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Research Article

A Classification of a Totally Umbilical Slant Submanifold of Cosymplectic Manifolds

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We study slant submanifolds of a cosymplectic manifold. It is shown that a totally umbilical slant submanifold M of a cosymplectic manifold $\overline{M}$ is either an anti-invariant submanifold or a 1–dimensional submanifold. We show that every totally umbilical proper slant submanifold of a cosymplectic manifold is totally geodesic.

1. Introduction

The study of slant submanifolds in complex spaces was initiated by Chen as a natural generalization of both holomorphic and totally real submanifolds [1, 2]. Since then, many research papers have appeared concerning the existence of these submanifolds as well as on the geometry of the existent slant submanifolds in different known spaces (cf. [3, 4]). The slant submanifolds of an almost contact metric manifold were defined and studied by Lotta [4]. Later on, these submanifolds were studied by Cabrerizo et al. in the setting of Sasakian manifolds [3].

Recently, Şahin proved that a totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic [5]. Our aim in the present paper is to investigate slant submanifolds in contact manifolds. Thus, we study slant submanifolds of a cosymplectic manifold. We have shown that a totally umbilical slant submanifold M of a cosymplectic manifold $\overline{M}$ is either an anti-invariant submanifold or the dim $M = 1$ or the mean curvature vector $H \in \Gamma(\mu)$, and then we have obtained an interesting result for a totally umbilical proper slant submanifold of a cosymplectic manifold.

2. Preliminaries

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold with $(1,1)$ tensor field ϕ satisfying [6]:

$$\phi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

where I is the identity transformation, ξ a vector field, and η a 1-form on $\overline{M}$ satisfying $\phi\xi = \eta \circ \phi = 0$ and $\eta(\xi) = 1$. Then $\overline{M}$ is said to have an almost contact structure. There always exists a Riemannian metric g on $\overline{M}$ such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields X, Y, on $\overline{M}$. From (2.2), it is easy to observe that

$$g(\phi X, Y) + g(X, \phi Y) = 0. \tag{2.3}$$

The fundamental 2-form Φ is defined as: $\Phi(X,Y) = g(X, \phi Y)$. If $[\phi,\phi] + d\eta \otimes \xi = 0$, then the almost contact structure is said to be normal, where $[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. If $\Phi = d\eta$, the almost contact structure is a contact structure. A normal almost contact structure such that Φ is closed and $d\eta = 0$ is called *cosymplectic structure*. It is well known [7] that the cosymplectic structure is characterized by

$$\left(\overline{\nabla}_X \phi\right) Y = 0, \qquad \left(\overline{\nabla}_X \eta\right) Y = 0, \tag{2.4}$$

for all vector fields X, Y, on $\overline{M}$, where $\overline{\nabla}$ is the Levi-Civita connection of g. From the formula $\overline{\nabla}_X \phi = 0$, it follows that $\overline{\nabla}_X \xi = 0$.

Let M be submanifold of an almost contact metric manifold $\overline{M}$ with induced metric g and let ∇ and $\nabla^\perp$ be the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M, respectively. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth sections of a vector bundle TM over M, then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{2.6}$$

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively for the immersion of M into $\overline{M}$. They are related as

$$g(h(X, Y), N) = g(A_N X, Y), \tag{2.7}$$

where g denotes the Riemannian metric on $\overline{M}$ as well as the one induced on M [8].

For any $X \in \Gamma(TM)$, we write

$$\phi X = PX + FX, \tag{2.8}$$

where PX is the tangential component and FX is the normal component of ϕX. Similarly for any $N \in \Gamma(T^{\perp}M)$, we write

$$\phi N = tN + fN, \tag{2.9}$$

where tN is the tangential component and tN is the normal component of ϕN. If we denote the orthogonal complementary distribution of $F(TM)$ in $T^{\perp}M$ by μ, then we have the direct sum

$$T^{\perp}M = F(TM) \oplus \mu. \tag{2.10}$$

We can see that μ is an invariant subbundle with respect to ϕ. Furthermore, the covariant derivatives of the tensor fields P and F are defined as

$$\begin{aligned} \left(\overline{\nabla}_X P\right)Y &= \nabla_X PY - P\nabla_X Y, \\ \left(\overline{\nabla}_X F\right)Y &= \nabla_X^{\perp} FY - F\nabla_X Y, \end{aligned} \tag{2.11}$$

for any $X, Y \in \Gamma(TM)$.

A submanifold M is said to be *invariant* if F is identically zero, that is, $\phi X \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. On the other hand, M is said to be *anti-invariant* if P is identically zero, that is, $\phi X \in \Gamma(T^{\perp}M)$, for any $X \in \Gamma(TM)$.

A submanifold M of an almost contact metric manifold $\overline{M}$ is called *totally umbilical* if

$$h(X, Y) = g(X, Y)H, \tag{2.12}$$

for any $X, Y \in \Gamma(TM)$. The mean curvature vector H is given by

$$H = \sum_{i=1}^{m} h(e_i, e_i), \tag{2.13}$$

where m is the dimension of M and $\{e_1, e_2, \ldots, e_m\}$ is the local orthonormal frame on M. A submanifold M is said to be *totally geodesic* if $h(X, Y) = 0$ for each $X, Y \in \Gamma(TM)$ and is *minimal* if $H = 0$ on M.

3. Slant Submanifolds

Throughout the section, we assume that M is a slant submanifold of a cosymplectic manifold $\overline{M}$. We always consider such submanifold tangent to the structure vector field ξ. For each nonzero vector X tangent to M at x, we denote by $0 \leq \theta(X) \leq \pi/2$, the angle between ϕX and T_xM, known as the *Wirtinger angle* of X. If the Wirtinger angle $\theta(X)$ is constant, that is, independent of the choice of $x \in M$ and $X \in T_xM - \{\xi\}$, then M is said to be a *slant submanifold* [4]. In this case the constant angle θ is called *slant angle* of the slant submanifold. Obviously if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant submanifold. If M is a slant submanifold of an almost contact metric manifold, then the tangent bundle TM is decomposed as

$$TM = \mathfrak{D} \oplus \langle \xi \rangle, \tag{3.1}$$

where $\langle \xi \rangle$ denotes the distribution spanned by the structure vector field ξ and $\mathfrak{D}$ is the complementary distribution of $\langle \xi \rangle$ in TM, known as the *slant distribution*.

We recall the following result for a slant submanifold.

Theorem 3.1 (see [3]). *Let M be a submanifold of an almost contact metric manifold $\overline{M}$, such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{3.2}$$

Furthermore, if θ is slant angle, then $\lambda = \cos^2\theta$.

The following relations are straightforward consequence of (3.2):

$$g(PX, PY) = \cos^2\theta[g(X,Y) - \eta(X)\eta(Y)], \tag{3.3}$$

$$g(FX, FY) = \sin^2\theta[g(X,Y) - \eta(X)\eta(Y)], \tag{3.4}$$

for any X, Y tangent to M.

Now, we prove the following.

Theorem 3.2. *Let M be a totally umbilical slant submanifold of a cosymplectic manifold $\overline{M}$. Then at least one of the following statements is ture:*

(i) *M is an anti-invariant submanifold;*

(ii) *M is a 1-dimensional submanifold;*

(iii) *If M is a proper slant submanifold, then $H \in \Gamma(\mu)$,*

where H is the mean curvature vector of the submanifold M.

Proof. Let M be a totally umbilical slant submanifold of a cosymplectic manifold $\overline{M}$, then for any $X, Y \in \Gamma(TM)$, we have

$$h(PX, PY) = g(PX, PY)H. \tag{3.5}$$

From (2.5) and (3.3), we deduce that

$$\overline{\nabla}_{PX}PX - \nabla_{PX}PX = \cos^2\theta\{g(X,X) - \eta(X)\eta(X)\}H. \tag{3.6}$$

Using (2.8) and the fact that $\overline{M}$ is cosymplectic we obtain that

$$\phi\overline{\nabla}_{PX}X - \overline{\nabla}_{PX}FX - \nabla_{PX}PX = \cos^2\theta\left\{\|X\|^2 - \eta^2(X)\right\}H. \tag{3.7}$$

Then from (2.5) and (2.6), we get

$$\phi\nabla_{PX}X + \phi h(X, PX) + A_{FX}PX - \nabla^{\perp}_{PX}FX - \nabla_{PX}PX = \cos^2\theta\left\{\|X\|^2 - \eta^2(X)\right\}H. \tag{3.8}$$

Thus by (2.8) and (2.12), we obtain

$$P\nabla_{PX}X + F\nabla_{PX}X + g(PX, X)\phi H + A_{FX}PX - \nabla^{\perp}_{PX}FX - \nabla_{PX}PX = \cos^2\theta\left\{\|X\|^2 - \eta^2(X)\right\}H. \tag{3.9}$$

Equating the normal components, we get

$$F\nabla_{PX}X - \nabla^{\perp}_{PX}FX = \cos^2\theta\left\{\|X\|^2 - \eta^2(X)\right\}H. \tag{3.10}$$

On the other hand, from (3.4), we have

$$g(FX, FX) = \sin^2\theta\{g(X, X) - \eta(X)\eta(X)\}, \tag{3.11}$$

for any $X \in \Gamma(TM)$. Taking the covariant derivative of the above equation with respect to PX, we obtain

$$2g\left(\overline{\nabla}_{PX}FX, FX\right) = 2\sin^2\theta g\left(\overline{\nabla}_{PX}X, X\right) - 2\sin^2\theta\eta(X)g\left(\overline{\nabla}_{PX}X, \xi\right) - 2\sin^2\theta\eta(X)g\left(X, \overline{\nabla}_{PX}\xi\right). \tag{3.12}$$

Using the property of metric connection $\overline{\nabla}$, the last two terms of the right-hand side are cancelling each other, thus we have

$$g\left(\overline{\nabla}_{PX}FX, FX\right) = \sin^2\theta g\left(\overline{\nabla}_{PX}X, X\right). \tag{3.13}$$

Then by (2.5) and (2.6), we derive

$$g\left(\nabla^{\perp}_{PX}FX, FX\right) = \sin^2\theta g(\nabla_{PX}X, X). \tag{3.14}$$

Now, taking the inner product in (3.10) with FX, for any $X \in \Gamma(TM)$, then

$$g(F\nabla_{PX}X, FX) - g\left(\nabla^{\perp}_{PX}FX, FX\right) = \cos^2\theta\left\{\|X\|^2 - \eta^2(X)\right\}g(H, FX). \tag{3.15}$$

Then from (3.4) and (3.14), we obtain

$$-\sin^2\theta\eta(X)\eta(\nabla_{PX}X) = \cos^2\theta\Big\{\|X\|^2 - \eta^2(X)\Big\}g(H, FX), \tag{3.16}$$

or

$$-\sin^2\theta\eta(X)g(\nabla_{PX}X, \xi) = \cos^2\theta\Big\{\|X\|^2 - \eta^2(X)\Big\}g(H, FX). \tag{3.17}$$

Using (2.5), we derive

$$-\sin^2\theta\eta(X)g\left(\overline{\nabla}_{PX}X, \xi\right) = \cos^2\theta\Big\{\|X\|^2 - \eta^2(X)\Big\}g(H, FX). \tag{3.18}$$

Since $\overline{\nabla}$ is the metric connection, then the above equation can be written as

$$\sin^2\theta\eta(X)g\left(X, \overline{\nabla}_{PX}\xi\right) = \cos^2\theta\Big\{\|X\|^2 - \eta^2(X)\Big\}g(H, FX). \tag{3.19}$$

As $\overline{M}$ is cosymplectic thus using the fact that $\overline{\nabla}_{PX}\xi = 0$, the left hand side of the above equation vanishes identically, then

$$\cos^2\theta\Big\{\|X\|^2 - \eta^2(X)\Big\}g(H, FX) = 0. \tag{3.20}$$

Thus from (3.20), it follows that either $\theta = \pi/2$ or $X = \xi$ or $H \in \Gamma(\mu)$, where μ is the invariant normal subbundle orthogonal to FTM. This completes the proof. □

Theorem 3.3. *Every totally umbilical proper slant submanifold M of a cosymplectic manifold $\overline{M}$ is totally geodesic, provided $\nabla_X^\perp H \in \Gamma(\mu)$, for any $X \in TM$.*

Proof. As $\overline{M}$ is cosymplectic, then we have

$$\overline{\nabla}_U\phi V = \phi\overline{\nabla}_U V, \tag{3.21}$$

for any $U, V \in \Gamma(T\overline{M})$. Using this fact and formulae (2.5) and (2.8) we obtain that

$$\overline{\nabla}_X PY + \overline{\nabla}_X FY = P\nabla_X Y + F\nabla_X Y + \phi h(X, Y), \tag{3.22}$$

for any $X, Y \in \Gamma(TM)$. Then from (2.5), (2.6) and (2.12), we get

$$\nabla_X PY + h(X, PY) - A_{FY}X + \nabla_X^\perp FY = P\nabla_X Y + F\nabla_X Y + g(X, Y)\phi H. \tag{3.23}$$

Taking the inner product in (3.23) with ϕH and using the fact that $H \in \Gamma(\mu)$ (by Theorem 3.2), we obtain

$$g(h(X,PY),\phi H) + g\left(\nabla_X^{\perp} FY,\phi H\right) = g(X,Y)g(\phi H,\phi H). \tag{3.24}$$

Then from (2.2) and (2.12), we derive

$$g(X,PY)g(H,\phi H) + g\left(\nabla_X^{\perp} FY,\phi H\right) = g(X,Y)g(H,H). \tag{3.25}$$

That is,

$$g\left(\nabla_X^{\perp} FY,\phi H\right) = g(X,Y)\|H\|^2. \tag{3.26}$$

Now, we consider

$$\overline{\nabla}_X \phi H = \phi \overline{\nabla}_X H, \tag{3.27}$$

for any $X \in \Gamma(TM)$. From (2.6), we obtain

$$-A_{\phi H}X + \nabla_X^{\perp}\phi H = \phi\left(-A_H X + \nabla_X^{\perp} H\right). \tag{3.28}$$

Thus, on using (2.8), (2.9), we get

$$-A_{\phi H}X + \nabla_X^{\perp}\phi H = -PA_H X - FA_H X + t\nabla_X^{\perp} H + f\nabla_X^{\perp} H. \tag{3.29}$$

Taking the inner product with FY, for any $Y \in \Gamma(TM)$, then

$$g\left(\nabla_X^{\perp}\phi H, FY\right) = -g(FA_H X, FY) + g\left(f\nabla_X^{\perp} H, FY\right). \tag{3.30}$$

Since $f\nabla_X^{\perp} H \in \Gamma(\mu)$, then by (3.4) the above equation takes the form

$$g\left(\nabla_X^{\perp}\phi H, FY\right) = -\sin^2\theta\{g(A_H X, Y) - \eta(A_H X)\eta(Y)\}. \tag{3.31}$$

Using (2.6), (2.7), and (2.12), we get

$$g\left(\overline{\nabla}_X \phi H, FY\right) = -\sin^2\theta\{g(X,Y) - \eta(X)\eta(Y)\}\|H\|^2. \tag{3.32}$$

The above equation can be written as

$$\mathrm{g}\left(\overline{\nabla}_X FY, \phi H\right) = \sin^2\theta\{g(X,Y) - \eta(X)\eta(Y)\}\|H\|^2. \tag{3.33}$$

Again using the fact that $H \in \Gamma(\mu)$, then by (2.6), we obtain

$$g\left(\nabla^{\perp}_{X} FY, \phi H\right) = \sin^2\theta\{g(X,Y) - \eta(X)\eta(Y)\}\|H\|^2. \tag{3.34}$$

From (3.26) and (3.34), we derive

$$\left\{\cos^2\theta g(X,Y) + \sin^2\theta\eta(X)\eta(Y)\right\}\|H\|^2 = 0. \tag{3.35}$$

Thus, (3.35) implies either $H = 0$ or $\theta = \tan^{-1}(\sqrt{-g(X,Y)/\eta(X)\eta(Y)})$, which is not possible, because the slant angle $\theta \in (0, \pi/2)$. Hence, M is totally geodesic in $\overline{M}$. □

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References

[1] B. Y. Chen, "Slant immersions," *Bulletin of the Australian Mathematical Society*, vol. 41, no. 1, pp. 135–147, 1990.
[2] B.-Y. Chen, *Geometry of Slant Submanifolds*, Katholieke Universiteit Leuven, Leuven, Belgium, 1990.
[3] J. L. Cabrerizo, A. Carriazo, L. M. Fernández, and M. Fernández, "Slant submanifolds in Sasakian manifolds," *Glasgow Mathematical Journal*, vol. 42, no. 1, pp. 125–138, 2000.
[4] A. Lotta, "Slant submanifolds in contact geometry," *Bulletin Mathematical Society Roumanie*, vol. 39, pp. 183–198, 1996.
[5] B. Şahin, "Every totally umbilical proper slant submanifold of a Kahler manifold is totally geodesic," *Results in Mathematics*, vol. 54, no. 1-2, pp. 167–172, 2009.
[6] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Springer-Verlag, New York, NY, USA, 1976.
[7] G. D. Ludden, "Submanifolds of cosymplectic manifolds," *Journal of Differential Geometry*, vol. 4, pp. 237–244, 1970.
[8] K. Yano and M. Kon, *Structures on Manifolds*, Series in Pure Mathematics, World Scientific Publishing, Singapore, Singapore, 1984.