On the ℓ-adic cohomology of Jacobian elliptic surfaces over finite fields

Gerald E. Welters

Abstract For a Jacobian elliptic surface S_0 over a finite field k and a prime ℓ different from the characteristic of k, the points of period ℓ^r on the smooth fibers of S_0 yield, for each $r \in \mathbb{Z}_{\geq 0}$, a smooth projective curve C_r over k by taking Zariski closure in S_0 and normalization. We consider the restriction map in ℓ-adic étale cohomology $H^2(S_0, \mathbb{Z}_\ell(1)) \to H^2(\bigsqcup_{r\geq 0} C_r, \mathbb{Z}_\ell(1)) = \prod_{r\geq 0} H^2(C_r, \mathbb{Z}_\ell(1))$. By using an earlier result of ours we prove that, except for at most a finite number of such primes ℓ, this map is faithful on the submodule $F^1H^2(S_0, \mathbb{Z}_\ell(1))^0$ of those classes vanishing on the geometric fibers and on the zero section of S_0, and that it gives an isomorphism between this submodule and the subgroup of $\mathrm{Pic}(\bigsqcup_{r\geq 0} C_r) = \prod_{r\geq 0} \mathrm{Pic}(C_r)$ of primitive elements in the sense of Serre.

1. Introduction

Throughout, unless otherwise specified, k will denote a finite field of positive characteristic p, and $K = k(B)$, with B a geometrically irreducible, smooth, projective curve over k, will denote a global field with k as its field of constants. First, we let E_0 be an elliptic curve (an abelian variety of dimension 1) over K and call $\pi : S_0 \to B$ its associated Kodaira–Néron surface, the minimal regular completion of its Néron model $\mathcal{E}_0 \to B$. Second, for any prime integer $\ell \neq p$, we let $\mathcal{E}_0(\ell)^c \to B$ be the regular completion of the Néron model $\mathcal{E}_0(\ell) \to B$ of the étale group scheme $E_0(\ell)$ over K given by the ℓ-primary component of the torsion subgroup of E_0. The inclusion map $E_0(\ell) \hookrightarrow E_0$ extends to a morphism $h : \mathcal{E}_0(\ell)^c \to S_0$, and we focus here on the restriction map induced in ℓ-adic étale cohomology, $h^* : H^2(S_0, \mathbb{Z}_\ell(1)) \to H^2(\mathcal{E}_0(\ell)^c, \mathbb{Z}_\ell(1))$.

Associated with the increasing filtration of $E_0(l)$ by the group subschemes ${}_{l^n}E_0$, $n \geq 0$, of l^n-torsion points we have an increasing filtration of $\mathcal{E}_0(\ell)^c$ given by the regular completions ${}_{l^n}\mathcal{E}_0^c$ of their respective Néron models ${}_{l^n}\mathcal{E}_0$. The scheme $\mathcal{E}_0(\ell)^c$ is the sum $\bigsqcup_{r\geq 0} C_r$ of smooth projective curves over k given by $C_r = {}_{l^r}\mathcal{E}_0^c \setminus_{l^{r-1}} \mathcal{E}_0^c$ if $r \geq 1$ and $C_0 = {}_1\mathcal{E}_0$, the zero section of S_0. Similarly, ${}_{l^n}\mathcal{E}_0^c = \bigsqcup_{r=0}^n C_r$ for all $n \geq 0$.

The addition map $s : \mathcal{E}_0(\ell) \times_B \mathcal{E}_0(\ell) \to \mathcal{E}_0(\ell)$ extends in a unique way to a morphism $s : (\mathcal{E}_0(\ell) \times_B \mathcal{E}_0(\ell))^c \to \mathcal{E}_0(\ell)^c$ between the respective regular comple-

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tions. We define $\mathrm{Pic}(\mathcal{E}_0(\ell)^c)^{\mathrm{inv}} = \{\xi \in \mathrm{Pic}(\mathcal{E}_0(\ell)^c) \mid s^*(\xi) = p_1^*(\xi) + p_2^*(\xi)\}$, where $p_1, p_2 : (\mathcal{E}_0(\ell) \times_B \mathcal{E}_0(\ell))^c \to \mathcal{E}_0(\ell)^c$ are the extensions of the respective projection maps $p_1, p_2 : \mathcal{E}_0(\ell) \times_B \mathcal{E}_0(\ell) \to \mathcal{E}_0(\ell)$. This is a subgroup of $\mathrm{Pic}(\mathcal{E}_0(\ell)^c)$, whose elements might be called the *primitive* elements of $\mathrm{Pic}(\mathcal{E}_0(\ell)^c)$, in imitation of [12, p. 181]. We define similarly the subgroup $\mathrm{Pic}({}_{l^n}\mathcal{E}_0^c)^{\mathrm{inv}}$ of $\mathrm{Pic}({}_{l^n}\mathcal{E}_0^c)$ for each $n \geq 0$. For $m \in \mathbb{Z}$, if $m : {}_{l^n}\mathcal{E}_0^c \to {}_{l^n}\mathcal{E}_0^c$ denotes the extension of the multiplication by m map $m : {}_{l^n}\mathcal{E}_0 \to {}_{l^n}\mathcal{E}_0$, then one has $m^*(\xi) = m\xi$ for all $\xi \in \mathrm{Pic}({}_{l^n}\mathcal{E}_0^c)^{\mathrm{inv}}$, since $(m_1 + m_2)^*(\xi) = m_1^*(\xi) + m_2^*(\xi)$ for all $m_1, m_2 \in \mathbb{Z}$. It follows from this that $\mathrm{Pic}({}_{l^n}\mathcal{E}_0^c)^{\mathrm{inv}} \subset {}_{l^n}\mathrm{Pic}({}_{l^n}\mathcal{E}_0^c)$ for all $n \geq 0$. In particular, $\mathrm{Pic}(\mathcal{E}_0(\ell)^c)^{\mathrm{inv}} = \varprojlim_n \mathrm{Pic}({}_{l^n}\mathcal{E}_0^c)^{\mathrm{inv}}$ has a natural structure of a $\mathbb{Z}_\ell$-module.

For all $r \geq 0$, the kernel of the Chern class map $\mathrm{Pic}(C_r) \to H^2(C_r, \mathbb{Z}_\ell(1))$ is $\mathrm{Pic}(C_r)$ (no ℓ), the prime-to-ℓ torsion subgroup of $\mathrm{Pic}(C_r)$; actually, since $Br(C_r) = 0$ (see [4, p. 96]), this map gives here an isomorphism $\mathrm{Pic}(C_r) \otimes \mathbb{Z}_\ell \simeq H^2(C_r, \mathbb{Z}_\ell(1))$. The kernel of the Chern class map $\mathrm{Pic}(\mathcal{E}_0(\ell)^c) \to H^2(\mathcal{E}_0(\ell)^c, \mathbb{Z}_\ell(1))$ therefore equals $\prod_{r \geq 0} \mathrm{Pic}(C_r)$ (no ℓ), and by the preceding, this map yields an embedding $\mathrm{Pic}(\mathcal{E}_0(\ell)^c)^{\mathrm{inv}} \hookrightarrow H^2(\mathcal{E}_0(\ell)^c, \mathbb{Z}_\ell(1))$. Similar remarks apply, of course, with $\mathcal{E}_0(\ell)^c$ replaced by ${}_{l^n}\mathcal{E}_0^c$, for any $n \geq 0$.

In studying the $\mathbb{Z}_\ell$-module $H^2(S_0, \mathbb{Z}_\ell(1))$ one can split off elementary, geometrically motivated submodules and leave an essential part, given by the direct summand $F^1H^2(S_0, \mathbb{Z}_\ell(1))^0$ consisting of those cohomology classes that vanish on the geometric fibers and on the zero section of S_0. Here $F^1H^2(S_0, \mathbb{Z}_\ell(1))$ is the first term (beyond the zeroth) of the filtration of $H^2(S_0, \mathbb{Z}_\ell(1))$ coming from the Leray spectral sequence for $\pi : S_0 \to B$ and the ℓ-adic sheaf $\mathbb{Z}_\ell(1)$ on S_0, and it is given by the kernel of the edge homomorphism $H^2(S_0, \mathbb{Z}_\ell(1)) \to H^0(B, R^2\pi_*\mathbb{Z}_\ell(1))$. The submodule $F^1H^2(S_0, \mathbb{Z}_\ell(1))^0$ is obtained then by taking the kernel of the restriction map $0^* : F^1H^2(S_0, \mathbb{Z}_\ell(1)) \to H^2(B, \mathbb{Z}_\ell(1))$ by the zero section of S_0.

The Tate conjecture for the surface S_0 asserts that the Chern class map induces isomorphisms $\mathrm{Pic}(S_0) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} H^2(S_0, \mathbb{Z}_\ell(1))$ for all primes ℓ (here different from p) or, equivalently, for a single such ℓ. And it can also be stated equivalently as yielding isomorphisms $\mathrm{Pic}^0(S_0/B) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} F^1H^2(S_0, \mathbb{Z}_\ell(1))^0$ for all (resp., for a single) ℓ, where $\mathrm{Pic}^0(S_0/B) \subset \mathrm{Pic}(S_0)$ is the subgroup of elements having degree 0 on the irreducible components of the geometric fibers of π and restricting to the trivial element on the zero section of S_0. We consider in this article the restriction map

$$h^* : F^1H^2\big(S_0, \mathbb{Z}_\ell(1)\big)^0 \longrightarrow H^2\big(\mathcal{E}_0(l)^c, \mathbb{Z}_\ell(1)\big) \tag{1.1}$$

on this submodule of $H^2(S_0, \mathbb{Z}_\ell(1))$.

The point is that there exists, canonically, an alternate way to go from $F^1H^2(S_0, \mathbb{Z}_\ell(1))^0$ to $H^2(\mathcal{E}_0(l)^c, \mathbb{Z}_\ell(1))$, described as follows. From the Leray spectral sequence one has an edgelike morphism $F^1H^2(S_0, \mathbb{Z}_\ell(1)) \to H^1(B, R^1\pi_*\mathbb{Z}_\ell(1))$, which is surjective due to the existence of a section for the map π. The kernel being $H^2(B, Z_\ell(1))$, we obtain therefore an isomorphism $F^1H^2(S_0, \mathbb{Z}_\ell(1))^0 \xrightarrow{\sim} H^1(B, R^1\pi_*\mathbb{Z}_\ell(1))$. One has furthermore, for all $n \geq 0$, canonical isomorphisms between sheaves on B for the étale topology: $R^1\pi_*\mu_{\ell^n,S_0} \xrightarrow{\sim} {}_{\ell^n}\operatorname{Pic}_{S_0/B} \xrightarrow{\sim} {}_{\ell^n}\mathcal{E}_0^0$, where $\mathcal{E}_0^0 \hookrightarrow \mathcal{E}_0$ denotes the (sheaf of sections of the) open group subscheme cutting out the identity components on the fibers of $\mathcal{E}_0$. The second of these isomorphisms is induced by the morphism of sheaves $\operatorname{Pic}_{S_0/B} \to \mathcal{E}_0$ given by abelian summation along the smooth fibers of S_0 (see Note (b) of Section 3). It follows a canonical isomorphism $R^1\pi_*\mathbb{Z}_\ell(1) \xrightarrow{\sim} T_\ell\mathcal{E}_0^0$. So far we thus have an isomorphism

$$F^1H^2\big(S_0,\mathbb{Z}_\ell(1)\big)^0 \xrightarrow{\sim} H^1(B,T_\ell\mathcal{E}_0^0). \tag{1.2}$$

Moreover, the inclusion map $\mathcal{E}_0^0 \hookrightarrow \mathcal{E}_0$ induces an isomorphism

$$H^1(B,T_\ell\mathcal{E}_0^0) \xrightarrow{\sim} H^1(B,T_\ell\mathcal{E}_0). \tag{1.3}$$

The next and main step is the class invariant morphism considered in [14] and [15]:

$$\gamma : H^1(B,T_\ell\mathcal{E}_0) \longrightarrow \operatorname{Pic}\big(\mathcal{E}_0(l)^c\big)^{\mathrm{inv}}. \tag{1.4}$$

It was shown in [14] (see also [15]) that this map is an isomorphism for all but at most a finite set of primes $\ell \neq p$. Finally, we take the inclusion map

$$\operatorname{Pic}\big(\mathcal{E}_0(\ell)^c\big)^{\mathrm{inv}} \hookrightarrow H^2\big(\mathcal{E}_0(\ell)^c,\mathbb{Z}_\ell(1)\big) \tag{1.5}$$

discussed above. Composition of the maps (1.2)–(1.5) gives a morphism

$$F^1H^2\big(S_0,\mathbb{Z}_\ell(1)\big)^0 \longrightarrow H^2\big(\mathcal{E}_0(l)^c,\mathbb{Z}_\ell(1)\big), \tag{1.6}$$

which, by the preceding, is injective and yields an isomorphism $F^1H^2(S_0, \mathbb{Z}_\ell(1))^0 \xrightarrow{\sim} \operatorname{Pic}(\mathcal{E}_0(\ell)^c)^{\mathrm{inv}}$ for all but at worst a finite set of primes $\ell \neq p$. The results in [14] imply, in particular, that the maps (1.1) and (1.6) coincide when restricted to $\operatorname{Pic}^0(S_0/B) \otimes \mathbb{Z}_\ell$. The Tate conjecture therefore suggests the equality of these two maps. This equality is proved in this article.

THEOREM 1.7

For all primes $\ell \neq p$, the maps (1.1) and (1.6) coincide.

COROLLARY 1.8

For all but at most a finite set of primes $\ell \neq p$, the map h^* in (1.1) is injective and yields an isomorphism $F^1H^2(S_0,\mathbb{Z}_\ell(1))^0 \xrightarrow{\sim} \operatorname{Pic}(\mathcal{E}_0(\ell)^c)^{\mathrm{inv}}$.

The proof of Theorem 1.7 is given in Section 6. In the remaining sections we gather the formulae used for that purpose. Most of this material is common in the literature; yet, specific features of the relative case and the need for having the signs in the formulae carefully checked brought us to write out the details.

NOTES

(a) It follows from Theorem 1.7 that the restriction map (1.1) is actually a map

$$F^1H^2\big(S_0,\mathbb{Z}_\ell(1)\big)^0 \longrightarrow \prod_{r\geq 0} H^2\big(C_r,\mathbb{Z}_\ell(1)\big)_{\mathrm{tors}} \subset \prod_{r\geq 0} H^2\big(C_r,\mathbb{Z}_\ell(1)\big),$$

so the corresponding map with rational ℓ-adic coefficients

$$F^1H^2\big(S_0,\mathbb{Q}_\ell(1)\big)^0 \longrightarrow \prod_{r\geq 0} H^2\big(C_r,\mathbb{Q}_\ell(1)\big) \simeq \prod_{r\geq 0} \mathbb{Q}_\ell^{n_r},$$

where n_r is the number of connected components of C_r, vanishes identically.

(b) For smooth projective varieties X over finite fields, the Tate conjecture in codimension 1, in the formulation used in this article, states that the $\mathbb{Z}_\ell$-module $H^2(X,\mathbb{Z}_\ell(1)) = \varprojlim_n H^2(X,\mu_{\ell^n})$ and its submodule $\mathrm{Pic}(X)\otimes\mathbb{Z}_\ell = \varprojlim_n \mathrm{Pic}(X)/\ell^n$ actually coincide. This is known to hold, in particular, for (smooth, projective) curves. So, an integral codimension 1 ℓ-adic Tate class $\xi\in H^2(X,\mathbb{Z}_\ell(1))$ induces on the normalization C of any projective curve in X an element of $\mathrm{Pic}(C)\otimes\mathbb{Z}_\ell$ which, in the case of being torsion, is an element of $\mathrm{Pic}(C)(\ell)$. That is, $\xi \mid C$ is then (the isomorphism class of) a line bundle on C annihilated by some power of ℓ. This is the case, on our surface S_0, for the Tate classes belonging to $F^1H^2(S_0,\mathbb{Z}_\ell(1))^0 \subset H^2(S_0,\mathbb{Z}_\ell(1))$ and the curves labeled C_r in this section, and also for the vertical curves in S_0, sums of irreducible components of fibers of the map $\pi: S_0\to B$. Let $\xi\in F^1H^2(S_0,\mathbb{Z}_\ell(1))^0$ be given, and suppose we know that ξ belongs to $\mathrm{Pic}(S_0)\otimes\mathbb{Z}_\ell$. Then it actually belongs to $\mathrm{Pic}^0(S_0/B)\otimes\mathbb{Z}_\ell$, $\xi = ([L_n])_{n\geq 0}$, $[L_n]\in \mathrm{Pic}^0(S_0/B)/\ell^n$. (To avoid double brackets, here we use the same symbol for the bundle L_n as for its class in $\mathrm{Pic}(S_0)$.) If $S_0(t)$, $t\in B_{\mathrm{cl}}$, is (say, for simplicity) a smooth fiber of the map π, then it is easily seen that $\xi \mid S_0(t)\in \mathrm{Pic}(S_0(t))$ is the ℓ-primary component of the class of the bundle $L_n \mid S_0(t)$, $n\gg 0$, in the finite abelian group $\mathrm{Pic}^0(S_0(t))$. In contrast with this, it follows from [14] (see also diagram (6.3) of the present article) that, for all $r\geq 0$, $\xi\mid C_r\in\mathrm{Pic}(C_r)$ is the (whole) class of the bundle $L_r\mid C_r$. This suggests trying to obtain more information about the Tate classes $\xi\in F^1H^2(S_0,\mathbb{Z}_\ell(1))^0$ from their associated line bundles $\xi\mid C_r$, $r\geq 0$. Corollary 1.8 of this article is a support for this quest.

2. Pairings, I: Abelian schemes

We review here in the relative setting standard pairings for abelian varieties and, in the next two sections, similar and related pairings in neighboring contexts. We use explicit recipe and cocycle manipulation; for a more conceptual framework we refer to [8]. Although our applications in this article concern Néron models $\mathcal{E}_0\to B$ of elliptic curves, we treat first, and separately, the case of abelian schemes, both as a tool and as a blueprint. When speaking of abelian schemes, we shall always mean projective abelian schemes $\mathcal{A}\to B$ over (arbitrary) noetherian base schemes.

NOTES (NOTATION AND CONVENTIONS)

(a) In this article, wherever it will be question of an integer $N\in\mathbb{Z}_{\geq 1}$, N will be supposed to be prime to the residue characteristics.

(b) As a rule, we shall not distinguish between sheaves (always for the étale topology, in this article) induced by group schemes and the group schemes themselves, the context indicating which interpretation is at work in each case. For example, when considering a pairing of an abelian group scheme $\mathcal{G}$ over a scheme B with values in a group scheme $\mathcal{H}$ over B, both $\mathcal{G} \times_B \mathcal{G} \to \mathcal{H}$ and $\mathcal{G} \times \mathcal{G} \to \mathcal{H}$ will make sense: in the first case one considers group schemes, and in the second case, one considers sheaves on B.

(c) Since they are mutual counterparts, we shall alternate the use of invertible sheaves $\mathcal{L}$ on a scheme B with that of line bundles $L \to B$, the latter being better suited for handling relative morphisms, namely, $(\tilde{f}, f) : (L, B) \to (L', B')$ versus $\tilde{f} : \mathcal{L} \to f^*(\mathcal{L}')$, especially when composing two or more of these.

(d) For an abelian scheme $\pi : \mathcal{A} \to B$ and due to the existence of the zero section, the obvious morphism $\mathrm{Pic}(\mathcal{A}) \to \Gamma(B, \mathrm{Pic}_{\mathcal{A}/B})$, coming from the Leray spectral sequence for the sheaf $\mathbb{G}_m$ on $\mathcal{A}$, is surjective and has $\pi^* \mathrm{Pic}(B) \simeq \mathrm{Pic}(B)$ as its kernel, so it induces, in particular, an isomorphism $\mathrm{Pic}^0(\mathcal{A}/B) \xrightarrow{\sim} \Gamma(B, \hat{\mathcal{A}})$ between (1) the subgroup $\mathrm{Pic}^0(\mathcal{A}/B) \subset \mathrm{Pic}(\mathcal{A})$ of isomorphism classes of invertible sheaves topologically trivial on the geometric fibers of π and trivial on the zero section of π, and (2) the group of global sections of the dual abelian scheme $\hat{\mathcal{A}} \to B$ of $\mathcal{A}$. Unless otherwise stated, we shall always represent global sections of $\hat{\mathcal{A}}$ by invertible sheaves (line bundles) with class in $\mathrm{Pic}^0(\mathcal{A}/B)$.

(e) Let $f : X \xrightarrow{\sim} X$ be an automorphism of a scheme X. If M is a line bundle on X such that $f^*(M) \simeq M$, then we have relative automorphisms $(\tilde{f}, f) : (M, X) \to (M, X)$, with $\tilde{f}$ determined up to a factor $\rho \in \Gamma(X, \mathbb{G}_m)$. This happens, in particular, for trivial line bundles $M \simeq \mathbb{1}_X$, $\mathbb{1} = \mathrm{Spec}(\mathbb{Z}[T]) \to \mathrm{Spec}(\mathbb{Z})$. Now, since there is a canonical choice $\tilde{f}_0$ of $\tilde{f}$ for $\mathbb{1}_X$, by composing this map with any isomorphism $M \xrightarrow{\sim} \mathbb{1}_X$ and its inverse one finds a canonical choice $\tilde{f}_0^{(M)}$ of $\tilde{f}$ for M, too. The symbol $\tilde{f}_0^{(M)}$ has obvious functorial properties in M and in f, which we shall use freely below. Also, always for trivial line bundles M, we may identify thus at our convenience relative isomorphisms $\tilde{f} = \rho \tilde{f}_0^{(M)}$ for f and M with invertible functions $\rho \in \Gamma(X, \mathbb{G}_m)$ on X. If not always, then we shall do this regularly, below, with X an abelian scheme and $f = \tau_\alpha$ the translation map with a section α of X. To lighten notation we shall then write $\tilde{\alpha}$ instead of $\tilde{\tau}_\alpha$.

(f) Last but not least, we shall always approach étale cohomology through the Čech theory. This puts a (weak) constraint on the schemes considered in this article (see [7, p. 104, Theorem 2.17]): through Artin's work, this is allowed for quasicompact schemes such that every finite subset is contained in an open affine set. We shall assume that our schemes satisfy this property, where this should not be the case by default.

(i) *The pairing* $\bar{e}_N$. Let $\mathcal{A} \to B$ be an abelian scheme over some base scheme B, and let $\hat{\mathcal{A}} \to B$ be its dual. Call ${}_N\mathcal{A}$, ${}_N\hat{\mathcal{A}}$ their respective N-torsion group subschemes. We shall use the following description of the standard pairing between the kernels of the mutually dual multiplication by N maps $N_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ and $N_{\hat{\mathcal{A}}} : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$:

$$\bar{e}_{N,\mathcal{A}} : {}_N\mathcal{A} \times {}_N\hat{\mathcal{A}} \longrightarrow \mu_{N,B} \subset \mathbb{G}_{m,B}. \tag{2.1}$$

Let $\alpha \in \Gamma(U, {}_N\mathcal{A})$ and $\lambda \in \Gamma(U, {}_N\hat{\mathcal{A}})$ be sections over some open $U \to B$ in the étale topology. The section λ is represented by a line bundle L on $\mathcal{A}_U$ such that $L^{\otimes N} \simeq \mathbb{1}_{\mathcal{A}_U}$. By refining $U \to B$ suitably we may assume that $\tau_\alpha^*(L) \simeq L$, where $\tau_\alpha : \mathcal{A}_U \to \mathcal{A}_U$ denotes translation by α. Then, as explained above, any relative automorphism $(\tilde{\alpha}, \tau_\alpha) : (L, \mathcal{A}_U) \to (L, \mathcal{A}_U)$ lifting τ_α yields an invertible function $\tilde{\alpha}^{\otimes N} \in \Gamma(\mathcal{A}_U, \mathbb{G}_m) = \Gamma(U, \mathbb{G}_m)$, and by refining $U \to B$ again we may choose $\tilde{\alpha}$ such that $\tilde{\alpha}^{\otimes N} = 1$. Indeed, take any $\tilde{\alpha}$, write $\tilde{\alpha}^{\otimes N}$ as an Nth power $\tilde{\alpha}^{\otimes N} = a^N$, $a \in \Gamma(U, \mathbb{G}_m)$, and then replace $\tilde{\alpha}$ with $a^{-1}\tilde{\alpha}$. Note that such an $\tilde{\alpha}$ is determined up to multiplication by elements from $\Gamma(U, \mu_N)$. Iterating N times the automorphism $(\tilde{\alpha}, \tau_\alpha)$ we obtain $(\tilde{\alpha}, \tau_\alpha)^{\circ N} = (\tilde{\alpha}^{\circ N}, 1_{\mathcal{A}_U})$; hence, $\tilde{\alpha}^{\circ N} \in \Gamma(U, \mathbb{G}_m)$, and as a matter of fact, $\tilde{\alpha}^{\circ N} \in \Gamma(U, \mu_N)$, since $(\tilde{\alpha}^{\circ N})^N = (\tilde{\alpha}^{\circ N})^{\otimes N} = (\tilde{\alpha}^{\otimes N})^{\circ N} = 1^{\circ N} = 1$.

PROPOSITION 2.2

In the preceding notation and with the given assumptions, if $(\tilde{\alpha}, \tau_\alpha) \in \mathrm{Aut}(L, \mathcal{A}_U)$ *satisfies* $\tilde{\alpha}^{\otimes N} = 1$, *then one has, in* $\Gamma(U, \mu_N)$, $\bar{e}_{N,\mathcal{A}}(\alpha, \lambda) = (\tilde{\alpha}^{\circ N})^{-1}$.

Proof

The line bundle $N^*L = N^*_{\mathcal{A}_U}(L)$ on $\mathcal{A}_U$ is trivial and so we have the canonical relative automorphism $(\tilde{\alpha}_0^{(N^*L)}, \tau_\alpha)$ of $(N^*L, \mathcal{A}_U)$ described in Note (e) above. A second relative automorphism $(\tilde{\alpha}_1^{(N^*L)}, \tau_\alpha)$ for $(N^*L, \mathcal{A}_U)$ is obtained from the canonical isomorphism $N^*L \xrightarrow{\sim} \tau_\alpha^*(N^*L)$ coming from the equality $N_{\mathcal{A}_U} = N_{\mathcal{A}_U} \circ \tau_\alpha$. Then (see [10, p. 184]), $\bar{e}_{N,\mathcal{A}}(\alpha, \lambda) \in \Gamma(U, \mu_N)$ is defined by the identity $\tilde{\alpha}_1^{(N^*L)} = \bar{e}_{N,\mathcal{A}}(\alpha, \lambda)\tilde{\alpha}_0^{(N^*L)}$. It suffices therefore to show that the function $h = \tilde{\alpha}^{\circ N}$ satisfies $\tilde{\alpha}_0^{(N^*L)} = h\tilde{\alpha}_1^{(N^*L)}$.

Upon refining $U \to B$ if necessary, we may write $\alpha = N\beta$ with $\beta \in \Gamma(U, \mathcal{A})$. The relative automorphism $(\tilde{\beta}_0^{(N^*L)}, \tau_\beta)$ of $(N^*L, \mathcal{A}_U)$ descends by the map $N_{\mathcal{A}_U} : \mathcal{A}_U \to \mathcal{A}_U$ to a relative automorphism $(\tilde{\alpha}_L, \tau_\alpha)$ of $(L, \mathcal{A}_U)$. This holds because, more generally, relative automorphisms $(\tilde{\beta}, \tau_\beta)$ and $(\tilde{\gamma}, \tau_\gamma)$ for a trivial line bundle on $\mathcal{A}_U$ always commute; hence, in particular, $(\tilde{\beta}_0^{(N^*L)}, \tau_\beta)$ commutes with $(\tilde{\gamma}_1^{(N^*L)}, \tau_\gamma)$ for all $\gamma \in \Gamma(U', {}_N\mathcal{A})$, with $U' \to U$ any refinement of $U \to B$. Iterating N times the relative automorphism and its descent we find that $(\tilde{\alpha}_0^{(N^*L)}, \tau_\alpha) = ((\tilde{\beta}_0^{(N^*L)})^{\circ N}, \tau_\beta^{\circ N}) : (N^*L, \mathcal{A}_U) \to (N^*L, \mathcal{A}_U)$ descends by the map $N_{\mathcal{A}_U} : \mathcal{A}_U \to \mathcal{A}_U$ to $(h', 1_{\mathcal{A}_U}) = (\tilde{\alpha}_L^{\circ N}, \tau_\alpha^{\circ N}) : (L, \mathcal{A}_U) \to (L, \mathcal{A}_U)$. Since on the other hand $(\tilde{\alpha}_1^{(N^*L)}, \tau_\alpha) : (N^*L, \mathcal{A}_U) \to (N^*L, \mathcal{A}_U)$ descends to the identity $(1_L, 1_{\mathcal{A}_U})$ of $(L, \mathcal{A}_U)$, it follows that $\tilde{\alpha}_0^{(N^*L)} = h'\tilde{\alpha}_1^{(N^*L)}$. It remains to see that $h' = h$. To this end, note that $(\tilde{\beta}_0^{(N^*L)})^{\otimes N} = \tilde{\beta}_0^{(N^*L)^{\otimes N}} = \tilde{\beta}_0^{N^*(L^{\otimes N})}$ and that $((\tilde{\beta}_0^{(N^*L)})^{\otimes N}, \tau_\beta)$ descends to $(\tilde{\alpha}_L^{\otimes N}, \tau_\alpha)$, while $(\tilde{\beta}_0^{N^*(L^{\otimes N})}, \tau_\beta)$ descends to $(\tilde{\alpha}_0^{(L^{\otimes N})}, \tau_\alpha)$. Therefore, $\tilde{\alpha}_L^{\otimes N} = \tilde{\alpha}_0^{(L^{\otimes N})} = \tilde{\alpha}^{\otimes N}$, so $\tilde{\alpha}_L$ and $\tilde{\alpha}$ differ by a factor in $\Gamma(U, \mu_N)$ and the functions $h = \tilde{\alpha}^{\circ N}$ and $h' = \tilde{\alpha}_L^{\circ N}$ coincide, as claimed. $\square$

For a different proof of this proposition we refer to the remark at the end of the proof of Proposition 2.16. Note also that it follows from Proposition 2.2 that, in the notation and with the assumptions preceding it but without the added assumption from the proposition, one has, in $\Gamma(U,\mu_N)$, $\bar{e}_{N,\mathcal{A}}(\alpha,\lambda) = \tilde{\alpha}^{\otimes N}/\tilde{\alpha}^{\circ N}$.

As a particular case of the canonical pairing between the kernels of two mutually dual morphisms of abelian schemes, the pairing $\bar{e}_N$ is antisymmetric. This means that one has

$$\bar{e}_{N,\hat{\mathcal{A}}}(\lambda,\alpha) = \bar{e}_{N,\mathcal{A}}(\alpha,\lambda)^{-1}, \tag{2.3}$$

for any sections $\alpha \in \Gamma(U, {}_N\mathcal{A})$ and $\lambda \in \Gamma(U, {}_N\hat{\mathcal{A}})$, with $U \to B$ open in the étale topology of B.

The effect of the pairing $\bar{e}_N$ in cohomology will be denoted by the same symbol, that is,

$$\bar{e}_{N,\mathcal{A}} : H^r(B, {}_N\mathcal{A}) \times H^s(B, {}_N\hat{\mathcal{A}}) \longrightarrow H^{r+s}(B,\mu_N) \tag{2.4}$$

for all $r, s \geq 0$. By (2.3) one has

$$\bar{e}_{N,\hat{\mathcal{A}}}(\lambda,\alpha) = (-1)^{rs+1}\bar{e}_{N,\mathcal{A}}(\alpha,\lambda) \tag{2.5}$$

for all $\alpha \in H^r(B, {}_N\mathcal{A})$ and $\lambda \in H^s(B, {}_N\hat{\mathcal{A}})$. Note that we switched to additive notation when dealing with cohomology classes.

PROPOSITION 2.6

Let ∂ denote the connecting homomorphisms in the cohomology exact sequences of the exact sequences deduced from the respective Kummer sequences for $\mathbb{G}_{m,B}$, $\mathcal{A}$, and $\hat{\mathcal{A}}$:

$$0 \longrightarrow \mu_{N,B} \longrightarrow \mu_{N^2,B} \xrightarrow{N} \mu_{N,B} \longrightarrow 0,$$

$$0 \longrightarrow {}_N\mathcal{A} \longrightarrow {}_{N^2}\mathcal{A} \xrightarrow{N} {}_N\mathcal{A} \longrightarrow 0,$$

$$0 \longrightarrow {}_N\hat{\mathcal{A}} \longrightarrow {}_{N^2}\hat{\mathcal{A}} \xrightarrow{N} {}_N\hat{\mathcal{A}} \longrightarrow 0.$$

For all $\alpha \in H^r(B, {}_N\mathcal{A})$ and $\lambda \in H^s(B, {}_N\hat{\mathcal{A}})$ the following identity holds in $H^{r+s+1}(B,\mu_N)$:

$$\partial\bar{e}_{N,\mathcal{A}}(\alpha,\lambda) = \bar{e}_{N,\mathcal{A}}(\partial\alpha,\lambda) + (-1)^r\bar{e}_{N,\mathcal{A}}(\alpha,\partial\lambda).$$

Proof

We choose a sufficiently fine open covering $(U_i \to B)_{i\in I}$ of B (see [7, Chapter III, Lemma 2.19]) so that the following holds: α is represented by a cocycle $\{\alpha_{i_0\cdots i_r}\}$, $\alpha_{i_0\cdots i_r} \in \Gamma(U_{i_0\cdots i_r}, {}_N\mathcal{A})$, with $\alpha_{i_0\cdots i_r} = N\tilde{\alpha}_{i_0\cdots i_r}$, $\tilde{\alpha}_{i_0\cdots i_r} \in \Gamma(U_{i_0\cdots i_r}, {}_{N^2}\mathcal{A})$, and λ is represented by a cocycle $\{\lambda_{i_0\cdots i_s}\}$, $\lambda_{i_0\cdots i_s} \in \Gamma(U_{i_0\cdots i_s}, {}_N\hat{\mathcal{A}})$, with $\lambda_{i_0\cdots i_s} = N\tilde{\lambda}_{i_0\cdots i_s}$, $\tilde{\lambda}_{i_0\cdots i_s} \in \Gamma(U_{i_0\cdots i_s}, {}_{N^2}\hat{\mathcal{A}})$. Then $\partial\alpha$ is represented by the cocycle given by $(\partial\alpha)_{i_0\cdots i_{r+1}} = \sum_0^{r+1}(-1)^\nu\tilde{\alpha}_{i_0\cdots\hat{i}_\nu\cdots i_{r+1}}$ and $\partial\lambda$ is represented by the cocycle given by $(\partial\lambda)_{i_0\cdots i_{s+1}} = \sum_0^{s+1}(-1)^\nu\tilde{\lambda}_{i_0\cdots\hat{i}_\nu\cdots i_{s+1}}$. The class $\bar{e}_{N,\mathcal{A}}(\alpha,\lambda)$ is represented by the cocycle given by $\bar{e}_{N,\mathcal{A}}(\alpha,\lambda)_{i_0\cdots i_{r+s}} = \bar{e}_{N,\mathcal{A}}(\alpha_{i_0\cdots i_r},\lambda_{i_r\cdots i_{r+s}}) = \bar{e}_{N^2,\mathcal{A}}(\tilde{\alpha}_{i_0\cdots i_r},\tilde{\lambda}_{i_r\cdots i_{r+s}})^N$. The last equality follows from the general fact that, given $\alpha\in\Gamma(U,{}_{N^2}\mathcal{A})$ and $\lambda\in\Gamma(U,{}_N\hat{\mathcal{A}})$, one has $\bar{e}_{N^2,\mathcal{A}}(\alpha,\lambda)=\bar{e}_{N,\mathcal{A}}(N\alpha,\lambda)$ (see [10, p. 185]).

So we have

$$\begin{aligned}
\bar{e}_{N,\mathcal{A}}(\partial\alpha,\lambda)_{i_0\cdots i_{r+s+1}} &= \bar{e}_{N,\mathcal{A}}\big((\partial\alpha)_{i_0\cdots i_{r+1}},\lambda_{i_{r+1}\cdots i_{r+s+1}}\big)\\
&= \bar{e}_{N,\mathcal{A}}\Big(\sum_0^{r+1}(-1)^\nu\tilde{\alpha}_{i_0\cdots\hat{i}_\nu\cdots i_{r+1}},\lambda_{i_{r+1}\cdots i_{r+s+1}}\Big)\\
&= \bar{e}_{N^2,\mathcal{A}}\Big(\sum_0^{r+1}(-1)^\nu\tilde{\alpha}_{i_0\cdots\hat{i}_\nu\cdots i_{r+1}},\tilde{\lambda}_{i_{r+1}\cdots i_{r+s+1}}\Big)\\
&= \prod_0^{r+1}\bar{e}_{N^2,\mathcal{A}}(\tilde{\alpha}_{i_0\cdots\hat{i}_\nu\cdots i_{r+1}},\tilde{\lambda}_{i_{r+1}\cdots i_{r+s+1}})^{(-1)^\nu},\\
\bar{e}_{N,\mathcal{A}}(\alpha,\partial\lambda)_{i_0\cdots i_{r+s+1}} &= \bar{e}_{N,\mathcal{A}}\big(\alpha_{i_0\cdots i_r},(\partial\lambda)_{i_r\cdots i_{r+s+1}}\big)\\
&= \bar{e}_{N,\mathcal{A}}\Big(\alpha_{i_0\cdots i_r},(-1)^r\sum_r^{r+s+1}(-1)^\nu\tilde{\lambda}_{i_r\cdots\hat{i}_\nu\cdots i_{r+s+1}}\Big)\\
&= \bar{e}_{N^2,\mathcal{A}}\Big(\tilde{\alpha}_{i_0\cdots i_r},\sum_r^{r+s+1}(-1)^\nu\tilde{\lambda}_{i_r\cdots\hat{i}_\nu\cdots i_{r+s+1}}\Big)^{(-1)^r}\\
&= \Big(\prod_r^{r+s+1}\bar{e}_{N^2,\mathcal{A}}(\tilde{\alpha}_{i_0\cdots i_r},\tilde{\lambda}_{i_r\cdots\hat{i}_\nu\cdots i_{r+s+1}})^{(-1)^\nu}\Big)^{(-1)^r},
\end{aligned}$$

and

$$\begin{aligned}
\partial\bar{e}_{N,\mathcal{A}}(\alpha,\lambda)_{i_0\cdots i_{r+s+1}} = &\prod_0^r\bar{e}_{N^2,\mathcal{A}}(\tilde{\alpha}_{i_0\cdots\hat{i}_\nu\cdots i_{r+1}},\tilde{\lambda}_{i_{r+1}\cdots i_{r+s+1}})^{(-1)^\nu}\\
&\cdot\prod_{r+1}^{r+s+1}\bar{e}_{N^2,\mathcal{A}}(\tilde{\alpha}_{i_0\cdots i_r},\tilde{\lambda}_{i_r\cdots\hat{i}_\nu\cdots i_{r+s+1}})^{(-1)^\nu},
\end{aligned}$$

hence

$$\begin{aligned}
\big(\partial\bar{e}_{N,\mathcal{A}}(\alpha,\lambda)-\bar{e}_{N,\mathcal{A}}(\partial\alpha,\lambda)\big)_{i_0\cdots i_{r+s+1}} &= \bar{e}_{N^2,\mathcal{A}}(\tilde{\alpha}_{i_0\cdots i_r},\tilde{\lambda}_{i_{r+1}\cdots i_{r+s+1}})^{-(-1)^{r+1}}\\
&\quad\cdot\prod_{r+1}^{r+s+1}\bar{e}_{N^2,\mathcal{A}}(\tilde{\alpha}_{i_0\cdots i_r},\tilde{\lambda}_{i_r\cdots\hat{i}_\nu\cdots i_{r+s+1}})^{(-1)^\nu}\\
&= \prod_r^{r+s+1}\bar{e}_{N^2,\mathcal{A}}(\tilde{\alpha}_{i_0\cdots i_r},\tilde{\lambda}_{i_r\cdots\hat{i}_\nu\cdots i_{r+s+1}})^{(-1)^\nu}\\
&= \big((-1)^r\bar{e}_{N,\mathcal{A}}(\alpha,\partial\lambda)\big)_{i_0\cdots i_{r+s+1}}. \qquad\square
\end{aligned}$$

(ii) *The pairing* e_N. Let $\mathcal{A} \to B$ be an abelian scheme with a principal polarization $\Phi : \mathcal{A} \xrightarrow{\sim} \hat{\mathcal{A}}$. One has $\hat{\Phi} = \Phi$; hence, $(N\Phi)\hat{} = N\Phi$. The pairing

$$e_{N,\mathcal{A}} : {}_N\mathcal{A} \times {}_N\mathcal{A} \longrightarrow \mu_{N,B} \subset \mathbb{G}_{m,B} \tag{2.7}$$

is the standard pairing on the kernel of the self-dual map $N\Phi$. It is also derived from $\bar{e}_{N,\mathcal{A}}$ by transport via Φ, $e_{N,\mathcal{A}}(\alpha,\beta) = \bar{e}_{N,\mathcal{A}}(\alpha,\Phi(\beta)) = \bar{e}_{N,\hat{\mathcal{A}}}(\Phi(\alpha),\beta)$ for $\alpha,\beta \in \Gamma(U, {}_N\mathcal{A})$. In the notation of [10], if Φ is given by a line bundle L on $\mathcal{A}$, $\Phi = \Phi_L$, then one has $e_{N,\mathcal{A}} = e^{L^{\otimes N}}$ (see [10, Section IV.23]). The pairing e_N is antisymmetric. Hence, using as before the same symbol to denote the pairings induced in cohomology for all $r,s \geq 0$,

$$e_{N,\mathcal{A}} : H^r(B, {}_N\mathcal{A}) \times H^s(B, {}_N\mathcal{A}) \longrightarrow H^{r+s}(B, \mu_N), \tag{2.8}$$

one has

$$e_{N,\mathcal{A}}(\beta,\alpha) = (-1)^{rs+1} e_{N,\mathcal{A}}(\alpha,\beta) \tag{2.9}$$

for $\alpha \in H^r(B, {}_N\mathcal{A})$ and $\beta \in H^s(B, {}_N\mathcal{A})$.

As the morphism Φ induces a morphism between the Kummer sequences for $\mathcal{A}$ and $\hat{\mathcal{A}}$, the result from Proposition 2.6 translates verbatim, so that for all $r,s \geq 0$ and $\alpha \in H^r(B, {}_N\mathcal{A})$, $\beta \in H^s(B, {}_N\mathcal{A})$ one has

$$\partial e_{N,\mathcal{A}}(\alpha,\beta) = e_{N,\mathcal{A}}(\partial\alpha,\beta) + (-1)^r e_{N,\mathcal{A}}(\alpha,\partial\beta) \tag{2.10}$$

in $H^{r+s+1}(B,\mu_N)$.

We shall be concerned in this work with the particular case of an abelian scheme $\mathcal{A} \to B$ of (relative) dimension 1, with the polarization defined by the invertible sheaf associated with the zero section, $\Phi = \Phi_{\mathcal{L}}$, $\mathcal{L} = \mathcal{O}_{\mathcal{A}}(0)$.

(iii) *The (opposite) Tate pairing* $\langle \cdot,\cdot \rangle_T$. Let $\mathcal{A} \to B$ be an abelian scheme. The opposite of the Tate pairing for abelian varieties is obtained as follows, in the more general setting of abelian schemes:

$$\langle \cdot,\cdot \rangle_{T,\mathcal{A}} : H^1(B,\mathcal{A}) \times H^0(B,\hat{\mathcal{A}}) \longrightarrow H^2(B,\mathbb{G}_m). \tag{2.11}$$

Given $\lambda \in H^0(B,\hat{\mathcal{A}})$, represented by the line bundle L with class in $\mathrm{Pic}^0(\mathcal{A}/B)$, one considers the extension of sheaves over B (actually of commutative group schemes; see [10, Section IV.23] and Note (a) of Section 3 in this article)

$$0 \longrightarrow \mathbb{G}_{m,B} \longrightarrow \mathcal{G}(\lambda) \longrightarrow \mathcal{A} \longrightarrow 0, \tag{2.12}$$

where the sections of $\mathcal{G}(\lambda)$ over $U \to B$ are given by the relative automorphisms of $(L_{\mathcal{A}_U}, \mathcal{A}_U)$ lifting translations by global sections of $\mathcal{A}_U$:

$$\Gamma\big(U,\mathcal{G}(\lambda)\big) = \big\{(\tilde{\alpha},\alpha) \bigm| \alpha \in \Gamma(U,\mathcal{A}), (\tilde{\alpha},\tau_\alpha) \in \mathrm{Aut}(L_{\mathcal{A}_U},\mathcal{A}_U)\big\}.$$

The group law is given by composition, $(\tilde{\beta},\beta) \circ (\tilde{\alpha},\alpha) = (\tilde{\beta} \circ \tilde{\alpha}, \alpha+\beta)$. Commutativity follows because, $\mathcal{A} \to B$ being proper, the commutator $e : \mathcal{A} \times_B \mathcal{A} \to \mathbb{G}_m$ for $\mathcal{G}(\lambda)$ must be constant and, hence, trivial (see [10, Section IV.23]). The image of $(\tilde{\alpha},\alpha)$ in $\Gamma(U,\mathcal{A})$ is α, and the image of $u \in \Gamma(U,\mathbb{G}_{m,B})$ in $\Gamma(U,\mathcal{G}(\lambda))$ is $(u,0)$. For later convenience we point out that if in the construction just given we replace the line bundle L by any line bundle on $\mathcal{A}$ isomorphic to $L \otimes M$, with

M the pullback of a line bundle from B, then we obtain an extension equal to (2.12) up to a (necessarily) unique isomorphism. The notation $\mathcal{G}(\lambda)$ reflects the fact (Weil–Barsotti isomorphism; see [11, Theorem (18.1)]) that this extension depends solely on the section λ of $\hat{\mathcal{A}}$ and not on the particular line bundle on $\mathcal{A}$ giving rise to it.

For $\alpha \in H^1(B,\mathcal{A})$ and $\lambda \in H^0(B,\hat{\mathcal{A}})$ the image $\langle \alpha,\lambda\rangle_{T,\mathcal{A}}$ of (α,λ) by the pairing $\langle\cdot,\cdot\rangle_{T,\mathcal{A}}$ is defined as the image of α by the connecting homomorphism $H^1(B,\mathcal{A}) \to H^2(B,\mathbb{G}_{m,B})$ in the cohomology sequence of (2.12).

To understand the relation between the pairing $\langle\cdot,\cdot\rangle_T$ and the pairing $\bar{e}_N$ from (2.4) for $r=s=1$, we introduce two intermediate pairings $\langle\cdot,\cdot\rangle_{T^0}$ and $\langle\cdot,\cdot\rangle_{T^1}$, and express these in terms of $\bar{e}_N$. Here and below, we shall always use the symbol ν to denote the (forgetful) maps induced in cohomology by the inclusion maps ${}_N\mathcal{A} \hookrightarrow \mathcal{A}$, ${}_N\hat{\mathcal{A}} \hookrightarrow \hat{\mathcal{A}}$, and $\mu_{N,B} \hookrightarrow \mathbb{G}_{m,B}$.

(iv) *The pairing* $\langle\cdot,\cdot\rangle_{T^0}$. For an abelian scheme $\mathcal{A} \to B$ we shall denote by

$$\langle\cdot,\cdot\rangle_{T^0,\mathcal{A}} : H^1(B,\mathcal{A}) \times H^0(B,{}_N\hat{\mathcal{A}}) \longrightarrow H^2(B,\mu_N), \tag{2.13}$$

the pairing defined similarly to $\langle\cdot,\cdot\rangle_{T,\mathcal{A}}$, but instead of taking (2.12) for $\nu\lambda$, we take, for $\lambda \in H^0(B,{}_N\hat{\mathcal{A}})$, the subextension

$$0 \longrightarrow \mu_{N,B} \longrightarrow \mathcal{G}_0(\lambda) \longrightarrow \mathcal{A} \longrightarrow 0 \tag{2.14}$$

defined by

$$\Gamma\big(U,\mathcal{G}_0(\lambda)\big) = \big\{(\tilde{\alpha},\alpha) \in \Gamma\big(U,\mathcal{G}(\nu\lambda)\big) \bigm| \tilde{\alpha}^{\otimes N} = 1\big\}.$$

Note that this makes sense since, in the above notation, one now has $L^{\otimes N} \simeq \mathbb{1}_{\mathcal{A}}$, by assumption.

The relationship between $\langle\cdot,\cdot\rangle_{T^0}$ and $\langle\cdot,\cdot\rangle_T$ is expressed by the formula for all $\alpha \in H^1(B,\mathcal{A})$ and $\lambda \in H^0(B,{}_N\hat{\mathcal{A}})$

$$\nu\langle\alpha,\lambda\rangle_{T^0,\mathcal{A}} = \langle\alpha,\nu\lambda\rangle_{T,\mathcal{A}}. \tag{2.15}$$

The relationship between the pairings $\langle\cdot,\cdot\rangle_{T^0}$ and $\bar{e}_N$ is described as follows.

PROPOSITION 2.16

For all $\alpha \in H^1(B,\mathcal{A})$ and $\lambda \in H^0(B,{}_N\hat{\mathcal{A}})$, letting ∂ denote here the connecting homomorphism for the Kummer sequence for $\mathcal{A}$ and multiplication by N, one has, in $H^2(B,\mu_N)$, $\langle\alpha,\lambda\rangle_{T^0,\mathcal{A}} = -\bar{e}_{N,\mathcal{A}}(\partial\alpha,\lambda)$.

Proof

This follows from the existence of a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & {}_N\mathcal{A} \arrow[r] \arrow[d, "\bar{e}_{N,\mathcal{A}}(-,\lambda)^{-1}"'] & \mathcal{A} \arrow[r, "N"] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d, "1"'] & 0 \\ 0 \arrow[r] & \mu_{N,B} \arrow[r] & \mathcal{G}_0(\lambda) \arrow[r] & \mathcal{A} \arrow[r] & 0 \end{tikzcd}$$

where the bottom sequence is (2.14), and the middle vertical arrow is given by the following recipe. Let λ be represented by L, with class in $\mathrm{Pic}^0(\mathcal{A}/B)$. Then, given

$\alpha \in \Gamma(U, \mathcal{A})$ one locally lifts τ_α to a relative automorphism $(\tilde{\alpha}, \tau_\alpha)$ of $(L_{\mathcal{A}_U}, \mathcal{A}_U)$ such that $\tilde{\alpha}^{\otimes N} = 1$, and then takes, as the image of α, $(\tilde{\alpha}^{\circ N}, N\alpha) \in \Gamma(U, \mathcal{G}_0(\lambda))$. Note that, indeed, $(\tilde{\alpha}^{\circ N})^{\otimes N} = (\tilde{\alpha}^{\otimes N})^{\circ N} = 1^{\circ N} = 1$. By Proposition 2.2, this restricts to ${}_N\mathcal{A}$ as indicated in the diagram. □

REMARK

We mention in passing that the preceding proof may be used to give another proof of Proposition 2.2. Namely, the commutative diagram above is already known, and since $\mathrm{Hom}_B(\mathcal{A}, \mu_{N,B}) = 0$, there is at most one arrow $\mathcal{A} \to \mathcal{G}_0(\lambda)$ and, hence, at most one arrow ${}_N\mathcal{A} \to \mu_{N,B}$ fitting in this diagram, so both definitions for $\bar{e}_{N,\mathcal{A}}$ in Proposition 2.2 must coincide.

(v) *The pairing* $\langle\cdot,\cdot\rangle_{T^1}$. For an abelian scheme $\mathcal{A} \to B$ we shall denote by

$$\langle\cdot,\cdot\rangle_{T^1,\mathcal{A}} : H^1(B, {}_N\mathcal{A}) \times H^0(B, \hat{\mathcal{A}}) \longrightarrow H^2(B, \mu_N), \tag{2.17}$$

the pairing defined similarly to $\langle\cdot,\cdot\rangle_{T,\mathcal{A}}$, but using, instead of (2.12), the subextension

$$0 \longrightarrow \mu_{N,B} \longrightarrow {}_N\mathcal{G}(\lambda) \longrightarrow {}_N\mathcal{A} \longrightarrow 0, \tag{2.18}$$

obtained by taking N-torsion everywhere in (2.12).

The relationship between $\langle\cdot,\cdot\rangle_{T^1}$ and $\langle\cdot,\cdot\rangle_T$ is expressed by the formula, for all $\alpha \in H^1(B, {}_N\mathcal{A})$ and $\lambda \in H^0(B, \hat{\mathcal{A}})$,

$$\nu\langle\alpha, \lambda\rangle_{T^1,\mathcal{A}} = \langle\nu\alpha, \lambda\rangle_{T,\mathcal{A}}. \tag{2.19}$$

The relationship between the pairings $\langle\cdot,\cdot\rangle_{T^1}$ and $\bar{e}_N$ is described as follows.

PROPOSITION 2.20

For all $\alpha \in H^1(B, {}_N\mathcal{A})$ and $\lambda \in H^0(B, \hat{\mathcal{A}})$, letting ∂ denote here the connecting homomorphism for the Kummer sequence for $\hat{\mathcal{A}}$ and multiplication by N, one has, in $H^2(B, \mu_N)$, $\langle\alpha, \lambda\rangle_{T^1,\mathcal{A}} = -\bar{e}_{N,\mathcal{A}}(\alpha, \partial\lambda)$.

REMARK 2.21

(a) In contrast with the proof of Proposition 2.16, which benefited from Proposition 2.2, our proof for this proposition is quite long. Later on (see Remark 3.25) we shall describe a way to circumvent this, in the case in which λ comes from $H^0(B, {}_N\hat{\mathcal{A}})$.

(b) We point out that we shall make repeated use, without further mention, of the commutativity of the composition of relative automorphisms of line bundles M with class in $\mathrm{Pic}^0(\mathcal{A}_U/U)$ over translation maps $\tau_a : \mathcal{A}_U \to \mathcal{A}_U$, $a \in \Gamma(U, \mathcal{A}_U)$ (see Part (iii) of this section).

Proof of Proposition 2.20

Let L, $[L] \in \mathrm{Pic}^0(\mathcal{A}/B)$, represent λ. Through successive refinements we find and choose a covering $\mathcal{U} = \{U_i \to B\}_{i \in I}$ fine enough so as to fulfill all the requirements listed below. For notational convenience we shall write $\mathcal{A}_i$ for $\mathcal{A}_{U_i}$, $\mathcal{A}_{ij}$ for $\mathcal{A}_{U_{ij}}$, with $U_{ij} = U_i \times_B U_j$, and so on. Similarly, we shall write L_i instead of $L_{\mathcal{A}_i}$ and L_{ij} instead of $L_{\mathcal{A}_{ij}}$.

The conditions that $\mathcal{U}$ is assumed to satisfy are the following. For all $i \in I$ we have $\lambda_{U_i} = N\mu_i$, with $\mu_i \in \Gamma(U_i, \hat{\mathcal{A}}_i)$. We call M_i, $[M_i] \in \mathrm{Pic}^0(\mathcal{A}_i/U_i)$, the line bundle representing μ_i, so that $M_i^{\otimes N} \simeq L_i$. Furthermore, $\alpha \in H^1(B, {}_N\mathcal{A})$ is represented by a 1-cocycle $\{a_{ij}\}_{i,j\in I}$, $a_{ij} \in \Gamma(U_{ij}, {}_N\mathcal{A}_{ij})$ for $\mathcal{U}$, and for all $i,j,k \in I$ the restriction of M_i to $\mathcal{A}_{ijk}$ is invariant under translation by the restriction of a_{jk}, that is, $\tau^*_{a_{jk}}(M_i) \simeq M_i$ on $\mathcal{A}_{ijk}$, for short. Moreover, for each $i, j \in I$ we have relative automorphisms $(\tilde{a}_{ij}, \tau_{a_{ij}}) \in \mathrm{Aut}(L_{ij}, \mathcal{A}_{ij})$ lifting the translation map by a_{ij} and such that $\tilde{a}_{ij}^{\circ N} = 1$.

We compute first $\langle \alpha, \lambda \rangle_{T^1, \mathcal{A}}$. This is the image of α by the connecting homomorphism for the exact sequence (2.18). The 1-cochain $\{(\tilde{a}_{ij}, a_{ij})\}_{i,j\in I}$ for $\mathcal{U}$ with values in ${}_N\mathcal{G}(\lambda)$ lifts the 1-cocycle $\{a_{ij}\}_{i,j\in I}$, so that $\langle \alpha, \lambda \rangle_{T^1, \mathcal{A}} \in H^2(B, \mu_N)$ is represented by the 2-cocycle

$$(ijk) \longmapsto \tilde{a}_{jk} \circ \tilde{a}_{ik}^{-1} \circ \tilde{a}_{ij}, \tag{2.22}$$

where the factors are taken to be restricted above $\mathcal{A}_{ijk}$. The expression on the right is an element of $\Gamma(U_{ijk}, \mathbb{G}_m)$ and belongs in fact to $\Gamma(U_{ijk}, \mu_N)$, since $(\tilde{a}_{jk} \circ \tilde{a}_{ik}^{-1} \circ \tilde{a}_{ij})^N = (\tilde{a}_{jk} \circ \tilde{a}_{ik}^{-1} \circ \tilde{a}_{ij})^{\circ N} = (\tilde{a}_{jk}^{\circ N}) \circ (\tilde{a}_{ik}^{\circ N})^{-1} \circ (\tilde{a}_{ij}^{\circ N}) = 1 \circ 1 \circ 1 = 1$.

Next we compute $\bar{e}_{N,\mathcal{A}}(\alpha, \partial\lambda)$. The class $\partial\lambda \in H^1(B, {}_N\hat{\mathcal{A}})$ is represented by the 1-cocycle $\{\mu_{ij}\}_{i,j\in I}$, $\mu_{ij} = \mu_j - \mu_i$ on U_{ij}. We write $M_{ij} = M_j \otimes M_i^{-1}$, $[M_{ij}] \in \mathrm{Pic}^0(\mathcal{A}_{ij}/U_{ij})$, for the line bundle representing $\mu_{ij} \in \Gamma(U_{ij}, {}_N\hat{\mathcal{A}})$; note that $M_{ij}^{\otimes N} \simeq \mathbb{1}_{\mathcal{A}_{ij}}$. The class $\bar{e}_{N,\mathcal{A}}(\alpha, \partial\lambda)$ is represented by the 2-cocycle

$$(ijk) \longmapsto \bar{e}_{N,\mathcal{A}}(a_{ij}, \mu_{jk}). \tag{2.23}$$

If $(\tilde{a}_{ijk}, \tau_{a_{ij}}) \in \mathrm{Aut}(M_{jk}, \mathcal{A}_{ijk})$ denotes (slightly abusively) an arbitrary automorphism of the restriction of M_{jk} to $\mathcal{A}_{ijk}$ over the translation map by the restriction of a_{ij} to U_{ijk} such that $\tilde{a}_{ijk}^{\otimes N} = 1$, then

$$\bar{e}_{N,\mathcal{A}}(a_{ij}, \mu_{jk}) = (\tilde{a}_{ijk}^{\circ N})^{-1}. \tag{2.24}$$

We shall choose $\tilde{a}_{ijk}$ linked to the $\tilde{a}_{ij}$'s so as to allow comparison between (2.22) and (2.23)–(2.24). To this end we choose first, for each $i, j, \upsilon \in I$, a relative automorphism

$$(\tilde{a}_{ij,\upsilon}, \tau_{a_{ij}}) \in \mathrm{Aut}(M_\upsilon, \mathcal{A}_{ij\upsilon})$$

of the restriction of M_υ to $\mathcal{A}_{ij\upsilon}$ over the translation map by the restriction of a_{ij} to $\mathcal{A}_{ij\upsilon}$, satisfying the property

$$\tilde{a}_{ij,\upsilon}^{\otimes N} = \tilde{a}_{ij}.$$

This may require yet a further refinement of our initial covering $\mathcal{U}$, and we assume this accomplished. We have, for the invertible function $\tilde{a}_{ij,\upsilon}^{\circ N}$ on $U_{ij\upsilon}$, $\tilde{a}_{ij,\upsilon}^{\circ N} \in \Gamma(U_{ij\upsilon}, \mu_N)$, since $(\tilde{a}_{ij,\upsilon}^{\circ N})^N = (\tilde{a}_{ij,\upsilon}^{\circ N})^{\otimes N} = (\tilde{a}_{ij,\upsilon}^{\otimes N})^{\circ N} = \tilde{a}_{ij}^{\circ N} = 1$. We choose now

$$\tilde{a}_{ijk} = \tilde{a}_{ij,k} \otimes \tilde{a}_{ij,j}^{\otimes(-1)}.$$

This choice fulfills $\tilde{a}_{ijk}^{\otimes N} = \tilde{a}_{ij,k}^{\otimes N} \otimes (\tilde{a}_{ij,j}^{\otimes N})^{\otimes(-1)} = \tilde{a}_{ij} \otimes \tilde{a}_{ij}^{\otimes(-1)} = 1$. We have, moreover, $\tilde{a}_{ijk}^{\circ N} = \tilde{a}_{ij,k}^{\circ N} \otimes (\tilde{a}_{ij,j}^{\circ N})^{\otimes(-1)} = (\tilde{a}_{ij,k}^{\circ N})(\tilde{a}_{ij,j}^{\circ N})^{-1}$.

To end the proof of the proposition we show that the product of the cocycles (2.22) and (2.23) is a coboundary. Multiplying and dividing this product by the coboundary of the 1-cochain $\{\tilde{a}_{ij,j}^{\circ N}\}_{i,j\in I}$, $(\delta\{\tilde{a}_{ij,j}^{\circ N}\})_{ijk} = (\tilde{a}_{jk,k}^{\circ N})(\tilde{a}_{ik,k}^{\circ N})^{-1}(\tilde{a}_{ij,j}^{\circ N})$, we have

$$\begin{aligned}
(\tilde{a}_{jk} \circ \tilde{a}_{ik}^{-1} \circ \tilde{a}_{ij})(\tilde{a}_{ijk}^{\circ N})^{-1} &= (\tilde{a}_{jk} \circ \tilde{a}_{ik}^{-1} \circ \tilde{a}_{ij})(\tilde{a}_{ij,k}^{\circ N})^{-1}(\tilde{a}_{ij,j}^{\circ N}) \\
&= (\tilde{a}_{jk} \circ \tilde{a}_{ik}^{-1} \circ \tilde{a}_{ij})(\tilde{a}_{jk,k}^{\circ N})^{-1}(\tilde{a}_{ik,k}^{\circ N})(\tilde{a}_{ij,k}^{\circ N})^{-1}\big(\delta\{\tilde{a}_{ij,j}^{\circ N}\}\big)_{ijk}, \\
\tilde{a}_{jk} \circ \tilde{a}_{ik}^{-1} \circ \tilde{a}_{ij} &= \tilde{a}_{jk,k}^{\otimes N} \circ (\tilde{a}_{ik,k}^{\otimes N})^{-1} \circ \tilde{a}_{ij,k}^{\otimes N} \\
&= (\tilde{a}_{jk,k} \circ \tilde{a}_{ik,k}^{-1} \circ \tilde{a}_{ij,k})^{\otimes N} \\
&= (\tilde{a}_{jk,k} \circ \tilde{a}_{ik,k}^{-1} \circ \tilde{a}_{ij,k})^{N},
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{a}_{jk,k}^{\circ N})^{-1}(\tilde{a}_{ik,k}^{\circ N})(\tilde{a}_{ij,k}^{\circ N})^{-1} &= (\tilde{a}_{jk,k}^{\circ N})^{-1} \circ (\tilde{a}_{ik,k}^{\circ N}) \circ (\tilde{a}_{ij,k}^{\circ N})^{-1} \\
&= (\tilde{a}_{jk,k}^{-1} \circ \tilde{a}_{ik,k} \circ \tilde{a}_{ij,k}^{-1})^{\circ N} \\
&= \big((\tilde{a}_{jk,k} \circ \tilde{a}_{ik,k}^{-1} \circ \tilde{a}_{ij,k})^{-1}\big)^{\circ N} \\
&= (\tilde{a}_{jk,k} \circ \tilde{a}_{ik,k}^{-1} \circ \tilde{a}_{ij,k})^{-N},
\end{aligned}$$

from which the result now follows. □

The relationship between the pairings $\langle\cdot,\cdot\rangle_{T^0}$ and $\langle\cdot,\cdot\rangle_{T^1}$ is deduced from Propositions 2.6, 2.16, and 2.20.

COROLLARY 2.25

For all $\alpha \in H^1(B, {}_N\mathcal{A})$ and $\lambda \in H^0(B, {}_N\hat{\mathcal{A}})$ one has, in $H^2(B, \mu_N)$,

$$\langle\alpha, \nu\lambda\rangle_{T^1,\mathcal{A}} - \langle\nu\alpha, \lambda\rangle_{T^0,\mathcal{A}} = \partial\bar{e}_{N,\mathcal{A}}(\alpha, \lambda).$$

3. Pairings, II: 1-dimensional Néron models

In the notation of Section 1, we now replace the abelian scheme $\mathcal{A} \to B$ from Section 2 by the Néron model $\mathcal{E}_0 \to B$ of the elliptic curve E_0 over K, as well as by its restrictions to arbitrary (Zariski) open subsets U of B, and we discuss a version in this context of the pairings of that section. To ease notation, except in Note (a) below, but otherwise throughout this section, the symbol B will stand for a fixed open subset U of our former B, and $\mathcal{E}_0$, S_0, .. will denote instead the restrictions $(\mathcal{E}_0)_U$, $(S_0)_U$, $(..)_U$ of these objects above U.

NOTES (PRELIMINARIES)

(a) We give first a description of the extensions (2.12) and (2.14) with $\mathcal{A}$ replaced by $\hat{\mathcal{A}}$, but viewed from the side of $\mathcal{A}$, that is, in the notation of Section 2, for

$\alpha \in H^0(B, \mathcal{A})$, a description of

$$0 \longrightarrow \mathbb{G}_{m,B} \longrightarrow \mathcal{G}(\alpha) \longrightarrow \hat{\mathcal{A}} \longrightarrow 0 \tag{3.1}$$

and, for $\alpha \in H^0(B, {}_N\mathcal{A})$, a description of

$$0 \longrightarrow \mu_{N,B} \longrightarrow \mathcal{G}_0(\alpha) \longrightarrow \hat{\mathcal{A}} \longrightarrow 0, \tag{3.2}$$

in both cases interpreted through the biduality isomorphism $\mathcal{A} \simeq \hat{\hat{\mathcal{A}}}$. The duality correspondence between the abelian schemes $\mathcal{A} \to B$ and $\hat{\mathcal{A}} \to B$ is given by the Poincaré torsor $\mathcal{P} \to \mathcal{A} \times_B \hat{\mathcal{A}}$, whose fiber $\mathcal{P}(\bar{a}, [\bar{L}])$ above a geometric point $(\bar{a}, [\bar{L}])$ of $\mathcal{A} \times_B \hat{\mathcal{A}}$ over a geometric point $\bar{t}$ of B is described canonically as the set of relative automorphisms of $\bar{L}$ lifting the translation map by $\bar{a}$:

$$\begin{array}{ccc} \bar{L} & \xrightarrow{\tilde{\bar{a}}} & \bar{L} \\ \downarrow & & \downarrow \\ \mathcal{A}(\bar{t}) & \xrightarrow{\tau_{\bar{a}}} & \mathcal{A}(\bar{t}) \end{array}$$

The group scheme $\mathcal{G}(\lambda)$ in (2.12) is the pullback of $\mathcal{P}$ according to the map $\mathcal{A} = \mathcal{A} \times_B B \xrightarrow{1 \times_B \lambda} \mathcal{A} \times_B \hat{\mathcal{A}}$. Changing roles between $\mathcal{A}$ and $\hat{\mathcal{A}}$ gives that, for $\alpha \in H^0(B, \mathcal{A})$, the group scheme $\mathcal{G}(\alpha)$ in (3.1) is the pullback of $\mathcal{P}$ by the map $\hat{\mathcal{A}} = B \times_B \hat{\mathcal{A}} \xrightarrow{\alpha \times_B 1} \mathcal{A} \times_B \hat{\mathcal{A}}$. Thus,

$$\Gamma\big(U, \mathcal{G}(\alpha)\big) = \big\{ \big(\tilde{\alpha}(\lambda), \lambda\big) \,\big|\, \lambda \in \Gamma(U, \hat{\mathcal{A}}),\ \big(\tilde{\alpha}(\lambda), \tau_\alpha\big) \in \operatorname{Aut}\big(L(\lambda), \mathcal{A}_U\big) \big\},$$

with $L(\lambda)$, $[L(\lambda)] \in \operatorname{Pic}^0(\mathcal{A}_U/U)$, denoting the representative of λ. The (commutative) group law of $\mathcal{G}(\alpha)$ is given by the tensor product, $(\tilde{\alpha}(\lambda), \lambda) \otimes (\tilde{\alpha}(\mu), \mu) = (\tilde{\alpha}(\lambda) \otimes \tilde{\alpha}(\mu), \lambda + \mu)$. The image of $(\tilde{\alpha}(\lambda), \lambda) \in \Gamma(U, \mathcal{G}(\alpha))$ in $\Gamma(U, \hat{\mathcal{A}})$ is λ, and the image of $u \in \Gamma(U, \mathbb{G}_{m,B})$ in $\Gamma(U, \mathcal{G}(\alpha))$ is $(u, 0)$. As for (3.2), we have

$$\Gamma\big(U, \mathcal{G}_0(\alpha)\big) = \big\{ \big(\tilde{\alpha}(\lambda), \lambda\big) \in \Gamma\big(U, \mathcal{G}(\nu\alpha)\big) \,\big|\, \tilde{\alpha}(\lambda)^{\circ N} = 1 \big\}.$$

(b) An important role is played in this article by the morphism of sheaves $\operatorname{Pic}_{S_0/B} \to \mathcal{E}_0$ obtained by taking the abelian sum along the smooth fibers of $S_0 \to B$. This map induces an isomorphism $\operatorname{Pic}^0_{S_0/B} \xrightarrow{\sim} \mathcal{E}^0_0$ between these two subsheaves, and this in turn gives, for all our N, the isomorphism ${}_N\operatorname{Pic}^0_{S_0/B} = {}_N\operatorname{Pic}_{S_0/B} \xrightarrow{\sim} {}_N\mathcal{E}^0_0$ appearing in Section 1.

Indeed, consider an open $U \to B$ in the étale topology; an element $\lambda \in \Gamma(U, \operatorname{Pic}_{S_0/B})$ maps to $\alpha \in \Gamma(U, \mathcal{E}_0)$ if and only if λ is representable by $[L] \in \operatorname{Pic}((S_0)_U)$ with $L = \mathcal{O}_{(S_0)_U}(D)$, $D = \Gamma_\alpha - \Gamma_{0_U} + D_v$, where 0_U indicates the zero section of $(S_0)_U$, $\Gamma_{(..)}$ denotes the graph of the corresponding section (..), and D_v is a divisor of $(S_0)_U$ whose components are nonneutral components of fibers of $(S_0)_U \to U$. On the other hand, this element λ belongs to $\Gamma(U, \operatorname{Pic}^0_{S_0/B})$ if and only if the divisor D has zero intersection multiplicity with all components of the geometric fibers of $(S_0)_U \to U$. For one thing, this implies that the abelian summation map induces an embedding $\operatorname{Pic}^0_{S_0/B} \hookrightarrow \mathcal{E}_0$, since, in the preceding notation, $D = D_v$ and $\bar{D}_v \cdot \bar{D}_v = 0$ (the bar indicating base change to an algebraic closure $\bar{k}$ of k) imply $D = 0$ (see, e.g., [13, Lemma 2.5]). Secondly, it shows that $\mathcal{E}_0^0 \subset \mathcal{E}_0$ is contained in the image of $\mathrm{Pic}^0_{S_0/B}$, by choosing in the above $D_v = 0$ for an arbitrarily given $\alpha \in \Gamma(U, \mathcal{E}_0^0)$. Finally, to show the opposite inclusion, suppose that, for a given $\lambda \in \Gamma(U, \mathrm{Pic}^0_{S_0/B})$, the section $\alpha \in \Gamma(U, \mathcal{E}_0)$ does not belong to $\Gamma(U, \mathcal{E}_0^0)$. Let $\bar{t}$ be a geometric point of B above which the graph Γ_α meets a nonneutral component of the fiber of $S_0 \to B$. Writing out the vanishing of the intersection multiplicities of $\bar{D}$ with the nonneutral components above $\bar{t}$, one finds that the -integral-coefficients of $\bar{D}_v$ in these components provide a column (of the opposite) of the inverse intersection matrix of these components, contradicting, for example, [13, Table 8.16].

It follows, in particular, that the sheaves $\mathrm{Pic}^0_{S_0/B}$ and ${}_N\mathrm{Pic}^0_{S_0/B}$ are representable by schemes. The image of an element $\lambda \in \Gamma(U, \mathrm{Pic}_{S_0/B})$ by the abelian summation map $\mathrm{Pic}_{S_0/B} \to \mathcal{E}_0$ will be denoted $\lambda_{\mathrm{Ab}} \in \Gamma(U, \mathcal{E}_0)$.

(c) Let $B^- \subset B$ be the open complement of the discriminant locus of the map $S_0 \to B$. We shall denote by $(..)^-$ the restriction above B^- of any relative object $(..)$ over B. So, in particular, $(\mathcal{E}_0^0)^- = \mathcal{E}_0^- = S_0^-$ is an abelian scheme of dimension 1 over B^-, and one has $(\mathrm{Pic}^0_{S_0/B})^- = \mathrm{Pic}^0_{\mathcal{E}_0^-/B^-} = (\mathcal{E}_0^-)\hat{}$. We note that the restriction $(\mathcal{E}_0^-)\hat{} \xrightarrow{\sim} \mathcal{E}_0^-$ of the abelian summation isomorphism is the opposite inverse of the standard isomorphism $\Phi_{\mathcal{L}} : \mathcal{E}_0^- \xrightarrow{\sim} (\mathcal{E}_0^-)\hat{}$ associated with the invertible sheaf $\mathcal{L} = \mathcal{O}_{\mathcal{E}_0^-}(0_{\mathcal{E}_0^-})$ as defined in [10, Sections II.6 and III.13].

(d) The action of $\mathcal{E}_0$ on S_0 by translations induces an action of $\mathcal{E}_0$ on $\mathrm{Pic}_{S_0/B}$ which is trivial on $\mathrm{Pic}^0_{S_0/B}$. In fact, this holds true when we restrict everything to $B^- \subset B$. Consider then $\mathrm{Pic}^0_{S_0/B}$ as the scheme $\mathcal{E}_0^0$, and let α and β be sections of $\mathcal{E}_0$ and $\mathcal{E}_0^0$, respectively, over some open $U \to B$. Then β itself and the transform of β by α coincide on the dense subset $U^- \subset U$, the inverse image of B^-; hence, they are equal.

(e) As in Note (d) of the preceding section, the existence of the zero section for $\pi : S_0 \to B$ and the Leray spectral sequence for the sheaf $\mathbb{G}_m$ on S_0 provide a canonically split exact sequence

$$0 \longrightarrow \mathrm{Pic}(B) \longrightarrow \mathrm{Pic}(S_0) \longrightarrow \Gamma(B, \mathrm{Pic}_{S_0/B}) \longrightarrow 0$$

and, in particular, a canonical isomorphism $\mathrm{Pic}^0(S_0/B) \xrightarrow{\sim} \Gamma(B, \mathrm{Pic}^0_{S_0/B})$ between (1) the subgroup $\mathrm{Pic}^0(S_0/B) \subset \mathrm{Pic}(S_0)$ of isomorphism classes of invertible sheaves which are topologically trivial on the geometric fibers of π and trivial on the zero section of π and (2) the group of global sections of $\mathrm{Pic}^0_{S_0/B}$. We shall always represent the latter by invertible sheaves or line bundles with class in $\mathrm{Pic}^0(S_0/B)$ through this isomorphism, unless otherwise stated. We do the same thing with $S_0 \to B$ replaced by its restriction $(S_0)_U \to U$, for any open $U \to B$ in the étale topology of B.

(i) *The pairing* $\bar{e}_N$. Due to the étaleness of ${}_N\mathrm{Pic}^0_{S_0/B} \simeq {}_N\mathcal{E}_0^0$ and ${}_N\mathcal{E}_0$ over B and the properness of $\mu_{N,B}$ over B, the pairing $\bar{e}_{N,(\mathcal{E}_0^-)\hat{}} : {}_N(\mathcal{E}_0^-)\hat{} \times_{B^-} {}_N\mathcal{E}_0^- \to \mu_{N,B^-}$ of group schemes over B^- extends uniquely to a pairing $\bar{e}_N : {}_N\mathrm{Pic}^0_{S_0/B} \times_B {}_N\mathcal{E}_0 \to$

$\mu_{N,B}$ of group schemes over B which yields, in the language of sheaves, a pairing

$$\bar{e}_N : {}_N\operatorname{Pic}^0_{S_0/B} \times {}_N\mathcal{E}_0 \longrightarrow \mu_{N,B} \tag{3.3}$$

inducing the pairing $\bar{e}_{N,(\mathcal{E}_0^-)\check{}} : {}_N(\mathcal{E}_0^-)\check{} \times {}_N\mathcal{E}_0^- \to \mu_{N,B^-}$ by restriction. By its very uniqueness, this pairing admits the following description, fitting over B^- with the description given in Section 2 for $\bar{e}_{N,\mathcal{A}}$ with $\mathcal{A} = (\mathcal{E}_0^-)\check{}$, as translated by the dictionary given in Note (a) of the present section. Given $\lambda \in \Gamma(U, {}_N\operatorname{Pic}^0_{S_0/B})$ and $\alpha \in \Gamma(U, {}_N\mathcal{E}_0)$ for $U \to B$ open in the étale topology of B, let λ be represented by the line bundle L on $(S_0)_U$. One has $L^{\otimes N} \simeq \mathbb{1}_{(S_0)_U}$ and, refining U if necessary, $\tau_\alpha^*(L) \simeq L$. Upon refining U again we find a relative automorphism $(\tilde{\alpha}(\lambda), \tau_\alpha) : (L, (S_0)_U) \to (L, (S_0)_U)$ such that $\tilde{\alpha}(\lambda)^{\circ N} = 1$. Then, in $\Gamma(U, \mu_N)$,

$$\bar{e}_N(\lambda, \alpha) = \big(\tilde{\alpha}(\lambda)^{\otimes N}\big)^{-1}. \tag{3.4}$$

(ii) *The pairing* e_N. In a similar way as in (i), the pairing of sheaves $e_{N,\mathcal{E}_0^-} : {}_N\mathcal{E}_0^- \times {}_N\mathcal{E}_0^- \to \mu_{N,B^-}$ extends to a pairing of sheaves

$$e_N : {}_N\mathcal{E}_0 \times {}_N\mathcal{E}_0 \longrightarrow \mu_{N,B}, \tag{3.5}$$

which is also antisymmetric, by a continuity argument applied to the corresponding pairings of group schemes. The relation between e_N and $\bar{e}_N$ is described for sections $\lambda \in \Gamma(U, {}_N\operatorname{Pic}^0_{S_0/B})$, $\alpha \in \Gamma(U, {}_N\mathcal{E}_0)$ by the formula

$$\bar{e}_N(\lambda, \alpha) = e_N(\lambda_{Ab}, \alpha)^{-1}. \tag{3.6}$$

Indeed, this is equivalent to the commutativity of the following diagram of group schemes:

$$\begin{array}{ccc} {}_N\operatorname{Pic}^0_{S_0/B} \times_B {}_N\mathcal{E}_0 & \xrightarrow{\bar{e}_N} & \mu_{N,B} \\ {\scriptstyle Ab\times_B 1}\big\downarrow & & \big\downarrow{\scriptstyle (..)^{-1}} \\ {}_N\mathcal{E}_0 \times_B {}_N\mathcal{E}_0 & \xrightarrow{e_N} & \mu_{N,B} \end{array}$$

a fact which in turn is equivalent, by continuity, to the commutativity of its restriction over B^-,

$$\begin{array}{ccc} {}_N(\mathcal{E}_0^-)\check{} \times_{B^-} {}_N\mathcal{E}_0^- & \xrightarrow{\bar{e}_{N,(\mathcal{E}_0^-)\check{}}} & \mu_{N,B^-} \\ {\scriptstyle Ab\times_{B^-} 1}\big\downarrow & & \big\downarrow{\scriptstyle (..)^{-1}} \\ {}_N\mathcal{E}_0^- \times_{B^-} {}_N\mathcal{E}_0^- & \xrightarrow{e_{N,\mathcal{E}_0^-}} & \mu_{N,B^-} \end{array}$$

To show the latter, let $\lambda \in \Gamma(U, {}_N(\mathcal{E}_0^-)\check{})$ and $\alpha \in \Gamma(U, {}_N\mathcal{E}_0^-)$. One has

$$\begin{aligned} \bar{e}_{N,(\mathcal{E}_0^-)\check{}}(\lambda, \alpha) &= \bar{e}_{N,\mathcal{E}_0^-}(\alpha, \lambda)^{-1} \quad \big((2.3)\big) \\ &= \bar{e}_{N,\mathcal{E}_0^-}(\alpha, -\lambda) \\ &= \bar{e}_{N,\mathcal{E}_0^-}\big(\alpha, \Phi(\lambda_{Ab})\big) \quad \big(\text{Section 3, Note (c); Section 2, Part (ii)}\big) \end{aligned}$$

$$= e_{N,\mathcal{E}_0^-}(\alpha, \lambda_{Ab}) \quad \bigl(\text{Section 2, Part (ii)}\bigr)$$
$$= e_{N,\mathcal{E}_0^-}(\lambda_{Ab}, \alpha)^{-1},$$

thereby finishing the proof of (3.6).

(iii) *The (opposite) Tate pairing* $\langle \cdot, \cdot \rangle_T$. This is a pairing

$$\langle \cdot, \cdot \rangle_T : H^1(B, \mathrm{Pic}^0_{S_0/B}) \times H^0(B, \mathcal{E}_0) \longrightarrow H^2(B, \mathbb{G}_m), \tag{3.7}$$

compatible by restriction with the pairing (2.11) for $\mathcal{A} = (\mathcal{E}_0^-)\hat{}$ and $B = B^-$. Given $\alpha \in H^0(B, \mathcal{E}_0)$ one has, canonically, an extension of sheaves of abelian groups for the étale topology on B

$$0 \longrightarrow \mathbb{G}_{m,B} \longrightarrow \mathcal{G}(\alpha) \longrightarrow \mathrm{Pic}^0_{S_0/B} \longrightarrow 0, \tag{3.8}$$

which restricts on B^- to the extension (3.1) for $\mathcal{A} = \mathcal{E}_0^-$ and $\alpha = \alpha^-$,

$$0 \longrightarrow \mathbb{G}_{m,B^-} \longrightarrow \mathcal{G}(\alpha^-) \longrightarrow (\mathcal{E}_0^-)\hat{} \longrightarrow 0.$$

For $U \to B$ open in the étale topology of B one defines

$$\Gamma\bigl(U, \mathcal{G}(\alpha)\bigr) = \bigl\{ \bigl(\tilde{\alpha}(\lambda), \lambda\bigr) \bigm| \lambda \in \Gamma(U, \mathrm{Pic}^0_{S_0/B}), \bigl(\tilde{\alpha}(\lambda), \tau_\alpha\bigr) \in \mathrm{Aut}\bigl(L(\lambda), (S_0)_U\bigr) \bigr\},$$

with $L(\lambda)$, $[L(\lambda)] \in \mathrm{Pic}^0((S_0)_U/U)$, representing λ. (Here too, as in Part (iii) of Section 2, the line bundle $L(\lambda)$ may be replaced by its tensor product with any line bundle coming from U. The relative automorphisms over τ_α for both line bundles are in canonical correspondence.) The group law for $\mathcal{G}(\alpha)$ and the morphisms in the sequence (3.8) are described as in Note (a) of this section.

For $\lambda \in H^1(B, \mathrm{Pic}^0_{S_0/B})$ and $\alpha \in H^0(B, \mathcal{E}_0)$ the image $\langle \lambda, \alpha \rangle_T$ of (λ, α) by $\langle \cdot, \cdot \rangle_T$ is defined as the image of λ by the connecting homomorphism $H^1(B, \mathrm{Pic}^0_{S_0/B}) \to H^2(B, \mathbb{G}_m)$ in the cohomology exact sequence of (3.8). As stated before, restriction to B^- gives, for λ and α as above,

$$\langle \lambda, \alpha \rangle_T^- = \langle \lambda^-, \alpha^- \rangle_{T, (\mathcal{E}_0^-)\hat{}}.$$

(iv) *The pairing* $\langle \cdot, \cdot \rangle_{T^0}$. Paralleling the derivation of (2.14) from (2.12), as translated in Note (a) of this section into the dual setting, given $\alpha \in H^0(B, {}_N\mathcal{E}_0)$ we deduce from (3.8) an extension

$$0 \longrightarrow \mu_{N,B} \longrightarrow \mathcal{G}_0(\alpha) \longrightarrow \mathrm{Pic}^0_{S_0/B} \longrightarrow 0, \tag{3.9}$$

which restricts on B^- to the extension (3.2) with $\mathcal{A} = \mathcal{E}_0^-$ and $\alpha = \alpha^-$. For $U \to B$ open in the étale topology one takes, in the notation from Part (iii) above,

$$\Gamma\bigl(U, \mathcal{G}_0(\alpha)\bigr) = \bigl\{ \bigl(\tilde{\alpha}(\lambda), \lambda\bigr) \in \Gamma\bigl(U, \mathcal{G}(\nu\alpha)\bigr) \bigm| \tilde{\alpha}(\lambda)^{\circ N} = 1 \bigr\}.$$

One then defines the pairing

$$\langle \cdot, \cdot \rangle_{T^0} : H^1(B, \mathrm{Pic}^0_{S_0/B}) \times H^0(B, {}_N\mathcal{E}_0) \longrightarrow H^2(B, \mu_N) \tag{3.10}$$

by sending (λ, α) to the image of λ by the connecting homomorphism $H^1(B, \mathrm{Pic}^0_{S_0/B}) \to H^2(B, \mu_N)$ in the cohomology exact sequence of (3.9).

One has, for $\lambda \in H^1(B, \mathrm{Pic}^0_{S_0/B})$ and $\alpha \in H^0(B, {}_N\mathcal{E}_0)$,

$$\nu \langle \lambda, \alpha \rangle_{T^0} = \langle \lambda, \nu\alpha \rangle_T\,. \tag{3.11}$$

Restriction to B^- yields, for λ and α as above,

$$\langle \lambda, \alpha \rangle_{T^0}^- = \langle \lambda^-, \alpha^- \rangle_{T^0,(\mathcal{E}_0^-)\hat{}}. \tag{3.12}$$

The relationship between $\langle \cdot, \cdot \rangle_{T^0}$ and $\bar{e}_N$ (resp., e_N) is similar to the one for abelian schemes:

PROPOSITION 3.13

For all $\lambda \in H^1(B, \mathrm{Pic}^0_{S_0/B})$ and $\alpha \in H^0(B, {}_N\mathcal{E}_0)$ one has, in $H^2(B, \mu_N)$, $\langle \lambda, \alpha \rangle_{T^0} = -\bar{e}_N(\partial\lambda, \alpha) = e_N(\partial\lambda_{Ab}, \alpha)$, where ∂ refers to the corresponding connecting homomorphism in the cohomology exact sequence of the Kummer sequence for $\mathrm{Pic}^0_{S_0/B}$ (resp., $\mathcal{E}_0^0$) and the integer N.

Proof

The last equality is inherited from (3.6), since the abelian summation isomorphism $\mathrm{Pic}^0_{S_0/B} \xrightarrow{\sim} \mathcal{E}_0^0$ identifies the Kummer sequences for both sheaves, and so $(\partial\lambda)_{Ab} = \partial\lambda_{Ab}$ holds. As for the first equality, this is deduced from a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & {}_N\mathrm{Pic}^0_{S_0/B} \arrow[r] \arrow[d, "\bar{e}_N(-,\alpha)^{-1}"'] & \mathrm{Pic}^0_{S_0/B} \arrow[r, "N"] \arrow[d] & \mathrm{Pic}^0_{S_0/B} \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \mu_{N,B} \arrow[r] & \mathcal{G}_0(\alpha) \arrow[r] & \mathrm{Pic}^0_{S_0/B} \arrow[r] & 0
\end{tikzcd}$$

the bottom sequence being (3.9). Only the middle vertical arrow needs explanation. The rest is a diagram of commutative group schemes (for $\mathcal{G}_0(\alpha)$ this follows as in Remark 4.23), which extends the analogous diagram from the proof of Proposition 2.16, with $B = B^-$, $\mathcal{A} = (\mathcal{E}_0^-)\hat{}$, and $\lambda = \alpha^-$. By continuity, the middle vertical arrow in that diagram extends over the whole of B, too, and commutativity is preserved as well. Alternatively, an explicit description of the middle vertical arrow in the diagram above is as follows. Given $\lambda \in \Gamma(U, \mathrm{Pic}^0_{S_0/B})$ for $U \to B$ open in the étale topology, we represent this element by a line bundle L on $(S_0)_U$ with $[L] \in \mathrm{Pic}^0((S_0)_U/U)$ and, upon refining U if necessary, such that $\tau_\alpha^*(L) \simeq L$. Refining U again we find a relative automorphism $(\tilde{\alpha}(\lambda), \tau_\alpha) : (L, (S_0)_U) \xrightarrow{\sim} (L, (S_0)_U)$ such that $\tilde{\alpha}(\lambda)^{\circ N} = 1$. Then $(\tilde{\alpha}(\lambda)^{\otimes N}, \tau_\alpha) \in \mathrm{Aut}(L^{\otimes N}, (S_0)_U)$ is a well-defined element of $\Gamma(U, \mathcal{G}_0(\alpha))$ mapping to $N\lambda \in \Gamma(U, \mathrm{Pic}^0_{S_0/B})$. The commutativity of the left-hand side square in the diagram follows from the description of $\bar{e}_N$ given in Part (i) of this section. $\square$

(v) *The pairing* $\langle \cdot, \cdot \rangle_{T^1}$. Given $\alpha \in H^0(B, \mathcal{E}_0)$ we deduce, by taking N-torsion everywhere in (3.8), an extension

$$0 \longrightarrow \mu_{N,B} \longrightarrow {}_N\mathcal{G}(\alpha) \longrightarrow {}_N\mathrm{Pic}^0_{S_0/B} \longrightarrow 0. \tag{3.14}$$

We use this to define a pairing

$$\langle \cdot, \cdot \rangle_{T^1} : H^1(B, {}_N\mathrm{Pic}^0_{S_0/B}) \times H^0(B, \mathcal{E}_0) \longrightarrow H^2(B, \mu_N) \tag{3.15}$$

by sending (λ, α) to the image of λ by the connecting homomorphism $H^1(B, {}_N \operatorname{Pic}^0_{S_0/B}) \to H^2(B, \mu_N)$ in the cohomology exact sequence of (3.14).

One has, for $\lambda \in H^1(B, {}_N \operatorname{Pic}^0_{S_0/B})$ and $\alpha \in H^0(B, \mathcal{E}_0)$,

$$\nu\langle\lambda, \alpha\rangle_{T^1} = \langle\nu\lambda, \alpha\rangle_T. \tag{3.16}$$

On the other hand, restriction to B^- gives, for λ and α as above,

$$\langle\lambda, \alpha\rangle^-_{T^1} = \langle\lambda^-, \alpha^-\rangle_{T^1, (\mathcal{E}_0^-)}. \tag{3.17}$$

The group $H^1(B, \operatorname{Pic}^0_{S_0/B})$ being a torsion group (see, e.g., [8, Section III.7]), it follows from (3.16) and the Kummer exact sequences for $\operatorname{Pic}^0_{S_0/B}$ that the pairing $\langle\cdot,\cdot\rangle_T$ in Part (iii) of this section is actually dominated by the pairing $\langle\cdot,\cdot\rangle_{T^1}$. We have included it nevertheless here as a logical link, to keep a comprehensive picture.

In contrast to what happens for abelian schemes, the multiplication by N map on $\mathcal{E}_0$ needs to not be an epimorphism. For sections $\alpha \in H^0(B, N\mathcal{E}_0)$, however, we may parallel—in the present dual setting—Proposition 2.20 in the same way as Proposition 3.13 parallels Proposition 2.16. We explain this in detail, for the sake of completeness.

PROPOSITION 3.18

Given $\lambda \in H^1(B, {}_N \operatorname{Pic}^0_{S_0/B})$ and $\alpha \in H^0(B, N\mathcal{E}_0)$ we have, in $H^2(B, \mu_N)$, $\langle\lambda, \alpha\rangle_{T^1} = -\bar{e}_N(\lambda, \partial\alpha) = e_N(\lambda_{Ab}, \partial\alpha)$, where ∂ refers to the corresponding connecting homomorphism in the cohomology exact sequence of the exact sequence $0 \to {}_N\mathcal{E}_0 \to \mathcal{E}_0 \xrightarrow{N} N\mathcal{E}_0 \to 0$.

Proof

As before, the last equality follows from (3.6). We prove the first equality, following the proof of Proposition 2.20. Let $\mathcal{U} = \{U_i \to B\}_{i \in I}$ be an open covering of B in the étale topology such that the element λ is given by a family of line bundles L_{ij} on $(S_0)_{ij}$, $i, j \in I$, with $[L_{ij}] \in \operatorname{Pic}^0((S_0)_{ij}/U_{ij})$, and such that $L_{ij}^{\otimes N} \simeq \mathbb{1}_{(S_0)_{ij}}$ and $L_{jk} \otimes L_{ik}^{-1} \otimes L_{ij} \simeq \mathbb{1}_{(S_0)_{ijk}}$ on $(S_0)_{ijk}$, for all $i, j, k \in I$. (Here and below we use the notational conventions introduced in the proof of Proposition 2.20.) We take $\mathcal{U}$ fine enough so that for all $i, j \in I$ one has $\tau_\alpha^*(L_{ij}) \simeq L_{ij}$ and so that there exists a relative automorphism $(\tilde{\alpha}_{ij}, \tau_\alpha)$ of $(L_{ij}, (S_0)_{ij})$ satisfying $\tilde{\alpha}_{ij}^{\otimes N} = 1$, that is, a section $(\tilde{\alpha}_{ij}, [L_{ij}]) \in \Gamma(U_{ij}, {}_N\mathcal{G}(\alpha))$ lifting $[L_{ij}] \in \Gamma(U_{ij}, {}_N \operatorname{Pic}^0_{S_0/B})$. Under these assumptions, $\langle\lambda, \alpha\rangle_{T^1} \in H^2(B, \mu_N)$ is represented by the 2-cocycle of $\mathcal{U}$ given by

$$(ijk) \mapsto \tilde{\alpha}_{jk} \otimes \tilde{\alpha}_{ik}^{\otimes(-1)} \otimes \tilde{\alpha}_{ij} \in \Gamma(U_{ijk}, \mu_N). \tag{3.19}$$

Next we compute $\bar{e}_N(\lambda, \partial\alpha)$. Refining $\mathcal{U}$ again if necessary, we may assume that $\alpha = N\eta_i$ on U_i, $\eta_i \in \Gamma(U_i, \mathcal{E}_0)$, for all $i \in I$. So $\partial\alpha \in H^1(B, {}_N\mathcal{E}_0)$ is represented by the cocycle $\{\beta_{ij}\}$ of $\mathcal{U}$ given by $\beta_{ij} = \eta_j - \eta_i \in \Gamma(U_{ij}, {}_N\mathcal{E}_0)$, $i, j \in I$. Up to refining $\mathcal{U}$ once more we find, for each $i, j, v \in I$, a relative automorphism

$$(\tilde{\eta}_{ij,v}, \tau_{\eta_v}) \in \operatorname{Aut}\big(L_{ij} \mid (S_0)_{ijv}, (S_0)_{ijv}\big)$$

such that

$$\tilde{\eta}_{ij,v}^{\circ N} = \tilde{\alpha}_{ij}.$$

One has $\tilde{\eta}_{ij,v}^{\otimes N} \in \Gamma(U_{ijv}, \mu_N) \subset \Gamma(U_{ijv}, \mathbb{G}_m)$, since $(\tilde{\eta}_{ij,v}^{\otimes N})^N = (\tilde{\eta}_{ij,v}^{\otimes N})^{\circ N} = (\tilde{\eta}_{ij,v}^{\circ N})^{\otimes N} = (\tilde{\alpha}_{ij})^{\otimes N} = 1$. We put then, for each $i, j, k \in I$,

$$\tilde{\beta}_{ijk} = \tilde{\eta}_{ij,k} \circ \tilde{\eta}_{ij,j}^{-1},$$

so $(\tilde{\beta}_{ijk}, \tau_{\beta_{jk}}) \in \mathrm{Aut}(L_{ij} \mid (S_0)_{ijk}, (S_0)_{ijk})$. We have $(\tilde{\beta}_{ijk})^{\circ N} = (\tilde{\eta}_{ij,k}^{\circ N}) \circ (\tilde{\eta}_{ij,j}^{\circ N})^{-1} = \tilde{\alpha}_{ij} \circ \tilde{\alpha}_{ij}^{-1} = 1$. Here we have used the fact that the commutative property described for abelian schemes in Remark 2.21 implies, by continuity, the same property in the present setting. By Part (i) of this section, the class $\bar{e}_N(\lambda, \partial\alpha) \in H^2(B, \mu_N)$ is therefore represented by the 2-cocycle

$$(ijk) \mapsto \bar{e}_N([L_{ij}], \beta_{jk}) = (\tilde{\beta}_{ijk}^{\otimes N})^{-1} = (\tilde{\eta}_{ij,j}^{\otimes N})(\tilde{\eta}_{ij,k}^{\otimes N})^{-1} \in \Gamma(U_{ijk}, \mu_N). \tag{3.20}$$

To end this proof, we show that the product of the cocycles (3.19) and (3.20) is a coboundary. Introducing the 1-cochain $\{\tilde{\eta}_{ij,j}^{\otimes N}\} \in C^1(\mathcal{U}, \mu_N)$, $(\delta\{\tilde{\eta}_{ij,j}^{\otimes N}\})_{ijk} = (\tilde{\eta}_{jk,k}^{\otimes N})(\tilde{\eta}_{ik,k}^{\otimes N})^{-1}(\tilde{\eta}_{ij,j}^{\otimes N})$, we have, for each $i, j, k \in I$,

$$\begin{aligned}(\tilde{\eta}_{ij,j}^{\otimes N})(\tilde{\eta}_{ij,k}^{\otimes N})^{-1} &= (\tilde{\eta}_{jk,k}^{\otimes N})^{-1}(\tilde{\eta}_{ik,k}^{\otimes N})(\tilde{\eta}_{ij,k}^{\otimes N})^{-1}\big(\delta\{\tilde{\eta}_{ij,j}^{\otimes N}\}\big)_{ijk} \\ &= (\tilde{\eta}_{jk,k} \otimes \tilde{\eta}_{ik,k}^{\otimes(-1)} \otimes \tilde{\eta}_{ij,k})^{-N}\big(\delta\{\tilde{\eta}_{ij,j}^{\otimes N}\}\big)_{ijk},\end{aligned}$$

and on the other hand,

$$\begin{aligned}\tilde{\alpha}_{jk} \otimes \tilde{\alpha}_{ik}^{\otimes(-1)} \otimes \tilde{\alpha}_{ij} &= (\tilde{\eta}_{jk,k}^{\circ N}) \otimes (\tilde{\eta}_{ik,k}^{\circ N})^{\otimes(-1)} \otimes (\tilde{\eta}_{ij,k}^{\circ N}) \\ &= (\tilde{\eta}_{jk,k} \otimes \tilde{\eta}_{ik,k}^{\otimes(-1)} \otimes \tilde{\eta}_{ij,k})^N,\end{aligned}$$

which implies the claimed statement. □

REMARK 3.21

Under a similar restriction as in Proposition 3.18, that is, working with cohomology classes of $N\mathcal{E}_0$ instead of $\mathcal{E}_0$, the Leibniz-type formulae in Proposition 2.6 and (2.10) can be established in the present setting as well. Namely, for abelian schemes of dimension 1 (or, more generally, for principally polarized abelian schemes) both formulae are equivalent, and the proof given for Proposition 2.6 can be translated so as to apply directly for (2.10). In this form, the proof carries over verbatim into the present setting, giving the following statement.

PROPOSITION 3.22

Let $\alpha \in H^r(B, {}_N N\mathcal{E}_0)$ and $\beta \in H^s(B, {}_N N\mathcal{E}_0)$. One has, in $H^{r+s+1}(B, \mu_N)$,

$$\partial e_N(\alpha, \beta) = e_N(\partial\alpha, \beta) + (-1)^r e_N(\alpha, \partial\beta).$$

One gets, in particular, by Note (b) of this section and (3.6), the following.

COROLLARY 3.23

Let $\lambda \in H^r(B, {}_N\operatorname{Pic}^0_{S_0/B})$ *and* $\alpha \in H^s(B, {}_N N\mathcal{E}_0)$. *One has, in* $H^{r+s+1}(B, \mu_N)$,

$$\partial \bar{e}_N(\lambda, \alpha) = \bar{e}_N(\partial\lambda, \alpha) + (-1)^r \bar{e}_N(\lambda, \partial\alpha).$$

As in Section 2, Propositions 3.13 and 3.18 and Corollary 3.23 together yield a formula similar to the formula in Corollary 2.25 expressing the relation between $\langle \cdot, \cdot \rangle_{T^0}$ and $\langle \cdot, \cdot \rangle_{T^1}$ under the present restrictive assumption (see Remark 3.21). It is now an essential fact for the purpose of this work that this relation—the analogue of Corollary 2.25—holds true without any restriction:

PROPOSITION 3.24

For all $\lambda \in H^1(B, {}_N\operatorname{Pic}^0_{S_0/B})$ *and* $\alpha \in H^0(B, {}_N\mathcal{E}_0)$ *one has, in* $H^2(B, \mu_N)$,

$$\langle \lambda, \nu\alpha \rangle_{T^1} - \langle \nu\lambda, \alpha \rangle_{T^0} = \partial \bar{e}_N(\lambda, \alpha).$$

REMARK 3.25

The proof which follows could have been given already for Corollary 2.25, making that result independent from Propositions 2.6, 2.16, and 2.20. We did not do so because we wanted to get into the latter two results. On the other hand, any of them, together with Corollary 2.25, implies again the other one, with a small restriction on the data (see Remark 2.21).

Proof of Proposition 3.24

For fixed $\alpha \in H^0(B, {}_N\mathcal{E}_0)$, the statement of the proposition becomes an equality involving three morphisms $H^1(B, {}_N\operatorname{Pic}^0_{S_0/B}) \to H^2(B, \mu_N)$, each of which is given by cup product with a fixed extension class of $\mathbb{Z}_B$-modules from $\operatorname{Ext}^1({}_N\operatorname{Pic}^0_{S_0/B}, \mu_{N,B})$. Let

$$0 \longrightarrow \mu_{N,B} \longrightarrow \mathcal{G}_0(\alpha)_N \longrightarrow {}_N\operatorname{Pic}^0_{S_0/B} \longrightarrow 0 \tag{3.26}$$

be the extension of sheaves of abelian groups deduced from (3.9) by taking pullback by the inclusion map ${}_N\operatorname{Pic}^0_{S_0/B} \hookrightarrow \operatorname{Pic}^0_{S_0/B}$. First, the composition $\langle \cdot, \alpha \rangle_{T^0}\nu$ is given by cup product with the class of (3.26). Second, the morphism $\langle \cdot, \nu\alpha \rangle_{T^1}$ is given by cup product with the class of the extension (3.14) for $\nu\alpha \in H^0(B, \mathcal{E}_0)$:

$$0 \longrightarrow \mu_{N,B} \longrightarrow {}_N\mathcal{G}(\nu\alpha) \longrightarrow {}_N\operatorname{Pic}^0_{S_0/B} \longrightarrow 0. \tag{3.27}$$

Finally, the composition $\partial \bar{e}_N(\cdot, \alpha)$ is given by cup product with the class of the pullback of the extension $0 \to \mu_{N,B} \to \mu_{N^2,B} \xrightarrow{N} \mu_{N,B} \to 0$ by the morphism $\bar{e}_N(\cdot, \alpha) : {}_N\operatorname{Pic}^0_{S_0/B} \to \mu_{N,B}$. In the language of Yoneda classes (see, e.g., [9, Section VII.1]), the proposition will be proved if we show that this last extension is equivalent to the sum of (3.27) with the opposite of (3.26). This is tantamount to the existence of a commutative diagram of sheaves

$$
\begin{tikzcd}
0 \arrow[r] & \mu_{N,B}\times\mu_{N,B} \arrow[r] \arrow[d, "{(1,1)}"'] & {}_N\mathcal{G}(\nu\alpha)\times_{{}_N\mathrm{Pic}^0_{S_0/B}}\mathcal{G}_0(\alpha)_N \arrow[r] \arrow[d] & {}_N\mathrm{Pic}^0_{S_0/B} \arrow[r] \arrow[d, "{\bar{e}_N(-,\alpha)}"] & 0 \\
0 \arrow[r] & \mu_{N,B} \arrow[r] & \mu_{N^2,B} \arrow[r, "N"] & \mu_{N,B} \arrow[r] & 0
\end{tikzcd}
$$

in which the bottom sequence is the standard one just mentioned, the right-hand side arrow in the upper sequence is the structure map for the fiber product, and the left-hand side arrow in that sequence is the product of the corresponding arrow in (3.27) with the (multiplicative) inverse of the corresponding arrow in (3.26). It remains to define the middle vertical arrow. The sheaves ${}_N\mathcal{G}(\nu\alpha)$ and $\mathcal{G}_0(\alpha)_N$ are subsheaves of $\mathcal{G}(\nu\alpha)$, and the map we consider is the restriction of the difference map $(\xi,\zeta)\mapsto\xi\zeta^{-1}$ of $\mathcal{G}(\nu\alpha)$. Clearly the image of this map lies in $\mathbb{G}_{m,B}$ (see (3.8)), and since it is of N^2-torsion, it actually lies in $\mu_{N^2,B}$. The commutativity of the left-hand side square is obvious. As for the right-hand side square, a section of ${}_N\mathcal{G}(\nu\alpha)\times_{{}_N\mathrm{Pic}^0_{S_0/B}}\mathcal{G}_0(\alpha)_N$ over an open $U\to B$ is given by a couple $((\tilde{\alpha}',[L]),(\tilde{\alpha}'',[L]))$ with $[L]\in\mathrm{Pic}^0((S_0)_U/U)$, $L^{\otimes N}\simeq\mathbb{1}_{(S_0)_U}$, and $(\tilde{\alpha}',\tau_\alpha),(\tilde{\alpha}'',\tau_\alpha)\in\mathrm{Aut}(L,(S_0)_U)$ such that $\tilde{\alpha}'^{\otimes N}=1$ and $\tilde{\alpha}''^{\circ N}=1$. One has $\tilde{\alpha}'=\rho\tilde{\alpha}''$ with $\rho\in\Gamma(U,\mathbb{G}_m)$. The image in $\Gamma(U,\mu_{N^2})\subset\Gamma(U,\mathcal{G}(\nu\alpha))$ of the chosen section is $\tilde{\alpha}'\otimes\tilde{\alpha}''^{\otimes(-1)}=\rho$ (see Part (iii) and Note (a) of the present section). Now, $1=\tilde{\alpha}'^{\otimes N}=\rho^N\tilde{\alpha}''^{\otimes N}$ implies $\rho^N=(\tilde{\alpha}''^{\otimes N})^{-1}=\bar{e}_N([L],\alpha)$ (see (3.4)), and this ends the proof. □

4. Pairings, III: 0-cycles

(i) *Abelian schemes.* The (opposite) Tate pairing (2.11) and its refinement (2.17) can be extended to deal with relative schemes $\pi:\mathcal{X}\to B$ such that $R^0\pi_*\mathbb{G}_{m,\mathcal{X}}=\mathbb{G}_{m,B}$ and relative 0-cycles $\mathfrak{z}$ of degree 0 on them. For simplicity, here in Part (i) we shall assume that the base scheme B is integral, regular, and of dimension at most 1. The relative 0-cycles on $\mathcal{X}$ are the elements of the free abelian group $Z_0(\mathcal{X}/B)$ generated by the irreducible closed subsets of $\mathcal{X}$ mapping finitely (in particular, properly) onto B. Given $\mathfrak{z}\in Z_0(\mathcal{X}/B)$, $\mathfrak{z}=\sum_{i=1}^r m_iZ_i$ we put $\deg(\mathfrak{z})=\sum_{i=1}^r m_i[Z_i:B]$, the (relative) degree of $\mathfrak{z}$, and we write $Z_0(\mathcal{X}/B)_0\subset Z_0(\mathcal{X}/B)$ for the subgroup of the degree 0 relative 0-cycles of $\mathcal{X}$.

One has a natural morphism

$$Z_0(\mathcal{X}/B)_0\longrightarrow\mathrm{Ext}^1(\mathrm{Pic}_{\mathcal{X}/B},\mathbb{G}_{m,B}) \tag{4.1}$$

into a group of 1-extension classes of sheaves of abelian groups, which gives, via cup product, a pairing

$$\langle\cdot,\cdot\rangle_T^\sharp: H^1(B,\mathrm{Pic}_{\mathcal{X}/B})\times Z_0(\mathcal{X}/B)_0\longrightarrow H^2(B,\mathbb{G}_m). \tag{4.2}$$

By further composing (4.1) with the map

$$\mathrm{Ext}^1(\mathrm{Pic}_{\mathcal{X}/B},\mathbb{G}_{m,B})\to\mathrm{Ext}^1({}_N\mathrm{Pic}_{\mathcal{X}/B},\mu_{N,B}),$$

obtained by taking N-torsion parts everywhere, one gets in a similar way a pairing

$$(4.3)\qquad \langle\cdot,\cdot\rangle^{\sharp}_{T^1} : H^1(B, {}_N\operatorname{Pic}_{\mathcal{X}/B}) \times Z_0(\mathcal{X}/B)_0 \longrightarrow H^2(B, \mu_N).$$

The pairings (4.2) and (4.3) are related by a formula analogous to (3.16), and for abelian schemes $\mathcal{A}$, they are compatible, respectively, with (2.11) and (2.17) for $\hat{\mathcal{A}}$, through the Albanese map. We give the details.

We recall the definition of the map (4.1). Given $\mathfrak{z} \in Z_0(\mathcal{X}/B)_0$, let $\mathcal{Z} = \bigcup_{i=1}^r Z_i$ be the support of $\mathfrak{z}$, considered as a reduced closed subscheme of $\mathcal{X}$. The exact sequence of higher direct images with respect to the structure map $\pi : \mathcal{X} \to B$ for the short exact sequence on $\mathcal{X}$

$$0 \longrightarrow \mathbb{G}_{m,\mathcal{X},\mathcal{Z}} \longrightarrow \mathbb{G}_{m,\mathcal{X}} \longrightarrow \mathbb{G}_{m,\mathcal{Z}} \longrightarrow 0$$

provides an extension of sheaves of abelian groups on B:

$$(4.4)\qquad 0 \longrightarrow R^0\pi_*\mathbb{G}_{m,\mathcal{Z}}/R^0\pi_*\mathbb{G}_{m,\mathcal{X}} \longrightarrow \operatorname{Pic}_{\mathcal{X},\mathcal{Z}/B} \longrightarrow \operatorname{Pic}_{\mathcal{X}/B} \longrightarrow 0.$$

The sheaf $\operatorname{Pic}_{\mathcal{X},\mathcal{Z}/B}$ is the sheaf associated to the presheaf that attaches to an open $U \to B$ in the étale topology the group $\operatorname{Pic}(\mathcal{X}_U, \mathcal{Z}_U)$ of isomorphism classes of pairs (L, θ) with L a line bundle on $\mathcal{X}_U$ and θ an isomorphism of line bundles $\mathbb{1}_{\mathcal{Z}_U} \xrightarrow{\sim} L|\mathcal{Z}_U$. We define a morphism

$$(4.5)\qquad z : R^0\pi_*\mathbb{G}_{m,\mathcal{Z}}/R^0\pi_*\mathbb{G}_{m,\mathcal{X}} \longrightarrow \mathbb{G}_{m,B}$$

by factoring the map $\tilde{z} = \prod_{i=1}^r (Nm_{Z_i/B} \circ \rho_{\mathcal{Z},Z_i})^{m_i} : R^0\pi_*\mathbb{G}_{m,\mathcal{Z}} \to \mathbb{G}_{m,B}$, where $\rho_{\mathcal{Z},Z_i}$ stands here for the restriction map $R^0\pi_*\mathbb{G}_{m,\mathcal{Z}} \to R^0\pi_*\mathbb{G}_{m,Z_i}$ and $Nm_{Z_i/B} : R^0\pi_*\mathbb{G}_{m,Z_i} \to \mathbb{G}_{m,B}$ is the norm map for the finite, locally free morphism of schemes $Z_i \to B$. To see that this map indeed factors, let f be a section of $R^0\pi_*\mathbb{G}_{m,\mathcal{X}} = \mathbb{G}_{m,B}$; one has $\prod_{i=1}^r (Nm_{Z_i/B} \circ \rho_{\mathcal{Z},Z_i})^{m_i}(f) = \prod_{i=1}^r Nm_{Z_i/B}(f)^{m_i} = \prod_{i=1}^r (f^{[Z_i:B]})^{m_i} = f^{\deg(\mathfrak{z})} = 1$. Taking now the pushout of the extension (4.4) by the morphism (4.5) we obtain an extension of sheaves of abelian groups on B

$$(4.6)\qquad 0 \longrightarrow \mathbb{G}_{m,B} \longrightarrow \mathcal{G}_{\mathfrak{z}} \longrightarrow \operatorname{Pic}_{\mathcal{X}/B} \longrightarrow 0,$$

and this represents the image of $\mathfrak{z}$ by the map (4.1). We refer to [16] for further details on this. The image of $\mathfrak{z}$ in $\operatorname{Ext}^1({}_N\operatorname{Pic}_{\mathcal{X}/B}, \mu_{N,B})$ is represented by the extension

$$(4.7)\qquad 0 \longrightarrow \mu_{N,B} \longrightarrow {}_N\mathcal{G}_{\mathfrak{z}} \longrightarrow {}_N\operatorname{Pic}_{\mathcal{X}/B} \longrightarrow 0$$

obtained by taking the N-torsion parts of the members of (4.6). Alternatively, one may apply a similar procedure as above directly to the sequences (5.4) and (5.3) from Section 5.

The claimed relationship between $\langle\cdot,\cdot\rangle^{\sharp}_{T}$ and $\langle\cdot,\cdot\rangle^{\sharp}_{T^1}$ is obvious in view of the definitions. We turn to the compatibility of these pairings with $\langle\cdot,\cdot\rangle_T$ and $\langle\cdot,\cdot\rangle_{T^1}$, respectively, for abelian schemes. Actually, we will consider this in a more general setting (see Remark 4.11 below). Suppose that $\pi : \mathcal{X} \to B$ is smooth and projective, with irreducible geometric fibers, and suppose that the sheaf $\operatorname{Pic}^0_{\mathcal{X}/B}$, now considered in the big étale site of B (see [5, Section 2]), is represented by an abelian scheme, a fact which happens if and only if the Picard varieties of the geometric fibers $\mathcal{X}(\bar{t})$ of $\pi:\mathcal{X}\to B$ have constant dimension equal to $h^1\mathcal{O}_{\mathcal{X}(\bar{t})}$ (see [5, Section 5]). Writing then $\mathrm{Alb}^0_{\mathcal{X}/B}=(\mathrm{Pic}^0_{\mathcal{X}/B})\check{}$ for the dual abelian scheme of the abelian scheme $\mathrm{Pic}^0_{\mathcal{X}/B}$, one has a natural map

$$\mathrm{alb}: Z_0(\mathcal{X}/B)_0 \longrightarrow \Gamma(B,\mathrm{Alb}^0_{\mathcal{X}/B}), \tag{4.8}$$

defined as follows. Assuming first that π has a global section, we may find a (Poincaré) line bundle M on $\mathcal{X}\times_B \mathrm{Pic}^0_{\mathcal{X}/B}$ such that, for any geometric point $\bar{\lambda}$ of $\mathrm{Pic}^0_{\mathcal{X}/B}$, over a geometric point $\bar{t}$ of B, the line bundle $M|(\mathcal{X}(\bar{t})\times\bar{\lambda})$ represents the class $\bar{\lambda}\in\mathrm{Pic}^0(\mathcal{X}(\bar{t}))$. For $\mathfrak{z}\in Z_0(\mathcal{X}/B)_0$, $\mathfrak{z}=\sum_{i=1}^r m_iZ_i$, the line bundle

$$\bigotimes_{i=1}^r Nm_{Z_i\times_B\mathrm{Pic}^0_{\mathcal{X}/B}/B\times_B\mathrm{Pic}^0_{\mathcal{X}/B}}\big(M\mid(Z_i\times_B\mathrm{Pic}^0_{\mathcal{X}/B})\big)^{\otimes m_i}$$

on $\mathrm{Pic}^0_{\mathcal{X}/B}$ defines a global section of $\mathrm{Alb}^0_{\mathcal{X}/B}$, and this is the image $\mathrm{alb}(\mathfrak{z})$ of $\mathfrak{z}$ in this case. Note that the result is indeed independent from the choice of M, since two such choices will differ by a factor line bundle coming from $\mathrm{Pic}^0_{\mathcal{X}/B}$, and this will have no effect on the above formula.

In the general case, a section of π exists locally in the étale topology, and by the previous remark, the above construction, performed over the members $U_i\to B$, $i\in I$, of a suitable étale open covering of B matches above the intersections $U_{ij}=U_i\times_B U_j$, $i,j\in I$, and descends to give a global section $\mathrm{alb}(\mathfrak{z})$ of $\mathrm{Alb}^0_{\mathcal{X}/B}$.

PROPOSITION 4.9

(a) *For all $\lambda^0\in H^1(B,\mathrm{Pic}^0_{\mathcal{X}/B})$ and $\mathfrak{z}\in Z_0(\mathcal{X}/B)_0$, one has in $H^2(B,\mathbb{G}_m)$, calling $\lambda\in H^1(B,\mathrm{Pic}_{\mathcal{X}/B})$ the image of λ^0, $\langle\lambda,\mathfrak{z}\rangle^\sharp_T=\langle\lambda^0,\mathrm{alb}(\mathfrak{z})\rangle_{T,\mathrm{Pic}^0_{\mathcal{X}/B}}$.*

(b) *For all $\lambda^0\in H^1(B,{}_N\mathrm{Pic}^0_{\mathcal{X},B})$ and $\mathfrak{z}\in Z_0(\mathcal{X}/B)_0$, one has in $H^2(B,\mu_N)$, calling $\lambda\in H^1(B,{}_N\mathrm{Pic}_{\mathcal{X}/B})$ the image of λ^0, $\langle\lambda,\mathfrak{z}\rangle^\sharp_{T^1}=\langle\lambda^0,\mathrm{alb}(\mathfrak{z})\rangle_{T^1,\mathrm{Pic}^0_{\mathcal{X}/B}}$.*

Proof

It suffices to exhibit a commutative diagram of sheaves of abelian groups on B (a similar diagram of corresponding N-torsion parts then follows from this):

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{G}_{m,B} \arrow[r] \arrow[d, "1"'] & \mathcal{G}^0_{\mathfrak{z}} \arrow[r] \arrow[d] & \mathrm{Pic}^0_{\mathcal{X}/B} \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \mathbb{G}_{m,B} \arrow[r] & \mathcal{G}\big(\mathrm{alb}(\mathfrak{z})\big) \arrow[r] & \mathrm{Pic}^0_{\mathcal{X}/B} \arrow[r] & 0
\end{tikzcd}$$

the top sequence being the pullback of (4.6) by the morphism $\mathrm{Pic}^0_{\mathcal{X}/B}\to\mathrm{Pic}_{\mathcal{X}/B}$ and the bottom sequence being (2.12). The existence of such a diagram is equivalent to the existence of a commutative diagram of sheaves on B:

$$\begin{tikzcd} 0 \arrow[r] & R^0\pi_*\mathbb{G}_{m,\mathcal{Z}}/R^0\pi_*\mathbb{G}_{m,\mathcal{X}} \arrow[r] \arrow[d, "z"'] & \mathrm{Pic}^0_{\mathcal{X},\mathcal{Z}/B} \arrow[r] \arrow[d] & \mathrm{Pic}^0_{\mathcal{X}/B} \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & \mathbb{G}_{m,B} \arrow[r] & \mathcal{G}\big(\mathrm{alb}(\mathfrak{z})\big) \arrow[r] & \mathrm{Pic}^0_{\mathcal{X}/B} \arrow[r] & 0 \end{tikzcd} \tag{4.10}$$

the top sequence coming from (4.4) by the pullback by the morphism $\mathrm{Pic}^0_{\mathcal{X}/B} \to \mathrm{Pic}_{\mathcal{X}/B}$. The existence of the middle vertical map can be proved locally. Namely, if it exists, then it is unique since $\mathcal{H}\mathrm{om}_B(\mathrm{Pic}^0_{\mathcal{X}/B}, \mathbb{G}_{m,B}) = 0$, and therefore, local constructions glue together and descend to a global one. So we may assume, without loss of generality, that $\pi : \mathcal{X} \to B$ admits a global section. We then let M be a Poincaré bundle on $\mathcal{X} \times_B \mathrm{Pic}^0_{\mathcal{X}/B}$, as before.

Consider a section of $\mathrm{Pic}^0_{\mathcal{X},\mathcal{Z}/B}$ over some open $U \to B$, given by a couple (L,θ) with L a line bundle on $\mathcal{X}_U$ yielding a section $\lambda \in \Gamma(U, \mathrm{Pic}^0_{\mathcal{X}/B})$ and θ an isomorphism of line bundles $\mathbb{1}_{\mathcal{Z}_U} \xrightarrow{\sim} L \mid \mathcal{Z}_U$. By writing as before $\mathfrak{z} = \sum_{i=1}^r m_i Z_i$ for the chosen $\mathfrak{z} \in Z_0(\mathcal{X}/B)_0$, the liftings of λ to a section of $\mathcal{G}(\mathrm{alb}(\mathfrak{z}))$ are given by the relative automorphisms of the line bundle

$$M_0 = \bigotimes_{i=1}^r Nm_{(Z_i)_U \times_U \mathrm{Pic}^0_{\mathcal{X}_U/U} / U \times_U \mathrm{Pic}^0_{\mathcal{X}_U/U}} \big(M \mid \big((Z_i)_U \times_U \mathrm{Pic}^0_{\mathcal{X}_U/U}\big)\big)^{\otimes m_i}$$

on $\mathrm{Pic}^0_{\mathcal{X}_U/U}$ over the translation map $\tau_\lambda : \mathrm{Pic}^0_{\mathcal{X}_U/U} \to \mathrm{Pic}^0_{\mathcal{X}_U/U}$. Denoting by $\eta : \mathrm{Pic}^0_{\mathcal{X}_U/U} \to U$ the structure map, we have from Note (d) of Section 2 canonical isomorphisms $\tau_\lambda^*(M_0) \otimes M_0^{-1} \xrightarrow{\sim} \eta^* 0_U^*(\tau_\lambda^*(M_0) \otimes M_0^{-1}) \xrightarrow{\sim} \eta^*(\lambda^*(M_0) \otimes 0_U^*(M_0)^{-1})$, with 0_U denoting the zero section of $\mathrm{Pic}^0_{\mathcal{X}_U/U}$. So the set of liftings of λ to a section of $\mathcal{G}(\mathrm{alb}(\mathfrak{z}))$ can be identified with the set $\mathrm{Iso}_U(0_U^*(M_0), \lambda^*(M_0))$ of isomorphisms of line bundles on U. Since $M \mid (\mathcal{X} \times_B 0_{\mathrm{Pic}^0_{\mathcal{X}/B}}) = \pi^*(Q)$ for some line bundle Q on B, one has, canonically,

$$\begin{aligned} 0_U^*(M_0) &= \bigotimes_{i=1}^r Nm_{(Z_i)_U/U} \big(\pi_U^*(Q_U) \bigm| (Z_i)_U\big)^{\otimes m_i} \\ &= \bigotimes_{i=1}^r Q_U^{\otimes m_i [Z_i:B]} = Q_U^{\otimes \sum_{i=1}^r m_i [Z_i:B]} = \mathbb{1}_U. \end{aligned}$$

Similarly, since $(1 \times_U \lambda)^*(M) = L \otimes \pi_U^*(R)$ on $\mathcal{X}_U$, with R a line bundle on U, one has, also canonically,

$$\begin{aligned} \lambda^*(M_0) &= \bigotimes_{i=1}^r Nm_{(Z_i)_U/U} \big((L \otimes \pi_U^*(R)) \bigm| (Z_i)_U\big)^{\otimes m_i} \\ &= \bigotimes_{i=1}^r Nm_{(Z_i)_U/U} \big(L \bigm| (Z_i)_U\big)^{\otimes m_i}. \end{aligned}$$

Therefore, the set of liftings of the section λ of $\mathrm{Pic}^0_{\mathcal{X}/B}$ to a section of $\mathcal{G}(\mathrm{alb}(\mathfrak{z}))$ is finally identified with the set $\mathrm{Iso}_U(\mathbb{1}_U, \bigotimes_{i=1}^r Nm_{(Z_i)_U/U}(L \mid (Z_i)_U)^{\otimes m_i})$ of isomorphisms of line bundles on U. The middle vertical arrow in diagram (4.10) is defined then by sending the section of $\mathrm{Pic}^0_{\mathcal{X},\mathcal{Z}/B}$ given by (L,θ) to the isomorphism

$$\bigotimes_{i=1}^{r} Nm_{(Z_i)_U/U}\big(\theta \,\big|\, (Z_i)_U\big)^{\otimes m_i} : \mathbb{1}_U \xrightarrow{\sim} \bigotimes_{i=1}^{r} Nm_{(Z_i)_U/U}\big(L \,\big|\, (Z_i)_U\big)^{\otimes m_i}.$$

This makes the diagram commutative, thereby ending this proof. □

We note that, due to our hypotheses on the scheme B, the group $H^1(B, \mathrm{Pic}^0_{\mathcal{X}/B})$ is here a torsion group (see, e.g., [8, Section II.5, the proof of Proposition 5.1(a)], together with [1, Section 1.2, Proposition 8]), and hence, Proposition 4.9(a) is in fact a consequence of Proposition 4.9(b).

REMARK 4.11

The foregoing applies, in particular, to abelian schemes $\pi: \mathcal{A} \to B$, the scheme $\mathrm{Pic}^0_{\mathcal{A}/B}$ being then the dual abelian scheme $\hat{\mathcal{A}}$ of $\mathcal{A}$. The biduality isomorphism $\mathcal{A} \xrightarrow{\sim} \hat{\hat{\mathcal{A}}}$ yields here a natural isomorphism $\mathcal{A} \xrightarrow{\sim} \mathrm{Alb}^0_{\mathcal{A}/B}$ and, with this identification, the map $\mathrm{alb}: Z_0(\mathcal{A}/B)_0 \to \Gamma(B, \mathcal{A})$ from (4.8) is just abelian summation along the fibers of π.

(ii) *Néron models*. Back in the notation of Section 1, we treat here the analogue of Proposition 4.9 of Part (i) of the present section, with $\pi: \mathcal{X} \to B$ now replaced by $\pi: S_0 \to B$, as well as by its restrictions to arbitrary open subsets U of B. We shall adopt in this Part (ii) the same notational conventions as in Section 3, and assume that an open subset U of B has been fixed, now denoted B, and $\mathcal{E}_0$, S_0, .. standing here for the restrictions $(\mathcal{E}_0)_U$, $(S_0)_U$, $(..)_U$ of these objects above U.

We show that the pairings $\langle\cdot,\cdot\rangle^\sharp_T$ and $\langle\cdot,\cdot\rangle^\sharp_{T^1}$ are similarly related to the pairings $\langle\cdot,\cdot\rangle_T$ (see (3.7)) and $\langle\cdot,\cdot\rangle_{T^1}$ (see (3.15)), respectively, at least if one restricts oneself to the subgroup $Z_0(\mathcal{E}_0/B) \subset Z_0(S_0/B)$ of relative 0-cycles supported inside $\mathcal{E}_0 \subset S_0$, that is, formal linear combinations with integral coefficients of irreducible closed subsets of $\mathcal{E}_0$ mapping finitely (and hence properly) onto B. Here appears, instead of the Albanese map from (4.8), the abelian summation map (restricted to $Z_0(\mathcal{E}_0/B)_0$)

$$Z_0(\mathcal{E}_0/B) \longrightarrow \Gamma(B, \mathcal{E}_0), \quad \mathfrak{z} \mapsto \mathfrak{z}_{Ab}. \tag{4.12}$$

By using the Néron universal property, this map extends to the whole of $Z_0(S_0/B)$, yielding the composition of the natural map $Z_0(S_0/B) \to \Gamma(B, \mathrm{Pic}_{S_0/B})$ with the map induced on global sections by the abelian summation map $\mathrm{Pic}_{S_0/B} \to \mathcal{E}_0$ from Note (b) of Section 3. We have not looked into the theme of the following proposition in this more extended setting.

PROPOSITION 4.13

(a) *For all $\lambda^0 \in H^1(B, \mathrm{Pic}^0_{S_0/B})$ and $\mathfrak{z} \in Z_0(\mathcal{E}_0/B)_0$ one has in $H^2(B, \mathbb{G}_m)$, calling $\lambda \in H^1(B, \mathrm{Pic}_{S_0/B})$ the image of λ^0, $\langle\lambda, \mathfrak{z}\rangle^\sharp_T = \langle\lambda^0, \mathfrak{z}_{Ab}\rangle_T$.*

(b) *For all* $\lambda^0 \in H^1(B, {}_N\operatorname{Pic}^0_{S_0/B})$ *and* $\mathfrak{z} \in Z_0(\mathcal{E}_0/B)_0$, *one has in* $H^2(B, \mu_N)$, *calling* $\lambda \in H^1(B, {}_N\operatorname{Pic}_{S_0/B})$ *the (isomorphic) image of* λ^0, $\langle \lambda, \mathfrak{z}\rangle^\sharp_{T^1} = \langle \lambda^0, \mathfrak{z}_{Ab}\rangle_{T^1}$.

Proof

We freely use notation from Part (i) of the present section; in particular, we write $\mathfrak{z} = \sum_{i=1}^r m_i Z_i$, with $\sum_{i=1}^r m_i [Z_i : B] = 0$, and $\mathcal{Z}$ stands for $\bigcup_{i=1}^r Z_i \subset \mathcal{E}_0$ with its reduced scheme structure. We put, moreover, $\alpha = \mathfrak{z}_{Ab} \in \Gamma(B, \mathcal{E}_0)$. By the definitions of $\langle \cdot,\cdot\rangle^\sharp_T$ and $\langle \cdot,\cdot\rangle^\sharp_{T^1}$ and the definitions of $\langle \cdot,\cdot\rangle_T$ and $\langle \cdot,\cdot\rangle_{T^1}$ given in (3.7) and (3.15), respectively, the result will follow if we show the existence of the middle vertical arrow making the following diagram of sheaves commutative:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{G}_{m,B} \arrow[r] \arrow[d, "1"'] & \mathcal{G}^0_{\mathfrak{z}} \arrow[r] \arrow[d] & \operatorname{Pic}^0_{S_0/B} \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \mathbb{G}_{m,B} \arrow[r] & \mathcal{G}(\alpha) \arrow[r] & \operatorname{Pic}^0_{S_0/B} \arrow[r] & 0
\end{tikzcd}$$

This is equivalent to the existence of the middle vertical arrow making the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & R^0\pi_*\mathbb{G}_{m,\mathcal{Z}}/R^0\pi_*\mathbb{G}_{m,S_0} \arrow[r] \arrow[d, "z"'] & \operatorname{Pic}^0_{S_0,\mathcal{Z}/B} \arrow[r] \arrow[d] & \operatorname{Pic}^0_{S_0/B} \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \mathbb{G}_{m,B} \arrow[r] & \mathcal{G}(\alpha) \arrow[r] & \operatorname{Pic}^0_{S_0/B} \arrow[r] & 0
\end{tikzcd} \tag{4.14}$$

To define this arrow, consider a section of $\operatorname{Pic}^0_{S_0,\mathcal{Z}/B}$ over some open $U \to B$, given by a line bundle L on $(S_0)_U$ defining a section λ of $\operatorname{Pic}^0_{S_0/B}$ on U, together with an isomorphism of line bundles $\theta : \mathbb{1}_{\mathcal{Z}_U} \xrightarrow{\sim} L \mid \mathcal{Z}_U$. The sections of $\mathcal{G}(\alpha)$ on U which lift $\lambda \in \Gamma(U, \operatorname{Pic}^0_{S_0/B})$ are the relative automorphisms $(\tilde{\alpha}, \tau_\alpha) \in \operatorname{Aut}(L, (S_0)_U)$ of L over the translation map of $(S_0)_U$ by (the restriction of) α. Note that the line bundle L may have nontrivial restriction on the zero section of $(S_0)_U$, but that this does not affect the description just given (see Part (iii) of Section 3). By Notes (d) and (e) of Section 3, we have canonically, as in the proof of Proposition 4.9 above, $\tau^*_\alpha(L) \otimes L^{-1} \simeq \pi^* 0^*_U(\tau^*_\alpha(L) \otimes L^{-1}) \simeq \pi^*(\alpha^*_U(L) \otimes 0^*_U(L)^{-1})$, with α_U and 0_U denoting the restrictions of α and of the zero section of S_0 to U, respectively. Hence, to give such an $\tilde{\alpha}$ is tantamount to giving a line bundle isomorphism on U, $0^*_U(L) \xrightarrow{\sim} \alpha^*_U(L)$. So, the set of liftings of $\lambda \in \Gamma(U, \operatorname{Pic}^0_{S_0/B})$ to a section of $\mathcal{G}(\alpha)$ is identified canonically with the set $\operatorname{Iso}_U(0^*_U(L), \alpha^*_U(L))$ of isomorphisms of line bundles on U. To continue, we are led to open the black box from Remark 4.11.

□

LEMMA 4.15

We keep the above notation. One has a canonical isomorphism of line bundles *on B*

$$\bigotimes_{i=1}^{r} Nm_{(Z_i)_U/U}\big(L \,\big|\, (Z_i)_U\big)^{\otimes m_i} \xrightarrow{\sim} \alpha_U^*(L)\otimes 0_U^*(L)^{-1}.$$

Proof of Lemma 4.15
The claim is equivalent to the existence of a canonical isomorphism of line bundles on U:

$$\bigotimes_{i=1}^{r} Nm_{(Z_i)_U/U}\big(L\otimes 0_U^*(L)^{-1} \,\big|\, (Z_i)_U\big)^{\otimes m_i} \xrightarrow{\sim} \alpha_U^*(L)\otimes 0_U^*(L)^{-1}. \tag{4.16}$$

Put in this way, the statement actually holds without the restriction $\sum_{i=1}^r m_i[Z_i : B] = 0$ on the relative degree of $\mathfrak{z}$, and consequently, we drop it.

(1) We review first the analogous statement for an abelian variety $\bar{A}$ over an algebraically closed field. Here $L\in \mathrm{Pic}^0(\bar{A})$, $\mathfrak{z} = \sum_{i=1}^r m_i\bar{a}_i$, $\bar{a}_i\in\bar{A}$, and (4.16) reads

$$\bigotimes_{i=1}^{r}\big(L(\bar{a}_i)\otimes L(\bar{0})^{-1}\big)^{\otimes m_i} \xrightarrow{\sim} L(\bar{a})\otimes L(\bar{0})^{-1} \tag{4.17}$$

canonically, with $\bar{a}\in\bar{A}$ the abelian sum of the 0-cycle $\mathfrak{z}$; thus, $\bar{a} = \sum_{i=1}^r m_i\bar{a}_i$, here considered not formally, but performed inside $\bar{A}$. The line bundle L is invariant by translations, and for all $\bar{b}\in\bar{A}$ one has $L(\bar{b})\otimes L(\bar{0})^{-1} = \mathrm{Hom}(L,\tau_{\bar{b}}^*(L)) = \mathrm{Hom}_{\tau_{\bar{b}}}(L,L)$, the latter term denoting the vector space of relative endomorphisms $(\tilde{\bar{b}},\tau_{\bar{b}}) : (L,\bar{A}) \to (L,\bar{A})$ over the translation map by $\bar{b}$. For all $\bar{b}_1,\bar{b}_2\in\bar{A}$ we have a canonical isomorphism of 1-dimensional vector spaces

$$\mathrm{Hom}_{\tau_{\bar{b}_1}}(L,L)\otimes \mathrm{Hom}_{\tau_{\bar{b}_2}}(L,L) \xrightarrow{\sim} \mathrm{Hom}_{\tau_{\bar{b}_1+\bar{b}_2}}(L,L) \tag{4.18}$$

given by the composition of relative endomorphisms. This map is associative and commutative in $\bar{b}_1,\bar{b}_2$, the latter fact being explained as in Part (iii) of Section 2 on the commutativity of the group scheme $\mathcal{G}(\lambda)$. From this, the canonical isomorphism (4.17) follows. The (yet to be defined) isomorphism (4.16) will have this description on geometric fibers at points of good reduction, underpinning therewith its canonical nature.

(2) We extend the foregoing to the case of an abelian scheme $\pi : \mathcal{A}\to B$ over a scheme B. Let L be a line bundle on $\mathcal{A}$, yielding a section belonging to $\Gamma(B,\hat{A})\subset \Gamma(B,\mathrm{Pic}_{\mathcal{A}/B})$, and suppose that $\mathfrak{z} = \sum_{i=1}^r m_i\Gamma_{\alpha_i}$ is given, with $\Gamma_{\alpha_i}\subset B\times_B\mathcal{A} = \mathcal{A}$ the graph of a section $\alpha_i\in\Gamma(B,\mathcal{A})$, $i=1,\ldots,r$. Call $\alpha = \sum_{i=1}^r m_i\alpha_i\in\Gamma(B,\mathcal{A})$. We have an (a fortiori canonical) isomorphism

$$\bigotimes_{i=1}^{r}\big(\alpha_i^*(L)\otimes 0_B^*(L)^{-1}\big)^{\otimes m_i} \xrightarrow{\sim} \alpha^*(L)\otimes 0_B^*(L)^{-1} \tag{4.19}$$

inducing (4.17) on the geometric fibers of π. For all $\beta\in\Gamma(B,\mathcal{A})$ the translation map $\tau_\beta : \mathcal{A}\to\mathcal{A}$ acts as the identity on $\hat{\mathcal{A}}$, and hence, as before, one has canonical isomorphisms $\tau_\beta^*(L)\otimes L^{-1} \xrightarrow{\sim} \pi^*(\beta^*(L)\otimes 0_B^*(L)^{-1})$. From this the following canonical isomorphisms ensue (here we alternate somewhat crudely between the notions of line bundle and invertible sheaf; see Note (c) of Section 1):

$$\begin{aligned}\beta^*(L)\otimes 0_B^*(L)^{-1} &\xrightarrow{\sim} R^0\pi_*\pi^*\big(\beta^*(L)\otimes 0_B^*(L)^{-1}\big)\\ &\xrightarrow{\sim} R^0\pi_*\big(\tau_\beta^*(L)\otimes L^{-1}\big)\\ &\xrightarrow{\sim} R^0\pi_*\big(\underline{\mathrm{Hom}}_{\mathcal{A}}\big(L,\tau_\beta^*(L)\big)\big)\\ &\xrightarrow{\sim} R^0\pi_*\big(\underline{\mathrm{Hom}}_{\tau_\beta}(L,L)\big),\end{aligned}$$

with $\underline{\mathrm{Hom}}_{\tau_\beta}(L,L)$ denoting here the sheaf of relative endomorphisms of L over the translation map $\tau_\beta:\mathcal{A}\to\mathcal{A}$. Given sections $\beta_1,\beta_2\in\Gamma(B,\mathcal{A})$, the composition of relative endomorphisms gives an isomorphism of line bundles on B

$$R^0\pi_*\big(\underline{\mathrm{Hom}}_{\tau_{\beta_1}}(L,L)\big)\otimes R^0\pi_*\big(\underline{\mathrm{Hom}}_{\tau_{\beta_2}}(L,L)\big)\xrightarrow{\sim} R^0\pi_*\big(\underline{\mathrm{Hom}}_{\tau_{\beta_1+\beta_2}}(L,L)\big)$$

restricting to (4.18) on geometric fibers. From this and from the preceding isomorphisms we derive the isomorphism (4.19), given by (4.17) on geometric fibers, as desired.

(3) Finally, we turn to the isomorphism (4.16). There will be no loss of generality in assuming U to be a (Zariski) open subset of B and it will cause no harm either to restrict ourselves to the case $U=B$ in order to simplify notation in what follows. Thus, L is now supposed to be a line bundle on S_0 yielding a section belonging to $\Gamma(B,\mathrm{Pic}^0_{S_0/B})\subset\Gamma(B,\mathrm{Pic}_{S_0/B})$, and the claimed isomorphism reads

$$\bigotimes_{i=1}^{r} Nm_{Z_i/B}\big(L\otimes 0_B^*(L)^{-1}\,\big|\,Z_i\big)^{\otimes m_i}\xrightarrow{\sim}\alpha^*(L)\otimes 0_B^*(L)^{-1}. \tag{4.20}$$

Suppose first that $Z_i=\Gamma_{\alpha_i}$, $\alpha_i\in\Gamma(B,S_0)$, $i=1,\dots,r$, as in (2). Then (4.20) is rewritten as

$$\bigotimes_{i=1}^{r}\big(\alpha_i^*(L)\otimes 0_B^*(L)^{-1}\big)^{\otimes m_i}\xrightarrow{\sim}\alpha^*(L)\otimes 0_B^*(L)^{-1} \tag{4.21}$$

with $\alpha=\sum_{i=1}^r m_i\alpha_i$. The existence of this isomorphism, restricting to (4.17) on the geometric fibers at points of good reduction, is established as in (2), bearing in mind (see Note (d) of Section 3) that for all $\beta\in\Gamma(B,S_0)=\Gamma(B,\mathcal{E}_0)$ the translation map $\tau_\beta:S_0\to S_0$ acts as the identity on $\mathrm{Pic}^0_{S_0/B}$.

In the general case, $\mathfrak{z}=\sum_{i=1}^r m_iZ_i$ with components Z_i of arbitrary degree over B, it will suffice now, by (4.21), to consider the case $\mathfrak{z}=Z$ of an irreducible cycle. Let $f:B'\to B$ be a finite morphism with B' a smooth projective curve such that the Cartier divisor $Z'=Z\times_B B'$ of $S_0'=S_0\times_B B'$ splits as $Z'=\sum_{i=1}^{r'} m_i'\Gamma_{\alpha_i'}$, with $\alpha_i'\in\Gamma(B',S_0')$, $i=1,\dots,r'$. Similarly, write $\mathcal{E}_0'=\mathcal{E}_0\times_B B'$ so that $Z'\subset\mathcal{E}_0'\subset S_0'$ and hence $\alpha_i'\in\Gamma(B',\mathcal{E}_0')$, $i=1,\dots,r'$. By [3, Proposition 6.5.8, p. 130], $f^*Nm_{Z/B}=Nm_{Z'/B'}g^*$, where $g:Z'\to Z$ is the map induced by f. Writing L' for the pullback of L to S_0' one has thus, canonically,

$$
\begin{aligned}
f^*Nm_{Z/B}\big(L\otimes 0_B^*(L)^{-1}\bigm| Z\big) &\xrightarrow{\sim} Nm_{Z'/B'}\big(L'\otimes 0_{B'}^*(L')^{-1}\bigm| Z'\big)\\
&\xrightarrow{\sim} \bigotimes_{i=1}^{r'} Nm_{\Gamma_{\alpha_i'}/B'}\big(L'\otimes 0_{B'}^*(L')^{-1}\bigm| \Gamma_{\alpha_i'}\big)^{\otimes m_i'}\\
&\xrightarrow{\sim} \bigotimes_{i=1}^{r'} \big(\alpha_i'^*(L')\otimes 0_{B'}^*(L')^{-1}\big)^{\otimes m_i'}.
\end{aligned}
$$

Call $\alpha' = \sum_{i=1}^{r'} m_i'\alpha_i' \in \Gamma(B',\mathcal{E}_0')$. We claim that the following formula, similar to (4.21), holds on B':

$$
\bigotimes_{i=1}^{r'}\big(\alpha'^*_i(L')\otimes 0_{B'}^*(L')^{-1}\big)^{\otimes m_i'} \xrightarrow{\sim} \alpha'^*(L')\otimes 0_{B'}^*(L')^{-1}, \tag{4.22}
$$

by an isomorphism inducing (4.17) on geometric fibers at points of good reduction. This is proved in the same way as in (2) and here above, upon settling the claim below. The action of $\mathcal{E}_0$ on S_0 by translations $\mathcal{E}_0\times_B S_0\to S_0$ pulls back to a similar action of $\mathcal{E}_0'$ on S_0', $\mathcal{E}_0'\times_{B'} S_0'\to S_0'$. Given a section $\beta'\in\Gamma(B',\mathcal{E}_0')$, we have therefore a translation isomorphism $\tau_{\beta'}: S_0'\to S_0'$. We claim that the line bundle $\tau_{\beta'}^*(L')\otimes(L')^{-1}$ is the pullback of some line bundle on B' by the structure map $\pi': S_0'\to B'$. Since the geometric fibers of π' are isomorphic to the geometric fibers of π and since the vector space of global regular functions of the latter ones is 1-dimensional, we may apply the seesaw principle (see [10, Sections II.5 and III.10]), and the statement reduces to showing that, for all geometric points $\bar t'$ of B', $\tau^*_{\beta(\bar t')}(L'(\bar t'))\simeq L'(\bar t')$ holds. If $\bar t = f(\bar t')$ denotes the geometric point of B obtained as the image of $\bar t'$ by the morphism f, then one has $\beta(\bar t')\in\mathcal{E}_0(\bar t)$, so $\beta(\bar t')$ is the value at a certain geometric point $\bar x_0\in U$ above $\bar t$ of a section $\beta_0\in\Gamma(U,\mathcal{E}_0)$ for a suitable open $U\to B$ in the étale topology. Since translation by β_0 leaves the line bundle L_U invariant up to tensor product with a line bundle coming from U, it follows that $\tau^*_{\beta_0(\bar x_0)}(L_U(\bar x_0))\simeq L_U(\bar x_0)$; hence, $\tau^*_{\beta(\bar t')}(L'(\bar t'))\simeq L'(\bar t')$.

Now having that $\alpha' = f^*(\alpha)$ and $0_{B'} = f^*(0_B)$ and consequently that $\alpha'^*(L') = f^*\alpha^*(L)$ and $0_{B'}^*(L') = f^*0_B^*(L)$, we obtain finally, from (4.22) and the chain of isomorphisms preceding it, an isomorphism of line bundles on B'

$$
f^*Nm_{Z/B}\big(L\otimes 0_B^*(L)^{-1}\bigm| Z\big)\xrightarrow{\sim} f^*\big(\alpha^*(L)\otimes 0_B^*(L)\big)
$$

fulfilling the descent conditions for the map f, namely, the equality of its two restrictions to $B'\times_B B'$, because this holds at geometric points of B of good reduction; hence, it holds everywhere. By descent we thus obtain the isomorphism $Nm_{Z/B}(L\otimes 0_B^*(L)^{-1}\mid Z)\xrightarrow{\sim}\alpha^*(L)\otimes 0_B^*(L)$ with the required property. This finishes the proof of Lemma 4.15. □

By Lemma 4.15, the set of liftings of $\lambda\in\Gamma(U,\mathrm{Pic}^0_{S_0/B})$ to a section of $\mathcal{G}(\alpha)$ becomes finally identified, as in Proposition 4.9 above, with the set $\mathrm{Iso}_U(\mathbb{1}_U,\bigotimes_{i=1}^r Nm_{(Z_i)_U/U}(L\mid (Z_i)_U)^{\otimes m_i})$ of isomorphisms of line bundles on U.

The image of our chosen section of $\mathrm{Pic}^0_{S_0,\mathcal{Z}/B}$ over U by the middle vertical map in (4.14) is defined then to be the isomorphism $\bigotimes_{i=1}^r Nm_{(Z_i)_U/U}(\theta \mid (Z_i)_U)^{\otimes m_i}$ deduced from θ. This makes the diagram (4.14) commutative, and ends the proof of Proposition 4.13.

REMARK 4.23

The price we paid with the proof of Lemma 4.15 for an explicit description of the middle vertical arrow in diagram (4.14) can be avoided as follows. As was the case with Proposition 4.9, here in Proposition 4.13 and by a similar reason (see Part (v) of Section 3), Part (a) follows from Part (b). To prove Part (b), one needs only the existence of the middle vertical arrow yielding a commutative diagram of sheaves of abelian groups

$$\begin{tikzcd}
0 \arrow[r] & \mu_{N,B} \arrow[r] \arrow[d, "1"'] & {}_N\mathcal{G}^0_{\mathfrak{z}} \arrow[r] \arrow[d] & {}_N\,\mathrm{Pic}^0_{S_0/B} \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \mu_{N,B} \arrow[r] & {}_N\mathcal{G}(\alpha) \arrow[r] & {}_N\,\mathrm{Pic}^0_{S_0/B} \arrow[r] & 0
\end{tikzcd} \tag{4.24}$$

deduced from the diagram above (4.14) by taking N-torsion parts everywhere. Except for the middle terms so far, the sheaves in this diagram are represented by commutative group schemes. It follows, as in [11, Proposition 17.4]—with the flat topology now replaced by the étale topology—that these two terms are also representable, so that (4.24) is actually a diagram of commutative group schemes over B. Then, as in the proof of Proposition 3.13, we may apply a continuous extension argument to the restriction of (4.24) above the dense open subset B^- of B, for which a middle vertical arrow with the required properties exists, by the proof of Proposition 4.9 together with Remark 4.11.

5. A formula of Manin

In this section we consider, as in Part (i) of Section 4, relative schemes $\pi : \mathcal{X} \to B$ with B integral, regular, and of dimension at most 1 and such that $R^0\pi_*\mathbb{G}_{m,\mathcal{X}} = G_{m,B}$ holds. Let $F^1H^2(\mathcal{X},\mu_N) = \mathrm{Ker}(H^2(\mathcal{X},\mu_N) \to H^0(B, R^2\pi_*\mu_{N,\mathcal{X}}))$ be the first term of the filtration of $H^2(\mathcal{X},\mu_N)$ coming from the Leray spectral sequence for π and the sheaf $\mu_{N,\mathcal{X}}$. Given a relative 0-cycle of $\mathcal{X}$, $\mathfrak{z} \in Z_0(\mathcal{X}/B)$, $\mathfrak{z} = \sum_{i=1}^r m_i Z_i$, we have an evaluation map

$$ev_{\mathfrak{z}} : H^2(\mathcal{X},\mu_N) \longrightarrow H^2(B,\mu_N), \tag{5.1}$$

obtained as the composition

$$H^2(\mathcal{X},\mu_N) \xrightarrow{\rho_{\mathcal{X},\mathcal{Z}}} H^2(\mathcal{Z},\mu_N) \xleftarrow{\sim} H^2(B, R^0\pi_*\mu_{N,\mathcal{Z}}) \xrightarrow{\tilde{z}} H^2(B,\mu_N),$$

where, as in Section 4, $\mathcal{Z} = \bigcup_{i=1}^r Z_i$ with its structure as a reduced closed subscheme of $\mathcal{X}$, $\rho_{\mathcal{X},\mathcal{Z}}$ is the restriction map, the intermediate morphism is the inverse of the edge isomorphism from the degenerating Leray spectral sequence for π and $\mu_{N,\mathcal{Z}}$, and $\tilde{z}$ is induced by the map $\tilde{z} = \prod_{i=1}^{r} (Nm_{Z_i/B} \circ \rho_{\mathcal{Z},Z_i})^{m_i} : R^0\pi_*\mu_{N,\mathcal{Z}} \to \mu_{N,B}$.

On the other side, if we assume in addition that $\mathfrak{z}$ has degree 0, $\mathfrak{z} \in Z_0(\mathcal{X}/B)_0$, then we have from Part (i) of Section 4 the map

$$\langle \cdot, \mathfrak{z}\rangle^{\sharp}_{T^1} : H^1(B, {}_N\operatorname{Pic}_{\mathcal{X}/B}) \longrightarrow H^2(B, \mu_N) \tag{5.2}$$

given by the composition of the first connecting homomorphism in the cohomology sequence of the exact sequence

$$0 \longrightarrow R^0\pi_*\mu_{N,\mathcal{Z}}/R^0\pi_*\mu_{N,\mathcal{X}} \longrightarrow R^1\pi_*\mu_{N,\mathcal{X},\mathcal{Z}} \longrightarrow R^1\pi_*\mu_{N,\mathcal{X}} \longrightarrow 0 \tag{5.3}$$

deduced from the sequence

$$0 \to \mu_{N,\mathcal{X},\mathcal{Z}} \to \mu_{N,\mathcal{X}} \to \mu_{N,\mathcal{Z}} \to 0, \tag{5.4}$$

with the morphism $H^2(B, z)$, deduced from the map $z : R^0\pi_*\mu_{N,\mathcal{Z}}/R^0\pi_*\mu_{N,\mathcal{X}} \to \mu_{N,B}$ that factors our map $\tilde{z}$ above. (Note that $R^1\pi_*\mu_{N,\mathcal{X}} = {}_N\operatorname{Pic}_{\mathcal{X}/B}$.)

We shall need the following result, which goes back to [6, Proposition 8, p. 407].

PROPOSITION 5.5

As above, let $\pi : \mathcal{X} \to B$ be a relative scheme with B integral, regular, and of dimension at most 1 and such that $R^0\pi_*\mathbb{G}_{m,\mathcal{X}} = \mathbb{G}_{m,B}$ holds. Let $\mathfrak{z} \in Z_0(\mathcal{X}/B)_0$ be a relative 0-cycle of degree 0 of $\mathcal{X}$, and let $\xi \in F^1H^2(\mathcal{X}, \mu_N)$ have image $\lambda \in H^1(B, {}_N\operatorname{Pic}_{\mathcal{X}/B})$ by the canonical map $F^1H^2(\mathcal{X}, \mu_N) \to H^1(B, {}_N\operatorname{Pic}_{\mathcal{X}/B})$ coming from the Leray spectral sequence for π and $\mu_{N,\mathcal{X}}$. One then has the equality, in $H^2(B, \mu_N)$, $ev_{\mathfrak{z}}(\xi) = -\langle \lambda, \mathfrak{z}\rangle^{\sharp}_{T^1}$.

Proof

In view of the relationship between the mappings z and $\tilde{z}$, it will be sufficient to show that the following diagram, associated with $\mathfrak{z}$, is anticommutative:

$$\begin{tikzcd} F^1H^2(\mathcal{X}, \mu_N) \arrow[r, "\rho_{\mathcal{X},\mathcal{Z}}"] \arrow[d] & H^2(\mathcal{Z}, \mu_N) & H^2(B, R^0\pi_*\mu_{N,\mathcal{Z}}) \arrow[l, "\sim"'] \arrow[d] \\ H^1(B, R^1\pi_*\mu_{N,\mathcal{X}}) \arrow[rr, "\partial"] & & H^2(B, R^0\pi_*\mu_{N,\mathcal{Z}}/R^0\pi_*\mu_{N,\mathcal{X}}) \end{tikzcd} \tag{5.6}$$

where the left-hand side vertical arrow is the aforementioned map and the right-hand side vertical arrow is the obvious map.

Let $0 \to (\mathcal{I}'^{\cdot}, \phi') \to (\mathcal{I}^{\cdot}, \phi) \to (\mathcal{I}''^{\cdot}, \phi'') \to 0$ be an injective resolution of the sequence (5.4), with $\mathcal{I}^{\cdot} = \mathcal{I}'^{\cdot} \oplus \mathcal{I}''^{\cdot}$ and

$$\phi = \begin{pmatrix} \phi' & \varepsilon \\ 0 & \phi'' \end{pmatrix},$$

where $\varepsilon : \mathcal{I}''^{\cdot} \to \mathcal{I}'^{\cdot+1}$ is a morphism of degree 1. We have the relations $\phi'^2 = 0$, $\phi''^2 = 0$, and $\phi'\varepsilon + \varepsilon\phi'' = 0$. For a sheaf of abelian groups $\mathcal{F}$ on $\mathcal{X}$ and an injective resolution $(\mathcal{J}^{\cdot}, \psi)$ of it, there are canonical isomorphisms, for all $q \geq 0$, $H^q(\mathcal{X}, \mathcal{F}) \simeq H^q(\Gamma(\mathcal{X}, \mathcal{J}^{\cdot})) \simeq \mathbb{H}^q(B, \pi_* \mathcal{J}^{\cdot}) \simeq \varinjlim_{\mathcal{U}} H^q(C^{\cdot}(\mathcal{U}, \pi_* \mathcal{J}^{\cdot})_{\mathrm{tot}})$, where $\mathcal{U}$ runs through the étale open coverings of B. The Leray spectral sequence for π and $\mathcal{F}$ is given by the second spectral sequence of hypercohomology for the complex $\pi_* \mathcal{J}^{\cdot}$ on B. The differential in the total complex $C^{\cdot}(\mathcal{U}, \pi_* \mathcal{J}^{\cdot})_{\mathrm{tot}}$ is given by $dx = (-1)^p \psi x + \delta x$ for $x \in C^p(\mathcal{U}, \pi_* \mathcal{J}^q)$. As usual, we shall denote kernels (resp., images) of $\pi_* \mathcal{J}^{\cdot}$ by $\mathcal{Z}^{\cdot}$ (resp., $\mathcal{B}^{\cdot}$).

The map ε sends, in particular, $\mathcal{Z}''^0$ into $\mathcal{Z}'^1$, and the morphism so obtained $\mathcal{Z}''^0 \to \mathcal{Z}'^1/\mathcal{B}'^1$ is the connecting homomorphism $\partial : R^0 \pi_* \mu_{N,\mathcal{Z}} \to R^1 \pi_* \mu_{N,\mathcal{X},\mathcal{Z}}$ of the cohomology sequence of (5.4); hence, it describes the first morphism in (5.3). Indeed, let u be a section of $R^0 \pi_* \mu_{N,\mathcal{Z}} = \mathcal{Z}''^0$ over some open $U \to \mathcal{B}$. This lifts to the section $(0, u)$ of $\mathcal{I}^0$ over U. Since $\phi(0, u) = (\varepsilon u, 0)$, it follows that $\varepsilon u \in \Gamma(U, \mathcal{Z}'^1)$ represents $\partial u \in \Gamma(U, R^1 \pi_* \mu_{N,\mathcal{X},\mathcal{Z}})$.

Now let $\xi \in F^1 H^2(\mathcal{X}, \mu_N)$ be given, represented in $H^2(\mathcal{X}, \mu_N)$ by $a = (a', a'') \in \Gamma(\mathcal{X}, \mathcal{B}^2)$. For a suitable open covering $\mathcal{U}$ of B we may write, for its image $a \in C^0(\mathcal{U}, \pi_* \mathcal{I}^2)$, $a = \phi b$ with $b = (b', b'') \in C^0(\mathcal{U}, \pi_* \mathcal{I}^1)$. So $a' = \phi' b' + \varepsilon b''$ and $a'' = \phi'' b''$. Then, as $db = a + \delta b$ in $C^{\cdot}(\mathcal{U}, \pi_* \mathcal{I}^{\cdot})_{\mathrm{tot}}$, it follows that $a \in Z^0(\mathcal{U}, \mathcal{Z}^2)$ is cohomologous to $-\delta b = (-\delta b', -\delta b'') \in Z^1(\mathcal{U}, \mathcal{Z}^1)$, and therefore, the latter cocycle also represents ξ in $H^2(\mathcal{X}, \mu_N) = \mathbb{H}^2(B, \pi_* \mathcal{I}^{\cdot})$ and visualizes the image of ξ in $H^1(B, R^1 \pi_* \mu_{N,\mathcal{X}})$ as the element represented by $-\delta b \in Z^1(\mathcal{U}, \mathcal{Z}^1)$.

In order to find the image of this class by the bottom map ∂ in (5.6), we write the sequence (5.3) as

$$0 \longrightarrow \mathcal{Z}''^0/\mathcal{Z}^0 \xrightarrow{\varepsilon} \mathcal{Z}'^1/\mathcal{B}'^1 \longrightarrow \mathcal{Z}^1/\mathcal{B}^1 \longrightarrow 0$$

and seek—upon refining $\mathcal{U}$ suitably—a lift of $-\delta b \in Z^1(\mathcal{U}, \mathcal{Z}^1)$ to a cochain in $C^1(\mathcal{U}, \mathcal{Z}'^1)$, modulo $C^1(\mathcal{U}, \mathcal{B}^1)$. The vanishing of $R^1 \pi_* \mu_{N,\mathcal{Z}}$ gives $\mathcal{Z}''^1 = \mathcal{B}''^1$, and so $\delta b'' \in Z^1(\mathcal{U}, \mathcal{Z}''^1) = Z^1(\mathcal{U}, \mathcal{B}''^1)$ can be written, after refining $\mathcal{U}$, as $\delta b'' = \phi'' c''$ for some $c'' \in C^1(\mathcal{U}, \pi_* \mathcal{I}''^0)$. Hence, $-\delta b + \phi(0, c'') = (-\delta b', -\delta b'') + (\varepsilon c'', \phi'' c'') = (\varepsilon c'' - \delta b', 0)$, and therefore, $\varepsilon c'' - \delta b' \in C^1(\mathcal{U}, \mathcal{Z}'^1)$ does the job. Then, since $\delta(\varepsilon c'' - \delta b') = \varepsilon(\delta c'')$, it follows finally that $\delta c'' \in Z^2(\mathcal{U}, \mathcal{Z}''^0)$ represents the image of $\xi \in F^1 H^2(\mathcal{X}, \mu_N)$ in $H^2(B, R^0 \pi_* \mu_{N,\mathcal{Z}}/R^0 \pi_* \mu_{N,\mathcal{X}})$ by the composition of the left-hand side vertical map and the bottom map in (5.6).

On the other hand, the image of ξ by the map $\rho_{\mathcal{X},\mathcal{Z}}$ is the class of $H^2(\mathcal{Z}, \mu_N)$ represented by $a'' \in \Gamma(\mathcal{X}, \mathcal{B}''^2)$. We have, for its image $a'' \in Z^0(\mathcal{U}, \mathcal{Z}''^2)$ in $C^{\cdot}(\mathcal{U}, \pi_* \mathcal{I}''^{\cdot})_{\mathrm{tot}}$, $d(b'' + c'') = a'' + \delta c''$; hence, $\rho_{\mathcal{X},\mathcal{Z}}(\xi)$ is also represented by the cocycle $-\delta c'' \in Z^2(\mathcal{U}, \mathcal{Z}''^0)$, and this says that $\rho_{\mathcal{X},\mathcal{Z}}(\xi)$ is the image, by the edge map, of the element represented by $-\delta c''$ in $H^2(B, R^0 \pi_* \mu_{N,\mathcal{Z}})$. The image of the latter element in $H^2(B, R^0 \pi_* \mu_{N,\mathcal{Z}}/R^0 \pi_* \mu_{N,\mathcal{X}})$ is also represented by $-\delta c''$, and this, being the opposite of the result that we found earlier, finishes the proof of Proposition 5.5. $\square$

REMARK 5.7

Everything in this section remains true if one replaces μ_N with $\mathbb{G}_m$, ${}_N \mathrm{Pic}_{\mathcal{X}/B}$ with $\mathrm{Pic}_{\mathcal{X}/B}$, and $\langle \cdot, \cdot \rangle^{\sharp}_{T^1}$ with $\langle \cdot, \cdot \rangle^{\sharp}_{T}$.

6. The comparison result

In this section we prove Theorem 1.7. In the notation of Section 1, it states that, for all prime integers $\ell \neq p$, the following diagram is commutative:

$$
\begin{tikzcd}
H^1(B, T_\ell \mathcal{E}_0^0) \arrow[r, "\sim"] & H^1(B, T_\ell \mathcal{E}_0) \arrow[r, "\gamma"] & \mathrm{Pic}\big(\mathcal{E}_0(\ell)^c\big)^{\mathrm{inv}} \arrow[d, hook] \\
F^1H^2\big(S_0, \mathbb{Z}_\ell(1)\big)^0 \arrow[u, "\simeq"', "\epsilon"] \arrow[rr, "h^*"] & & H^2\big(\mathcal{E}_0(\ell)^c, \mathbb{Z}_\ell(1)\big)
\end{tikzcd}
\tag{6.1}
$$

where ϵ (edge) denotes the composition

$$F^1H^2\big(S_0, \mathbb{Z}_\ell(1)\big)^0 \xrightarrow{\sim} H^1\big(B, R^1\pi_*\mathbb{Z}_\ell(1)\big) \xrightarrow{\sim} H^1(B, T_\ell \,\mathrm{Pic}_{S_0/B}) \xrightarrow{\sim} H^1(B, T_\ell \mathcal{E}_0^0)$$

and where γ is the class invariant homomorphism (see [14], [15]).

The restriction of this diagram to $\mathrm{Pic}^0(S_0/B) \otimes \mathbb{Z}_\ell \subset F^1H^2(S_0, \mathbb{Z}_\ell(1))^0$ can be written

$$
\begin{tikzcd}
H^0(B, \mathcal{E}_0^0) \otimes \mathbb{Z}_\ell \arrow[r] & H^1(B, T_\ell \mathcal{E}_0) \arrow[r, "\gamma"] & \mathrm{Pic}\big(\mathcal{E}_0(\ell)^c\big)^{\mathrm{inv}} \arrow[d, hook] \\
\mathrm{Pic}^0(S_0/B) \otimes \mathbb{Z}_\ell \arrow[u, "\simeq"', "Ab"] \arrow[rr, "h^*"] & & H^2\big(\mathcal{E}_0(\ell)^c, \mathbb{Z}_\ell(1)\big)
\end{tikzcd}
\tag{6.2}
$$

with the first upper arrow equal to the composition

$$H^0(B, \mathcal{E}_0^0) \otimes \mathbb{Z}_\ell \xrightarrow{\partial} H^1(B, T_\ell \mathcal{E}_0^0) \xrightarrow{\sim} H^1(B, T_\ell \mathcal{E}_0).$$

This is because the restriction of ϵ to $\mathrm{Pic}^0(S_0/B) \otimes \mathbb{Z}_\ell$ equals the composition ∂Ab due to the commutative diagram (with standard maps), for all N prime to p,

$$
\begin{tikzcd}
H^0(B, \mathrm{Pic}^0_{S_0/B}) \arrow[r] & H^1(B, {}_N\mathrm{Pic}_{S_0/B}) \\
\mathrm{Pic}^0(S_0/B) \arrow[u] \arrow[r] & F^1H^2(S_0, \mu_N) \arrow[u]
\end{tikzcd}
$$

Diagram (6.2) is the projective limit of the following ones tensored with $\mathbb{Z}/N\mathbb{Z}$, for $N = \ell^r$, $r \geq 0$:

$$
\begin{tikzcd}
H^0(B, \mathcal{E}_0^0) \arrow[r] & H^1(B, {}_N\mathcal{E}_0) \arrow[r, "\gamma_N"] & \mathrm{Pic}({}_N\mathcal{E}_0^c)^{\mathrm{inv}} \arrow[d, hook] \\
\mathrm{Pic}^0(S_0/B) \arrow[u, "\simeq"', "Ab"] \arrow[rr, "h^*"] & & \mathrm{Pic}({}_N\mathcal{E}_0^c)
\end{tikzcd}
\tag{6.3}
$$

The diagrams (6.3) were shown in [14] to be commutative, for all N prime to p, using geometric arguments. (The restrictions imposed in that article do not affect the proof of this fact, which holds in the present generality.)

Our diagram (6.1) is the projective limit of the following ones, for $N=\ell^r$, $r\geq 0$:

$$
\begin{tikzcd}
H^1(B, {}_N\mathcal{E}_0^0) \arrow[r] & H^1(B, {}_N\mathcal{E}_0) \arrow[r, "\gamma_N"] & \mathrm{Pic}({}_N\mathcal{E}_0^c)^{\mathrm{inv}} \arrow[d] \\
F^1H^2(S_0, \mu_N)^0 \arrow[u, "\simeq"', "\epsilon_N"] \arrow[rr, "h^*"] & & H^2({}_N\mathcal{E}_0^c, \mu_N)
\end{tikzcd}
\tag{6.4}
$$

with ϵ_N equal to the composition

$$F^1H^2(S_0,\mu_N)^0 \xrightarrow{\sim} H^1(B, R^1\pi_*\mu_{N,S_0}) \xrightarrow{\sim} H^1(B, {}_N\mathrm{Pic}_{S_0/B}) \xrightarrow{\sim} H^1(B, {}_N\mathcal{E}_0^0).$$

The following will then prove the commutativity of diagram (6.1).

CLAIM 6.5

For all N prime to p, the diagram (6.4) is commutative for all elements in the image of the map $F^1H^2(S_0,\mu_{N^2})^0 \to F^1H^2(S_0,\mu_N)^0$ deduced from the multiplication by N map (i.e., the Nth power map) $\mu_{N^2,S_0}\to\mu_{N,S_0}$.

We put $\tilde{B} = {}_N\mathcal{E}_0^c$ and $f=\pi h:\tilde{B}\to B$. Let $\tilde{\pi}:\tilde{S}_0\to\tilde{B}$ be the Kodaira–Néron surface of the pullback of the curve E_0 to the generic points of the (irreducible) connected components of $\tilde{B}$, and let $\tilde{\mathcal{E}}_0\subset\tilde{S}_0$ (resp., $\tilde{\mathcal{E}}_0^0\subset\tilde{S}_0$) be the Néron model of that curve (resp., the open group subscheme of the identity components of the fibers of the latter). One has a natural morphism $(\mathcal{E}_0)_{\tilde{B}}\to\tilde{\mathcal{E}}_0$ of group schemes over $\tilde{B}$, and the tautological section of ${}_N(\mathcal{E}_0)_{\tilde{B}}\to\tilde{B}$ over the open subset $f^{-1}(B^-)\subset\tilde{B}$ extends uniquely to a section $\tilde{\omega}$ of ${}_N\tilde{\mathcal{E}}_0\to\tilde{B}$. We rewrite diagram (6.4) as follows, by writing out γ_N according to its definition (see [15, p. 404]) and using elementary functoriality properties:

$$
\begin{tikzcd}
H^1(B, {}_N\mathcal{E}_0^0) \arrow[r, "f^*"] & H^1(\tilde{B}, ({}_N\mathcal{E}_0^0)_{\tilde{B}}) \arrow[r] & H^1(\tilde{B}, {}_N\tilde{\mathcal{E}}_0^0) \arrow[r] & H^1(\tilde{B}, {}_N\tilde{\mathcal{E}}_0) \arrow[r, "e_N(-,\tilde{\omega})"] & H^1(\tilde{B}, \mu_N) \arrow[d, "\nu"] \\
& & & & \mathrm{Pic}(\tilde{B}) \arrow[d, "\partial"] \\
F^1H^2(S_0,\mu_N)^0 \arrow[uu, "\simeq"', "\epsilon_N"] \arrow[rrrr, "h^*"] & & & & H^2(\tilde{B}, \mu_N)
\end{tikzcd}
$$

(6.6)

The map ∂ on the right-hand side is a connecting homomorphism from the cohomology exact sequence of the Kummer sequence for $\mathbb{G}_{m,\tilde{B}}$ and the integer N, and the composition $\partial\nu$ is therefore a connecting homomorphism (also written as ∂ in this article) for the restriction of the latter to $\mu_{N,\tilde{B}}\subset\mathbb{G}_{m,\tilde{B}}$ (see the first exact sequence in Proposition 2.6).

For its study we split (6.6) into two other diagrams. The first one is

$$
\text{(6.7)} \qquad
\begin{tikzcd}
H^1(B, {}_N\mathcal{E}_0^0) \arrow[r, "f^*"] & H^1\big(\tilde{B}, ({}_N\mathcal{E}_0^0)_{\tilde{B}}\big) \arrow[r] & H^1(\tilde{B}, {}_N\tilde{\mathcal{E}}_0^0) \\
 & & F^1H^2(\tilde{S}_0, \mu_N)^0 \arrow[u, "\simeq \tilde{\epsilon}_N"'] \arrow[d, "\tilde{\omega}^*"] \\
F^1H^2(S_0, \mu_N)^0 \arrow[uu, "\simeq", "\epsilon_N"'] \arrow[rr, "h^*"] & & H^2(\tilde{B}, \mu_N)
\end{tikzcd}
$$

the upper side being the first half of the upper side of (6.6), and the map $\tilde{\epsilon}_N$ being defined similarly to the map ϵ_N using the Leray spectral sequence for $\tilde{\pi} : \tilde{S}_0 \to \tilde{B}$ and the sheaf $\mu_{N,\tilde{S}_0}$.

The second diagram is

$$
\text{(6.8)} \qquad
\begin{tikzcd}
H^1(\tilde{B}, {}_N\tilde{\mathcal{E}}_0^0) \arrow[r] & H^1(\tilde{B}, {}_N\tilde{\mathcal{E}}_0) \arrow[r, "e_N(-,\tilde{\omega})"] & H^1(\tilde{B}, \mu_N) \arrow[d, "\nu"] \\
 & & \mathrm{Pic}(\tilde{B}) \arrow[d, "\partial"] \\
F^1H^2(\tilde{S}_0, \mu_N)^0 \arrow[uu, "\simeq", "\tilde{\epsilon}_N"'] \arrow[rr, "\tilde{\omega}^*"] & & H^2(\tilde{B}, \mu_N)
\end{tikzcd}
$$

the upper side being the second half of the upper side of (6.6). Diagram (6.6) is then obtained from diagrams (6.7) and (6.8) by joining them along their common arrows $\tilde{\epsilon}_N$ and $\tilde{\omega}^*$. Claim 6.5 will follow from the next lemma.

LEMMA 6.9

(a) *The diagram (6.7) is commutative, replacing the arrow $\tilde{\epsilon}_N$ by its inverse $\tilde{\epsilon}_N^{-1}$.*

(b) *The diagram (6.8) commutes for all elements belonging to the image of the morphism $F^1H^2(\tilde{S}_0, \mu_{N^2})^0 \to F^1H^2(\tilde{S}_0, \mu_N)^0$, deduced from the multiplication by N map $\mu_{N^2,\tilde{S}_0} \to \mu_{N,\tilde{S}_0}$.*

Proof

We prove Part (a). The morphism $(\mathcal{E}_0)_{\tilde{B}} \to \tilde{\mathcal{E}}_0$ embeds $(S_0^-)_{\tilde{B}}$ as an open subset of $\tilde{S}_0$, and the projection map $(S_0^-)_{\tilde{B}} \to S_0^-$ defines a rational map $\tilde{S}_0 \dashrightarrow S_0$, compatible with $f : \tilde{B} \to B$. We may complete this picture into a commutative diagram (see [2])

$$
\begin{tikzcd}
S_0^- & (S_0^-)_{\tilde{B}} \arrow[l] \arrow[r, "\mathrm{id}"] & (S_0^-)_{\tilde{B}} \\
S_0 \arrow[d, "\pi"] \arrow[u, hookleftarrow] & \tilde{S}_0' \arrow[l, "g"'] \arrow[r, "\delta"] \arrow[d, "\tilde{\pi}'"] \arrow[u, hookleftarrow] & \tilde{S}_0 \arrow[d, "\tilde{\pi}"] \arrow[u, hookleftarrow] \\
B & \tilde{B} \arrow[l, "f"'] \arrow[r, "\mathrm{id}"] & \tilde{B}
\end{tikzcd}
\tag{6.10}
$$

with $\tilde{S}_0'$ a smooth projective surface over k and morphisms g and δ, the latter one birational, and the upper vertical arrows being open embeddings. The lower half of this diagram yields, by the functorial properties of the Leray spectral sequence, a commutative diagram

$$
\begin{tikzcd}
H^1(B, {}_N\mathcal{E}_0^0) & & H^1(\tilde{B}, {}_N\tilde{\mathcal{E}}_0^0) \\
H^1(B, {}_N\operatorname{Pic}_{S_0/B}) \arrow[r] \arrow[u, "\simeq"'] & H^1(\tilde{B}, {}_N\operatorname{Pic}_{\tilde{S}_0'/\tilde{B}}) & H^1(\tilde{B}, {}_N\operatorname{Pic}_{\tilde{S}_0/\tilde{B}}) \arrow[l, "\simeq"'] \arrow[u, "\simeq"'] \\
H^1(B, R^1\pi_*\mu_{N,S_0}) \arrow[r] \arrow[u, "\simeq"'] & H^1(\tilde{B}, R^1\tilde{\pi}'_*\mu_{N,\tilde{S}_0'}) \arrow[u, "\simeq"'] & H^1(\tilde{B}, R^1\tilde{\pi}_*\mu_{N,\tilde{S}_0}) \arrow[l, "\simeq"'] \arrow[u, "\simeq"'] \\
F^1H^2(S_0,\mu_N)^0 \arrow[r, "g^*"] \arrow[u, "\simeq"'] & F^1H^2(\tilde{S}_0',\mu_N)^0 \arrow[u, "\simeq"'] & F^1H^2(\tilde{S}_0,\mu_N)^0 \arrow[l, "\simeq", "\delta^*"'] \arrow[u, "\simeq"']
\end{tikzcd}
\tag{6.11}
$$

The right-hand side horizontal morphisms are isomorphisms, for instance, because the upper first one is as well: for all open $\tilde{U} \to \tilde{B}$ in the étale topology, the map $(\tilde{S}_0')_{\tilde{U}} \to (\tilde{S}_0)_{\tilde{U}}$ is a proper birational morphism of smooth varieties; hence, $\operatorname{Pic}((\tilde{S}_0')_{\tilde{U}})$ is the direct sum of $\operatorname{Pic}((\tilde{S}_0)_{\tilde{U}})$ and a (finitely generated) free abelian group. Hence, these groups have the same torsion subgroup. It follows, in particular, that ${}_N\operatorname{Pic}_{\tilde{S}_0'/\tilde{B}} \xleftarrow{\simeq} {}_N\operatorname{Pic}_{\tilde{S}_0/\tilde{B}}$.

Next, the diagram obtained from (6.11) by inserting the upper side of diagram (6.7) remains commutative. This follows from the commutativity of the diagram of étale group schemes over $\tilde{B}$:

$$
\begin{tikzcd}
({}_N\mathcal{E}_0^0)_{\tilde{B}} \arrow[rr] & & {}_N\tilde{\mathcal{E}}_0^0 \\
({}_N\operatorname{Pic}_{S_0/B})_{\tilde{B}} \arrow[r, "g^*"] \arrow[u, "\simeq"'] & {}_N\operatorname{Pic}_{\tilde{S}_0'/\tilde{B}} & {}_N\operatorname{Pic}_{\tilde{S}_0/\tilde{B}} \arrow[l, "\simeq", "\delta^*"'] \arrow[u, "\simeq"']
\end{tikzcd}
\tag{6.12}
$$

which is due to the commutativity of its restriction to the open dense subset $f^{-1}(B^-) \subset \tilde{B}$, where all the horizontal arrows become identities (see the lower half of diagram (6.10)).

In this way, the commutativity of diagram (6.7) amounts to the equality between h^* and $\tilde{\omega}^*(\delta^*)^{-1}g^*$. To prove this equality, note that the section $\tilde{\omega}$ of $\tilde{\pi}: \tilde{S}_0 \to \tilde{B}$ lifts to a section $\tilde{\omega}'$ of $\tilde{\pi}': \tilde{S}_0' \to \tilde{B}$, $\tilde{\omega} = \delta\tilde{\omega}'$, and that $h = g\tilde{\omega}'$, because this equality holds above the open dense subset $f^{-1}(B^-) \subset \tilde{B}$. So $\tilde{\omega}^* = \tilde{\omega}'^*\delta^*$, and hence, $\tilde{\omega}^*(\delta^*)^{-1}g^* = \tilde{\omega}'^*g^* = (g\tilde{\omega}')^* = h^*$. This proves Part (a) of Lemma 6.9.

We prove Part (b). Let $\tilde{\xi} \in F^1H^2(\tilde{S}_0, \mu_N)^0$, and call $\tilde{\lambda} \in H^1(\tilde{B}, {}_N\operatorname{Pic}_{\tilde{S}_0/\tilde{B}})$ its image by the edgelike morphism in the Leray spectral sequence for $\tilde{\pi}: \tilde{S}_0 \to \tilde{B}$ and the sheaf $\mu_{N,\tilde{S}_0}$. The image of $\tilde{\xi}$ in $H^1(\tilde{B}, {}_N\tilde{\mathcal{E}}_0^0)$ by the composition of $\tilde{\epsilon}_N$ with the first upper arrow in diagram (6.8) is $\tilde{\lambda}_{Ab}$, and so its image by the composition of the left-hand side, the top side, and the right-hand side in that diagram equals

$$\begin{aligned} \partial e_N(\tilde{\lambda}_{Ab}, \tilde{\omega}) &= -\partial\bar{e}_N(\tilde{\lambda}, \tilde{\omega}) \quad \big((3.6)\big) \\ &= \langle \nu\tilde{\lambda}, \tilde{\omega}\rangle_{T^0} - \langle \tilde{\lambda}, \nu\tilde{\omega}\rangle_{T^1} \quad \text{(Proposition 3.24)}. \end{aligned}$$

(Here and below, the propositions and formulae referred to from earlier sections are being applied over each connected (i.e., irreducible) component of $\tilde{B}$ separately.)

Secondly, call $\Gamma_{\tilde{0}}$ and $\Gamma_{\tilde{\omega}}$, respectively, the graphs of the sections $\tilde{0}$ and $\tilde{\omega}$ of $\tilde{\pi}: \tilde{\mathcal{E}}_0 \to \tilde{B}$, and let $\tilde{\mathfrak{z}} = \Gamma_{\tilde{\omega}} - \Gamma_{\tilde{0}} \in Z_0(\tilde{S}_0/\tilde{B})_0$. In the notation of Section 5, we have $ev_{\tilde{\mathfrak{z}}} = \tilde{\omega}^* - \tilde{0}^*$ and $\tilde{\mathfrak{z}}_{Ab} = \nu\tilde{\omega}$. Thus, bearing in mind that $\tilde{0}^*(\tilde{\xi}) = 0$, we have that the image of $\tilde{\xi}$ by the bottom arrow in diagram (6.8) equals

$$\begin{aligned} \tilde{\omega}^*(\tilde{\xi}) &= ev_{\tilde{\mathfrak{z}}}(\tilde{\xi}) \\ &= -\langle \tilde{\lambda}, \tilde{\mathfrak{z}}\rangle^{\sharp}_{T^1} \quad \text{(Proposition 5.5)} \\ &= -\langle \tilde{\lambda}, \nu\tilde{\omega}\rangle_{T^1} \quad \big(\text{Proposition 4.13(b)}\big). \end{aligned}$$

The discrepancy between these two results is given by the term

$$\langle \nu\tilde{\lambda}, \tilde{\omega}\rangle_{T^0} = -\bar{e}_N(\partial\nu\tilde{\lambda}, \tilde{\omega}) \quad \text{(Proposition 3.13)}.$$

This term vanishes, and diagram (6.8) consequently becomes commutative, for instance, for those $\tilde{\xi} \in F^1H^2(\tilde{S}_0, \mu_N)^0$ such that $\partial\nu\tilde{\lambda} = 0$. This condition is equivalent to $\tilde{\lambda}$ belonging to the image of the morphism $H^1(\tilde{B}, {}_{N^2}\operatorname{Pic}_{\tilde{S}_0/\tilde{B}}) \to H^1(\tilde{B}, {}_N\operatorname{Pic}_{\tilde{S}_0/\tilde{B}})$ induced by the multiplication by N map ${}_{N^2}\operatorname{Pic}_{\tilde{S}_0/\tilde{B}} \to {}_N\operatorname{Pic}_{\tilde{S}_0/\tilde{B}}$, which in turn is equivalent to $\tilde{\xi} \in F^1H^2(\tilde{S}_0, \mu_N)^0$ fulfilling the condition in the statement of Part (b) of Lemma 6.9. The proof of Lemma 6.9(a) and the analog of diagram (6.11) with N replaced by N^2 imply that, in diagram (6.7), the elements mentioned in the statement of Claim 6.5 are mapped to the elements mentioned in the statement of Lemma 6.9(b). Together with Lemma 6.9 this proves Claim 6.5 and therefore finishes the proof of Theorem 1.7. □

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Facultad de Matemáticas, Universidad de Barcelona, Barcelona, Spain; welters@ub.edu