(\alpha-1)}{(\alpha-s_0)/\Gamma(\alpha)\Gamma(2-\beta) - (1-s_0)^{1-\beta}/\Gamma(\alpha-\beta)} \\
&= \frac{2/\Gamma(\alpha) + 1/\Gamma(\alpha-1)}{\left((\alpha-1)(1-\beta) - \beta\left((1-\beta)\Gamma(\alpha)\Gamma(2-\beta)/\Gamma(\alpha-\beta)\right)^{1/\beta}\right)/(1-\beta)\Gamma(\alpha)\Gamma(\alpha-\beta)} \\
&= \gamma_0 > 0.
\end{aligned} \tag{3.15}$$

Thus,

$$\max_{0\leq t\leq 1}|u(t)| = \max_{0\leq t\leq 1}\int_0^1 G(t,s)y(s)ds \leq \int_0^1 K(s)y(s)ds \leq \int_0^1 \gamma_0 k(s)y(s)ds \leq \gamma_0 \max_{0\leq t\leq 1}\left|D^{\beta}u(t)\right|. \tag{3.16}$$

Let the Banach space $E = \{u(t) \in C[0,1],\ D^{\beta}u(t) \in C[0,1]\}$ be endowed with the norm

$$\|u\| = \max\left\{\max_{0\leq t\leq 1}|u(t)|, \max_{0\leq t\leq 1}\left|D^{\beta}u(t)\right|\right\}, \quad u \in E. \tag{3.17}$$

We define the cone $P \subset E$ by

$$P = \left\{u \in E \mid u(t) \geq 0,\ \min_{1/3\leq t\leq 2/3} u(t) \geq \frac{1}{6}\max_{0\leq t\leq 1} u(t), \max_{0\leq t\leq 1} u(t) \leq \gamma_0 \max_{0\leq t\leq 1}\left|D^{\beta}u(t)\right|\right\}. \tag{3.18}$$

Denote the positive constants

$$M = \int_0^1 K(s)ds = \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)},$$
$$N = \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{\Gamma(2-\beta)}\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha+1)}\right). \tag{3.19}$$

□

Lemma 3.5. *Let $T : P \to E$ be the operator defined by*

$$Tu(t) := \int_0^1 G(t,s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds. \tag{3.20}$$

Then $T : P \to P$ is completely continuous.

Proof. The operator T is nonnegative and continuous in view of the nonnegativeness and continuity of functions $G(t,s)$ and $f(t,u(t),D_{0+}^{\beta}u(t))$. Let $\Omega \subset K$ be bounded. Then there exists a positive constant $R_1 > 0$ such that $\|u\| \le R_1$, $u \in \Omega$. Denote

$$R = \max_{0\le t\le 1,\, u\in\Omega} \left| f\left(t, u(t), D_{0+}^{\beta} u(t)\right)\right| + 1. \tag{3.21}$$

Then for $u \in \Omega$, we have

$$\begin{aligned}
|(Tu)(t)| &\le \int_0^1 G(t,s)\left|f\left(s,u,D_{0+}^{\beta}u\right)\right|ds \le \int_0^1 K(s)\left|f\left(s,u,D_{0+}^{\beta}u\right)\right|ds \\
&\le R\int_0^1 K(s)ds = MR, \\
\left|D_{0+}^{\beta}(Tu)(t)\right| &= \left|\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} f\left(s,u(s),D_{0+}^{\beta}u(s)\right)ds \right. \\
&\quad - \frac{t^{1-\beta}}{\Gamma(2-\beta)}\left(\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} f\left(s,u,D_{0+}^{\beta}u\right)ds \right. \\
&\quad \left.\left. + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f\left(s,u,D_{0+}^{\beta}u\right)ds\right)\right| \\
&\le \left[\frac{1}{(\alpha-\beta)\Gamma(\alpha-\beta)}\right. \\
&\quad \left. + \frac{1}{\Gamma(2-\beta)}\left(\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}ds\right)\right] \times R \\
&= NR.
\end{aligned} \tag{3.22}$$

Hence $T(\Omega)$ is bounded. On the other hand, for $u \in \Omega$, $t_1, t_2 \in [0,\ 1]$, one has

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| &= \left| \int_0^1 G(t_1, s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \right. \\
&\quad \left. - \int_0^1 G(t_2, s) f\left(s, u(s), D_{0+}^{\beta} u(s)\right) ds \right| \\
&\le \int_0^{t_1} |G(t_2, s) - G(t_1, s)| \left| f\left(s, u(s), D_{0+}^{\beta} u(s)\right) \right| ds \\
&\quad + \int_{t_1}^{t_2} |G(t_2, s) - G(t_1, s)| \left| f\left(s, u(s), D_{0+}^{\beta} u(s)\right) \right| ds \\
&\quad + \int_{t_2}^{1} |G(t_2, s) - G(t_1, s)| \left| f\left(s, u(s), D_{0+}^{\beta} u(s)\right) \right| ds \\
&\le \left[\frac{2(t_2 - t_1)}{\Gamma(\alpha)} + \frac{t_2^{\alpha} - t_1^{\alpha}}{\Gamma(\alpha + 1)} \right] \times R, \\
\left| D_{0+}^{\beta}(Tu(t_2)) - D_{0+}^{\beta}(Tu(t_1)) \right| &= \left| \frac{1}{\Gamma(\alpha - \beta)} \left(\int_0^{t_2} (t_2 - s)^{\alpha - \beta - 1} f\left(s, u, D_{0+}^{\beta} u\right) ds \right. \right. \\
&\quad \left. - \int_0^{t_1} (t_1 - s)^{\alpha - \beta - 1} f\left(s, u, D_{0+}^{\beta} u\right) ds \right) \\
&\quad + \frac{\left(t_1^{1-\beta} - t_2^{1-\beta}\right)}{\Gamma(2 - \beta)} \left(\frac{1}{\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha - 2} f\left(s, u, D_{0+}^{\beta} u\right) ds \right. \\
&\quad \left. \left. + \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 1} f\left(s, u, D_{0+}^{\beta} u\right) ds \right) \right| \\
&\le \frac{R \times \left| t_2^{\alpha - \beta} - t_1^{\alpha - \beta} \right|}{(\alpha - \beta)\Gamma(\alpha - \beta)} \\
&\quad + \frac{R}{\Gamma(2 - \beta)} \left(\frac{1}{\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha - 2} ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 1} \right) \\
&\quad \times \left| t_2^{1-\beta} - t_1^{1-\beta} \right| \\
&\le \frac{R}{(\alpha - \beta)\Gamma(\alpha - \beta)} \left| t_2^{\alpha - \beta} - t_1^{\alpha - \beta} \right| \\
&\quad + \frac{R}{\Gamma(2 - \beta)} \left(\frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha)} \right) \left| t_2^{1-\beta} - t_1^{1-\beta} \right|.
\end{aligned} \tag{3.23}$$

Thus,

$$\|Tu(t_2) - Tu(t_1)\| \longrightarrow 0, \quad \text{for } t_1 \longrightarrow t_2. \tag{3.24}$$

By means of the Arzela-Ascoli theorem, T is completely continuous. Furthermore, for $u \in P$, we have

$$\begin{aligned}
\min_{1/3\leq t\leq 2/3} Tu(t) &= \min_{1/3\leq t\leq 2/3} \int_0^1 G(t,s) f\left(s,u(s),D_{0+}^{\beta}u(s)\right)ds \\
&\geq \frac{1}{6}\int_0^1 K(s) f\left(s,u(s),D_{0+}^{\beta}u(s)\right)ds \\
&= \frac{1}{6}\max_{0\leq t\leq 1} Tu(t), \\
\max_{0\leq t\leq 1}|u(t)| &= \int_0^1 G(t,s) f\left(s,u(s),D_{0+}^{\beta}u(s)\right)ds \\
&\leq \int_0^1 K(s) f\left(s,u(s),D_{0+}^{\beta}u(s)\right)ds \\
&\leq \int_0^1 \gamma_0 k(s) f\left(s,u(s),D_{0+}^{\beta}u(s)\right)ds \\
&= \gamma_0 \max_{0\leq t\leq 1}\left|D^{\beta}u(t)\right|.
\end{aligned} \tag{3.25}$$

Thus, $T : P \rightarrow P$ is completely continuous.

Let the continuous nonnegative concave functional α, the the continuous nonnegative convex functionals γ, θ, and the continuous nonnegative functional ψ be defined on the cone by

$$\gamma(u) = \max_{0\leq t\leq 1}\left|D^{\beta}u(t)\right|, \qquad \theta(u) = \psi(u) = \max_{0\leq t\leq 1}|u(t)|, \qquad \alpha(u) = \min_{1/3\leq t\leq 2/3}|u(t)|. \tag{3.26}$$

By Lemmas 3.3 and 3.4, the functionals defined above satisfy that

$$\frac{1}{6}\theta(u) \leq \alpha(u) \leq \theta(u) = \psi(u), \quad \|u\| \leq \gamma_1\gamma(u), \quad u \in P, \tag{3.27}$$

where $\gamma_1 = \max\{\gamma_0, 1\}$. Therefore condition (2.10) of Lemma 2.9 is satisfied.

Assume that there exist constants $0 < a, b, d$ with $a < b < (M/6N)d$, $c = 6b$ such that

(A_1) $f(t,u,v) \leq d/N$, $(t,u,v) \in [0,1] \times [0,\gamma_1 d] \times [-d,d]$,

(A_2) $f(t,u,v) > 6b/M$, $(t,u,v) \in [0,1] \times [b,6b] \times [-d,d]$,

(A_3) $f(t,u,v) < a/M$, $(t,u,v) \in [0,1] \times [0,a] \times [-d,d]$.

□

Theorem 3.6. *Under assumptions* (A_1)*–*(A_3)*, problem* (1.1) *has three positive solutions* u_1*,* u_2*,* u_3 *satisfying*

$$\begin{aligned} &\max_{0\le t\le 1}\left|D^{\beta}u(t)\right| \le d, \qquad i=1,2,3; \qquad b<\min_{1/3\le t\le 2/3}|u_1(t)|; \\ &a<\max_{0\le t\le 1}|u_2(t)|, \qquad \min_{1/3\le t\le 2/3}|u_2(t)|<b; \qquad \max_{0\le t\le 1}|u_3(t)|\le a. \end{aligned} \tag{3.28}$$

Proof. Problem (1.1) has a solution $u=u(t)$ if and only if u solves the operator equation

$$u(t)=\int_0^1 G(t,s)f\left(s,u(s),D^{\beta}u(s)\right)ds=(Tu)(t). \tag{3.29}$$

For $u\in\overline{P(\gamma,d)}$, we have $\gamma(u)=\max_{0\le t\le 1}|D^{\beta}u(t)|<d$. From assumption (A_1), we obtain

$$f\left(t,u(t),D^{\beta}u(t)\right)\le \frac{d}{N}. \tag{3.30}$$

Thus

$$\begin{aligned} \gamma(Tu) &= \max_{0\le t\le 1}\left|D^{\beta}(\mathrm{T}u)\right| \\ &= \Bigg|\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}f\left(s,u,D_{0+}^{\beta}u\right)ds \\ &\qquad -\frac{t^{1-\beta}}{\Gamma(2-\beta)}\left(\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}f\left(s,u,D_{0+}^{\beta}u\right)ds\right. \\ &\qquad\qquad \left.+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f\left(s,u,D_{0+}^{\beta}u\right)ds\right)\Bigg| \\ &\le \Bigg[\frac{1}{(\alpha-\beta)\Gamma(\alpha-\beta)} \\ &\qquad +\frac{1}{\Gamma(2-\beta)}\left(\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}ds+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}ds\right)\Bigg]\times\frac{d}{N}=d. \end{aligned} \tag{3.31}$$

Hence, $T:\overline{P(\gamma,d)}\to\overline{P(\gamma,d)}$.

The fact that the constant function $u(t)=6b\in P(\gamma,\theta,\alpha,b,c,d)$ and $\alpha(6b)>b$ implies that $\{u\in P(\gamma,\theta,\alpha,b,c,d|\alpha(u)>b)\}\neq\emptyset$.

For $u\in P(\gamma,\theta,\alpha,b,c,d)$, we have $b\le u(t)\le 6b$ and $|D^{\beta}u(t)|<d$ for $0\le t\le 1$. From assumption (A_2),

$$f\left(t,u(t),D^{\beta}u(t)\right)>\frac{6b}{M}. \tag{3.32}$$

Thus

$$\alpha(Tu) = \min_{1/3\le t\le 2/3}\int_0^1 G(t,s) f\left(s, u(s), D_{0+}^{\beta}u(s)\right) ds$$
$$\ge \frac{1}{6}\int_0^1 K(s) f\left(s, u(s), D^{\beta}u(s)\right) ds \tag{3.33}$$
$$\ge \frac{1}{6}\int_0^1 K(s) ds \times \frac{6b}{M} = b,$$

which means $\alpha(Tu) > b$, for all $u \in P(\gamma, \theta, \alpha, b, 6b, d)$. These ensure that condition (S_1) of Lemma 2.9 is satisfied. Secondly, for all $u \in P(\gamma, \alpha, b, d)$ with $\theta(Tu) > 6b$,

$$\alpha(Tu) \ge \frac{1}{6}\theta(Tu) > \frac{1}{6} \times c = \frac{1}{6} \times 6b = b. \tag{3.34}$$

Thus, condition (S_2) of Lemma 2.3 holds. Finally we show that (S_3) also holds. We see that $\psi(0) = 0 < a$ and $0 \notin R(\gamma, \psi, a, d)$. Suppose that $u \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$. Then by assumption (A_3),

$$\psi(Tu) = \max_{0\le t\le 1}|(Tu)(t)| = \int_0^1 G(t,s) f\left(s, u(s), D^{\beta}u(s)\right) ds$$
$$\le \int_0^1 K(s) ds \times \frac{a}{M} = a. \tag{3.35}$$

Thus, all conditions of Lemma 2.9 are satisfied. Hence problem (1.1) has at least three positive concave solutions u_1, u_2, u_3 satisfying (3.28). □

4. Example

Consider the nonlinear FBVPs

$$\begin{gathered} D_{0+}^{\alpha}u(t) = f\left(t, u(t), D_{0+}^{\beta}u(t)\right), \quad t \in (0,1), \\ u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0, \end{gathered} \tag{4.1}$$

where $\alpha = 1.6$, $\beta = 0.5$, $n = 2$ and

$$f(t,u,v) = \begin{cases} \dfrac{1}{\pi^4}e^t + \dfrac{1}{10}u^4 + \dfrac{1}{100}\sin\left(\dfrac{v}{10000}\right), & 0 \le u \le 10, \\ \dfrac{1}{\pi^4}e^t + 1000 + \dfrac{1}{100}\sin\left(\dfrac{v}{10000}\right), & u > 10. \end{cases} \tag{4.2}$$

Choose $a = 1$, $b = 3$, $d = 10000$. By a simple computation, we obtain that

$$\gamma_0 \approx 8.7149, \qquad \gamma_1 \approx 8.7149, \qquad M \approx 2.5181, \qquad N \approx 2.8672. \tag{4.3}$$

We can check that the nonlinear term $f(t,u,v)$ satisfies

(1) $f(t,u,v) < d/N$, $(t,u,v) \in [0,1] \times [0,87149] \times [-10000,10000]$,

(2) $f(t,u,v) > 6b/M$, $(t,u,v) \in [0,1] \times [3,18] \times [-10000,10000]$,

(3) $f(t,u,v) < a/M$, $(t,u,v) \in [0,1] \times [0,1] \times [-10000,10000]$.

Then all assumptions of Theorem 3.6 are satisfied. Thus problem (4.1) has at least three positive solutions $u_1(t)$, $u_2(t)$, $u_3(t)$ satisfying

$$\begin{gathered}
\max_{0\leq t\leq 1}\left|D_{0+}^{\beta}u(t)\right| \leq 10000, \qquad i = 1,2,3; \\
3 < \min_{1/2\leq t\leq 1}|u_1(t)|, \qquad 1 < \max_{0\leq t\leq 1}|u_2(t)|, \\
\min_{1/2\leq t\leq 1}|u_2(t)| < 3, \qquad \max_{0\leq t\leq 1}|u_3(t)| \leq 1.
\end{gathered} \tag{4.4}$$

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References

[1] D. Delbosco, "Fractional calculus and function spaces," *Journal of Fractional Calculus*, vol. 6, pp. 45–53, 1994.

[2] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley & Sons, New York, NY, USA, 1993.

[3] V. Lakshmikantham and A. S. Vatsala, "Theory of fractional differential inequalities and applications," *Communications in Applied Analysis*, vol. 11, no. 3-4, pp. 395–402, 2007.

[4] V. Lakshmikantham and J. V. Devi, "Theory of fractional differential equations in a Banach space," *European Journal of Pure and Applied Mathematics*, vol. 1, no. 1, pp. 38–45, 2008.

[5] V. Lakshmikantham and S. Leela, "Nagumo-type uniqueness result for fractional differential equations," *Nonlinear Analysis*, vol. 71, no. 7-8, pp. 2886–2889, 2009.

[6] V. Lakshmikantham and S. Leela, "A Krasnoselskii-Krein-type uniqueness result for fractional differential equations," *Nonlinear Analysis*, vol. 71, no. 7-8, pp. 3421–3424, 2009.

[7] V. Lakshmikantham, "Theory of fractional functional differential equations," *Nonlinear Analysis*, vol. 69, no. 10, pp. 3337–3343, 2008.

[8] C. Yu and G. Gao, "Existence of fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 310, no. 1, pp. 26–29, 2005.

[9] S. Zhang, "The existence of a positive solution for a nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 252, no. 2, pp. 804–812, 2000.

[10] S. Zhang, "Existence of positive solution for some class of nonlinear fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 1, pp. 136–148, 2003.

[11] A. Babakhani and V. Daftardar-Gejji, "Existence of positive solutions of nonlinear fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 2, pp. 434–442, 2003.

[12] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[13] S. Zhang, "Positive solutions for boundary-value problems of nonlinear fractional differential equations," *Electronic Journal of Differential Equations*, vol. 36, pp. 1–12, 2006.

[14] N. Kosmatov, "A singular boundary value problem for nonlinear differential equations of fractional order," *Journal of Applied Mathematics and Computing*, vol. 29, no. 1-2, pp. 125–135, 2009.

[15] X. Xu, D. Jiang, and C. Yuan, "Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation," *Nonlinear Analysis*, vol. 71, no. 10, pp. 4676–4688, 2009.
[16] X. Su, "Boundary value problem for a coupled system of nonlinear fractional differential equations," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 64–69, 2009.
[17] S. Liang and J. Zhang, "Positive solutions for boundary value problems of nonlinear fractional differential equation," *Nonlinear Analysis*, vol. 71, no. 11, pp. 5545–5550, 2009.
[18] R. P. Agarwal, D. O'Regan, and S. Staněk, "Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 371, no. 1, pp. 57–68, 2010.
[19] C. S. Goodrich, "Existence of a positive solution to a class of fractional differential equations," *Applied Mathematics Letters*, vol. 23, no. 9, pp. 1050–1055, 2010.
[20] D. Jiang and C. Yuan, "The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application," *Nonlinear Analysis*, vol. 72, no. 2, pp. 710–719, 2010.
[21] J. Caballero, J. Harjani, and K. Sadarangani, "Positive solutions for a class of singular fractional boundary value problems," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1325–1332, 2011.
[22] B. Ahmad and J. J. Nieto, "Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions," *Computers & Mathematics with Applications*, vol. 58, no. 9, pp. 1838–1843, 2009.
[23] Z. Bai, "On positive solutions of a nonlocal fractional boundary value problem," *Nonlinear Analysis*, vol. 72, no. 2, pp. 916–924, 2010.
[24] B. Ahmad and S. Sivasundaram, "On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order," *Applied Mathematics and Computation*, vol. 217, no. 2, pp. 480–487, 2010.
[25] C. F. Li, X. N. Luo, and Y. Zhou, "Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1363–1375, 2010.
[26] Y. Wang, L. Liu, and Y. Wu, "Positive solutions for a nonlocal fractional differential equation," *Nonlinear Analysis*, vol. 74, no. 11, pp. 3599–3605, 2011.
[27] H. A. H. Salem, "On the fractional calculus in abstract spaces and their applications to the Dirichlet-type problem of fractional order," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1278–1293, 2010.
[28] M. ur Rehman and R. A. Khan, "Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations," *Applied Mathematics Letters*, vol. 23, no. 9, pp. 1038–1044, 2010.
[29] Y. Guo and W. Ge, "Positive solutions for three-point boundary value problems with dependence on the first order derivative," *Journal of Mathematical Analysis and Applications*, vol. 290, no. 1, pp. 291–301, 2004.
[30] X. Yang, Z. Wei, and W. Dong, "Existence of positive solutions for the boundary value problem of nonlinear fractional differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 1, pp. 85–92, 2012.
[31] R. I. Avery and A. C. Peterson, "Three positive fixed points of nonlinear operators on ordered Banach spaces," *Computers & Mathematics with Applications*, vol. 42, no. 3–5, pp. 313–322, 2001.
[32] L. Yang, X. Liu, and M. Jia, "Multiplicity results for second-order m-point boundary value problem," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 1, pp. 532–542, 2006.
[33] H. Lian, H. Pang, and W. Ge, "Triple positive solutions for boundary value problems on infinite intervals," *Nonlinear Analysis*, vol. 67, no. 7, pp. 2199–2207, 2007.
[34] N. Kosmatov, "Symmetric solutions of a multi-point boundary value problem," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 1, pp. 25–36, 2005.
[35] L. Yang, C.-f. Shen, and X.-p. Liu, "Existence of three positive solutions for some second-order m-point boundary value problems," *Acta Mathematicae Applicatae Sinica. English Series*, vol. 24, no. 2, pp. 253–264, 2008.
[36] X.-B. Shu and Y.-T. Xu, "Triple positive solutions for a class of boundary value problem of second-order functional differential equations," *Nonlinear Analysis*, vol. 61, no. 8, pp. 1401–1411, 2005.
[37] H. Feng and W. Ge, "Existence of three positive solutions for m-point boundary-value problems with one-dimensional p-Laplacian," *Nonlinear Analysis*, vol. 68, no. 7, pp. 2017–2026, 2008.