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ON SOAP BUBBLES AND ISOPERIMETRIC REGIONS IN NON-COMPACT SYMMETRIC SPACES, I

Dedicated to Professor Ichiro Satake

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1. Introduction. In a given Riemannian manifold M^n, a *soap bubble* is a closed hypersurface of constant mean curvature; an *isoperimetric region* is a compact region which achieves the minimal $(n-1)$-dimensional Hausdorff measure of the boundary among all compact regions of the same n-dimensional Hausdorff measure. The former is a type of distinguished global geometric object characterized by one of the simplest local condition and the latter is a solution to one of the most fundamental geometric variation problems. Moreover, they are conceptually closely related in such a way that the *regular part* of the boundary of an isoperimetric region is a hypersurface of constant mean curvature.

In the simplest basic case of the Euclidean spaces, E^n, $n \geq 3$, the existence of *homothety transformations* enable us to *normalize* the constant of mean curvature to be equal to 1, i.e. $\operatorname{tr} \mathrm{II} = n-1$, and the spheres of unit radius are obvious examples of such soap bubbles in E^n. In fact, up to 1981, the round spheres were the only known examples of soap bubbles in E^n and the hyperbolic spaces H^n; and the beautiful uniqueness theorems of Hopf [Ho] and Alexandrov [Al-1], [Al-2] had, somehow, misled many geometers to believe that they are most likely the only possible ones (cf. the remark of Alexandrov in [Al-2]). The discovery of infinitely many congruent classes of *spherical* soap bubbles of mean curvature 1 in E^n for all $n \geq 4$ in 1982 [HTY-1], [HTY-2], [Hs-1] and the recent examples of Wente [We], Abresh [Ab], Kapouleas [Ka-1], [Ka-2] and Pinkall–Sterling [PS] on soap bubbles in E^3 with non-zero genus clearly indicate that there are abundant varieties of soap bubbles in E^n, $n \geq 3$, for us to discover and to understand. On the other hand, the situation of isoperimetric regions in E^n is quite different. Namely, it has already been neatly settled by the unique-existence theorem of Dinghas–Schmidt in the 1940's [DS].

Of course, the Euclidean spaces are just natural starting testing spaces for the study of both the soap bubbles and the isoperimetric regions. Therefore, it is rather natural to broaden the scope to include other fundamental Riemannian manifolds as the ambient spaces. For example, those simply connected, *non-positively curved* symmetric spaces are natural generalizations of E^n and H^n and they form an interesting family of testing spaces for the study of soap bubbles and isoperimetric regions. This family consists of those symmetric spaces of non-compact type and the products of such spaces with Euclidean spaces. We shall simply call them *non-compact symmetric spaces*.

In this paper we shall mainly concern ourselves with the *existence aspect* of soap bubbles and the *isoperimetric profiles* of non-compact symmetric spaces. Notice that, in spaces other than the Euclidean spaces, one no longer has the homothety transformation to normalize the values of the mean curvature of soap bubbles and the "distance spheres" are *not* soap bubbles in symmetric spaces of ranks ≥ 2. Therefore, the actual *range of values* of the constant of mean curvatures of all soap bubbles in a *given ambient space* becomes a basic issue on the existence aspect and, moreover, "easy examples" of soap bubbles are no longer readily available for non-compact symmetric spaces of ranks ≥ 2. In the following discussions, it is slightly more convenient to use the sum of the principal curvatures, i.e. $\operatorname{tr} \mathrm{II}$, instead of their mean value. Therefore, from now on, the mean curvature of a hypersurface is *redefined* to be just the *trace* of *its second fundamental form*. We state the main results of this paper as the following theorems:

THEOREM 1. *To each given non-compact symmetric space M, there exists a lower bound $b(M)$ such that*

(i) *the mean curvature of any soap bubble in M is strictly greater than $b(M)$.*

(ii) *The mean curvature of the (regular part) boundary of any isoperimetric region in M is strictly greater than $b(M)$.*

(iii) *In the case that M is* irreducible *and equipped with the normalized metric (cf. §2),*

$$b(M) = \max\left\{ \sum_{\alpha \in \Delta(M)} m(\alpha) \cdot |\langle \alpha, v \rangle|;\ |v| = 1 \right\}$$

where $\Delta(M)$ is the root system of M and $m(\alpha)$ is the multiplicity of α (cf. Table I *at the end of* §2 *for the actual value of $b(M)$ for each irreducible non-compact symmetric space of rank* 2).

(iv) $b(\lambda M) = \lambda^{-1} b(M)$, *where λM denotes the magnification of M by a factor λ*

(v) $b(\prod_{j=1}^{l} M_j) = \{\sum_{j=1}^{l} b(M_j)^2\}^{1/2}$.

THEOREM 2. *If the rank of the non-compact symmetric space M is less than or equal to* 2, *then every value $h > b(M)$ can be realized as the mean curvature of an imbedded, spherical soap bubble in M.*

THEOREM 3. *If* $\operatorname{rk}(M) = 2$ *and the dimension of its euclidean factor is not equal to* 1, *then there exist infinitely many congruence classes of immersed, spherical soap bubbles in M with each $h > b(M)$ as the constant of mean curvature.*

THEOREM 4. *If $M = \prod_{j=1}^{l} M_j$ and* $\operatorname{rk}(M_j) = 1$, *then to each given value of $v > 0$, there always exists an imbedded, spherical K-invariant soap bubble in M which bounds a region of volume equal to v.*

CONJECTURE 1. *Every value* $h > b(M)$ *can always be realized as the mean curvature of an* imbedded, spherical, stable soap bubble *in* M; *namely, the actual range of the mean curvatures of soap bubbles in* M *should be exactly the open interval* $(b(M), \infty)$.

CONJECTURE 2. *Let* Ω *be an isoperimetric region in* M *and* $h(\partial\Omega)$ *be the (constant) mean curvature of* $\partial\Omega$ *at its regular points. We conjecture that* $h(\partial\Omega) \to b(M)$ *as* $\mathrm{vol}(\Omega) \to \infty$.

REMARKS. (i) The generalization of the uniqueness theorem of A. D. Alexandrov on *imbedded* soap bubbles in E^n and H^n to the realm of non-compact symmetric spaces is, indeed, a rather attractive prospect and a challenging task. Of course one should first establish the existence of such objects before one proceeds to prove their uniqueness.

(ii) If isoperimetric regions in a given non-compact symmetric space M can be proved to be always *regular*, then the boundary of isoperimetric regions provides a natural source of imbedded soap bubbles. Such a regularity theorem has already been established for products of euclidean and hyperbolic spaces in [Hs-2] and we believe that it should hold for all non-compact symmetric spaces.

(iii) The *isoperimetric profile* of a given space M is, by definition, the function which records the "areas" of the boundaries of isoperimetric regions in M as a function of their "volumes". It is actually the generalization of "*optimal isoperimetric inequality*" for the given space M (cf. § 5).

2. The geometry of K-orbits and the proof of Theorem 1. Let M be a simply connected, non-positively curved symmetric space, G be the identity component of its isometry group, and K be the isotropy subgroup of M fixing a chosen base point 0. In this section we shall first determine the *orbital geometry* of (K, M) that the proof of Theorem 1 is based upon. Let (K, T_0M) be the isotropy representation of K on the tangent space of M at 0. Then the exponential map, $\mathrm{Exp}: T_0M \to M$, is a K-equivariant diffeomorphism. Therefore, the orbital geometry of (K, T_0M) is a natural convenient stepping stone in the study of the orbital geometry of (K, M).

2.1. The orbital geomery of (K, T_0M). Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively, σ be the involution of $\mathfrak{g}$ induced by the central symmetry of M at the base point 0 and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into the ± 1 eigenspaces of σ. Then

$$(1) \qquad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k} \quad \text{and} \quad (K, T_0M) \cong (K, \mathfrak{p})$$

where $(K, \mathfrak{p})$ is the restriction of Ad_K to $\mathfrak{p}$. Let

$$G/K = M = \prod_{j=1}^{l} M_j = \prod_{j=1}^{l} (G_j/K_j)$$

be the decomposition of M into the product of irreducible symmetric spaces. Then

$$(2) \qquad K=\prod_{j=1}^{l} K_j\,, \quad \mathfrak{p}=\oplus\sum\mathfrak{p}_j\,, \quad (K,\mathfrak{p})=\oplus\sum(K_j,\mathfrak{p}_j)\cong\oplus\sum(K_j,T_0M_j)$$

is the corresponding decomposition of $(K, \mathfrak{p})$ into the outer direct sum of irreducible components. Therefore, it is straightforward to reduce the orbital geometry of $(K, \mathfrak{p})$ to that of its irreducible components $(K_j, \mathfrak{p}_j)$. Hence, for the discussion of K-orbits, one may assume that $M=G/K$ is itself an irreducible symmetric space to begin with.

Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. An element $x_0\in\mathfrak{a}$ is called regular (resp. singular) if

$$(3) \qquad \mathfrak{z}(x_0,\mathfrak{p})=\{x\in\mathfrak{p};\ [x_0,x]=0\}=\mathfrak{a}(\text{resp.}\neq\mathfrak{a})\,.$$

$\mathfrak{a}$ intersects all K-orbits *perpendicularly* and $K(x_0)$, $x_0\in\mathfrak{a}$, is a *principal* K-orbit if and only if x_0 is a regular element. The singular set of $\mathfrak{a}$ is the union of a finite collection of hyperplanes in $\mathfrak{a}$. Let $\tilde{\Delta}$ be the set of such hyperplanes of singular elements, say $\tilde{\Delta}=\{\mathfrak{b}_i;\ 1\le i\le m\}$. To each $\mathfrak{b}_i\in\tilde{\Delta}$, set

$$(4) \qquad \begin{aligned} &\mathfrak{p}_i=\mathfrak{z}(\mathfrak{b}_i,\mathfrak{p})=\{x\in\mathfrak{p};\ [\mathfrak{b}_i,x]=0\}\\ &\mathfrak{k}_i=[\mathfrak{p}_i,\mathfrak{p}_i]\,, \quad \mathfrak{g}_i=\mathfrak{k}_i\oplus\mathfrak{p}_i\,,\\ &\mathfrak{p}_i'\text{: the orthogonal complement of } \mathfrak{b}_i \text{ in } \mathfrak{p}_i\,,\\ &\mathfrak{g}_i'=\mathfrak{k}_i\oplus\mathfrak{p}_i'\,. \end{aligned}$$

Then it is straightforward to verify that both $\mathfrak{g}_i$ and $\mathfrak{g}_i'$ are involutive subalgebras of $\mathfrak{g}$, $\mathfrak{g}_i=\mathfrak{b}_i\oplus\mathfrak{g}_i'$, and one has the following orthogonal decomposition of $\mathfrak{p}$

$$(5) \qquad \mathfrak{p}=\mathfrak{a}\oplus\sum_{i=1}^{m}\tilde{\mathfrak{p}}_i$$

where $\tilde{\mathfrak{p}}_i$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}_i$.

Geometrically, the orbits of $(K, \mathfrak{p})$ forms an *isoparametric foliation* with the principal K-orbits as the isoparametric submanifolds (cf. [Te], [HPT]) and the singular K-orbits as the focal varieties. The associated Coxeter group W of the above isoparametric foliation is generated by the reflections with respect to the hyperplanes of $\tilde{\Delta}$. In fact, the set of maximal abelian subalgebras of $\mathfrak{p}$ can also be geometrically characterized simply as the normal planes to a *fixed principal K-orbit*, and hence they are clearly K-conjugate to each other. Let C_0 (resp. $\bar{C}_0$) be an open (resp. closed) Weyl chamber of $(W, \mathfrak{a})$ and K_i be the connected subgroup of K with $\mathfrak{k}_i$ as its Lie algebra. Then

(i) the orbit space $\mathfrak{p}/K$, equipped with the orbital distance metric, can be identified with $\bar{C}_0$ isometrically, namely, $\mathfrak{p}/K\cong\mathfrak{a}/W\cong\bar{C}_0$,

(ii) to each point $x_0\in C_0$, $K_i(x_0)$ is a sphere of radius $r_i=d(x_0,\mathfrak{b}_i)$ in $\mathfrak{p}_i'$,

(iii) the tangent space of $K(x_0)$ at x_0 splits into the *orthogonal direct sum* of that of $K_i(x_0)$.

2.2. The orbital geometry of (K, M). The exponential map at the base point $0 \in M$ is a K-equivariant diffeomorphism, $\mathrm{Exp}: (K, \mathfrak{p}) \to (K, M)$. Moreover, one has the following clean-cut correspondence between $(K, \mathfrak{p})$ and (K, M):

(i) $A = \mathrm{Exp}(\mathfrak{a})$ is a *maximal flat totally geodesic* submanifold which intersects all K-orbits in M perpendicularly.

(ii) $M/K \cong A/W \cong \bar{C}_0$ isometrically, namely, the orbital distance metric of M/K is a flat cone $\bar{C}_0$.

(iii) Let G_i, G_i' and K_i, $1 \leq i \leq m$, be subgroups of G with $\mathfrak{g}_i$, $\mathfrak{g}_i'$ and $\mathfrak{k}_i$ as their Lie algebras respectively. Then

$$M_i = \mathrm{Exp}(\mathfrak{p}_i) = G_i/K_i\,, \quad M_i' = \mathrm{Exp}(\mathfrak{p}_i') = G_i'/K_i$$

are non flat totally geodesic submanifolds of M and, moreover,

$$\mathrm{rk}(M_i') = 1 \quad \text{and} \quad M_i = M_i' \times B_i\,, \quad B_i = \mathrm{Exp}(\mathfrak{b}_i)\,.$$

(iv) Identifying $\mathrm{Exp}(\bar{C}_0)$ with $\bar{C}_0$, then, to each $x \in \bar{C}_0$, $K(x)$ is a principal orbit if and only if x is an interior point of $\bar{C}_0$, namely, $x \in C_0$.

(v) To each $x_0 \in C_0$, $K_i(x_0)$ is a "distance sphere" in the rank one symmetric space M_i' with $d(x_0, B_i)$ as its "radius". The tangent space of $K(x_0)$ at x_0 is the *orthogonal direct sum* of that of $K_i(x_0)$ at x_0.

Next let us proceed to determine the volume function, $v: M/K \cong \bar{C}_0 \to \boldsymbol{R}^+$, which records the volumes of *pincipal* K-orbits in M (the volumes of singular orbits are defined to be zero because they are of lower dimensions). Recall that M is assumed to be an irreducible symmetric space of non-compact type. We refer to §2, Ch. VII of [He] for the definition of the root system of M, denoted by $\Delta(M)$. It is easy to check that $\Delta(M)$ is the disjoint union of $\Delta(M_i')$, $1 \leq i \leq m$, as sets with multiplicities, namely

$$\Delta(M) = \bigcup_{i=1}^{m} \Delta(M_i')\,. \tag{6}$$

In the special case of rank one symmetric spaces of non-compact type, one usually normalizes the metrics so that the sectional curvature is equal to -1 for H^n and lies between -4 and -1 for HCP^n, HQP^n, $n \geq 2$, and $HCaP^2$. It is not difficult to show that the volumes of the "distance spheres" of radius r in the above spaces are as follows:

$$\begin{aligned} &H^n\colon v(r) = c_n \cdot \sinh^{n-1} r \\ &HCP^n\colon v(r) = c_n' \cdot \sinh^{2n-2} r \cdot \sinh 2r \\ &HQP^n\colon v(r) = c_n'' \cdot \sinh^{4n-4} r \cdot \sinh^3 2r \\ &HCaP^2\colon v(r) = c \cdot \sinh^8 r \cdot \sinh^7 2r\,. \end{aligned} \tag{7}$$

It follows from (v) that

$$\text{vol}(K(x_0)) = c_M \cdot \prod_{i=1}^{m} \text{vol}(K_i(x_0)) \tag{8}$$

where c_M is a positive constant depending only on M. Hence it follows from (6), (7) and (8) that, for a suitably normalized metric on M, the volume of principal K-orbits in M can be given as follows, namely,

$$v(x) = \text{vol}(K(\text{Exp}\, x)) = c'_M \cdot \prod_{\alpha \in \Delta(M)} \sinh^{m(\alpha)} |\langle \alpha, x \rangle| , \qquad x \in C_0 \tag{9}$$

where $m(\alpha)$ is the multiplicity of α in $\Delta(M)$ and c'_M is a positive constant depending only on M.

2.3. The proof of Theorem 1. (i) Let M be a given non-compact symmetric space; $K = \text{ISO}(M, 0)$ is the isotropy subgroup of the chosen base point $0 \in M$ and $K(x_0)$ is an arbitrary but fixed *principal* K-orbit. Then there exists a (unique) flat, complete, totally geodesic submanifold A which intersects $K(x_0)$ at x_0 perpendicularly and transversally, namely, $T_{x_0}M = T_{x_0}A \oplus T_{x_0}K(x_0)$. Let $N_K(A)$ (resp. $Z_K(A)$) be the subgroup of K which keeps A invariant (resp. pointwise fixed) and set $W = N_K(A)/Z_K(A)$. Then (W, A) is a group generated by reflections and $M/K \cong A/W \cong \bar{C}_0$ where $\bar{C}_0$ is a chosen closed Weyl chamber of (W, A).

Let Σ be a given soap bubble in M, $0'$ be its center of gravity, and $r = \text{Max}\{d(0', x); x \in \Sigma\}$. The open Weyl chamber C_0 is an open cone in $A \cong \boldsymbol{R}^k$, $k = \text{rk}(M)$. To a given ray $\overrightarrow{0a}$ in C_0, let v be the unit vector in the direction of $\overrightarrow{0a}$ and $D_v(s, t)$ be the normal disc of $\overrightarrow{0a}$ centered at the point on $\overrightarrow{0a}$ with distance t from 0 and with radius s. Set $s = 2r$ and t sufficiently large so that the above $D_v(s, t)$ is far away from the boundary of $\bar{C}_0$. Let $P_v(s, t) = K(D_v(s, t))$ be the K-invariant hypersurface in M generated by $D_v(s, t)$ under the action of K and T_v be the 1-parameter subgroup of *transvections* along the geodesic ray $\overrightarrow{0a}$.

Let us first move the given soap bubble Σ by a transvection which maps $0'$ to 0, and then push it by the action of T_v until it *touches* $P_v(s, t)$. Suppose x'_0 is one of the touching points and set $x_0 = K(x'_0) \cap D_v(s, t)$. Then it is easy to see that the mean curvature of Σ, $h(\Sigma)$, must be greater than or equal to the mean curvature of $P_v(s, t)$ at x_0. Let $v\colon \bar{C}_0 \to \boldsymbol{R}^+$ be the *volume function* of (K, M). By (9),

$$v(x) = c_M \cdot \prod_{\alpha \in \Delta(M)} \sinh^{m(\alpha)} \lambda_\alpha \cdot |\langle x, \alpha \rangle| \tag{9$'$}$$

where c_M and λ_α are constants solely depending on the metric of M and $\alpha \in \Delta(M)$. Since the mean curvature of $P_v(s, t)$ at x_0 with respect to the inward normal is equal to $(d/dv) \ln v(x)|_{x_0}$, one has

$$h(\Sigma)\geq\frac{d}{dv}\ln v(x)|_{x_0}=\sum_{\alpha\in\Delta(M)} m(\alpha)\lambda_\alpha\cdot|\langle\alpha, v\rangle|\cdot\coth\lambda_\alpha\cdot|\langle\alpha, x_0\rangle| \tag{10}$$

$$>\sum_{\alpha\in\Delta(M)} m(\alpha)\cdot\lambda_\alpha\cdot|\langle\alpha, v\rangle|$$

because the values of coth are always >1. Therefore,

$$h(\Sigma)>\operatorname{Max}\left\{\sum_{\alpha\in\Delta(M)} m(\alpha)\cdot\lambda_\alpha\cdot|\langle\alpha, v\rangle|;\ v\in\mathfrak{a}\text{ and }|v|=1\right\}=b(M) \tag{11}$$

(cf. §2.4 for the computation of the above $b(M)$).

(ii) Next let Ω be an isoperimetric region in M, $\partial\Omega$ be its boundary and $0'$ be its center of gravity. Then the singular set of $\partial\Omega$ (may be empty) is of codimension at least 7 in $\partial\Omega$ and the regular set, denoted by $R(\partial\Omega)$, is a connected hypersurface of constant mean curvature. Again, one may move Ω by suitable transvections to have a touching contact with $P_v(s, t)$. We claim that such a touching point *cannot be singular*, because the tangent cone at a singular point cannot be confined in a half space! Therefore, the touching point must be regular and hence the constant mean curvature of $R(\partial\Omega)$ must be strictly greater than $b(M)$.

In case M is *irreducible* and equipped with the *normalized* metric, all the above coefficients λ_α are equal to 1. Therefore, the expression of $b(M)$ is simplified to that of (iii).

Finally, suppose $M=\prod_{j=1}^{l} M_j$. Then

$$\begin{aligned}(W, A)&=\bigoplus\sum_{j=1}^{l}(W_j, A_j)\,, \quad \text{[orthogonal direct sum]}\,,\\ \Delta(M)&=\bigcup_{j=1}^{l}\{\Delta(M_j)\}\,, \quad \text{[orthogonal disjoint union]}\,.\end{aligned} \tag{12}$$

Let v_j be the component of v in A_j. Then

$$\sum_{\alpha\in\Delta(M)} m(\alpha)\lambda_\alpha\cdot|\langle\alpha, v\rangle|=\sum_{j=1}^{l}\sum_{\alpha_j\in\Delta(M_j)} m(\alpha_j)\cdot\lambda_{\alpha_j}\cdot|\langle\alpha_j, v_j\rangle| \tag{13}$$

and hence it follows that

$$\begin{aligned}b(M)&=\operatorname{Max}\left\{\sum_{\alpha\in\Sigma(M)} m(\alpha)\cdot\lambda_\alpha\cdot|\langle\alpha, v\rangle|;\ |v|=1\right\}\\ &=\operatorname{Max}\left\{\sum_{j=1}^{l}|v_j|\cdot b(M_j);\ \Sigma|v_j|^2=1\right\}=\left\{\sum_{j=1}^{l} b(M_j)^2\right\}^{1/2}.\end{aligned} \tag{14}$$

This completes the proof of Theorem 1.

REMARK. The orbital geometry of (K, M) enables us to construct the large K-invariant hypersurface $P_v(s, t)$ whose mean curvature function is easily computable and very close to a constant. Moreover, the transvections of M enable us to produce a touching contact point between a given soap bubble Σ and the above $P_v(s, t)$, thus providing a lower bound estimate on $h(\Sigma)$.

2.4. The computation of $b(M)$ for irreducible symmetric spaces of non-compact type with normalized metrics. The lower bound, $b(M)$, of a general non-compact symmetric space can easily be expressed in terms of that of irreducible non-compact symmetric spaces with normalized metrics (cf. (iv) and (v) of Theorem 1). Therefore it is useful to work out a table of $b(M)$ for those irreducible normalized ones.

Let M be a given irreducible non-compact symmetric space with normalized metric, $\Delta(M)$ be its root system and C_0 be a chosen Weyl chamber of $(W, \mathfrak{a})$. Then

$$F(v) = \sum_{\alpha \in \Delta(M)} m(\alpha) \cdot |\langle \alpha, v \rangle| \tag{15}$$

is clearly a W-invariant function defined on the unit sphere of $\mathfrak{a}$ and hence

$$\mathrm{Max}\{F(v)\} = \mathrm{Max}\{F(v);\ v \in \bar{C}_0 \text{ and } |v| = 1\}. \tag{16}$$

Let $\Delta^+(M)$ (resp. $\Delta^-(M)$) be the system of positive (resp. negative) roots of M *with respect to* C_0 and v be a unit vector in $\bar{C}_0$. Then

$$\begin{aligned} F(v) &= \sum_{\alpha \in \Delta^+(M)} m(\alpha) \langle \alpha, v \rangle - \sum_{\beta \in \Delta^-(M)} m(\beta) \cdot \langle \beta, v \rangle \\ &= \left\langle \sum_{\alpha \in \Delta^+(M)} m(\alpha) \cdot \alpha - \sum_{\beta \in \Delta^-(M)} m(\beta) \cdot \beta, v \right\rangle, \qquad v \in \bar{C}_0. \end{aligned} \tag{17}$$

Therefore it is easy to see that

$$b(M) = \mathrm{Max}\{F(v);\ v \in \bar{C}_0 \text{ and } |v| = 1\} = \left| \sum_{\alpha \in \Delta^+(M)} m(\alpha) \cdot \alpha - \sum_{\beta \in \Delta^-(M)} m(\beta) \cdot \beta \right|. \tag{18}$$

For example, straightforward actual computation of (18) for each irreducible non-compact symmetric space of rank 2 will produce Table I.

3. *K*-Invariant soap bubbles and the proofs of Theorems 2 and 3. As it was already pointed out in § 1, one of the basic issues in the *existence* aspect of soap bubbles in a given ambient space M is the determination of the actual *range of values* of their mean curvatures. For the family of non-compact symmetric spaces, we believe that the lower bound, $b(M)$, of Theorem 1 should be exactly the greatest lower bound and, moreover, the whole open interval $(b(M), \infty)$ should be the actual range of mean curvatures of soap bubbles in M. The rank one case is rather easy to varify because the "*distance spheres*" provide a family of simple examples of soap bubbles in a *rank one* symmetric space. However, such simple examples of soap bubbles are *no longer*

TABLE I

Diagram with multi.	Isotropy subgroup K	$\dim M$	$b(M)$
$\overset{1}{\circ}\!-\!\overset{1}{\circ}$	$SO(3)$	5	4
$\overset{2}{\circ}\!-\!\overset{2}{\circ}$	$SU(3)$	8	8
$\overset{4}{\circ}\!-\!\overset{4}{\circ}$	$Sp(3)$	14	16
$\overset{8}{\circ}\!-\!\overset{8}{\circ}$	F_4	26	32
$\overset{2}{\circ}\!=\!\overset{2}{\circ}$	$SO(5)$	10	$2\sqrt{10}$
$\overset{m-2}{\circ}\!=\!\overset{1}{\circ}$	$SO(2)\times SO(m)$	$2m$	$\sqrt{2m^2-4m+4}$
$\overset{2m-3}{\circ}\!=\!\overset{2}{\circ}$	$S(U(2)\times U(m))$	$4m$	$2\cdot\sqrt{2m^2+2}$
$\overset{4m-5}{\circ}\!=\!\overset{4}{\circ}$	$Sp(2)\times_{\mathbf{Z}_2}Sp(m)$	$8m$	$2\cdot\sqrt{8m^2+8m+10}$
$\overset{5}{\circ}\!=\!\overset{4}{\circ}$	$U(5)$	20	$2\cdot\sqrt{58}$
$\overset{9}{\circ}\!=\!\overset{6}{\circ}$	$U(1)\times_{\mathbf{Z}_2}Spin(10)$	32	$2\cdot\sqrt{146}$
$\overset{2}{\circ}\!\equiv\!\overset{2}{\circ}$	G_2	14	$2\cdot\sqrt{28/3}$
$\overset{1}{\circ}\!\equiv\!\overset{1}{\circ}$	$SO(4)$	8	$\sqrt{28/3}$

available for non-compact symmetric spaces of ranks ≥ 2.

In this section we shall prove the existence of spherical soap bubbles for every given value of mean curvature $h>b(M)$ for all cases of *rank two* non-compact symmetric spaces, namely, Theorems 2 and 3 stated in § 1. The basic idea of the proof is to exploit the orbital geometry of K-orbits to construct those *K-invariant soap bubbles.*

3.1. K-invariant hypersurfaces of constant mean curvature in a rank two symmetric space. Let Σ be a K-invariant hypersurface of constant mean curvature in a given non-compact symmetric space M of rank two. Then the orbit space, M/K, equipped with the orbital distance metric is a *flat*, two-dimensional linear cone of angle π/g, $g=2$, 3, 4 or 6, and Σ/K is a curve in M/K which we shall call it the *generating curve* of Σ. [In the special case of $E^1\times M_2$, we shall take $K=O(1)\times K_2$ instead of $K=K_2$ thus making $g=2$ instead of $g=1$ for this special case.] It is convenient to parametrize M/K by the following coordinate system, namely,

$$M/K=\{(x, y);\ x\geq 0,\ 0\leq y\leq x\cdot\tan\pi/g\}\ .$$

Then the generating curves of those K-invariant hypersurfaces in M of constant mean curvature h can be neatly characterized by the following ODE [HsHu], namely

$$\frac{d\sigma}{ds}-\frac{d}{d\boldsymbol{n}}\ln v(x, y)=h \tag{19}$$

where σ is the directional angle, $d/d\boldsymbol{n}=(-\sin\sigma(\partial/\partial x)+\cos\sigma(\partial/\partial y))$ and $v(x, y)$ is the volume function which records the volume of the principal K-orbits (cf. Figure 1).

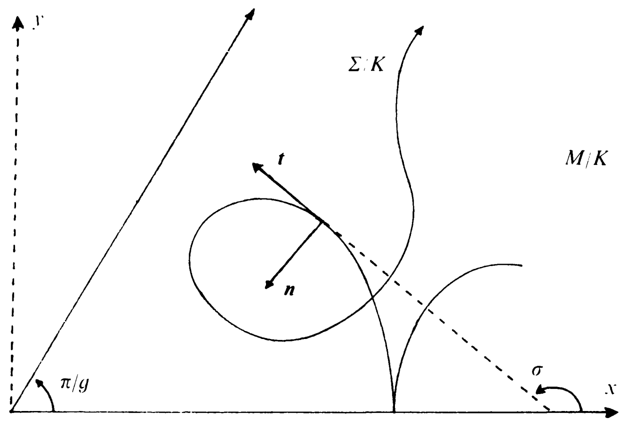

FIGURE 1

REMARKS. (i) The boundary points represent singular K-orbits, i.e. orbits of dimensions lower than that of the principal K-orbits and hence the volume function, $v(x, y)$, vanishes at all the boundary points. Therefore, the above ODE becomes singular at the boundary.

(ii) The singularities of the above ODE are of regular type. To each boundary point $B \neq (0, 0)$, there exists a unique solution curve of (19) which terminates (resp. starts) at such a singular point and, moreover, it is *automatically perpendicular* to the boundary line and *necessarily analytic* (we refer to Proposition 1 of [HH-1] for a proof of the above fact).

(iii) Σ is a K-invariant soap bubble if and only if its generating curves both start and terminate at the boundary lines, and Σ is of the diffeomorphism type of sphere if and only if it starts and terminates at *different* boundary lines. Therefore, the proof of Theorem 2 (resp. 3) amounts to showing the existence of an imbedded (resp. infinitely many immersed) solution curves of the ODE (19) of the above geometric type.

3.2. The proof of Theorems 2 and 3 for the reducible cases. Let M be a given reducible non-compact symmetric space of rank 2, namely, $M = M_1 \times M_2$ and

$$M_i = \begin{cases} E^n, & n \geq 1 \\ H^n, & n \geq 2 \\ HCP^n, & n \geq 2 \quad (i=1, 2) . \\ HQP^n, & n \geq 2 \\ HCaP^2 . & \end{cases}$$

Then $K = K_1 \times K_2$, $M/K = (M_1/K_1) \times (M_2/K_2) = \boldsymbol{R}^2_+$ and the volume function $v(x, y) = v_1(x) \cdot v_2(y)$. Therefore, the ODE (19) is of the following simple form, namely

(20)
$$\frac{d\sigma}{ds}+\sin\sigma\cdot\frac{v_1'(x)}{v_1(x)}-\cos\sigma\cdot\frac{v_2'(y)}{v_2(y)}=h$$

where the lower bound of h is given by

(21)
$$b(M)=\{b(M_1)^2+b(M_2)^2\}^{1/2}, \qquad b(M_i)=\lim_{r\to\infty}\left(\frac{v_i'(r)}{v_i(r)}\right).$$

Set

(22)
$$g_1(x)=\int_0^x v_1(t)dt, \qquad g_2(y)=\int_0^y v_2(t)dt$$
$$I=v_1(x)\sin\sigma-hg_1(x), \qquad J=v_2(y)\cos\sigma+hg_2(y).$$

Then, along an arbitrary solution curve $\gamma=\{(x(s), y(s))\}$, one has

(23)
$$\frac{dI}{ds}=v_1(x)\cos\sigma\left\{\frac{d\sigma}{ds}+\frac{v_1'(x)}{v_1(x)}\sin\sigma-h\right\}=v_1(x)\frac{v_2'(y)}{v_2(y)}\cos^2\sigma\geq 0$$
$$\frac{dJ}{ds}=v_2(y)\sin\sigma\left\{-\frac{d\sigma}{ds}+\frac{v_2'(y)}{v_2(y)}\cos\sigma+h\right\}=v_2(y)\frac{v_1'(x)}{v_1(x)}\sin^2\sigma\geq 0.$$

A *global* solution curve is, by definition, a solution curve of (20) which is infinitely extendable in both directions, i.e. $\gamma=\{(x(s), y(s)), -\infty<s<+\infty\}$. If a solution curve γ has a boundary point, then it is quite natural to continue it in the reverse direction with a *cusp point*. Therefore it is not difficult to show that every solution curve of (20) can be *uniquely* extended to a global one, possibly with cusp points on the *singular* boundary. Moreover, it follows from the monotonicity of I and J that a global solution curve can have *at most* one cusp on the y-axis and the x-axis respectively.

The analysis and the geometry of global solution curves of ODE of the above type had already been quite thoroughly discussed in [Hs-1] and [HH-2], namely, that of the special case of $E^p\times E^q$ in [Hs-1] and that of the special cases of $E^p\times H^q$ and $H^p\times H^q$ in [HH-2]. In fact, the following basic facts on the geometry of global solution curves of (20) can be shown by essentially the same proofs as that of [Hs-1] and [HH-2]:

(i) There are exactly two straight line solutions (i.e. solution curves with $d\sigma/ds\equiv 0$), namely

the horizontal line $y=y_0$ with $\frac{v_2'(y_0)}{v_2(y_0)}=h$

the vertical line $x=x_0$ with $\frac{v_1'(x_0)}{v_1(x_0)}=h.$

(ii) An *arbitrary global* solution curve of (21) must tend to the above horizontal (resp. vertical) line *asymptotically* as s tends to $-\infty$ (resp. $+\infty$).

(iii) On each given global solution curve γ there exists a unique "direction function", $\sigma_\gamma(s)$, such that $\lim_{s\to-\infty}\sigma_\gamma(s)=\pi$ and $\sigma_\gamma(s)$ is continuous except having a jump of π at each cusp point. It follows from (ii) that $\lim_{s\to+\infty}\sigma_\gamma(s)$ exists and is equal to $\pi/2$ *modulo an integral multiple of* 2π, namely

$$\lim_{s\to+\infty}\sigma_\gamma(s)=N(\gamma)\cdot 2\pi+\frac{\pi}{2} \tag{24}$$

where $N(\gamma)$ is a non-negative integer that we shall call the *winding number* of γ.

(iv) There exist global solution curves of (20) with arbitrarily large winding numbers.

It is natural to classify the global solution curves of (20) into the following five types according to the cusp points they contain, namely:

Type A: without cusp point
Type B: with exactly one cusp point on the x-axis
Type C: with exactly one cusp point on the y-axis
Type D: with two cusp points (must have one on each axis)
Type E: with one cusp point at the origin.

Following the same kind of arguments as that of [Hs-1], it is not difficult to prove the following basic existence result.

THEOREM 5. *Let γ be a global solution curve of* (20) *with $h>b(M)$. Then the winding number of γ*

$$N(\gamma)=1 \qquad \textit{if} \quad \gamma \textit{ is of type } \mathrm{E}$$

$$N(\gamma)\geq\begin{Bmatrix}0\\1\\2\end{Bmatrix} \qquad \textit{if} \quad \gamma \textit{ is of type } \begin{Bmatrix}\mathrm{A}\\ \mathrm{B} \textit{ or } \mathrm{C}\\ \mathrm{D}\end{Bmatrix}.$$

Conversely, any integer k satisfying the above inequality can always be realized as the winding number of a global solution curve of the respective type.

It is then easy to deduce both Theorems 2 and 3 for the *reducible cases* from the above existence of global solution curves of type D of arbitrary winding number $k\geq 2$. In fact, the middle segments between the two cusp points of those type D curves are exactly the generating curves of those K-invariant spherical soap bubbles in M.

3.3. The proof of Theorems 2 and 3 for the irreducible cases. Let M be an irreducible, rank two, symmetric space of non-compact type equipped with the normalized metric. Then the orbit space M/K equipped with the orbital distance metric is a *flat* two-dimensional linear cone of angle π/g, $g=3$, 4 or 6, and the volume function is given in terms of its root system as follows, namely

$$v(\xi)=c\cdot\prod_{\alpha\in\Delta(M)}\sinh^{m(\alpha)}|\alpha(\xi)|,\qquad \xi\in M/K\cong\bar{C}_0\,. \tag{25}$$

In terms of the cartesian coordinate system of $\bar{C}_0$ (cf. §3.1), the *modified root systems*, i.e., the set with multiplicities $\bar{\Delta}=\Delta^+(M)\cup\{-\beta,\,\beta\in\Delta^-(M)\}$, of the irreducible rank two cases can be listed as follows:

(i) $\overset{k}{\circ}\!-\!\!\!-\!\!\!-\overset{k}{\circ}$, $k=1, 2, 4, 8$:

$$\bar{\Delta}=\left\{y,\ \cos\frac{\pi}{6}x\pm\sin\frac{\pi}{6}y,\ \text{multi.}=k\right\}$$

(ii) $\overset{k}{\circ}\!\equiv\!\overset{k}{\circ}$, $k=1, 2$:

$$\bar{\Delta}=\left\{y,\ \cos\frac{\pi}{6}x\pm\sin\frac{\pi}{6}y;\ \frac{x}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\left(\cos\frac{\pi}{3}x\pm\sin\frac{\pi}{3}y\right);\ \text{multi.}=1, 2\right\}$$

(iii) $\overset{2}{\circ}\!=\!\overset{2}{\circ}$, $K=SO(5)$

$$\bar{\Delta}=\{x,\ y,\ x\pm y,\ \text{multi.}=2\}$$

(iv) $\overset{m-2}{\circ}\!=\!\overset{1}{\circ}$, $K=SO(2)\times SO(m)$:

$$\bar{\Delta}=\{x,\ y,\ \text{multi.}=(m-2);\ x\pm y,\ \text{multi.}=1\}$$

(v) $\overset{2m-3}{\circ}\!=\!\overset{2}{\circ}$, $K=S(U(2)\times U(m))$:

$$\bar{\Delta}=\{x,\ y,\ \text{multi.}=2(m-2);\ 2x,\ 2y,\ \text{multi.}=1;\ x\pm y,\ \text{multi.}=2\}$$

(vi) $\overset{4m-5}{\circ}\!=\!\overset{4}{\circ}$, $K=Sp(2)\times_{Z_2}Sp(m)$:

$$\bar{\Delta}=\{x,\ y,\ \text{multi.}=4(m-2);\ 2x,\ 2y,\ \text{multi.}=3;\ x\pm y,\ \text{multi.}=4\}$$

(vii) $\overset{5}{\circ}\!=\!\overset{4}{\circ}$, $K=U(5)$:

$$\bar{\Delta}=\{x,\ y,\ \text{multi.}=4,\ 2x,\ 2y,\ \text{multi.}=1;\ x\pm y,\ \text{multi.}=4\}$$

(viii) $\overset{9}{\circ}\!=\!\overset{6}{\circ}$, $K=U(1)\times_{Z_2}Spin(10)$:

$$\bar{\Delta}=\{x,\ y,\ \text{multi.}=8;\ 2x,\ 2y,\ \text{multi.}=1;\ x\pm y,\ \text{multi.}=6\}\,.$$

The *generating curves* of those *K-invariant* hypersurfaces in M with constant mean curvature $h>b(M)$ are characterized by the following ODE, namely

$$h=\frac{d\sigma}{ds}-\frac{d}{dn}\ln v(\xi)=\frac{d\sigma}{ds}-\sum_{\alpha\in\bar{\Delta}(M)} m(\alpha)\cdot\coth\alpha(\xi)\cdot\frac{d}{dn}\alpha(\xi) \tag{26}$$

where σ is the directional angle, $d/d\boldsymbol{n}=-\sin\sigma(\partial/\partial x)+\cos\sigma(\partial/\partial y)$ and $\xi\in\bar{C}_0$ which is the region defined by $\alpha_1=y\geq 0$ and $\alpha_2\geq 0$. The proof of Theorems 2 and 3 in the above cases can again be reduced to that of the existence of *infinitely many global solution curves of type* D for each of the above ODE (26).

Notice that the value of $\coth r$ is already very close to 1 for $r\geq 20$, namely

$$\coth r=1+2e^{-2r}\cdot(1-e^{-2r})^{-1}\,. \tag{27}$$

Therefore, in a region sufficiently bounded away from the boundary $\partial\bar{C}_0$, the ODE (27) is, in fact, closely approximated by the following ODE, namely

$$h=\frac{d\sigma}{ds}-\sum_{\alpha\in\bar{\Delta}(M)} m(\alpha)\cdot\frac{d}{dn}\alpha(\xi)=\frac{d\sigma}{ds}+A\cos\sigma+B\sin\sigma \tag{28}$$

where A, B are constants and $\sqrt{A^2+B^2}=b(M)$. Hence, the same proof as that of Lemma 2 in §5 of [HH-2] will show the existence of global solution curves of (26) with *arbitrarily large winding numbers.*

Let d_1 (resp. d_2) be the unique positive constant determined by the following equation, namely

$$\begin{gathered} h=m(\alpha_1)\coth d_1+\sum_{\substack{\alpha\in\bar{\Delta}(M)\\ \alpha\neq\alpha_1}} m(\alpha)\cdot\frac{\partial}{\partial y}\alpha(\xi) \\ \left(\text{resp. } h=m(\alpha_2)\cdot\coth d_2+\sum_{\substack{\alpha\in\bar{\Delta}(M)\\ \alpha\neq\alpha_2}} m(\alpha)\cdot\left(\sin\frac{\pi}{g}\frac{\partial\alpha}{\partial x}-\cos\frac{\pi}{g}\frac{\partial\alpha}{\partial y}\right)\right). \end{gathered} \tag{29}$$

Let Ω_1 (resp. Ω_2) be the following region along the boundary line $\alpha_1=0$ (resp. $\alpha_2=0$), (cf. Figure 2)

$$\Omega_1=\{\xi\in\bar{C}_0;\ |\xi|\geq 40+6d_1 \text{ and } \alpha_1(\xi)=y\leq 3d_1\}$$
$$(\text{resp. } \Omega_2=\{\xi\in\bar{C}_0;\ |\xi|\geq 40+6d_2 \text{ and } \alpha_2(\xi)\leq 3d_2\})\,.$$

Inside the above region Ω_1 (resp. Ω_2), the ODE (26) is closely approximated by the following stabilized ODE, namely

$$h=\frac{d\sigma}{ds}-m(\alpha_1)\cdot\coth y\cdot\cos\sigma-\sum_{\substack{\alpha\in\bar{\Delta}(M)\\ \alpha\neq\alpha_1}} m(\alpha)\cdot\frac{d}{dn}\alpha(\xi) \tag{30_1}$$

respectively,

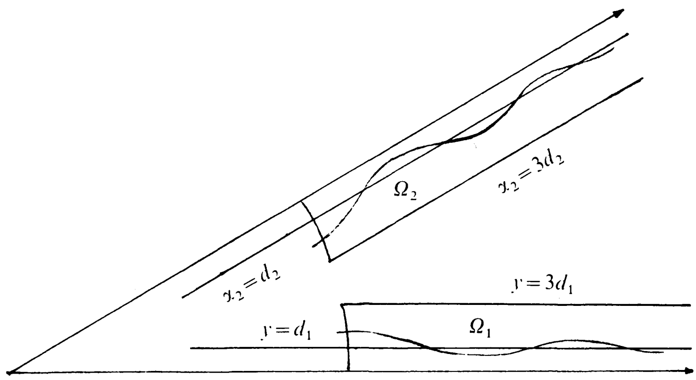

FIGURE 2

$$(30_2) \qquad h = \frac{d\sigma}{ds} - m(\alpha_2)\coth\alpha_2(\xi)\cdot\frac{d}{dn}\alpha_2(\xi) - \sum_{\substack{\alpha\in\tilde{\Delta}(M)\\ \alpha\neq\alpha_2}} m(\alpha)\cdot\frac{d}{dn}\alpha(\xi)$$

because $\coth\alpha(\xi)$, $\alpha\neq\alpha_1$ (resp. $\alpha\neq\alpha_2$) are all closely approximated by 1 inside Ω_1 (resp. Ω_2).

The following lemmas constitute the key steps in the proof of Theorems 2 and 3.

LEMMA 1. *If a global solution curve of* (26) γ *enters* (*resp. backs*) *into* Ω_2 (*resp.* Ω_1) *then*

$$(31) \qquad \lim_{s\to+\infty}\alpha_2(\gamma(s)) = d_2 \quad \left(\text{resp. } \lim_{s\to-\infty}\alpha_1(\gamma(s)) = d_1\right)$$

i.e., γ *tends to the line* $\alpha_2(\xi)=d_2$ (*resp.* $\alpha_1(\xi)=d_1$) *asymptotically as* s *tends to* $+\infty$ (*resp.* $-\infty$).

PROOF OF LEMMA 1. Since the proof of the above two cases are essentially the same, we shall only exhibit here the latter case, i.e. the case of Ω_1 and α_1. Set

$$(32) \qquad \begin{aligned} g(y) &= \int_0^y \sinh^{m(\alpha_1)} t\, dt\,, \\ J &= \sinh^{m(\alpha_1)} y\cdot\cos\sigma + h\cdot g(y)\,. \end{aligned}$$

Then, along arbitrary solution curve $\gamma=\{(x(s), y(s))\}$, one has

$$(33) \qquad \begin{aligned} \frac{dJ}{ds} &= \sinh^{m(\alpha_1)} y\cdot\sin\sigma\cdot\left\{-\frac{d\sigma}{ds} + m(\alpha_1)\coth y\cos\sigma + h\right\} \\ &= \sinh^{m(\alpha_1)} y\cdot\sin\sigma\cdot\left\{-\sum_{\substack{\alpha\in\tilde{\Delta}(M)\\ \alpha\neq\alpha_1}} m(\alpha)\coth\alpha(\xi)\frac{d}{dn}\alpha(\xi)\right\}. \end{aligned}$$

Recall that $\{\coth\alpha(\xi);\ \alpha\in\bar{\Delta}(m),\ \alpha\neq\alpha_1$ and $\xi\in\Omega_1\}$ are all very close to 1 and the set with multiplicity $\bar{\Delta}(m)\setminus\{\alpha_1, m(\alpha_1)\}$ is reflectionally symmetric with respect to the y-axis. Therefore

$$(34)\qquad -\sum_{\substack{\alpha\in\bar{\Delta}(M)\\ \alpha\neq\alpha_1}} m(\alpha)\coth\alpha(\xi)\frac{d}{dn}\alpha(\xi)=(C_M+\lambda(\xi))\sin\sigma+\mu(\xi)\cos\sigma$$

where C_M is a positive constant given by

$$(35)\qquad C_M=\sum_{\alpha\in\bar{\Delta}(M)} m(\alpha)\frac{\partial\alpha}{\partial x}$$

and both $|\lambda(\xi)|$ and $|\mu(\xi)|$ are extremely small. Hence, along a solution curve of (26) inside Ω_1, the range of σ for which $dJ/ds<0$ appears only for a glimpse, and in fact it is not difficult to modify the proof of Lemma 3.4 of [HsHu] to show that the amount of J decreased on such a very short segment can always be *adequately compensated* on a succeeding segment with the same y-level.

Observe that the auxiliary function $J(y,\sigma)$ has $J(d_1,\pi)$ as its *absolute minimum*. Based on the above overall monotonicity of J along a solution curve γ inside Ω_1, it is rather straightforward to adapt the arguments of [Hs-1] and [HsHu] to show that

$$(36)\qquad \lim_{s\to-\infty} J_\gamma(y,\sigma)=J(d_1,\pi)$$

which clearly implies that

$$(37)\qquad \lim_{s\to-\infty}\alpha_1(\gamma(s))=d_1\,.$$

□

LEMMA 2. *Any two global solution curves of* (26), *say*, γ_1 *and* γ_2, *can always be linked by a suitable continuous deformation which is* C^1*-continuous except at those cusp points.*

The proof of Lemma 2 is essentially the same as that of Lemma 3.2 of [HsHu].

It follows from Lemma 1 that the above *asymptoticity property* is both an open property and a closed property. Therefore, it follows from Lemma 2 that the asymptoticity property of Lemma 1 holds for all global solution curves of (26). Hence, one again has a *winding number* $N(\gamma)$ defined for each global solution curve by setting $\lim_{s\to+\infty}\sigma_\gamma(s)=N(\gamma)\cdot 2\pi+\pi/g$.

Finally, on the one hand, the condition $h>b(M)$ enables us to produce global solution curves of arbitrarily large winding number and, on the other hand, it is easy to produce a global solution curve of type A with zero winding number. Therefore, the following existence theorem can again be proved by tracing the evolution geometry of the global solution curves (cf. [Hs-1]).

THEOREM 5′. *Let γ be a global solution curve of* (26) *with $h > b(M)$. Then the winding number $N(\gamma)$ of γ satisfies*

$$N(\gamma) \geq \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix} \quad \textit{if} \quad \gamma \textit{ is of type} \begin{Bmatrix} \mathrm{A} \\ \mathrm{B} \ \textit{or} \ \mathrm{C} \\ \mathrm{D} \end{Bmatrix}.$$

Conversely, any integer k satisfying the above inequality can always be realized as the winding number of a global solution curve of the respective type.

Theorems 2 and 3 for the irreducible cases follow readily from the above Theorem 5′. This completes the proof of Theorems 2 and 3.

4. The proof of Theorem 4. Recent advances in geometric measure theory provide a quite satisfactory proof of the *general existence* of isoperimetric regions (possibly with boundary singularities) for a wide range of ambient spaces which, for example, include all closed manifolds and all homogeneous Riemannian manifolds (cf. [Alm]). If one can, in addition to the above existence result, prove that all isoperimetric regions *in a given family of ambient spaces are automatically everywhere regular, then the boundaries of isoperimetric regions* in such spaces certainly provide a natural source of examples of stable soap bubbles in those ambient spaces. For example, such a strong *regularity theorem* on isoperimetric regions in an arbitrary product of euclidean spaces and hyperbolic spaces, namely $E^{n_0} \times H^{n_1} \times \cdots \times H^{n_k}$, $n_0 \geq 0$; $n_1, \ldots, n_k \geq 2$, has already been proved in [Hs-2], thus providing a simple way of producing examples of stable, spherical, imbedded soap bubbles in the above family of spaces. Although such a strong regularity theorem is still beyond reach for many other non-compact symmetric spaces, the above method can be modified to produce spherical, imbedded soap bubbles for a broader family of spaces, namely, those arbitrary products of rank one non-compact symmetric spaces.

THE PROOF OF THEOREM 4. Let $M = \prod_{j=1}^{l} M_j$, $\mathrm{rk}(M_j) = 1$, be a product of rank one non-compact symmetric spaces and $v_j(r)$ be the volume of the distance sphere of radius r in M_j. Then the orbit space M/K is isometric to the l-dimensional quadrant, namely

$$M/K \cong \prod_{j=1}^{l} M_j/K_j \cong \boldsymbol{R}^l_+ \tag{38}$$

and the volume function of K-orbits is given by

$$v(x_1, \ldots, x_l) = \prod_{j=1}^{l} v_j(x_j) \,. \tag{39}$$

Let Ω be a K-invariant region in M, $\partial\Omega$ be its boundary, and Ω/K, $\partial\Omega/K$ be the image of Ω, $\partial\Omega$ in $\boldsymbol{R}^l_+$ respectively. Then the volume of Ω and the surface area of $\partial\Omega$

can be expressed by the following integrals,

$$\mathrm{vol}(\Omega)=\int_{\Omega/K} v(x)\cdot dx_1\wedge\cdots\wedge dx_l\,, \tag{40}$$

$$\mathrm{Area}(\partial\Omega)=\int_{\partial\Omega/K} v(x)dA \tag{41}$$

where dA is the "area element" of the hypersurface $\partial\Omega/K$ in $\boldsymbol{R}^l_+$. It is rather natural to consider the following "*K-equivariant isoperimetric problem*" on M, namely, the variational problem of seeking those *K-invariant regions* in M which minimize the above boundary integral among all K-invariant regions with a given fixed value of the above volume integral. Of course, such a K-equivariant isoperimetric problem is actually the isoperimetric variational problem of regions in $\boldsymbol{R}^l_+$ with respect to the above two integrals with $v(x)$ as the *weight function*. Therefore, it is rather straightforward to apply the geometric measure theory (cf. [Alm], [Gi]) to establish the *existence* of solutions for each given value of the volume integral.

Let $p_j\colon \boldsymbol{R}^l_+\to\boldsymbol{R}^{l-1}_+$ be the orthogonal projection along the j-th coordinate direction, namely, $p_j(x_1,\ldots,x_l)=(x_1,\ldots,\hat{x}_j,\ldots,x_l)$. A region $\bar{\Omega}\subset\boldsymbol{R}^l_+$ is said to be p_j-graphic if there exists a suitable function ϕ defined on the domain $D=p_j(\bar{\Omega})$ such that

$$\bar{\Omega}=\{x\in\boldsymbol{R}^l_+ : p_j(x)\in D\,,\ 0\le x_j\le\phi(x_1,\ldots,\hat{x}_j,\ldots,x_l)\}\,. \tag{42}$$

Now, suppose $\bar{\Omega}$ is an arbitrary solution of the above isoperimetric variational problem with respect to the $v(x)$-weighted integrals. We claim that $\bar{\Omega}$ must be p_j-graphic for $j=1,2,\ldots,l$. Since the proofs for each j, $j=1,2,\ldots,l$, are obviously identical, it is notationally simpler to verify only the case of $j=l$ in the following.

Suppose the contrary, that $\bar{\Omega}$ is not p_l-graphic. Then there exists a unique function, $\phi(y)$, defined on the domain $D=p_l(\bar{\Omega})$ such that

$$\int_0^{\phi(y)} v_l(t)dt=\int_{\bar{\Omega}\cap p_l^{-1}(y)} v_l(t)dt \tag{43}$$

for every point $y\in D=p_l(\bar{\Omega})$. Set

$$\bar{\Omega}'=\{(y,x_l);\ y\in D \text{ and } 0\le x_l\le\phi(y)\}\,. \tag{44}$$

Then it is easy to check that it follows from (43) that

$$\int_{\bar{\Omega}'} v(x)dx_1\wedge\cdots\wedge dx_l=\int_{\bar{\Omega}} v(x)dx_1\wedge\cdots\wedge dx_l\,. \tag{45}$$

For a generic point $y_0\in D$, i.e., modulo a possibly measure zero exception, there exists a sufficiently small neighborhood U of y_0 such that

$$\bar{\Omega}\cap p_l^{-1}(y)=\{(y,t);\ t\in\bigcup_i\{[\psi_i^-(y),\psi_i^+(y)]\}\}\,,\qquad y\in U \tag{46}$$

for suitable pairs of C^1-functions $\{\psi_i^{\pm}(y);\ y\in U\}$. Using the above expression of $\bar{\Omega}\cap p_l^{-1}(y)$, $y\in U$, to compute the partial differentation of (43) at y_0, one gets

$$v_l(\phi(y_0))\frac{\partial\phi}{\partial x_j}(y_0)=\sum_i\left\{v_l(\psi_i^+(y_0))\frac{\partial\psi_i^+}{\partial x_j}(y_0)-v_l(\psi_i^-(y_0))\frac{\partial\psi_i^-}{\partial x_j}(y_0)\right\}. \tag{47}$$

On the other hand, it is easy to check that

$$v_l(\phi(y_0))\leq\sum_i[v_l(\psi_i^+(y_0))+v_l(\psi_i^-(y_0))] \tag{48}$$

and the equality holds only when $\bar{\Omega}\cap p_l^{-1}(y_0)=[0,\phi(y_0)]$. Therefore it follows from (47) and (48) and the Schwarz inequality that

$$\begin{aligned} v_l(\phi(y_0))\cdot\left\{1+\sum_{j=1}^{l-1}\frac{\partial\phi}{\partial x_j}(y_0)^2\right\}^{1/2} &\leq\sum_i v_l(\psi_i^+(y_0))\left\{1+\sum_{j=1}^{l-1}\frac{\partial\psi_i^+}{\partial x_j}(y_0)^2\right\}^{1/2}\\ &\quad+\sum_i v_l(\psi_i^-(y_0))\left\{1+\sum_{j=1}^{l-1}\frac{\partial\psi_i^-}{\partial x_j}(y_0)^2\right\}^{1/2}\end{aligned} \tag{49}$$

and the equality holds only when $\bar{\Omega}\cap p_l^{-1}(y_0)=[0,\phi(y_0)]$. Hence, it follows easily that

$$\int_{\partial\Omega'}v(x)dA\leq\int_{\partial\Omega}v(x)dA \tag{50}$$

and the equality holds only when $\bar{\Omega}\cap p_l^{-1}(y)=[0,\phi(y)]$ for all generic points $y\in D=p_l(\bar{\Omega})$, namely, $\bar{\Omega}$ is, in fact, p_l-graphic to begin with.

Finally, it follows from the minimizing property of $\bar{\Omega}$ and the above graphic property that $\partial\bar{\Omega}$ must be a smooth hypersurface in $\boldsymbol{R}_+^l$ *perpendicular* to the boundary. Therefore the inverse image of $\partial\Omega$ is a *smooth closed hypersurface* in M *with constant mean curvature*. This completes the proof of Theorem 4.

5. The mean curvature of $\partial\Omega$ and the isoperimetric profile. Let M be a given non-compact symmetric space. Then, to each given positive value of volume v_0, there always exist isoperimetric regions Ω with $\mathrm{vol}(\Omega)=v_0$. The areas of all such isoperimetric regions are, by definition, the unique, absolute minimal value of the areas of all regions of volume v_0 in M and hence must be solely dependent on v_0. Therefore, there is a unique function $f_M\colon \boldsymbol{R}^+\to\boldsymbol{R}^+$ which records the area of the boundaries of isoperimetric regions in M as a function of their volumes; it is usually called the *isoperimetric profile* of M. The following are some basic facts of the isoperimetric profile of a non-compact symmetric space.

LEMMA 3. *The isoperimetric profile, $f_M(v)$, of a noncompact symmetric space M is a monotonically increasing function and hence it is almost everywhere differentiable.*

PROOF. Let v_0 be an arbitrarily given value of volume and Ω_0 be an isoperimetric region in M of volume v_0. By Theorem 1, $h(\partial\Omega_0)>b(M)>0$ and hence small inward deformations in the neighborhood of regular points of Ω_0 will certainly decrease the area of $\partial\Omega_0$. Therefore,

$$f_M(v)<f_M(v_0)=\text{Area}(\partial\Omega_0)$$

for $v=v_0-\delta$ and sufficiently small $\delta>0$, namely, $f_M(v)$ is everywhere a strictly increasing function defined on $[0, \infty)$. Now, it follows from a remarkable theorem of Lebesgue that $f_M(v)$ is almost everywhere differentiable. □

LEMMA 4. *If v_0 is a differentiable point of f_M and Ω_0 is an arbitrary isoperimetric region in M with* $\text{vol}(\Omega_0)=v_0$, *then* $h(\partial\Omega_0)=f_M'(v_0)$.

PROOF. Let h_0 be the constant mean curvature of $R(\partial\Omega_0)$ and ξ (resp. η) be an *outward* (resp. *inward*) *normal* variational vector field with compact support in $R(\partial\Omega_0)$. Then the first variations of the volumes and the boundary areas are as follows

(51)
$$\left.\frac{d}{dt}V(\Omega_t)\right|_{t=0}=\int_{\partial\Omega_0}|\xi|\cdot dA$$
$$\left(\text{resp. } \left.\frac{d}{dt}V(\Omega_t)\right|_{t=0}=-\int_{\partial\Omega_0}|\eta|\cdot dA\right)$$

(52)
$$\left.\frac{d}{dt}A(\partial\Omega_t)\right|_{t=0}=h_0\cdot\int_{\partial\Omega_0}|\xi|\cdot dA$$
$$\left(\text{resp. } \left.\frac{d}{dt}A(\partial\Omega_t)\right|_{t=0}=-h_0\int_{\partial\Omega_0}|\eta|\cdot dA\right).$$

It follows from (51) and (52) that

(53)
$$\lim_{\Delta v\to 0_+}\frac{1}{\Delta v}[f_M(v_0+\Delta v)-f_M(v_0)]\leq h_0$$
$$\left(\text{resp. } \lim_{\Delta v\to 0_-}\frac{1}{\Delta v}[f_M(v_0+\Delta v)-f_M(v_0)]\geq h_0\right)$$

and hence, by the differentiability assumption at v_0, $f_M'(v_0)=h_0$. □

REMARK. It follows from Lemma 4 that all isoperimetric regions in M of the same volume v_0 must also have the same mean curvature if v_0 is a differentiable point of f_M. In fact, the converse is also true, namely, v_0 is a differentiable point of f_M if all isoparametric regions in M with volume v_0 are of the same mean curvature. The following Lemma 5 is a refinement of such a converse.

LEMMA 5. *To any given value $v_0>0$, the left (resp. right) derivative of f_M at v_0 always exist, namely*

(54)
$$\lim_{\Delta v\to 0_-}\frac{1}{\Delta v}[f_M(v_0+\Delta v)-f_M(v_0)]=D_l f_M(v_0)$$
$$\left(\textit{resp. } \lim_{\Delta v\to 0_+}\frac{1}{\Delta v}[f_M(v_0+\Delta v)-f_M(v_0)]=D_r f_M(v_0)\right)$$

and, moreover,

(55)
$$D_l f_M(v_0)=\max\{h(\partial\Omega);\ \Omega \textit{ isop. and } \mathrm{vol}(\Omega)=v_0\}$$
$$(\textit{resp. } D_r f_M(v_0)=\min\{h(\partial\Omega);\ \Omega \textit{ isop. and } \mathrm{vol}(\Omega)=v_0\}) .$$

PROOF. Let

(56)
$$\lambda=\limsup_{\Delta v\to 0_-}\frac{1}{\Delta v}[f(v_0+\Delta v)-f(v_0)]$$
$$\left(\text{resp. } \rho=\liminf_{\Delta v\to 0_+}\frac{1}{\Delta v}[f(v_0+\Delta v)-f(v_0)]\right)$$

and $\{\Omega_n, n\in N\}$ (resp. $\{\Omega'_n; n\in \boldsymbol{n}\}$) be a sequence of isoperimetric regions in M such that

(i) their centers of gravity are all located at the base point 0,

(ii) $\mathrm{vol}(\Omega_n)\nearrow v_0$ (resp. $\mathrm{vol}(\Omega'_n)\searrow v_0$),

(iii) $\displaystyle\lim_{n\to\infty}\frac{\mathrm{Area}(\partial\Omega_n)-f(v_0)}{\mathrm{vol}(\Omega_n)-v_0}=\lambda \quad \left(\text{resp. } \lim_{n\to\infty}\frac{\mathrm{Area}(\partial\Omega'_n)-f(v_0)}{\mathrm{vol}(\Omega'_n)-v_0}=\rho\right).$

It follows from the above proof of Lemma 4 that $\lambda\geq h(\partial\Omega)\geq\rho$ holds for an arbitrary isoperimetric region Ω in M with $\mathrm{vol}(\Omega)=v_0$. Hence

$$\lambda=\sup\{h(\partial\Omega);\ \Omega \text{ isop. and } \mathrm{vol}(\Omega)=v_0\} ,$$
$$\rho=\inf\{h(\partial\Omega);\ \Omega \text{ isop. and } \mathrm{vol}(\Omega)=v_0\} .$$

Applying the basic compactness result of geometric measure theory, there exists a suitable subsequence of $\{\Omega_n\}$ (resp. $\{\Omega'_n\}$) which *converges* to a region Ω_0 (resp. Ω'_0). It is clear that both Ω_0 and Ω'_0 are isoperimetric and with volume v_0. For simplicity of notation, one may assume that $\Omega_n\to\Omega_0$ and $\Omega'_n\to\Omega_0$, namely Ω_n (resp. Ω'_n) are, in fact, small deformations of Ω_0 (resp. Ω'_0) for sufficiently large n. Therefore, one may again use the first variation formulas for both the volume and the area to show that

(57) $$\lambda=h(\partial\Omega_0) \qquad (\text{resp. } \rho=h(\partial\Omega'_0))$$

and, moreover,

$$(58)\quad \begin{aligned} h(\partial\Omega_0) &= \max\{h(\partial\Omega);\ \Omega \text{ isop. and } \mathrm{vol}(\Omega)=v_0\} = \lim_{\Delta v\to 0_-} \frac{1}{\Delta v}\,[f_M(v_0+\Delta v)-f_M(v_0)] \\ h(\partial\Omega_0') &= \min\{h(\partial\Omega);\ \Omega \text{ isop. and } \mathrm{vol}(\Omega)=v_0\} = \lim_{\Delta v\to 0_+} \frac{1}{\Delta v}\,[f_M(v_0+\Delta v)-f_M(v_0)]\,. \end{aligned}$$

□

DEFINITION. The mean curvature function of M, denoted by $h_M(v)$, is defined at a point v if all isoperimetric regions in M with volume v have the same mean curvature of their boundaries and its value at v is defined to be that common value of $h(\partial\Omega)$ for all isoperimetric regions Ω in M with $\mathrm{vol}(\Omega)=v$.

THEOREM 6. *The mean curvature function of M, $h_M(v)$, is defined almost everywhere on* $[0, \infty)$ *and*

$$(59)\qquad f_M(v_0)=\int_0^{v_0} h_M(v)dv\,.$$

PROOF. It follows readily from the above three lemmas. □

REMARKS. (i) To an arbitrarily given continuous family of compact regions $\{\Omega_t;\ t\in \boldsymbol{R}_+\}$ with $\mathrm{vol}(\Omega_t)\nearrow\infty$, there is an associated function $a=\phi(v)$, which records the area of $\partial\Omega_t$ as a function of the volume of Ω_t. It follows from the definition of the isoperimetric profile that $\phi(v)\geq f_M(v)$ for all $v\in\boldsymbol{R}_+$.

(ii) An "isoperimetric inequality" for regions in M usually estimates a lower bound of the area of $\partial\Omega$ in terms of the volume of Ω for an arbitrary region Ω in M. Therefore, such inequalities are, in fact, lower bound estimates of f_M. For example, it follows easily from Theorems 1 and 6 that

$$(60)\qquad f_M(v)>f_M(v_0)+b(M)\cdot(v-v_0)\,,\qquad v_0<v<\infty\,.$$

COROLLARY. $\liminf_{v_0\to\infty}\{h(\partial\Omega);$ *Ω isop. and* $\mathrm{vol}(\Omega)\geq v_0\}$ *is exactly equal to* $b(M)$.

PROOF. It follows from Theorem 1 that

$$(61)\qquad \liminf_{v_0\to\infty}\{h(\partial\Omega);\ \Omega \text{ isop. and } \mathrm{vol}(\Omega)\geq v_0\}\geq b(M)\,.$$

Suppose the contrary that it is not equal. Then there exist a sufficiently large v_0 and a sufficiently small $\delta>0$ such that $h(\partial\Omega)\geq b(M)+\delta$ for all isoperimetric regions Ω with $\mathrm{vol}(\Omega)\geq v_0$. Hence it follows from Theorem 6 that

$$(62)\qquad f_M(v)\geq f_M(v_0)+(b(M)+\delta)(v-v_0)$$

for all $v>v_0$.

Now we shall construct an explicit family of regions $\{\Omega_t\}$ such that its associated function $a=\phi(v)$ *does not* satisfy the inequality

(63) $$\phi(v) \geq f_M(v_0) + (b(M)+\delta)\cdot(v - v_0)\,, \qquad v > v_0\,.$$

Such a contradiction proves that

$$\liminf_{v_0 \to \infty} \{h(\partial\Omega);\ \Omega \text{ isop.}, \operatorname{vol}(\Omega) \geq v_0\} = b(M)\,.$$

Let v_0 be the direction in C_0 such that

(64) $$\sum_{\alpha \in \Delta(M)} m(\alpha)\lambda_\alpha \cdot |\langle \alpha, v_0 \rangle| = b(M)$$

and $P(v_0, t)$ be the piece of perpendicular hyperplane of v_0 given by

(65) $$P(v_0, t) = \{\xi \in C_0;\ \langle \xi, v_0 \rangle = t,\ \alpha_i(\xi) \geq 20,\ \alpha_i \in \Pi\}$$

where Π is the set of simple roots defining the walls of C_0. Let $\bar{\Omega}_t$ be the region in $\bar{C}_0$ bounded by the above linear piece $P(v_0, t)$ and k linear pieces perpendicular to the k walls of $\bar{C}_0$. For example, in the rank 2 cases, the above $\bar{\Omega}_t$ is a region indicated in Figure 3.

Finally, set $\Omega_t = K(\bar{\Omega}_t)$, $v(t) = \operatorname{vol}(\Omega_t)$ and $a(t) = \operatorname{area}(\partial\Omega_t)$. Let h_t be the mean curvature function of $\partial\Omega_t$ which is defined except at those corner points and can be expressed as linear combinations of coth's (cf. (10) of §2). Moreover, for sufficiently large t, all the coth's in the above expression are very close to 1. Observe that

(66) $$\phi'(v(t_0)) = \frac{a'(t_0)}{v'(t_0)}$$

and it is rather straightforward to estimate the above quotient in terms of the average value of h_t. For sufficiently large t_0, it is not difficult to show that

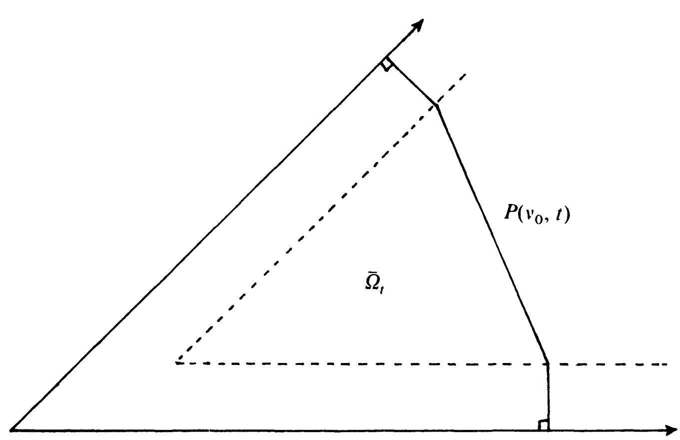

FIGURE 3

(67) $$\phi'(v(t_0)) = \frac{a'(t_0)}{v'(t_0)} < b(M) + \frac{\delta}{2}$$

and hence

(68) $$\phi(v) < \phi(v(t_0)) + \left(b(M) + \frac{\delta}{2}\right) \cdot (v - v(t_0))$$

for all $v > v(t_0)$. This clearly contradicts the inequality of (63). □

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