

HYPERBOLICITY OF CIRCULAR DOMAINS

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(Received August 9, 1982)

1. Introduction. If a domain D in C^n is hyperbolic in the sense of Kobayashi [5], then every holomorphic mapping from C into D is constant. In general the converse is not true. In this paper, we show that the converse holds if D is a strictly starlike circular domain in C^n . More strongly, if D is a starlike circular domain in C^n with $\bar{D} \subset \lambda D$ for any real $\lambda > 1$, and if every C -linear mapping from C into D is zero, then D is bounded (Proposition 4.4). Geometrically convex, circular domains or complete Reinhardt domains in C^n are strictly starlike (Propositions 4.2 and 4.3). Next we obtain equivalent conditions for a starlike circular domain in C^n to be pseudoconvex (Proposition 5.1 and Theorem 5.4). Finally, modifying the example in Barth [2], we construct a non-hyperbolic pseudoconvex circular domain in C^2 into which every holomorphic mapping from C is constant (Proposition 6.5).

In subsequent sections, we call a subset X of C^n or R^n *convex*, for brevity, when X is geometrically convex, i.e., $\{\lambda x + (1 - \lambda)y; 0 < \lambda < 1\} \subset X$ for any $x, y \in X$.

The author would like to express his thanks to Professors T. Kuroda and M. Suzuki for valuable discussions during the preparation of the present paper. The author would also like to thank the referee for helpful suggestions, which led to the improvement of the original manuscript.

2. Hyperbolicity of domains in C^n . Throughout this paper we consider the following conditions on a domain D in C^n :

(H. 1) D is bounded.

(H. 2) D is biholomorphic to a bounded domain in C^n .

(H. 3) D is C -hyperbolic, i.e., the Carathéodory pseudodistance of D is a distance.

(H. 4) D is K -hyperbolic, i.e., the Kobayashi pseudodistance of D is a distance.

(H. 5) D contains no entire holomorphic curve, i.e., there does not

Partly supported by the Grant-in-Aid for Encouragement of Young Scientists, the Ministry of Education, Science and Culture, Japan.

exist any non-constant holomorphic mapping from C into D .

(H. 6) D contains no complex affine line.

(H. 7) D contains no complex homogeneous line.

Here a *complex homogeneous line* in C^n is a complex affine line passing through the origin. We identify the complex projective space $P^{n-1}(C)$ with the family of all complex homogeneous lines in C^n . Then the condition (H. 7) is rewritten as follows: There does not exist any element ζ of $P^{n-1}(C)$ such that $\zeta \subset D$.

In general, (H. i) is stronger than (H. $i + 1$) for $i = 1, \dots, 6$ (cf. Kobayashi [5] and Remark in the last paragraph of this section). If we impose some restrictions on D , some of (H. i) are equivalent. For example, Barth [2] showed the following:

(B₁) For a convex domain D in C^n , (H. i) ($i = 2, \dots, 6$) are equivalent.

(B₂) There exists a pseudoconvex domain D in C^2 which satisfies (H. 5) but not (H. 4).

Recently the following was shown in Kodama [6]:

(K) For a starlike circular domain D in C^n , the conditions (H. i) ($i = 1, \dots, 4$) are equivalent.

REMARK. Sibony [11; Lemme 6] showed the following:

(S) A domain D in C is C -hyperbolic if and only if there exists a non-constant, bounded holomorphic function on D .

Combining (S) with Ahlfors and Beurling [1; Theorem 15], we conclude that there exists a C -hyperbolic domain in C which is not biholomorphic to any bounded domain in C . This shows that in general the condition (H. 2) is stronger than (H. 3).

3. Starlike circular domains in C^n . A domain D in C^n is called *circular* (resp. *starlike circular*) if D contains the origin 0 and $\lambda D \subset D$ for all $\lambda \in C$ with $|\lambda| = 1$ (resp. if $\lambda D \subset D$ for all $\lambda \in C$ with $|\lambda| \leq 1$).

From now on, we use the symbol π to denote the canonical projection of $C^n - \{0\}$ to $P^{n-1}(C)$.

To a starlike circular (not necessarily bounded) domain D in C^n , we associate a $(0, +\infty]$ -valued function R (called the *defining function* of D) on $P^{n-1}(C)$ as follows: If $\zeta \in P^{n-1}(C)$ is a complex homogeneous line in C^n , we define $R(\zeta)$ by $\sup\{|z|; z \in D \cap \zeta\}$ (cf. Sadullaev [10]), where $|\cdot|$ denotes the Euclidean norm on C^n . Then R is a lower semicontinuous mapping from $P^{n-1}(C)$ into $(0, +\infty]$, endowed with the topology induced by that of the two-point-compactification $[-\infty, +\infty]$ of R , and D is reproduced in terms of R as follows:

$$(3.1) \quad D = \{z \in \mathbb{C}^n - \{0\}; |z| < R \circ \pi(z)\} \cup \{0\} .$$

Conversely, given a $(0, +\infty]$ -valued lower semicontinuous function R on $\mathbb{P}^{n-1}(\mathbb{C})$, the set D defined by (3.1) is a starlike circular domain in \mathbb{C}^n .

It is clear that a starlike circular domain D satisfies the condition (H. 1) (resp. (H. 7)) if and only if the defining function R of D is bounded (resp. real valued) on $\mathbb{P}^{n-1}(\mathbb{C})$.

4. Strictly starlike circular domains in \mathbb{C}^n . A starlike circular domain D in \mathbb{C}^n is called *strictly starlike* if $\bar{D} \subset \lambda D$ for any real $\lambda > 1$ (cf. [9; p. 125]). As we see in the following proposition, this concept is equivalent to the continuity of the mapping $R: \mathbb{P}^{n-1}(\mathbb{C}) \rightarrow (0, +\infty]$.

PROPOSITION 4.1. *Let D be a starlike circular domain in \mathbb{C}^n defined by R . Then D is strictly starlike if and only if R is upper semicontinuous on $\mathbb{P}^{n-1}(\mathbb{C})$.*

PROOF. Suppose R is not upper semicontinuous on $\mathbb{P}^{n-1}(\mathbb{C})$. Then we can find a sequence $\{\zeta_j\}$ and a point ζ in $\mathbb{P}^{n-1}(\mathbb{C})$ such that $\zeta_j \rightarrow \zeta$ and $\lim_{j \rightarrow \infty} R(\zeta_j) > R(\zeta)$. Pick, for each j , a point z_j from $S^{2n-1} := \{z \in \mathbb{C}^n; |z| = 1\}$ so that $\pi(z_j) = \zeta_j$. Taking a subsequence, if necessary, we may assume the sequence $\{z_j\}$ converges to a point $z \in S^{2n-1}$; then $\pi(z) = \zeta$ and $\lim_{j \rightarrow \infty} R \circ \pi(z_j) > R \circ \pi(z)$. Let r be the real number with $\lim_{j \rightarrow \infty} R \circ \pi(z_j) \geq r > R \circ \pi(z)$. Then $rz \in \bar{D}$. Indeed, if (r_1, r_2) and V are arbitrary neighborhoods of r in $(0, +\infty)$ and of z in S^{2n-1} , respectively, it follows that $rz_j \in (r_1, r_2)V, j \geq j_1$ for some j_1 . Since $\lim_{j \rightarrow \infty} R \circ \pi(z_j) \geq r$, we can find $j_2 \geq j_1$ so that $R \circ \pi(z_j) > (r_1 + r)/2, j \geq j_2$. Therefore we have $((r_1 + r)/2)z_j \in D \cap (r_1, r_2)V, j \geq j_2$. This shows $rz \in \bar{D}$. Moreover, taking λ with $r/R \circ \pi(z) \geq \lambda > 1$, we have $rz \notin \lambda D$. Thus D is not strictly starlike.

Conversely, suppose R is upper semicontinuous on $\mathbb{P}^{n-1}(\mathbb{C})$. Then R is a continuous $(0, +\infty]$ -valued function. Hence we have

$$(4.1) \quad \bar{D} \subset \{z \in \mathbb{C}^n - \{0\}; |z| \leq R \circ \pi(z)\} \cup \{0\} .$$

But $\lambda D = \{z \in \mathbb{C}^n - \{0\}; |z| < \lambda R \circ \pi(z)\} \cup \{0\}$ for any real $\lambda > 1$. Combining this with (4.1), we obtain $\bar{D} \subset \lambda D$. This completes the proof.

COROLLARY. *Let D be a starlike circular domain in \mathbb{C}^n defined by R . Then the following assertions are equivalent:*

- (i) D is strictly starlike.
- (ii) $\bar{D} = \{z \in \mathbb{C}^n - \{0\}; |z| \leq R \circ \pi(z)\} \cup \{0\}$.
- (iii) $\partial D = \{z \in \mathbb{C}^n - \{0\}; |z| = R \circ \pi(z)\}$.

PROOF. First we note that

$$(4.2) \quad \bar{D} \supset \{z \in \mathbb{C}^n - \{0\}; |z| \leq R \circ \pi(z)\} \cup \{0\}.$$

Indeed, for $z \in \mathbb{C}^n$, $|z| = R \circ \pi(z)$ implies $\lambda z \in D$ for any $\lambda \in (0, 1)$; therefore $z \in \bar{D}$. Now suppose D is strictly starlike. Then, since R is continuous by Proposition 4.1, (4.1) holds. Combining this with (4.2), we obtain the equality (ii). The converse implication (ii) \Rightarrow (i) is clear by the last argument in the proof of Proposition 4.1. Since D is open in \mathbb{C}^n , the equivalence (ii) \Leftrightarrow (iii) is clear. q.e.d.

There are the following two typical families of strictly starlike circular domains as in Propositions 4.2 and 4.3.

PROPOSITION 4.2. *Convex circular domains in \mathbb{C}^n are strictly starlike.*

PROOF. Let D be a convex circular domain in \mathbb{C}^n . Clearly D is starlike circular; let R be the defining function of D . Suppose $R(\zeta) < \lambda$ for $\zeta \in P^{n-1}(\mathbb{C})$, $\lambda \in \mathbb{R}$. Take a point $z \in \partial D \cap \zeta$. Since by convexity there exists a supporting real hyperplane of D at z , it follows that $R \circ \pi < \lambda$ in a neighborhood U of z in $\mathbb{C}^n - \{0\}$. Therefore $R < \lambda$ in the neighborhood $\pi(U)$ of ζ in $P^{n-1}(\mathbb{C})$. This means that R is upper semicontinuous, as desired.

PROPOSITION 4.3. *Complete Reinhardt domains in \mathbb{C}^n are strictly starlike circular.*

PROOF. Let D be a complete Reinhardt domain in \mathbb{C}^n , and let $z = (z^1, \dots, z^n) \in \bar{D} - \{0\}$ and $\lambda > 1$. Then the set $U: = \{(w^1, \dots, w^n) \in \mathbb{C}^n; \lambda |w^j| > |z^j| \text{ if } z^j \neq 0\}$ is a neighborhood of z in \mathbb{C}^n with the property that $(w^1, \dots, w^n) \in \lambda U \cap (\mathbb{C} - \{0\})^n$ implies $|z^j| < |w^j|$ for all j . Take $(w^1, \dots, w^n) \in U \cap D \cap (\mathbb{C} - \{0\})^n$. Then $|z^j| < \lambda |w^j|$ for all j . Hence $\lambda^{-1}z \in D$ by completeness. This shows that D is strictly starlike, as desired.

On the hyperbolicity of strictly starlike circular domains in \mathbb{C}^n , we have:

PROPOSITION 4.4. *For a strictly starlike circular domain D in \mathbb{C}^n , the conditions (H. i) ($i = 1, \dots, 7$) are equivalent.*

PROOF. Let R be the defining function of D . By what is noted in the last paragraph in §3, it is sufficient to show the following:

(*) If R is real valued, then R is bounded.

But by virtue of Proposition 4.1, the assertion (*) holds, because $P^{n-1}(\mathbb{C})$ is compact. q.e.d.

Making use of Propositions 4.2 and 4.3, we obtain the following:

COROLLARY. *For a convex circular or complete Reinhardt domain D in \mathbb{C}^n , the conditions (H. i) ($i = 1, \dots, 7$) are equivalent.*

5. Pseudoconvexity of starlike circular domains in \mathbb{C}^n . Let D be a starlike circular domain in \mathbb{C}^n defined by R . In this section we shall establish some criteria for the pseudoconvexity of D in terms of R .

We define a $[-\infty, +\infty)$ -valued function Φ on \mathbb{C}^n by

$$(5.1) \quad \Phi(z) = \begin{cases} -\log R \circ \pi(z) + \log |z|, & z \neq 0 \\ -\infty, & z = 0. \end{cases}$$

Then the expression (3.1) becomes $D = \{z \in \mathbb{C}^n; \Phi(z) < 0\}$. Since R is lower semicontinuous on $\mathbb{P}^{n-1}(\mathbb{C})$, Φ is upper semicontinuous on $\mathbb{C}^n - \{0\}$. Moreover, Φ is upper semicontinuous also at 0. Indeed, it can be seen that

$$(5.2) \quad \limsup_{z \rightarrow 0, z \neq 0} \Phi(z) = -\infty,$$

because R is bounded from below. For each $i \in \{1, \dots, n\}$ and any $u = (u^1, \dots, u^{i-1}, u^{i+1}, \dots, u^n) \in \mathbb{C}^{n-1}$, let

$$(5.3) \quad \begin{aligned} \Phi_i(u) &:= \Phi(u^1, \dots, u^{i-1}, 1, u^{i+1}, \dots, u^n) \\ &= -\log R \circ \pi(u^1, \dots, u^{i-1}, 1, u^{i+1}, \dots, u^n) \\ &\quad + \log(1 + |u|^2)^{1/2}. \end{aligned}$$

Then, we have

$$(5.4) \quad \Phi(z) = \Phi_i((1/z^i)(z^1, \dots, z^{i-1}, z^{i+1}, \dots, z^n)) + \log |z^i|$$

for $z = (z^1, \dots, z^n) \in \mathbb{C}^n$ with $z^i \neq 0$.

Our key criterion is the following:

PROPOSITION 5.1. *Let D be a starlike circular domain in \mathbb{C}^n defined by R , and Φ_i the functions given by (5.3). Then D is pseudoconvex if and only if Φ_i ($i = 1, \dots, n$) are plurisubharmonic on \mathbb{C}^{n-1} .*

PROOF. Fixing $i \in \{1, \dots, n\}$ arbitrarily, we set $A_i := \{(z^1, \dots, z^n) \in \mathbb{C}^n; z^i = 0\}$. By the holomorphic mapping from $\mathbb{C}^n - A_i$ into \mathbb{C}^n defined by

$$(z^1, \dots, z^n) \mapsto (z^1/z^i, \dots, z^{i-1}/z^i, z^i, z^{i+1}/z^i, \dots, z^n/z^i),$$

the domain $D - A_i$ is mapped bijectively to a Hartogs domain $G_i - A_i$, where G_i is a domain in \mathbb{C}^n consisting of all points $(w^1, \dots, w^n) \in \mathbb{C}^n$ such that

$$|w^i| < \exp(-\Phi_i(w^1, \dots, w^{i-1}, w^{i+1}, \dots, w^n)).$$

Thereby we obtain the following implications: $D - A_i$ is pseudoconvex $\Leftrightarrow G_i - A_i$ is pseudoconvex $\Leftrightarrow G_i$ is pseudoconvex $\Leftrightarrow \Phi_i$ is plurisubharmonic.

The middle equivalence follows from Lelong [8; Proposition 14] and Hörmander [4; Theorem 2.5.14].

But, since pseudoconvexity for a domain in \mathbb{C}^n is a local property with respect to its boundary (cf. [4; Theorem 2.6.10]), $D - A_i$ is pseudoconvex for each $i = 1, \dots, n$ if and only if D is pseudoconvex. This completes the proof.

We can simplify the above criterion using the following extension theorem for plurisubharmonic functions, due to Grauert and Remmert [3; Satz 6]:

(G-R) Let A be a principal analytic set of a domain D in \mathbb{C}^n and f a plurisubharmonic function on $D - A$. Suppose that for any $a \in A$, we can find a neighborhood U of a in D so that f is bounded from above on $U - A$. Then the function

$$\tilde{f}(z) = \begin{cases} f(z), & z \in D - A \\ \limsup_{w \rightarrow z, w \in D - A} f(w), & z \in A \end{cases}$$

is the uniquely extended plurisubharmonic function of f on D .

We can rearrange the theorem (G - R) as follows:

PROPOSITION 5.2. *Let D, A be as in the theorem (G - R) and f a $[-\infty, +\infty)$ -valued function on D . Suppose that f satisfies the following two conditions:*

- (a) f is plurisubharmonic on $D - A$.
- (b) $\limsup_{w \rightarrow a, w \in D - A} f(w) = f(a)$ for any $a \in A$.

Then f is plurisubharmonic on D .

Using the uniqueness part of the theorem (G - R), we obtain the following:

PROPOSITION 5.3. *Let D, A be as in the theorem (G - R) and f a plurisubharmonic function on D . Then $\limsup_{w \rightarrow a, w \in D - A} f(w) = f(a)$ for any given $a \in A$.*

Now the main criterion can be stated as follows:

THEOREM 5.4. *Let D be a starlike circular domain in \mathbb{C}^n defined by R , and Φ be the function given by (5.1). Set $\varphi := \Phi_n$ (cf. (5.3)), $H := \{(z^1, \dots, z^n) \in \mathbb{C}^n; z^n = 0\}$. Consider the following condition on Φ :*

$$(\#) \quad \limsup_{w \rightarrow z, w \in \mathbb{C}^n - H} \Phi(w) = \Phi(z) \quad \text{for any } z \in H - \{0\}.$$

Then the following assertions are equivalent:

- (i) D is pseudoconvex.

- (ii) φ is plurisubharmonic on C^{n-1} and (#) holds.
- (iii) Φ is plurisubharmonic on C^n .

PROOF. (i) \Rightarrow (ii): Suppose D is pseudoconvex. By Proposition 5.1, $\varphi = \Phi_n$ is plurisubharmonic on C^{n-1} . To show the condition (#) to hold, we fix $z_0 = (z_0^1, \dots, z_0^n) \in H - \{0\}$. Take i with $z_0^i \neq 0$. Given $z = (z^1, \dots, z^n) \in C^n$ with $z^i \neq 0$, using (5.4), we have

$$(5.5) \quad \limsup_{z \rightarrow z_0, z \in C^n - H} \Phi(z) = \limsup_{z \rightarrow z_0, z \in C^n - H} \Phi_i((1/z^i)(z^1, \dots, z^{i-1}, z^{i+1}, \dots, z^n)) + \log |z_0^i|.$$

Since Φ_i is plurisubharmonic on C^{n-1} by Proposition 5.1, the first term of the right hand side of (5.5) coincides with $\Phi_i((1/z_0^i)(z_0^1, \dots, z_0^{i-1}, z_0^{i+1}, \dots, z_0^n))$ by Proposition 5.3. Hence we have $\limsup_{z \rightarrow z_0, z \in C^n - H} \Phi(z) = \Phi(z_0)$, as desired.

(iii) \Rightarrow (i): Suppose Φ is plurisubharmonic on C^n . Since the restriction of a plurisubharmonic function to a complex hyperplane is also plurisubharmonic, it follows that $\Phi_i (i = 1, \dots, n)$ are plurisubharmonic. Hence the assertion (i) follows from Proposition 5.1.

(ii) \Rightarrow (iii): We assume that (ii) holds. By (5.4), we have $\Phi(z) = \varphi((1/z^n)(z^1, \dots, z^{n-1})) + \log |z^n|$ for $z = (z^1, \dots, z^n) \in C^n - H$. This equation shows that Φ is plurisubharmonic on $C^n - H$, because so is φ on C^{n-1} . On the other hand, by (5.2) we have $\limsup_{z \rightarrow 0, z \in C^n - H} \Phi(z) = -\infty = \Phi(0)$. Combining this with the assumption (#), by virtue of Proposition 5.2, we conclude that Φ is plurisubharmonic on C^n . The proof is completed.

We can apply Theorem 5.4 to a complete Reinhardt domain and prove the following well-known criterion for pseudoconvexity:

(**) Let D be a complete Reinhardt domain in C^n and Γ the subset of R^n given by

$$\Gamma: = \{(\log |z^1|, \dots, \log |z^n|); (z^1, \dots, z^n) \in D \cap (C - \{0\})^n\}.$$

Then D is pseudoconvex if and only if Γ is convex.

Indeed, let R be the defining function of D , and Φ, φ the functions given by (5.1) and in Theorem 5.4, respectively. We consider the function r on $(0, +\infty)^{n-1}$ given by

$$r(t) := \sup \{\lambda > 0; \lambda(t, 1) \in D\}.$$

Then we have

$$R \circ \pi(u, 1) = r(|u^1|, \dots, |u^{n-1}|)(1 + |u|^2)^{1/2}$$

for $u = (u^1, \dots, u^{n-1}) \in C^{n-1}$, and so

$$(5.6) \quad \varphi(u) = -\log r(|u^1|, \dots, |u^{n-1}|).$$

By the definitions of Γ and r , it follows that

$$\Gamma = \{(x^1, \dots, x^n) \in \mathbf{R}^n; x^n < \log r(e^{x^1-x^n}, \dots, e^{x^{n-1}-x^n})\}.$$

Now, since φ is given by (5.6), and depends only on the absolute values $|w^1|, \dots, |w^{n-1}|$, by virtue of [7; Théorème 9] it follows that φ is plurisubharmonic on $(\mathbf{C} - \{0\})^{n-1}$ if and only if the subset

$$\tilde{\Gamma}: = \{(y^1, \dots, y^n) \in \mathbf{R}^n; y^n > -\log r(e^{y^1}, \dots, e^{y^{n-1}})\}$$

of \mathbf{R}^n is convex. Since convexity of a subset in \mathbf{R}^n is invariant by a non-singular linear transformation and since Γ can be transformed to $\tilde{\Gamma}$ by such, it follows that φ is plurisubharmonic on $(\mathbf{C} - \{0\})^{n-1}$ if and only if Γ is convex.

On the other hand, since R is continuous by Propositions 4.1 and 4.1, so are Φ and φ . By the continuity of φ and Proposition 5.2, $\varphi|_{(\mathbf{C}-\{0\})^{n-1}}$ is plurisubharmonic if and only if so is φ . By the continuity of Φ , the condition (#) in Theorem 5.4 holds trivially. Combining these with the assertion stated at the end of the preceding paragraph, and with Theorem 5.4, (i) \Leftrightarrow (ii), we obtain the desired assertion (**).

6. Examples. In the category of starlike circular domains D in \mathbf{C}^n , we have seen that

$$(H. 1) \Leftrightarrow (H. 2) \Leftrightarrow (H. 3) \Leftrightarrow (H. 4) \Rightarrow (H. 5) \Rightarrow (H. 6) \Rightarrow (H. 7)$$

(cf. §2). For the sake of completeness, we give counterexamples to the converse of the last three implications as follows: Denoting $L_1: = \{(z, 0); z \in \mathbf{C}, |z| \geq 1\}$, $L_2: = \{(0, w); w \in \mathbf{C}, |w| \geq 1\} \subset \mathbf{C}^2$, we consider the domains

$$\begin{aligned} D_5: &= \{(z, w) \in \mathbf{C}^2; |z|e^{|w|} < 1\} - L_2, \\ D_6: &= \{(z, w) \in \mathbf{C}^2; |zw| < 1\} - (L_1 \cup L_2) \quad \text{and} \\ D_7: &= \{(z, w) \in \mathbf{C}^2; |z| < 1\} - L_2 \end{aligned}$$

in \mathbf{C}^2 . It can be seen that each D_i satisfies (H. i) but not (H. $i - 1$) for $i = 5, 6, 7$. We note that the above three domains are not pseudoconvex (cf. Remark in the last paragraph of this section).

Next, modifying the example in Barth [2] which asserts (B₂) in §2, we shall give a pseudoconvex circular domain in \mathbf{C}^2 which satisfies (H. 5) but not (H. 4). For this, we set for $\lambda \in \mathbf{C}$,

$$v(\lambda): = \max \{ \log |\lambda|, \sum_{k=2}^{\infty} k^{-2} \log |\lambda - 1/k| \},$$

and set for $(z, w) \in \mathbf{C}^2 - \{0\}$,

$$R(\pi(z, w)): = \begin{cases} (1 + |z/w|^2)^{1/2} \exp(-v(z/w)), & w \neq 0 \\ 1, & w = 0. \end{cases}$$

Then $D := \{(z, w) \in \mathbb{C}^2 - \{0\}; |(z, w)| < R \circ \pi(z, w)\} \cup \{0\}$ satisfies the desired properties. We shall show this in several steps.

LEMMA 6.1. (a) v is subharmonic on \mathbb{C} .

(b) $v(1/k) = -\log k (k = 2, 3, \dots)$, $v(0) = -\sum_{k=2}^{\infty} k^{-2} \log k \in \mathbb{R}$.

(c) R is positive real valued and lower semicontinuous on $P^1(\mathbb{C})$.

Moreover, $R \circ \pi(1, \cdot)$ is continuous in a neighborhood of 0 in \mathbb{C} .

PROOF. For $\lambda \in \mathbb{C}$, let $v_1(\lambda) := \sum_{k=2}^{\infty} k^{-2} \log |\lambda - 1/k|$.

(a) Fixing r , with $0 < r < 1$, we can select a number k_0 so that for each $k \geq k_0$, $\log |\lambda - 1/k|$ is harmonic on $r \leq |\lambda| \leq 1/r$ and the series $\sum_{k=k_0}^{\infty} k^{-2} \log |\lambda - 1/k|$ converges uniformly there. Hence v_1 is subharmonic on $\mathbb{C} - \{0\}$. On the other hand, since $|\lambda| < 1/2$ implies $|\lambda - 1/k| < 1$, v_1 is the limit of a decreasing sequence consisting of subharmonic function on $|\lambda| < 1/2$. Therefore v_1 is subharmonic on $|\lambda| < 1/2$.

(b) The first is seen straightforwards and the second follows from the integrability of the function $\lambda^{-2} \log \lambda$.

(c) By part (a), $\log R \circ \pi(\cdot, 1) = -v + \log(1 + |\cdot|^2)^{1/2}$ is lower semicontinuous on \mathbb{C} and so is $R \circ \pi(\cdot, 1)$. On the other hand, since $v_1(1/w) = \sum_{k=2}^{\infty} k^{-2} \log |1 - w/k| - (\pi^2/6 - 1) \times \log |w|$ for $w \in \mathbb{C} - \{0\}$, it follows that for $w \neq 0$,

$$\log R \circ \pi(1, w) = -\max \left\{ 0, \sum_{k=2}^{\infty} k^{-2} \log |1 - w/k| + (2 - \pi^2/6) \log |w| \right\} + \log(1 + |w|^2)^{1/2}.$$

Thereby, $R \circ \pi(1, \cdot)$ coincides with $(1 + |\cdot|^2)^{1/2}$ in some deleted neighborhood of 0 in \mathbb{C} , because $2 > \pi^2/6$. Hence $R \circ \pi(1, \cdot)$ is continuous in a neighborhood of 0, as desired.

LEMMA 6.2. D is a pseudoconvex circular domain in \mathbb{C}^2 .

PROOF. By Lemma 6.1, (c), D is a starlike circular domain with defining function R . Let Φ and φ be the functions given by (5.1) and in Theorem 5.4, respectively. Then $\varphi = v$ on \mathbb{C} , so φ is subharmonic, by Lemma 6.1, (a). To show the condition (#) in Theorem 5.4 to hold, fix $z_0 \in \mathbb{C} - \{0\}$. For $(z, w) \in \mathbb{C}^2$ with $z \neq 0$, we have

$$\Phi(z, w) = -\log R \circ \pi(1, w/z) + \log(|z|^2 + |w|^2)^{1/2}.$$

Since $R \circ \pi(1, \cdot)$ is continuous in a neighborhood of 0, by Lemma 6.1, (c), it follows that $\lim_{(z,w) \rightarrow (z_0,0)} \Phi(z, w) = \Phi(z_0, 0)$, so (#) holds. By Theorem 5.4, we obtain the pseudoconvexity of D .

LEMMA 6.3. D contains no entire holomorphic curve.

PROOF. Let $f = (f^1, f^2)$ be an arbitrary holomorphic mapping from C into D . We shall show that f is constant. Since $v \geq \log |\cdot|$, D is contained in the cylinder $\{z \in C; |z| < 1\} \times C$, so $|f^1| < 1$ on C . Hence f^1 is constant, say $f^1 = a, |a| < 1$. Set $D_a := \{\lambda \in C; (a, \lambda) \in D\}$. Then

$$(6.1) \quad f^2(C) \subset D_a .$$

First suppose $a = 0$. Since $D_0 = \{\lambda \in C; |\lambda| < e^{-v(0)}\}$ and since $v(0) \in R$ by Lemma 6.1, (b), f^2 is bounded (cf. (6.1)), so f^2 is constant.

Next suppose $a \neq 0$. If $\lambda \in D_a$, then

$$(6.2) \quad |\lambda| < e^{-v(a/\lambda)} \leq \exp\left(-\sum_{k=2}^{\infty} k^{-2} \log |a/\lambda - 1/k|\right) .$$

We consider the balls $\{\lambda \in C; |a/\lambda - 1/k| < 1/2k^3\}_{k=2,3,\dots}$, which are mutually disjoint. If $\lambda \in C$ lies outside all the balls, i.e., if

$$(6.3) \quad |a/\lambda - 1/k| \geq 1/2k^3 \quad \text{for all } k = 2, 3, \dots ,$$

then

$$-\sum_{k=2}^{\infty} k^{-2} \log |a/\lambda - 1/k| \leq \sum_{k=2}^{\infty} k^{-2}(\log 2 + 3 \log k) .$$

Hence, if this λ moreover satisfies an additional condition

$$(6.4) \quad |\lambda| \geq \exp\left(\sum_{k=2}^{\infty} k^{-2}(\log 2 + 3 \log k)\right) ,$$

then λ does not satisfy the condition (6.2). Therefore we have

$$D_a \subset \left\{ \lambda \in C; |\lambda| < \exp\left(\sum_{k=2}^{\infty} k^{-2}(\log 2 + 3 \log k)\right) \right\} \\ \cup \left(\bigcup_{k=2}^{\infty} \{\lambda \in C; |a/\lambda - 1/k| < 1/2k^3\} \right) .$$

This asserts that every connected component of D_a is a bounded subset in C . Since $f^2(C)$ lies in a connected component of D_a (cf. (6.1)), f^2 must be constant, as desired.

LEMMA 6.4. *D is not K-hyperbolic.*

PROOF. By virtue of (K) in §2, the assertion is equivalent to the unboundedness of D . But since $R(\pi(1/k, 1)) = (1 + k^2)^{1/2}$ for $k = 2, 3, \dots$ by Lemma 6.1, (b), D is not bounded. q.e.d.

Hence we have shown the following:

PROPOSITION 6.5. *There exists a pseudoconvex circular domain in C^2 which contains no entire holomorphic curve and is not K-hyperbolic.*

REMARK. For a pseudoconvex circular domain D in C^n , (H. 6) is

equivalent to (H. 7). Indeed, if $n = 1$ the assertion is trivial. Suppose $n \geq 2$ and assume (H. 6) does not hold. We shall show (H. 7) does not hold, either. By assumption there exists an affine line $f(\lambda) = \lambda a + b$, $\lambda \in \mathbb{C}$ ($a \in \mathbb{C}^n - \{0\}$, $b \in \mathbb{C}^n$) such that $f(\mathbb{C}) \subset D$, i.e., $\mathbb{C}a + b \subset D$. Here we may assume without loss of generality that

$$(6.5) \quad \{a, b\} \text{ is linearly independent over } \mathbb{C}.$$

Since D is starlike circular, we obtain

$$(6.6) \quad \mathbb{C}a + \{\mu \in \mathbb{C}; 0 < |\mu| < 1\}b \subset D.$$

Moreover, since the origin is an interior point of D , we have

$$(6.7) \quad \{\lambda \in \mathbb{C}; |\lambda| < \varepsilon\}a + \{\mu \in \mathbb{C}; |\mu| < \varepsilon\}b \subset D$$

for some $\varepsilon > 0$. By (6.5), (6.6) and (6.7) we obtain

$$\mathbb{C}a + \{\mu \in \mathbb{C}; |\mu| < 1\}b \subset D,$$

because D is a domain of holomorphy. It follows that $R(\pi(a)) = +\infty$, hence (H. 7) does not hold, as desired.

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