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Research Article

Weighted Approximation for Jackson-Matsuoka Polynomials on the Sphere

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We consider the best approximation by Jackson-Matsuoka polynomials in the weighted L_p space on the unit sphere of $\mathbb{R}^d$. Using the relation between K-functionals and modulus of smoothness on the sphere, we obtain the direct and inverse estimate of approximation by these polynomials for the h-spherical harmonics.

1. Introduction and Notations

Let $\mathbb{S} := \mathbb{S}^{d-1} = \{x : \|x\| = 1\}$ denote the unit sphere in $\mathbb{R}^d$ $(d \geq 3)$, $d \in \mathbb{N}$, where $\|x\|$ denotes the usual Euclidean norm, $\mathbb{R}$ the set of real numbers. For a nonzero vector $v \in \mathbb{R}^d$, let σ_v denote the reflection with respect to the hyperplane perpendicular to $v, x\sigma_v := x - 2(\langle x, v\rangle / \|v\|^2)v,\ x \in \mathbb{R}^d$, where $\langle x, v\rangle$ denote the usual Euclidean inner product. Let G be a finite reflection group on $\mathbb{R}^d$ with a fixed positive root system $\mathbb{R}_+$, normalized so that $\langle v, v\rangle = 2$ for all $v \in \mathbb{R}_+$. Then G is a subgroup of the orthogonal group generated by the reflections $\{\sigma_v : v \in \mathbb{R}_+\}$. Let κ be a nonnegative multiplicity function $v \mapsto \kappa_v$ defined on $\mathbb{R}_+$ with the property that $\kappa_u = \kappa_v$ whenever σ_u is conjugate to σ_v in G, then $v \mapsto \kappa_v$ is a G-invariant function. We consider the weighted best L_p approximation with respect to the measure $h_\kappa^2 d\omega$ on $\mathbb{S}$, where h_κ^2 is defined by

$$h_\kappa = \prod_{v \in \mathbb{R}_+} |\langle x, v\rangle|^{\kappa_v}, \qquad x \in \mathbb{R}^d, \tag{1.1}$$

$d\omega$ is the surface (Lebesgue) measure on $\mathbb{S}$. The function h_κ is a positive homogeneous function of degree $\gamma_\kappa := \sum_{v \in R_+} \kappa_v$, and it is invariant under the reflection group. We denote by a_κ the normalization constant of h_κ, $a_\kappa^{-1} = \int_{\mathbb{S}} h_\kappa^2(y)d\omega$ and denote by $L_p(h_\kappa^2)$, $1 \le p \le \infty$, the space of functions defined on $\mathbb{S}$ with the finite norm

$$\|f\|_{\kappa,p} := \left(a_\kappa \int_{\mathbb{S}} |f(y)|^p h_\kappa^2(y)d\omega(y)\right)^{1/p}, \qquad 1 \le p < \infty, \tag{1.2}$$

and for $p = \infty$ we assume that L_∞ is replaced by $C(\mathbb{S})$ the space of continuous functions on $\mathbb{S}$ with the usual uniform norm $\|f\|_\infty$.

Δ_h denote the h-Laplacian. $\Delta_{h,0}$ is the Laplace-Beltrami operator on the sphere. $\mathcal{P}_n^d$ denote the subspace of homogeneous polynomials of degree n in d variables. The h-harmonics are defined as the homogeneous polynomials satisfying the equation $\Delta_h P = 0$, $P \in \mathcal{P}_n^d$. Furthermore, let $\mathcal{H}_n^d(h_\kappa^2)$ denote the space of h-spherical harmonics of degree n in d variables. The spherical h-harmonics are the restriction of h-harmonics on the unit sphere. It is well known that spherical h-harmonics are eigenfunctions of $\Delta_{h,0}$; that is,

$$\Delta_{h,0} Y(x) = -n(n+2\lambda)Y(x), \qquad x \in \mathbb{S},\ Y \in \mathcal{H}_n^d\left(h_\kappa^2\right). \tag{1.3}$$

The standard Hilbert space theory shows that $L_2(h_\kappa^2) = \sum_{n=0}^{\infty} \oplus \mathcal{H}_n^d(h_\kappa^2)$. That is, with each $f \in L_2(h_\kappa^2)$ we can associate its h-harmonic expansion

$$f(x) = \sum_{n=0}^{\infty} Y_n\left(h_\kappa^2; f, x\right), \qquad x \in \mathbb{S}, \tag{1.4}$$

in $L_2(h_\kappa^2)$ norm. For the surface measure ($\kappa = 0$), such a series is called the Laplace series (see [1]). The orthogonal projection $Y_n(h_\kappa^2) : L_2(h_\kappa^2) \to \mathcal{H}_n^d(h_\kappa^2)$ takes the form

$$Y_n\left(h_\kappa^2; f, x\right) := \int_{\mathbb{S}} f(y) P_n\left(h_\kappa^2; x, y\right) h_\kappa^2(y)d\omega(y), \tag{1.5}$$

where $P_n(h_\kappa^2; x, y)$ is the reproducing kernel of the space of h-harmonics $\mathcal{H}_n^d(h_\kappa^2)$, which is given by (see [2])

$$P_n\left(h_\kappa^2; x, y\right) = \frac{n+\lambda}{\lambda} V_\kappa\left[C_n^\lambda(\langle \cdot, y\rangle)\right](x). \tag{1.6}$$

C_n^λ is the ultraspherical polynomial of degree n, $\lambda := \gamma_\kappa + (d-2)/2$, $\gamma_\kappa = \sum_{v \in \mathbb{R}_+} \kappa_v$, and the intertwining operator V_κ is a linear operator uniquely determined by

$$V_\kappa \mathcal{P}_n \subset \mathcal{P}_n, \qquad V_\kappa 1 = 1, \qquad \mathfrak{D}_i V_\kappa = V_\kappa \partial_i, \quad 1 \le i \le d. \tag{1.7}$$

The spherical means are denoted by

$$T_\theta(f) = \frac{1}{\left|\mathbb{S}^{d-2}\right|(\sin\theta)^{d-2}} \int_{\langle x,y\rangle=\cos\theta} f(y)\,d\omega(y), \tag{1.8}$$

where $|\mathbb{S}^{d-2}| = \int_{\mathbb{S}^{d-2}} d\omega = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$.

The spherical means associated with $h_\kappa^2 d\omega$, in which $T_\theta^\kappa(f)$ is defined by

$$c_\lambda \int_0^\pi T_\theta^\kappa(f,x) g(\cos\theta)(\sin\theta)^{2\lambda} d\theta = a_\kappa \int_{\mathbb{S}} f(y) V_\kappa g(\langle x,y\rangle) h_\kappa^2(y)\,d\omega(y), \tag{1.9}$$

where g is any function $[-1,1] \mapsto \mathbb{R}$ such that the integral in the right-hand side is finite, $c_\lambda^{-1} = \int_{-1}^1 (1-t^2)^{\lambda-1/2} dt = \Gamma(\lambda+1/2)\sqrt{\pi}/\Gamma(\lambda+1)$. $T_\theta^\kappa(f)$ is a proper extension of $T_\theta(f)$, since $T_\theta(f)$ satisfies $T_\theta^\kappa(f)$ when $\kappa = 0$ and $V_\kappa = id$, and the properties of T_θ^κ are well known (see [2]). In particular, the function $T_\theta^\kappa f(x)$ has the expansion

$$T_\theta^\kappa(f) \sim \sum_{n=0}^\infty \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} Y_n\left(h_\kappa^2; f\right) := \sum_{n=0}^\infty Q_n^\lambda(\cos\theta) Y_n\left(h_\kappa^2; f\right). \tag{1.10}$$

Simultaneously, they lead to the following definition of an analog of the modulus of smoothness.

Definition 1.1 (see [2]). For $f \in L_p(h_\kappa^2)$, $1 \le p < \infty$, or $f \in C(\mathbb{S})$, the modulus of smoothness on the sphere is given by

$$\omega(f;t)_{\kappa,p} := \sup_{0<\theta\le t} \left\| f - T_\theta^\kappa(f) \right\|_{\kappa,p}. \tag{1.11}$$

The K-functional of the sphere is given by

$$K\left(f;t^2\right)_{\kappa,p} = \inf_{g\in W_p\left(h_\kappa^2\right)} \left\{ \|f-g\|_{\kappa,p} + t^2 \|\Delta_{h,0} g\|_{\kappa,p} \right\}, \tag{1.12}$$

where $W_p(h_\kappa^2) := \{f : f \in L_p(h_\kappa^2), -k(k+2\lambda)P_k(h_\kappa^2; f) = P_k(h_\kappa^2; g) \text{ for some } g \in L_p(h_\kappa^2)\}$, $0 < t < t_0$, t_0 is a positive constant.

In [2], Xu proved the weak equivalence relation

$$C^{-1}\omega(f;t)_{\kappa,p} \le K\left(f;t^2\right)_{\kappa,p} \le C\omega(f;t)_{\kappa,p}. \tag{1.13}$$

Throughout this paper, C denotes a positive constant independent on n and f and $C(a)$ denotes a positive constant dependent on a, which may be different according to the circumstances.

Based on the classical Jackson-Matsuoka kernel (see [3]), we define a new kernel

$$M_{n;j,i,s}(\theta) := \frac{1}{\Omega_{n;j,i,s}}\left(\frac{\sin^{2j} n\theta/2}{\sin^{2i}\theta/2}\right)^{2s}, \qquad n = 1,2,\dots,\ \theta \in \mathbb{R}, \tag{1.14}$$

where $j,i,s \in \mathbb{N}$, $\Omega_{n;j,i,s}$ is a constant chosen such that $c_\lambda \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta = 1$. It is known that $M_{n;j,i,s}(\theta)$ is an even nonnegative operator. In particular, it is an even nonnegative trigonometric polynomial of degree at most $2s(nj+2j-2i)$ for $j > i$ and the Jackson polynomial for $j = i$. Using $M_{n;j,i,s}(\theta)$ we consider the spherical convolution

$$J_{n;j,i,s}(f;x) := (f * M_{n;j,i,s})(x) := c_\lambda \int_0^\pi T_\theta^\kappa(f;x) M_{n;j,i,s}(\theta)(\theta)\sin^{2\lambda}\theta\, d\theta. \tag{1.15}$$

It is called the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel. In particular, $(f_0 * M_{n;j,i,s})(x) = 1$ for $f_0(x) = 1$. The classical Jackson-Matsuoka polynomials in the classical L_p space have been studied by many authors (see [3, 4]).

The purpose of this paper is to consider approximation by h-harmonic polynomials, which in the L_p metric can be viewed as weighted approximation, in which the measure $d\omega$ on the sphere is replaced by $h_\kappa^2 d\omega$. It is well known that the situation can be quite different from that of ordinary harmonics; the weighted approximation is not a simple extension. Since the orthogonal group acts transitively on the sphere $\mathbb{S}$, much of the results for the ordinary harmonics can be proved by considering just one point; the reflection groups do not act transitively on the sphere.

In this paper, we consider weighted approximation of the Jackson-Matsuoka polynomials on the sphere. With the help of the relation between K-functionals and modulus of smoothness of sphere and the properties of the spherical means, we obtain the direct and inverse estimate for the best approximation by Jackson-Matsuoka polynomials in the weighted L_p space on the unit sphere of $\mathbb{R}^d$. We only consider best weighted approximation by Jackson-Matsuoka polynomials, and for the other polynomials on the unit sphere of $\mathbb{R}^d$, the methods and the results are similar.

2. Auxiliary Lemmas

We need the following lemmas.

Lemma 2.1. *Let* $\Omega_{n;,j,i,s} = \int_0^\pi ((\sin^{2j} n\theta/2)/(\sin^{2i}\theta/2))^{2s}\sin^{2\lambda}\theta\, d\theta$. *Then, the weak equivalence*

$$\Omega_{n;j,i,s} \asymp n^{4is-2\lambda-1} \tag{2.1}$$

holds true for $4si > 2\lambda + 1$, $j \geq i$, *where the weak equivalence relation* $A(n) \asymp B(n)$ *means that* $A(n) \ll B(n)$ *and* $B(n) \ll A(n)$, *and relation* $A_n \ll B_n$ *means that there is a positive constant* C *independent on* n *such that* $A(n) \leq CB(n)$ *holds.*

The proof is similar to that of Lemma 2.2 and we omit it.

Lemma 2.2. *For* $4is > r + 2\lambda + 1$, $j \geq i$, $r \in \mathbb{R}$, *there is a constant* $C(\lambda, j, i, s)$ *such that*

$$\int_0^{\pi} \theta^r M_{n;j,i,s}(\theta) \sin^{2\lambda}\theta \, d\theta \leq C(\lambda, j, i, s) n^{-r}. \tag{2.2}$$

Proof. Since $\theta/\pi \leq \sin(\theta/2) \leq \theta/2$ and $\sin\theta \leq \theta$ hold for $0 \leq \theta \leq \pi$, by $\Omega_{n;j,i,s} \asymp n^{4is-2\lambda-1}$, we have

$$\begin{aligned}
\int_0^{\pi} \theta^r M_{n;j,i,s}(\theta) \sin^{2\lambda}\theta d\theta &\leq C(\lambda, j, i, s) n^{-4is+2\lambda+1} \int_0^{\pi} \theta^r \left(\frac{\sin^{2j} n\theta/2}{\sin^{2i}\theta/2} \right)^{2s} \sin^{2\lambda}\theta \, d\theta \\
&\leq C(\lambda, j, i, s) n^{-4is+2\lambda+1} n^{4is-r-2\lambda-1} \int_0^{n\pi/2} t^{r+2\lambda} \left(\frac{\sin^{2j} t}{t^{2i}} \right)^{2s} dt \\
&\leq C(\lambda, j, i, s) n^{-r} \left(\int_0^{\pi/2} t^{r+2\lambda} \left(\frac{\sin^{2j} t}{t^{2i}} \right)^{2s} dt + \int_{\pi/2}^{\infty} t^{r+2\lambda} \left(\frac{\sin^{2j} t}{t^{2i}} \right)^{2s} dt \right) \\
&\leq C(\lambda, j, i, s) C_2 n^{\lambda} \leq C(\lambda, j, i, s) n^{\lambda},
\end{aligned} \tag{2.3}$$

where

$$C_2 = \int_0^{\pi/2} t^{\lambda} \left(\frac{\sin^{2j} t}{t^{2i}} \right)^{2s} dt + \int_{\pi/2}^{\infty} t^{\lambda} \left(\frac{\sin^{2j} t}{t^{2i}} \right)^{2s} dt, \quad 4is > r + 2\lambda + 1, \ j \geq i. \tag{2.4}$$

Lemma 2.2 has been proved. □

Lemma 2.3 (see [2]). *For* $0 \leq \theta \leq \pi$, *one has*

$$\begin{aligned}
T_\theta^\kappa(g; x) - g(x) &= \int_0^{\theta} \sin^{-2\lambda} t \, dt \int_0^t T_u^\kappa(\Delta_{h,0} g) \sin^{2\lambda} u \, du \\
&= \int_0^{\theta} \sin^{-2\lambda} t \Phi(t) B_t(\Delta_{h,0} g, x) dt,
\end{aligned} \tag{2.5}$$

where

$$B_t(\Delta_{h,0} g, x) = \frac{1}{\Phi(t)} \int_0^t T_u^\kappa(\Delta_{h,0} g) \sin^{2\lambda} u \, du, \tag{2.6}$$

and $\Phi(t) = c_\lambda^{-1} \int_0^t \sin^{2\lambda} u \, du$.

Lemma 2.4. *Let $g, \Delta_{h,0}g, \Delta_{h,0}^2 g \in L_p(h_\kappa^2)$, $1 \le p \le \infty$, $J_{n;j,i,s}(f;x)$ be the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel, $4is > 2\lambda + 5$, $j \ge i$. Then, there is a constant $C(\lambda, j, i, s)$ such that*

$$\left\| J_{n;j,i,s}g - g - \alpha(n)\Delta_{h,0}g \right\|_{\kappa,p} \le C(\lambda, j, i, s) n^{-4} \left\| \Delta_{h,0}^2 g \right\|_{\kappa,p}, \tag{2.7}$$

where $\alpha(n) \asymp n^{-2}$.

Proof. By Lemma 2.3, we have

$$\begin{aligned}
J_{n;j,i,s}(g;x) - g(x) &= c_\lambda \int_0^\pi M_{n;j,i,s}(\theta)\left(T_\theta^\kappa(g;x) - g(x)\right)\sin^{2\lambda}\theta \, d\theta \\
&= c_\lambda \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \frac{\Phi(t)}{\sin^{2\lambda}t} B_t(\Delta_{h,0}g, x) dt \\
&= c_\lambda \Delta_{h,0}g(x) \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta \, d\theta \int_0^\theta \frac{\Phi(t)}{\sin^{2\lambda}t} dt \\
&\quad + c_\lambda \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta \, d\theta \int_0^\theta \frac{\Phi(t)}{\sin^{2\lambda}t}\left(B_t(\Delta_{h,0}g, x) - \Delta_{h,0}g(x)\right) dt \\
&= \Delta_{h,0}g(x) \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta \, d\theta \int_0^\theta \frac{dt}{\sin^{2\lambda}t} \int_0^t \sin^{2\lambda}u du \\
&\quad + \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta \, d\theta \int_0^\theta \frac{dt}{\sin^{2\lambda}t} \int_0^t \sin^{2\lambda}u\left(B_t(\Delta_{h,0}g, x) - \Delta_{h,0}g(x)\right) du \\
&:= \alpha(n)\Delta_{h,0}g(x) + \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta \Psi_\theta(g, x) d\theta,
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
\alpha(n) &:= \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta \, d\theta \int_0^\theta \frac{dt}{\sin^{2\lambda}t} \int_0^t \sin^{2\lambda}u \, du, \\
\Psi_\theta(g, x) &:= \int_0^\theta \frac{dt}{\sin^{2\lambda}t} \int_0^t \sin^{2\lambda}u\left(B_t(\Delta_{h,0}g, x) - \Delta_{h,0}g(x)\right) du.
\end{aligned} \tag{2.9}$$

By Lemma 2.1, we have

$$\begin{aligned}
\alpha(n) &= \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta \, d\theta \int_0^\theta \frac{dt}{\sin^{2\lambda}t} \int_0^t \sin^{2\lambda}u \, du \\
&\asymp \int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta \, d\theta \int_0^\theta \frac{t\sin^{2\lambda}\xi}{\sin^{2\lambda}t} dt \\
&\asymp \int_0^\pi \theta^2 M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta \, d\theta \asymp n^{-2}, \qquad (0 < \xi < t).
\end{aligned} \tag{2.10}$$

We now estimate, using Lemma 2.3 again, the expression $B_t(\Delta_{h,0}g, x) - \Delta_{h,0}g(x)$, and obtain

$$\|\Psi_\theta(g)\|_{\kappa,p} \leq C(\lambda, j, i, s)\theta^4 \left\|\Delta^2_{h,0}g\right\|_{\kappa,p}. \tag{2.11}$$

By Lemma 2.2 and Hölder-Minkowski inequality shows that

$$\begin{aligned}\left\|\int_0^\pi M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta\Psi_\theta(g,x)d\theta\right\|_{\kappa,p} &\leq C(\lambda, j, i, s)\left\|\Delta^2_{h,0}g\right\|_{\kappa,p}\int_0^\pi \theta^4 M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta\, d\theta \\ &\leq C(\lambda, j, i, s)n^{-4}\left\|\Delta^2_{h,0}g\right\|_{\kappa,p}.\end{aligned} \tag{2.12}$$

Consequently, by (2.8), (2.10), and (2.12) we complete the proof of this lemma. □

Lemma 2.5. *For $t \geq 0$, there is a constant C such that*

$$\omega(f;t\delta)_{\kappa,p} \leq C\max\left\{1, t^2\right\}\omega(f;\delta)_{\kappa,p}. \tag{2.13}$$

Proof. By the equivalence relation between the modulus of smoothness and K-functional, and the definition of $K(f;t^2)_{\kappa,p}$, we have

$$\begin{aligned}\omega(f;t\delta)_{\kappa,p} &\leq CK\left(f;(t\delta)^2\right)_{\kappa,p} \leq C\left(\|f-g\|_{\kappa,p} + t^2\delta^2\|\Delta_{h,0}g\|_{\kappa,p}\right) \\ &\leq C\max\left\{1,t^2\right\}\left(\|f-g\|_{\kappa,p} + \delta^2\|\Delta_{h,0}g\|_{\kappa,p}\right) \\ &\leq C\max\left\{1,t^2\right\}K\left(f;\delta^2\right)_{\kappa,p} \leq C\max\left\{1,t^2\right\}\omega(f;\delta)_{\kappa,p}.\end{aligned} \tag{2.14}$$

Lemma 2.5 has been proved. □

3. Main Results

Our main results are the following.

Theorem 3.1. *Suppose that $f \in L_p(h_\kappa^2)$, $1 \leq p \leq \infty$, $J_{n;j,i,s}(f;x)$ is the Jackson-Matsuoka polynomials on the sphere based on the Jackson-Matsuoka kernel, $4is > 2\lambda + 5$, $j \geq i$. Then*

$$\|J_{n;j,i,s}(f) - f\|_{\kappa,p} \asymp \omega\left(f;n^{-1}\right)_{\kappa,p}. \tag{3.1}$$

Proof. First we prove $\|J_{n;j,i,s}(f)-f\|_{\kappa,p} \ll \omega(f;n^{-1})_{\kappa,p}$. Since $(f_0 * M_{n;j,i,s})(x)=1$ for $f_0(x)=1$, therefore, we have that

$$\begin{aligned}\|J_{n;j,i,s}(f)-f\|_{\kappa,p} &= \left\|\int_0^{\pi} M_{n;j,i,s}(\theta)\left(f(x)-T_\theta^{\kappa}(f;x)\right)\sin^{2\lambda}\theta\, d\theta\right\|_{\kappa,p} \\ &\le \int_0^{\pi} \|f-T_\theta^{\kappa}(f)\|_{\kappa,p} M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta\, d\theta.\end{aligned} \tag{3.2}$$

Splitting the integral over $[0,\pi]$ into two integrals over $[0,1/n]$ and $[1/n,\pi]$, respectively, and using the definition of $\omega(f;t)_{\kappa,p}$, we conclude that

$$\|f-T_\theta^{\kappa}(f)\|_{\kappa,p} \le \omega\left(f;n^{-1}\right)_{\kappa,p} + \int_{1/n}^{\pi} \omega(f;\theta)_{\kappa,p} M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta. \tag{3.3}$$

From Lemma 2.5 it follows that, for $\theta \ge n^{-1}$,

$$\omega(f;\theta)_{\kappa,p} = \omega\left(f;n\frac{\theta}{n}\right)_{\kappa,p} \le C\max\left\{1,n^2\theta^2\right\}\omega(f;\theta)_{\kappa,p} \le Cn^2\theta^2\omega(f;\theta)_{\kappa,p}. \tag{3.4}$$

Therefore, it follows that

$$\|J_{n;j,i,s}(f)-f\|_{\kappa,p} \le \omega(f;\theta)_{\kappa,p}\left(1+Cn^2\int_{1/n}^{\pi}\theta^2 M_{n;j,i,s}(\theta)\sin^{2\lambda}\theta d\theta\right). \tag{3.5}$$

From Lemma 2.2, we get

$$\|J_{n;j,i,s}(f)-f\|_{\kappa,p} \le C(\lambda,j,i,s)\,\omega\left(f;n^{-1}\right)_{\kappa,p}. \tag{3.6}$$

Next we prove $\omega(f;n^{-1})_{\kappa,p} \ll \|J_{n;j,i,s}(f)-f\|_{\kappa,p}$. Let m be a fixed positive integer Denote by

$$J^m_{n;j,i,s}(f) := \sum_{k=0}^{m}\left(\int_0^{\pi} M_{n;j,i,s}(\theta)Q_k^{\lambda}(\cos\theta)\sin^{2\lambda}\theta d\theta\right)^m Y_k\left(h_\kappa^2;f\right). \tag{3.7}$$

By orthogonality of the orthogonal projector Y_k, we have that

$$\begin{aligned} J^{m+l}(f) &= \sum_{k=0}^{m}\left(\int_0^{\pi} M_{n;j,i,s}(\theta)Q_k^{\lambda}(\cos\theta)\sin^{2\lambda}\theta\, d\theta\right)^m \\ &\quad\times Y_k\left(h_\kappa^2;\sum_{v=0}^{m}\left(\int_0^{\pi} M_{n;j,i,s}(\theta)Q_v^{\lambda}(\cos\theta)\sin^{2\lambda}\theta\, d\theta\right)^l Y_v\left(h_\kappa^2;f\right)\right) \\ &= J^m_{n;j,i,s}\left(J^l_{n;j,i,s}(f)\right).\end{aligned} \tag{3.8}$$

Leting $g = J^m_{n;j,i,s}(f)$, by (3.8) we get

$$\begin{aligned}\|f-g\|_{\kappa,p} &= \left\|f - J^m_{n;j,i,s}(f)\right\|_{\kappa,p} \\ &\le \sum_{k=1}^{m}\left\|J^{k-1}_{n;j,i,s}(f) - J^{k}_{n;j,i,s}(f)\right\|_{\kappa,p} \\ &\le C(\lambda,j,i,s)\sum_{k=1}^{m}\left\|J^{k-1}_{n;j,i,s}((f) - J_{n;j,i,s}(f))\right\|_{\kappa,p} \\ &\le C(\lambda,j,i,s)m\|f - J_{n;j,i,s}(f)\|_{\kappa,p}\end{aligned} \tag{3.9}$$

where $J^0_{n;j,i,s}(f) = f$.

On the other hand,

$$\left\|\Delta_{h,0}J^m_{n;j,i,s}(f)\right\|_{\kappa,p} \le \sum_{k=0}^{m} k(k+2\lambda)\left(\int_0^\pi M_{n;j,i,s}(\theta)\left|Q^\lambda_k(\cos\theta)\right|\sin^{2\lambda}\theta d\theta\right)^m Y_k\left(h^2_\kappa; f\right). \tag{3.10}$$

Note that [5]

$$\left|Q^\lambda_k(\cos\theta)\right| \equiv \left|\frac{C^\lambda_k(\cos\theta)}{C^\lambda_k(1)}\right| \le C\min\left\{(k\theta)^{-1}, 1\right\}. \tag{3.11}$$

For $k\theta \ge 1$, from (2.2) it follows that

$$\begin{aligned}\left\|\Delta_{h,0}J^m_{n;j,i,s}(f)\right\|_{\kappa,p} &\le C(\lambda,j,i,s)\left\|\sum_{k=0}^{m} k(k+2\lambda)k^{-m\lambda}\left(\int_0^\pi M_{n;j,i,s}(\theta)\theta^{-\lambda}\sin^{2\lambda}\theta d\theta\right)^m Y_k\left(h^2_\kappa; f\right)\right\|_{\kappa,p} \\ &\le C(\lambda,j,i,s)n^{m\lambda}\|f\|_{\kappa,p}\sum_{k=0}^{\infty}k^{2-m\lambda} \le C(\lambda,j,i,s)n^{m\lambda}\|f\|_{\kappa,p}.\end{aligned} \tag{3.12}$$

holds for $m > 3/\lambda$. For $k\theta < 1$, by (2.2), we get

$$\begin{aligned}
&\left\| \Delta_{h,0} J^m_{n;j,i,s}(f) \right\|_{\kappa,p} \\
&\quad \le \left\| \sum_{k=0}^{m} \left(\int_0^{\pi} M_{n;j,i,s}(\theta)\theta^{-2/m} (\theta^2 k(k+2\lambda))^{1/m} |Q_k^{\lambda}(\cos\theta)| \sin^{2\lambda}\theta d\theta \right)^m Y_k\left(h_\kappa^2; f\right) \right\|_{\kappa,p} \\
&\quad \le C(\lambda,j,i,s) \left\| \sum_{k=0}^{m} \left(\int_0^{\pi} M_{n;j,i,s}(\theta)\theta^{-2/m} ((k\theta)^2)^{2/m} \sin^{2\lambda}\theta d\theta \right)^m Y_k\left(h_\kappa^2; f\right) \right\|_{\kappa,p} \\
&\quad \le C(\lambda,j,i,s) \left\| \sum_{k=0}^{m} \left(\int_0^{\pi} M_{n;j,i,s}(\theta)\theta^{-2/m} \sin^{2\lambda}\theta d\theta \right)^m Y_k\left(h_\kappa^2; f\right) \right\|_{\kappa,p} \\
&\quad \le C(\lambda,j,i,s) n^2 \left\| \sum_{k=0}^{\infty} Y_k\left(h_\kappa^2; f\right) \right\|_{\kappa,p} \le C n^2 \|f\|_{\kappa,p}.
\end{aligned} \tag{3.13}$$

Consequently, the inequality

$$\left\| \Delta_{h,0} J^m_{n;j,i,s}(f) \right\|_{\kappa,p} \le C(\lambda,j,i,s) n^2 \|f\|_{\kappa,p} \tag{3.14}$$

holds uniformly for $m > 3/\lambda$. Without loss of generality, we may assume $m_1 > 3/\lambda$, $m > m_1 + 3/\lambda$. Using Lemma 2.4 and (3.8), we have

$$\begin{aligned}
\alpha(n) \left\| \Delta_{h,0} J^m_{n;j,i,s}(f) \right\|_{\kappa,p} &= \left\| \alpha(n) \Delta_{h,0} J^m_{n;j,i,s}(f) \right\|_{\kappa,p} \\
&\le \left\| J^m_{n;j,i,s}(f) - f \right\|_{\kappa,p} + C(\lambda,j,i,s) n^{-4} \left\| \Delta^2_{h,0} J^m_{n;j,i,s}(f) \right\|_{\kappa,p} \\
&\le m \left\| J_{n;j,i,s}(f) - f \right\|_{\kappa,p} + C(\lambda,j,i,s) n^{-2} \left\| \Delta^2_{h,0} J^{m-m_1}_{n;j,i,s}(f) \right\|_{\kappa,p} \\
&\le m \left\| J_{n;j,i,s}(f) - f \right\|_{\kappa,p} \\
&\quad + C(\lambda,j,i,s) \left(n^{-2} \left\| \Delta_{h,0} J^m_{n;j,i,s}(f) \right\|_{\kappa,p} + n^{-2} \left\| J^m_{n;j,i,s}(f) - J^{m-m_1}_{n;j,i,s}(f) \right\|_{\kappa,p} \right) \\
&\le m \left\| J_{n;j,i,s}(f) - f \right\|_{\kappa,p} \\
&\quad + C(\lambda,j,i,s) \left(n^{-2} \left\| \Delta_{h,0} J^m_{n;j,i,s}(f) \right\|_{\kappa,p} + \left\| J^{m_1}_{n;j,i,s}(f) - f \right\|_{\kappa,p} \right) \\
&\le C(\lambda,j,i,s) \left(\left\| J_{n;j,i,s}(f) - f \right\|_{\kappa,p} + n^{-2} \left\| \Delta_{h,0} J^m_{n;j,i,s}(f) \right\|_{\kappa,p} \right) \\
&\le C(\lambda,j,i,s) \left(\left\| J_{n;j,i,s}(f) - f \right\|_{\kappa,p} + \|(f)\|_{\kappa,p} \right).
\end{aligned} \tag{3.15}$$

Consequently, $n^{-2}\|\Delta_{h,0}J^{m}_{n;j,i,s}(f)\|_{\kappa,p} \leq C(\lambda,j,i,s)\|f-J_{n;j,i,s}(f)\|_{\kappa,p}$, by the definition of $K(f;t^2)_{\kappa,p}$ and (1.13) shows that

$$\begin{aligned}\omega\left(f;n^{-1}\right)_{\kappa,p} &\leq CK\left(f;n^{-2}\right)_{\kappa,p}\\ &\leq C\left(\left\|f-J^{m}_{n;j,i,s}(f)\right\|_{\kappa,p}+n^{-2}\left\|\Delta_{h,0}J^{m}_{n;j,i,s}(f)\right\|_{\kappa,p}\right)\\ &\leq C(\lambda,j,i,s)\left\|f-J_{n;j,i,s}(f)\right\|_{\kappa,p},\end{aligned} \tag{3.16}$$

that is, $\omega(f;n^{-1})_{\kappa,p} \ll \|f-J_{n;j,i,s}(f)\|_{\kappa,p}$.

The proof is completed. □

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