

## Research Article

# The Global Behavior of a Periodic Epidemic Model with Travel between Patches

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We establish an SIS (susceptible-infected-susceptible) epidemic model, in which the travel between patches and the periodic transmission rate are considered. As an example, the global behavior of the model with two patches is investigated. We present the expression of basic reproduction ratio  $R_0$  and two theorems on the global behavior: if  $R_0 < 1$  the disease-free periodic solution is globally asymptotically stable and if  $R_0 > 1$ , then it is unstable; if  $R_0 > 1$ , the disease is uniform persistence. Finally, two numerical examples are given to clarify the theoretical results.

## 1. Introduction

Epidemic models have been paid intensive attention for recent decades. In the models, population is divided into several compartments, for example, susceptible (S), infected (I), and recovery (R) by individual state. The classic epidemic models, including SIS model and SIR model, generally aim at the basic reproduction ratio (the epidemic threshold) and the global behavior [1–6].

With the development of transportation, the travel becomes more and more easy for people. It has been observed that the travel can affect the spread of infectious disease. In [7, 8], authors showed that international travel is one of the major factors associated with the global spread of infectious disease. Ruan et al. investigated the effect of global travel on the spread of SARS [9] and pointed out that the basic reproduction ratio is independent upon the travel but the travel increase the number of infected individuals.

On the other hand, many infectious diseases show seasonal behavior, such as measles, chickenpox, rubella, and influenza. Zhang and Zhao [10] presented a periodic SIS epidemic model with individuals immigration among  $n$  patches. By employing the persistence theory,

they gave the expression of the epidemic threshold and obtained the conditions under which the positive periodic solution is globally asymptotically stable. In [11], Wang and Zhao showed that the threshold parameter is the basic reproduction ratio for a wide class of compartmental epidemic model in periodic environments. Applying the method in [10, 11], Nakata and Kuniya [12] and Bai and Zhou [13] examined the threshold dynamics of a periodic SEIRS epidemic model.

Combining the mobility and seasonality, we consider an SIS epidemic model, in which people can travel among  $n$  patches and the transmission rate is a periodic function. Our SIS epidemic model with mobility and seasonality is as follows:

$$\begin{aligned}
 \frac{dS_{ii}}{dt} &= B_i(t, N_i^p) N_i^p + \sum_{k=1, k \neq i}^n \rho_{ik}(t) S_{ik} - (\sigma_i(t) + d_{ii}(t)) S_{ii} - \sum_{k=1}^n \beta_{iik}(t) S_{ii} I_{ki} + \gamma_{ii}(t) I_{ii}, \\
 \frac{dI_{ii}}{dt} &= \sum_{k=1, k \neq i}^n \rho_{ik}(t) I_{ik} + \sum_{k=1}^n \beta_{iik}(t) S_{ii} I_{ki} - (\sigma_i(t) + d_{ii}(t) + \gamma_{ii}(t)) I_{ii}, \\
 \frac{dS_{ij}}{dt} &= \sigma_i(t) V_{ij}(t) S_{ii} - (\rho_{ij}(t) + d_{ij}(t)) S_{ij} - \sum_{k=1}^n \beta_{ijk}(t) S_{ij} I_{ki} + \gamma_{ij}(t) I_{ij}, \quad i \neq j \\
 \frac{dI_{ij}}{dt} &= \sigma_i(t) V_{ij}(t) I_{ii} + \sum_{k=1}^n \beta_{ijk}(t) S_{ij} I_{ki} - (\rho_{ij}(t) + d_{ij}(t) + \gamma_{ij}(t)) I_{ij}, \quad i \neq j,
 \end{aligned} \tag{1.1}$$

where  $S_{ij}(t)$  and  $I_{ij}(t)$  are the number of susceptible and infected individuals whose current location is the  $j$ th patch and home location is the  $i$ th patch at time  $t$ , respectively. Denote  $N_{ij} = S_{ij}(t) + I_{ij}(t)$ .  $N_i^p = \sum_{j=1}^n N_{ji}$ ,  $N_i^p$  is the number of individuals who are physically present in the  $i$ th patch at time  $t$ .  $N = \sum_{i,j=1}^n (S_{ij} + I_{ij})$ .  $B(t, N_i^p)$  is the birth rate of the population in the  $i$ th patch.  $d_{ij}(t)$  is the death rate of the individuals whose current location is the  $j$ th patch and home location is the  $i$ th patch at time  $t$ . Individuals are assumed to leave a patch  $i$  at a certain constant rate,  $\sigma_i(t)$ . The probability that a person travels from patch  $i$  to any other patch  $j$  is given by  $V_{ij}(t)$ . So  $\sigma_i(t) V_{ij}(t)$  is the travel rate of individuals from the  $i$ th patch to the  $j$ th patch at time  $t$ . A person from patch  $i$  who travels to patch  $j$  returns home at a rate  $\rho_{ij}(t)$ .  $\beta_{ikj}(t)$  is the disease transmission coefficient in patch  $k$  that a susceptible individual from patch  $i$  contacts with an infectious individual from patch  $j$ . The recovery rate of infectious individuals from  $i$ th patch who are present in region  $j$  is  $\gamma_{ij}(t)$ . In [14], the birth rate  $B_i(t, N_i^p)$  satisfies the following basic assumptions for  $N_i^p \in (0, \infty)$ :

- (A1)  $B_i(t, N_i^p) > 0$ ,  $i = 1, 2, \dots, n$ ;
- (A2)  $B_i(t, N_i^p)$  is continuously differentiable with  $dB_i(t, N_i^p)/dN_i^p < 0$ ,  $i = 1, 2, \dots, n$ ;
- (A3)  $B_i(t, \infty) < d_{ii}(t)$ ,  $i = 1, 2, \dots, n$ ;

and the birth function  $B_i(t, N_i^p) = B(t)/N_i^p + C(t)$  can be found in the biological literature.

We assume that these coefficients are functions being continuous, positive  $\omega$ -periodic in  $t$  and we can obtain a periodic SIS epidemic model, in which individuals can travel among  $n$  patches. For simplicity, we consider an SIS model with travel among two patches, that is,  $n = 2$ . In this paper, we assume that  $B_i(t, N_i^p) = B(t)/(N_i^p) + C(t)$ ,  $d_{ij}(t) = d(t)$ ,  $C(t) < d(t)$ .  $\beta_{ikj}(t) = \beta(t)$ ,  $\gamma_{ij}(t) = \gamma(t)$ ,  $i, j = 1, 2$ . Hence  $\sum_{j=1, j \neq i}^2 V_{ij}(t) = 1$ , we have  $V_{ij}(t) = 1$ .  $\sigma_1(t)$  and  $\sigma_2(t)$  are the travel rate from the 1st patch to the 2nd patch and from the 2nd patch to the

1st patch, respectively.  $B(t)$ ,  $C(t)$ ,  $d(t)$ ,  $\beta(t)$ ,  $\gamma(t)$ , and  $\rho_{ij}(t)$  are continuous, positive  $\omega$ -periodic functions of  $t$ . We have the following system:

$$\begin{aligned}
\frac{dS_{11}}{dt} &= \left( \frac{B(t)}{N_1^p} + C(t) \right) N_1^p + \rho_{12}(t)S_{12} - (\sigma_1(t) + d(t))S_{11} - \beta(t)S_{11}(I_{11} + I_{21}) + \gamma(t)I_{11}, \\
\frac{dS_{12}}{dt} &= \sigma_1(t)S_{11} - (\rho_{12}(t) + d(t))S_{12} - \beta(t)S_{12}(I_{12} + I_{22}) + \gamma(t)I_{12}, \\
\frac{dS_{21}}{dt} &= \sigma_2(t)S_{22} - (\rho_{21}(t) + d(t))S_{21} - \beta(t)S_{21}(I_{11} + I_{21}) + \gamma(t)I_{21}, \\
\frac{dS_{22}}{dt} &= \left( \frac{B(t)}{N_2^p} + C(t) \right) N_2^p + \rho_{21}(t)S_{21} - (\sigma_2(t) + d(t))S_{22} - \beta(t)S_{22}(I_{12} + I_{22}) + \gamma(t)I_{22}, \\
\frac{dI_{11}}{dt} &= \rho_{12}(t)I_{12} + \beta(t)S_{11}(I_{11} + I_{21}) - (\sigma_1(t) + d(t) + \gamma(t))I_{11}, \\
\frac{dI_{12}}{dt} &= \sigma_1(t)I_{11} + \beta(t)S_{12}(I_{12} + I_{22}) - (\rho_{12}(t) + d(t) + \gamma(t))I_{12}, \\
\frac{dI_{21}}{dt} &= \sigma_2(t)I_{22} + \beta(t)S_{21}(I_{11} + I_{21}) - (\rho_{21}(t) + d(t) + \gamma(t))I_{21}, \\
\frac{dI_{22}}{dt} &= \rho_{21}(t)I_{21} + \beta(t)S_{22}(I_{12} + I_{22}) - (\sigma_2(t) + d(t) + \gamma(t))I_{22}.
\end{aligned} \tag{1.2}$$

In this paper, we will study the basic reproduction ratio and global behavior of system (1.2). This paper is organized as follows. In Section 2, we show the existence of the disease-free periodic solution of (1.2) and define the basic reproduction ratio. In Section 3, we show the global asymptotical stability of the periodic disease-free solution and the uniform persistence of the disease. In Section 4, two numerical examples are given to clarify the theoretical results.

## 2. The Basic Reproduction Ratio

Let  $(R^n, R_+^n)$  be the standard ordered  $n$ -dimensional Euclidian space with a norm  $\|\cdot\|$ . For  $u, v \in R^n$ , we write  $u \geq v$  if  $u - v \in R_+^n$ ,  $u > v$ , if  $u - v \in R_+^n \setminus \{0\}$ , and  $u \gg v$  if  $u - v \in \text{Int}(R_+^n)$ . Let  $A(t)$  be a continuous, cooperative, irreducible, and  $\omega$ -periodic  $n \times n$  matrix function, and  $\Phi_A(t)$  is the fundamental solution matrix of the linear ordinary differential system

$$\frac{dX}{dt} = A(t)X, \tag{2.1}$$

and  $r(\Phi_A(\omega))$  be the spectral radius of  $\Phi_A(\omega)$ . By the Perron-Frobenius theorem,  $r(\Phi_A(\omega))$  is the principal eigenvalue of  $\Phi_A(\omega)$  in the sense that it is simple and admits an eigenvector  $v^* \gg 0$ . The following result is useful for our subsequent comparison arguments.

**Lemma 2.1** (see [10]). *Let  $P = (1/\omega) \ln r(\Phi_A(\omega))$ . Then there exists a positive,  $\omega$ -periodic function  $V(t)$  such that  $e^{Pt} V(t)$  is a solution of (2.1).*

**Lemma 2.2.** *Every forward solution of (1.2) eventually into*

$$\Gamma = \left\{ (S(t), I(t)) \in \mathbb{R}_+^8 \mid 0 \leq \sum_{i,j=1}^2 (S_{ij} + I_{ij}) \leq \frac{2b}{c} \right\}, \quad (2.2)$$

where  $b = \max_{t \in [0, \omega]} (B(t))$ ,  $c = \min_{t \in [0, \omega]} (d(t) - C(t))$ , and for each  $N(t) \geq 2b/c$ ,  $\Gamma$  is a positively invariant set for (1.2).

*Proof.* By the method of variation of constant, it is obvious that any solution of (1.2) with nonnegative initial values is nonnegative. From (1.2), we have

$$\frac{dN}{dt} = 2B(t) - (d(t) - C(t))N \leq 2b - cN \leq 0 \quad \text{if } N(t) \geq \frac{2b}{c}. \quad (2.3)$$

This implies that  $\Gamma$  is a forward invariant compact absorbing set of (1.2). Hence, the proof is complete.  $\square$

Next, we show the existence of the disease-free periodic solution of (1.2). To find the disease-free periodic solution of (1.2), we consider

$$\begin{aligned} \frac{dS_{11}}{dt} &= B(t) + C(t)(S_{11} + S_{21}) + \rho_{12}(t)S_{12} - (\sigma_1(t) + d(t))S_{11}, \\ \frac{dS_{12}}{dt} &= \sigma_1(t)S_{11} - (\rho_{12}(t) + d(t))S_{12}, \\ \frac{dS_{21}}{dt} &= \sigma_2(t)S_{22} - (\rho_{21}(t) + d(t))S_{21}, \\ \frac{dS_{22}}{dt} &= B(t) + C(t)(S_{12} + S_{22}) + \rho_{21}(t)S_{21} - (\sigma_2(t) + d(t))S_{22}. \end{aligned} \quad (2.4)$$

Denote

$$M(t) = \begin{pmatrix} C(t) - (\sigma_1(t) + d(t)) & \rho_{12}(t) & C(t) & 0 \\ \sigma_1(t) & -(\rho_{12}(t) + d(t)) & 0 & 0 \\ 0 & 0 & -(\rho_{21}(t) + d(t)) & \sigma_2(t) \\ 0 & C(t) & \rho_{21}(t) & C(t) - (\sigma_2(t) + d(t)) \end{pmatrix}. \quad (2.5)$$

Let  $\Psi : \mathbb{R}_+^1 \times \mathbb{R}_+^4 \rightarrow \mathbb{R}_+^4$  be defined by the right-hand side of (2.4).  $\Psi_i(t, S) \geq 0$  for every  $S \geq 0$  with  $S_i = 0, t \in \mathbb{R}_+, 1 \leq i \leq 4$ .  $\Psi(t, S)$  is strongly subhomogeneous for  $S \in \mathbb{R}_+^4$  in the sense that  $\Psi(t, \alpha S) \gg \alpha \Psi(t, S)$  for any  $t \geq 0, S \in \mathbb{R}_+^4$  and  $\alpha \in (0, 1)$ .  $M(t)$  is a continuous, cooperative, irreducible, and  $\omega$ -periodic  $4 \times 4$  matrix function. By Lemma 2.2, the solution of (2.4) is ultimately bounded in  $\mathbb{R}_+^4$ . By Theorem 2.3.2 of [15], applying the Poincaré map associated with (2.4), it follows that system (2.4) has a unique positive periodic solution

$$S^*(t) = (S_{11}^*(t), S_{12}^*(t), S_{21}^*(t), S_{22}^*(t)). \quad (2.6)$$

We need to assume that  $r(\phi_M(\omega)) > 1, r(\Phi_M(\omega))$  be the spectral radius of  $M(\omega)$ . By Theorem 2.1.2 of [15], it then follows that the unique positive periodic solution  $S^*(t)$  of (2.4) is globally attractive for  $S^0 \in R_+^4 \setminus \{0\}$ . Hence, (1.2) has a unique disease-free periodic state  $(S^*, 0, 0, 0)$ .

For convenience, we denote

$$S(t) = (S_{11}(t), S_{12}(t), S_{21}(t), S_{22}(t)), \quad I(t) = (I_{11}(t), I_{12}(t), I_{21}(t), I_{22}(t)). \quad (2.7)$$

Consider the following system:

$$\begin{aligned} \frac{dI_{11}}{dt} &= \rho_{12}(t)I_{12} + \beta(t)S_{11}(I_{11} + I_{21}) - (\sigma_1(t) + d(t) + \gamma(t))I_{11}, \\ \frac{dI_{12}}{dt} &= \sigma_1(t)I_{11} + \beta(t)S_{12}(I_{12} + I_{22}) - (\rho_{12}(t) + d(t) + \gamma(t))I_{12}, \\ \frac{dI_{21}}{dt} &= \sigma_2(t)I_{22} + \beta(t)S_{21}(I_{11} + I_{21}) - (\rho_{21}(t) + d(t) + \gamma(t))I_{21}, \\ \frac{dI_{22}}{dt} &= \rho_{21}(t)I_{21} + \beta(t)S_{22}(I_{12} + I_{22}) - (\sigma_2(t) + d(t) + \gamma(t))I_{22}. \end{aligned} \quad (2.8)$$

Define function matrix

$$\begin{aligned} F(t) &= \begin{pmatrix} \beta(t)S_{11}^*(t) & 0 & \beta(t)S_{11}^*(t) & 0 \\ 0 & \beta(t)S_{12}^*(t) & 0 & \beta(t)S_{12}^*(t) \\ \beta(t)S_{21}^*(t) & 0 & \beta(t)S_{21}^*(t) & 0 \\ 0 & \beta(t)S_{22}^*(t) & 0 & \beta(t)S_{22}^*(t) \end{pmatrix}, \\ V(t) &= \begin{pmatrix} \sigma_1(t) + d(t) + \gamma(t) & -\rho_{12}(t) & 0 & 0 \\ -\sigma_1(t) & \rho_{12}(t) + d(t) + \gamma(t) & 0 & 0 \\ 0 & 0 & \rho_{21}(t) + d(t) + \gamma(t) & -\sigma_2(t) \\ 0 & 0 & -\rho_{21}(t) & \sigma_2(t) + d(t) + \gamma(t) \end{pmatrix}. \end{aligned} \quad (2.9)$$

Then (2.8) can be rewritten as

$$\frac{dZ}{dt} = (F(t) - V(t))Z, \quad (2.10)$$

where  $Z = I(t)^T$ .

Assume that  $Y(t, s)$ ,  $t \geq s$  is the evolution operator of the linear periodic system

$$\frac{dy}{dt} = -V(t)y. \quad (2.11)$$

That is, for each  $s \in R$ , the  $4 \times 4$  matrix  $Y(t, s)$  satisfies

$$\frac{dY(t, s)}{dt} = -V(t)Y(t, s), \quad \forall t \geq s, Y(s, s) = I, \quad (2.12)$$

where  $I$  is a  $2 \times 2$  identity matrix.

Let  $C_\omega$  be the ordered Banach space of all  $\omega$ -periodic function  $R \rightarrow R^4$ , which is equipped with norm  $\|\cdot\|_\infty$  and the positive cone  $C_\omega^+ = \{\phi \in C_\omega : \phi(t) \geq 0, \forall t \in R\}$ .

Consider the following operator  $L : C_\omega \rightarrow C_\omega$  by

$$(L\phi)(t) = \int_0^{+\infty} Y(t, t-a)\phi(t-a)da, \quad \forall t \in R, \phi \in C_\omega. \quad (2.13)$$

We can define the basic reproduction ratio  $R_0 = r(L)$ , the spectral of radius of  $L$ .

**Theorem 2.3** (see [11, Theorem 2.2]). *The following statements are valid:*

- (i)  $R_0 = 1$  if and only if  $r(\Phi_{F-V}(\omega)) = 1$ .
- (ii)  $R_0 > 1$  if and only if  $r(\Phi_{F-V}(\omega)) > 1$ .
- (iii)  $R_0 < 1$  if and only if  $r(\Phi_{F-V}(\omega)) < 1$ .

Thus,  $(S^*, 0, 0, 0)$  of (1.2) is asymptotically stable if  $R_0 < 1$  and it is unstable if  $R_0 > 1$ .

### 3. The Threshold Dynamics

In this section, we show  $R_0$  as a threshold parameter between the extinction and the uniform persistence of the disease.

**Theorem 3.1.** *If  $R_0 < 1$ , the disease-free periodic solution  $(S^*, 0, 0, 0)$  is globally asymptotically stable and if  $R_0 > 1$ , it is unstable.*

*Proof.* By Theorem 2.3, if  $R_0 > 1$ , the disease-free periodic solution  $(S^*, 0, 0, 0)$  is unstable. If  $R_0 < 1$ , the disease-free periodic solution  $(S^*, 0, 0, 0)$  is locally stable. Hence, it is sufficient to show the global attractivity of  $(S^*, 0, 0, 0)$  when  $R_0 < 1$ .

By (1.2), we have

$$\begin{aligned} \frac{dN_{11}}{dt} &= B(t) + C(t)(N_{11} + N_{21}) + \rho_{12}(t)N_{12} - (\sigma_1(t) + d(t))N_{11}, \\ \frac{dN_{12}}{dt} &= \sigma_1(t)N_{11} - (\rho_{12}(t) + d(t))N_{12}, \\ \frac{dN_{21}}{dt} &= \sigma_2(t)N_{22} - (\rho_{21}(t) + d(t))N_{21}, \\ \frac{dN_{22}}{dt} &= B(t) + C(t)(N_{12} + N_{22}) + \rho_{21}(t)N_{21} - (\sigma_2(t) + d(t))N_{22}. \end{aligned} \quad (3.1)$$

By the aforementioned conclusion, the above system has a unique positive fixed point  $S^*(t)$  which is globally attractive in  $R_+^4 \setminus \{0\}$ . It then follows that for any  $\varepsilon_1 > 0$ , there exists  $T_1 > 1$  such that

$$N_{ij}(t) = S_{ij}(t) + I_{ij}(t) \leq S_{ij}^*(t) + \varepsilon_1, \quad \forall t > T_1. \quad (3.2)$$

Obviously,  $S_{ij}(t) \leq S_{ij}^*(t) + \varepsilon_1$ ,  $(i, j = 1, 2)$ . Hence, we have

$$\begin{aligned} \frac{dI_{11}}{dt} &\leq \rho_{12}(t)I_{12} + \beta(t)(S_{11}^* + \varepsilon_1)(I_{11} + I_{21}) - (\sigma_1(t) + d(t) + \gamma(t))I_{11}, \\ \frac{dI_{12}}{dt} &\leq \sigma_1(t)I_{11} + \beta(t)(S_{12}^* + \varepsilon_1)(I_{12} + I_{22}) - (\rho_{12}(t) + d(t) + \gamma(t))I_{12}, \\ \frac{dI_{21}}{dt} &\leq \sigma_2(t)I_{22} + \beta(t)(S_{21}^* + \varepsilon_1)(I_{11} + I_{21}) - (\rho_{21}(t) + d(t) + \gamma(t))I_{21}, \\ \frac{dI_{22}}{dt} &\leq \rho_{21}(t)I_{21} + \beta(t)(S_{22}^* + \varepsilon_1)(I_{12} + I_{22}) - (\sigma_2(t) + d(t) + \gamma(t))I_{22}. \end{aligned} \quad (3.3)$$

Denote

$$M_1(t) = \begin{pmatrix} \beta(t) & 0 & \beta(t) & 0 \\ 0 & \beta(t) & 0 & \beta(t) \\ \beta(t) & 0 & \beta(t) & 0 \\ 0 & \beta(t) & 0 & \beta(t) \end{pmatrix}. \quad (3.4)$$

By Theorem 2.3, we have  $r(\Phi_{F-V}(\omega)) < 1$ . We restrict  $\varepsilon_1 > 0$  such that  $r(\Phi_{F-V+\varepsilon M_1}(\omega)) < 1$ .

Consider the system

$$\begin{aligned} \frac{d\tilde{I}_{11}}{dt} &= \rho_{12}(t)I_{12} + \beta(t)(S_{11}^* + \varepsilon_1)(I_{11} + I_{21}) - (\sigma_1(t) + d(t) + \gamma(t))I_{11}, \\ \frac{d\tilde{I}_{12}}{dt} &= \sigma_1(t)I_{11} + \beta(t)(S_{12}^* + \varepsilon_1)(I_{12} + I_{22}) - (\rho_{12}(t) + d(t) + \gamma(t))I_{12}, \\ \frac{d\tilde{I}_{21}}{dt} &= \sigma_2(t)I_{22} + \beta(t)(S_{21}^* + \varepsilon_1)(I_{11} + I_{21}) - (\rho_{21}(t) + d(t) + \gamma(t))I_{21}, \\ \frac{d\tilde{I}_{22}}{dt} &= \rho_{21}(t)I_{21} + \beta(t)(S_{22}^* + \varepsilon_1)(I_{12} + I_{22}) - (\sigma_2(t) + d(t) + \gamma(t))I_{22}. \end{aligned} \quad (3.5)$$

Applying Lemma 2.1 and the standard comparison principle, there exists a positive  $\omega$ -positive function  $V_1(t)$  such that  $I(t) \leq V_1(t)e^{p_1 t}$ , where  $p_1 = \ln r(\Phi_{F-V+\varepsilon M_1}(\omega))/\omega < 0$ . Hence, we have that  $\lim_{t \rightarrow \infty} I_{ij}(t) = 0$ ,  $(i, j = 1, 2)$ . Consequently, we obtain that

$$\lim_{t \rightarrow \infty} (S(t) - S^*(t)) = \lim_{t \rightarrow \infty} (\tilde{N}(t) - I(t) - S^*(t)) = 0, \quad (3.6)$$

where  $\tilde{N}(t) = (N_{11}(t), N_{12}(t), N_{21}(t), N_{22}(t))$ .

Hence, the disease free periodic solution  $(S^*, 0, 0, 0)$  is globally attractive and the proof is complete.  $\square$

The following result shows that  $R_0$  is the threshold parameter for the extinction and the uniform persistence of the disease.

We define

$$X = R_+^8, \quad X_0 = R_+^4 \times \text{Int}(R_+^4), \quad \partial X_0 = X \setminus X_0. \quad (3.7)$$

Let  $P : R_+^8 \rightarrow R_+^8$  be the Poincare map associated with (1.2),  $P(x^0) = \mu(\omega, x^0)$ ,  $\forall x^0 \in R_+^8$ , where  $\mu(t, x^0)$  is the solution of (1.2) with  $\mu(0, x^0) = x^0$ .

It is obvious that both  $X$  and  $X_0$  are positively invariant and  $\partial X_0$  is relatively closed in  $X$ . Set

$$M_\partial = \left\{ (S^0, I^0) \in \partial X_0 : P^m(S^0, I^0) \in \partial X_0, \forall m \geq 0 \right\}. \quad (3.8)$$

We now show that

$$M_\partial = \{(S, 0) : S \geq 0\}. \quad (3.9)$$

Obviously,  $\{(S, 0) : S \geq 0\} \subseteq M_\partial$ . To show that  $M_\partial \setminus \{(S, 0) : S \geq 0\} = \emptyset$ , we consider for any  $(S^0, I^0) \in \partial X_0 \setminus \{(S, 0) : S \geq 0\}$ .

Firstly, if one element of  $I^0 = (I_{11}^0, I_{12}^0, I_{21}^0, I_{22}^0)$  is 0, say  $I_{11}^0$ , that is,  $I_{11}^0 = 0$ ,  $I_{12}^0 > 0$ ,  $I_{21}^0 > 0$ ,  $I_{22}^0 > 0$ , then  $I_{21}(t) > 0$ ,  $I_{22}(t) > 0$  for any  $t > 0$ . From (1.2)

$$\left. \frac{dI_{11}}{dt} \right|_{t=0} = \rho_{12}(0)I_{12}(0) + \beta(0)S_{11}(0)I_{21}(0) > 0. \quad (3.10)$$

It is clear that  $I_{11}(t) > 0$ ,  $I_{12}(t) > 0$ ,  $I_{21}(t) > 0$ ,  $I_{22}(t) > 0$ .

Secondly, if two elements of  $I^0 = (I_{11}^0, I_{12}^0, I_{21}^0, I_{22}^0)$  is 0, for example,  $I_{11}^0 = 0$ ,  $I_{12}^0 = 0$ ,  $I_{21}^0 > 0$ ,  $I_{22}^0 > 0$ . From (1.2), using the method of variation of constant, it is clear that  $S_{11}(t) > 0$ ,  $S_{12}(t) > 0$ ,  $S_{21}(t) > 0$ ,  $S_{22}(t) > 0$ , for any  $t > 0$ ,

$$I_{11} = \left( I_{11}^0 + \int_0^t (\rho_{12}(s)I_{12}(s) + \beta(s)S_{11}(s)I_{21}(s)) e^{\int_0^s (\sigma_1(u) + d(u) + \gamma(u) - \beta(s)S_{11}(s)) du} ds \right) \times e^{-\int_0^t (\sigma_1(s) + d(s) + \gamma(s) - \beta(s)S_{11}(s)) ds}. \quad (3.11)$$

Then  $I_{11}(t) > 0$  for any  $t > 0$ .  $I_{12}(t) > 0$  can be proven similarly. So that  $I_{11}(t) > 0$ ,  $I_{12}(t) > 0$ ,  $I_{21}(t) > 0$ ,  $I_{22}(t) > 0$ .

Thirdly, if three elements of  $I^0 = (I_{11}^0, I_{12}^0, I_{21}^0, I_{22}^0)$  are 0, for example, we chose  $I_{11}^0 = 0$ ,  $I_{12}^0 = 0$ ,  $I_{21}^0 = 0$ ,  $I_{22}^0 > 0$ . From (1.2),

$$\left. \frac{dI_{21}}{dt} \right|_{t=0} = \sigma_2(0)I_{22}(0) > 0. \quad (3.12)$$



So  $I_{21}(t) > 0$  for some small  $t$ . From (1.2), using the method of variation of constant, it is clear that  $I_{11}(t) > 0, I_{12}(t) > 0, I_{21}(t) > 0, I_{22}(t) > 0$ .

It follows that  $(S(t), I(t)) \notin \partial X_0$ , for  $0 < t \ll 1$ . Thus, the positive invariance of  $X_0$  implies (3.9). It is clear that there are two fixed points of  $P$  in  $M_{\partial}$ , which are  $M_0 = (0, 0)$  and  $M_1 = (S^*(0), 0)$ .

Now we see  $R_0$  as a threshold parameter between the extinction and the uniform persistence of the disease.

**Theorem 3.2.** *If  $R_0 > 1$ , then there exists some  $\varepsilon > 0$  such that any solution  $(S(t), I(t))$  of (1.2) with initial value  $(S(0), I(0)) = (S^0, I^0) \in R_+^4 \times \text{Int}(R_+^4)$ , satisfies  $\lim_{t \rightarrow \infty} \inf I(t) \geq \varepsilon$ . Furthermore, (1.2) admits at least one positive periodic solution.*

*Proof.* First we prove that  $P$  is uniformly persistent with respect to  $(X_0, \partial X)$ . By Theorem 2.3, we have that  $R_0 > 1$  if and only if  $r(\Phi_{F-V}(\omega)) > 1$ . Then we choose  $\eta > 0$  small enough such that  $r(\Phi_{F-V-\eta M_1}(\omega)) > 1$ . Note that the perturbed system of (2.4),

$$\begin{aligned} \frac{d\hat{S}_{11}}{dt} &= B(t) + C(t)(\hat{S}_{11} + \hat{S}_{21}) + \rho_{12}(t)\hat{S}_{12} - (\sigma_1(t) + d(t) + \beta(t)\delta)\hat{S}_{11}, \\ \frac{d\hat{S}_{12}}{dt} &= \sigma_1(t)\hat{S}_{11} - (\rho_{12}(t) + d(t) + \beta(t)\delta)\hat{S}_{12}, \\ \frac{d\hat{S}_{21}}{dt} &= \sigma_2(t)\hat{S}_{22} - (\rho_{21}(t) + d(t) + \beta(t)\delta)\hat{S}_{21}, \\ \frac{d\hat{S}_{22}}{dt} &= B(t) + C(t)(\hat{S}_{12} + \hat{S}_{22}) + \rho_{21}(t)\hat{S}_{21} - (\sigma_2(t) + d(t) + \beta(t)\delta)\hat{S}_{22}. \end{aligned} \quad (3.13)$$

As in our previous analysis of system (2.4), we can choose  $\delta > 0$  small enough such that the Poincare map associated with (3.13) admits a unique positive fixed point  $S^*(0, \delta)$  which is globally attractive in  $R_+^4 \setminus \{0\}$ . By the implicit function theorem, it follows that  $S^*(0, \delta)$  is continuous in  $\delta$ . Thus, we can fix a small number  $\delta > 0$  such that  $S^*(t, \delta) > S^*(t) - \bar{\eta}$ , where  $\bar{\eta} = \{\eta, \eta, \eta, \eta\}$ . By the continuity of solutions with respect to the initial values, there exists  $\delta_0^* > 0$  such that for all  $(S^0, I^0) \in X_0$ , with  $\|(S^0, I^0) - M\| \leq \delta_0^*$ . We have  $\|\mu(t, (S^0, I^0)) - \mu(t, M_i)\| < \delta, \forall t \in [0, \omega], i = 0, 1$ . We now claim that

$$\lim_{m \rightarrow \infty} \sup d(P^m(S^0, I^0), M_i) \geq \delta_0^*. \quad (3.14)$$

Suppose, by contradiction, that  $\lim_{m \rightarrow \infty} \sup d(P^m(S^0, I^0), M_i) < \delta_0^*$ , for some  $(S^0, I^0) \in X_0$ , and  $i = 0, 1$ . Without loss of generality, we can assume that  $d(P^m(S^0, I^0), M_i) < \delta_0^*, \forall m \geq 0$ . Then, we have  $\|\mu(t, P^m(S^0, I^0)) - \mu(t, M_i)\| < \delta, \forall m \geq 0, \forall t \in [0, \omega]$ .

For any  $t \geq 0$ , let  $t = m\omega + t'$ , where  $t' \in [0, \omega)$  and  $m = [t/\omega]$  is the greatest integer less than or equal to  $t/\omega$ . Then we get

$$\|\mu(t, (S^0, I^0)) - \mu(t, M_i)\| = \|\mu(t', P^m(S^0, I^0)) - \mu(t', M_i)\| < \delta, \quad \forall t \geq 0. \quad (3.15)$$

Let  $(S(t), I(t)) = \mu(t, (S^0, I^0))$ . It then follows that  $0 \leq I_{ij}(t) \leq \delta, \forall t \geq 0, \forall i, j = 1, 2$ .

We have

$$\begin{aligned}
 \frac{dS_{11}}{dt} &\geq B(t) + C(t)(S_{11} + S_{21}) + \rho_{12}(t)S_{12} - (\sigma_1(t) + d(t) + \beta(t)\delta)S_{11}, \\
 \frac{dS_{12}}{dt} &\geq \sigma_1(t)S_{11} - (\rho_{12}(t) + d(t) + \beta(t)\delta)S_{12}, \\
 \frac{dS_{21}}{dt} &\geq \sigma_2(t)S_{22} - (\rho_{21}(t) + d(t) + \beta(t)\delta)S_{21}, \\
 \frac{dS_{22}}{dt} &\geq B(t) + C(t)(S_{12} + S_{22}) + \rho_{21}(t)S_{21} - (\sigma_2(t) + d(t) + \beta(t)\delta)S_{22}.
 \end{aligned} \tag{3.16}$$

Since the fixed point  $S^*(0, \delta)$  of the Poincaré map associated with (3.13) is globally attractive and  $S^*(t, \delta) > S^*(t) - \bar{\eta}$ , there is  $T > 0$ , such that  $S(t) > S^*(t) - \bar{\eta}$  for  $t > T$ , there holds

$$\begin{aligned}
 \frac{dI_{11}}{dt} &\geq \rho_{12}(t)I_{12} + \beta(t)(S_{11}^* - \eta)(I_{11} + I_{21}) - (\sigma_1(t) + d(t) + \gamma(t))I_{11}, \\
 \frac{dI_{12}}{dt} &\geq \sigma_1(t)I_{11} + \beta(t)(S_{12}^* - \eta)(I_{12} + I_{22}) - (\rho_{12}(t) + d(t) + \gamma(t))I_{12}, \\
 \frac{dI_{21}}{dt} &\geq \sigma_2(t)I_{22} + \beta(t)(S_{21}^* - \eta)(I_{11} + I_{21}) - (\rho_{21}(t) + d(t) + \gamma(t))I_{21}, \\
 \frac{dI_{22}}{dt} &\geq \rho_{21}(t)I_{21} + \beta(t)(S_{22}^* - \eta)(I_{12} + I_{22}) - (\sigma_2(t) + d(t) + \gamma(t))I_{22}.
 \end{aligned} \tag{3.17}$$

Since  $r(\Phi_{F-V-\eta M_1}(\omega)) > 1$ , by Lemma 2.1, it is obvious that  $\lim_{t \rightarrow \infty} I_{ij}(t) = \infty, \forall i, j = 1, 2$ . This leads to a contradiction. Then (3.14) holds. Note that  $S^*(0)$  is globally attractive in  $R_+^4 \setminus \{0\}$ . By the aforementioned claim, it follows that  $M_0$  and  $M_1$  are isolated invariance sets in  $X$ ,  $W^s(M_0) \cap X_0 = \emptyset$ , and  $W^s(M_1) \cap X_0 = \emptyset$ . Clearly, every orbit in  $M_\partial$  converges to either  $M_0$  or  $M_1$ ,  $M_0$  and  $M_1$  are acyclic in  $M_\partial$ . By [15, Theorem 1.3.1],  $P$  is uniformly persistent with respect to  $(X_0, \partial X)$ . This implies the uniform persistence of the solutions of system (1.2) with respect to  $(X_0, \partial X)$ . By [6, Theorem 1.3.6],  $P$  has a fixed point  $P(\bar{S}(0), \bar{I}(0)) \in X_0$ . Then,  $\bar{S}(0) \in R_+^4, \bar{I}(0) \in \text{Int}(R_+^4)$ . We further claim that  $\bar{S}(0) \in R_+^4 \setminus \{0\}$ , suppose that  $\bar{S}(0) = 0$ , by (2.8), we can obtain  $-4(d(t) + \gamma(t))(\bar{I}_{11}(0) + \bar{I}_{12}(0) + \bar{I}_{21}(0) + \bar{I}_{22}(0)) = 0$ . And hence  $\bar{I}_{ij}(0) = 0, i, j = 1, 2$ , a contradiction. Thus,  $\bar{S}(0) \geq 0$ . Then  $(\bar{S}(0), \bar{I}(0))$  is a positive  $\omega$ -periodic solution of (1.2). The proof is complete.  $\square$

#### 4. Numerical Simulations

In this section, we give the numerical solutions (1.2) to clarify the correctness of our theoretical results. We set  $B(t) = 0.4, C(t) = 0.12, \rho_{21} = 0.16, \rho_{12} = 0.051, d(t) = 0.3, \gamma(t) = 0.365, \sigma_1(t) = 0.65 + 0.04 \cos(\pi t/6), \sigma_2(t) = 0.3 + 0.04 \cos(\pi t/6), \beta(t) = \beta + 0.06588 \cos(\pi t/6)$ . The initial value of the model is  $S_{11}(0) = 0.8, S_{12}(0) = 0.02, S_{21}(0) = 0.03, S_{22}(0) = 0.4, I_{11}(0) = 0.1, I_{12}(0) = 0.061, I_{21}(0) = 0.03, I_{22}(0) = 0.6$ . Figure 1 shows the numerical solutions of (1.2) when  $\beta = 0.4$ . Because basic reproduction ratio  $R_0 > 1$ , a positive periodic solution exists, and the disease is uniform persistence. In Figure 2,  $\beta = 0.25$ , the disease dies out because  $R_0 < 1$ .

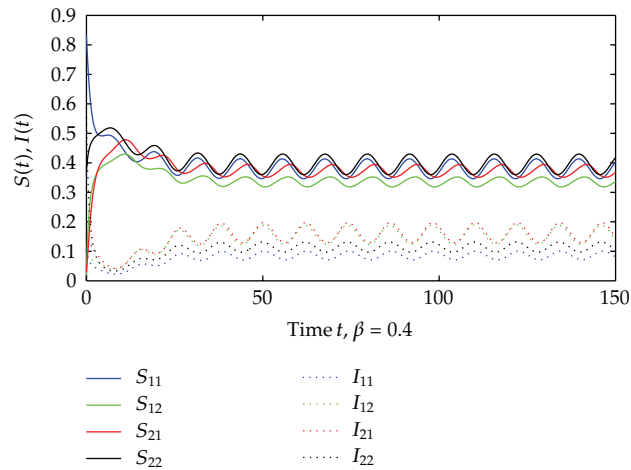


Figure 1: When  $R_0 > 1$ , a positive periodic solution exists and the disease will be uniform persistence.

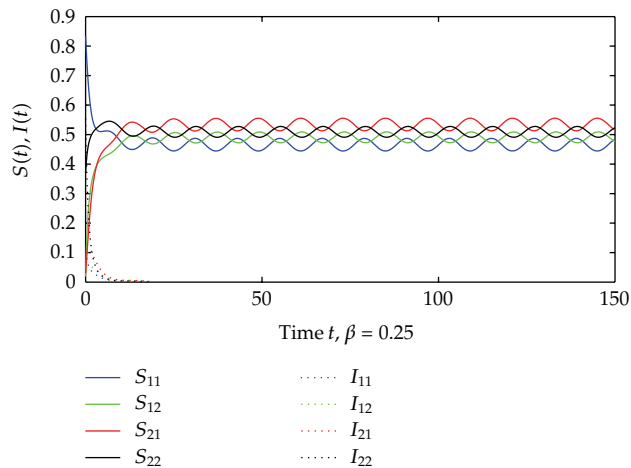


Figure 2: When  $R_0 < 1$ , a periodic disease-free solution exists and the disease dies out.

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