Erratum for “An affine version of a theorem of Nagata”

Gene Freudenburg

The main result of the author’s paper [1] is the following.

THEOREM 2.1

Let k be a field, and let R be an affine k-algebra. Suppose that there exists a field K with $R \subset K^{[1]}$ and $R \not\subset K$. Then there exists a field F and an algebraic extension $R \subset F^{[1]}$.

The subsequent Theorem 3.1 deals with the case where the transcendence degree of R over k is one, but without assuming that R is affine.

THEOREM 3.1

Let k be a field, and let R be a k-algebra with $\operatorname{tr.deg}_k R = 1$. Suppose that there exists a field K with $R \subset K^{[1]}$ and $R \not\subset K$. Then R is k-affine, and there exists a field F algebraic over k with $R \subset F^{[1]}$. If k is algebraically closed, then there exists $t \in \operatorname{frac}(R)$ with $R \subset k[t]$.

However, when the ground field k is not algebraically closed, the ring R may fail to be affine. For example, let K/k be an extension of fields that is not a finite extension. Define $W \subset K[s] = K^{[1]}$ by $W = \{as \mid a \in K\}$, and define $R = k[W]$. Then $R \subset K[s]$ is an algebraic extension and $R \not\subset K$, but R is not affine over k.

The purpose of this note is to give the correct version of Theorem 3.1, as follows.

THEOREM 3.1 (CORRECTED)

Let k be a field, and let R be a k-algebra with $\operatorname{tr.deg}_k R = 1$. Suppose that there exists a field K with $R \subset K^{[1]}$ and $R \not\subset K$. Then there exists a field F algebraic over k with $R \subset F^{[1]}$. If k is algebraically closed, then R is k-affine, and there exists $t \in \operatorname{frac}(R)$ with $R \subset k[t]$.

Kyoto Journal of Mathematics, Vol. 57, No. 1 (2017), 233–234
DOI 10.1215/21562261-3795903,

Proof
The proof of Theorem 3.1 given in the article is correct for the case in which the ground field k is algebraically closed. So the statement of the theorem is true in this case.

For the general case, suppose that $R \subset K[t] = K^{[1]}$. Let $\hat{K}$ be the algebraic closure of K, and let $\hat{k}$ be the algebraic closure of k in $\hat{K}$, noting that $\hat{k}$ is an algebraically closed field. Let $\hat{K}[t]$ be the natural extension of $K[t]$, and define the subring $\hat{R} = \hat{k}[R]$. Then $\mathrm{tr.deg}_{\hat{k}} \hat{R} = 1$ and $\hat{R} \not\subset \hat{K}$ (since $\hat{k}$ is algebraically closed). Therefore, $\hat{R} \subset \hat{k}^{[1]}$. It follows that $R \subset \hat{k}^{[1]}$, and this is an algebraic extension. $\square$

In the article, Theorem 3.1 is used to prove the Abhyankar–Eakin–Heinzer theorem (Theorem 4.3) in the characteristic zero case. Thus, although the Abhyankar–Eakin–Heinzer theorem is true over any field, the proof of this result given in the article is only valid when the ground field k is algebraically closed and of characteristic zero.

Acknowledgment. The author wishes to thank Daniel Daigle of the University of Ottawa for bringing the error in Theorem 3.1 to his attention and for pointing out the example cited above.

Reference

[1] G. Freudenburg, *An affine version of a theorem of Nagata*, Kyoto J. Math. **55** (2015), 663–672. MR 3395985. DOI 10.1215/21562261-3089136.

Department of Mathematics, Western Michigan University, Kalamazoo, Michigan, USA; gene.freudenburg@wmich.edu