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FUNCTIONS ON THE REAL LINE WITH NONNEGATIVE FOURIER TRANSFORMS

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Abstract. Unlike an integrable function on the unit circle which has the nonnegative Fourier coefficients and is square-integrable near the origin, an integrable function on the real line which has the nonnegative Fourier transform and is square-integrable near the origin is not always square-integrable on the real line. We give some examples, and consider an additional condition which guarantees the global square-integrability. Moreover, we treat an analogous problem for an integrable function on the real line which has non-negative wavelet coefficients of the Fourier transform and is square-integrable near the origin.

1. Introduction. In this paper we consider the following:

QUESTION. *Let* $f \in L^1(\boldsymbol{R})$ *with the Fourier transform* $\hat{f} \geq 0$ *and* f *restricted to a neighborhood* $(-\delta, \delta)$ of $x=0$ *belongs to* $L^2(\boldsymbol{R})$. *Then, does* f *belong to* $L^2(\boldsymbol{R})$?

A similar question in which we replace the Euclidean space $\boldsymbol{R}$ by a compact group G has an affirmative answer. For example, when G is a compact abelian group, $f \in L^1(G)$ with the nonnegative Fourier *coefficients* which is p-th $(1<p\leq 2)$ power integrable near the identity of G has the Fourier coefficients in l^q $(q=p/(p-1))$. For $p=2$ this conclusion is equivalent to $f \in L^2(G)$, and was obtained by N. Wiener for $G=\boldsymbol{T}$ (cf. Boas [2] and Shapiro [8]) and by Rains [7] for arbitrary compact abelian groups. For $1<p<2$ it was proved by Ash, Rains and Vági [1]. Moreover, when G is a compact semisimple Lie group, an analogue of this result for central and zonal functions on G was obtained by the first author and Miyazaki [5].

The answer to our question is unfortunately negative on the Euclidean space $\boldsymbol{R}$. In §2 we shall give two counterexamples: one is constructed by using step functions and the other by applying wavelets. Therefore, for a function f satisfying the assumption of the Question to be in $L^2(\boldsymbol{R})$, we need an additional condition of f. In §3 we replace the condition $f \in L^2(-\delta, \delta)$ by a stronger one, under which we can deduce the global square-integrability of f. In the last section we treat an analogue of the Question in which the assumption $\hat{f} \geq 0$ is replaced by the nonnegativity of the wavelet coefficients of $\hat{f}$. The second counterexample in §2 and the last section were announced by the first

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author in [4].

2. Counterexamples.

COUNTEREXAMPLE 1. Let $0<\gamma<1/2$ and α, β positive numbers satisfying

$$(1)\quad \alpha<\beta-1\,,\quad (2)\quad \alpha\geq 3(\beta-1)/4\,,\quad \text{and}\quad (3)\quad \alpha<\beta/2\,.$$

For each $n\in \boldsymbol{N}$ we define

$$g^n(x)=g^n_{\alpha,\beta,\gamma}(x)=\begin{cases} n^\alpha & \text{if}\quad n-\gamma n^{-\beta}\leq x\leq n+\gamma n^{-\beta}\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

and we put $g(x)=\sum_{n=1}^{\infty} g^n(x)$. Since $\operatorname{supp}(g^i)\cap\operatorname{supp}(g^j)=\varnothing$ $(i\neq j)$, it follows that $\|g\|_1=2\gamma\sum_{n=1}^{\infty} n^{\alpha-\beta}<\infty$ by (1) and $\|g\|_2=2\gamma\sum_{n=1}^{\infty} n^{2\alpha-\beta}=\infty$, because $2\alpha-\beta\geq(\beta-3)/2>\alpha/2-1>-1$ by (1) and (2). We put

$$f(x)=f_{\alpha,\beta,\gamma}(x)=g*\tilde{g}(x)\,,$$

where $\tilde{g}(x)=g(-x)$. It is easy to see that

$$\|f\|_1\leq\|g\|_1^2<\infty \tag{4}$$

and

$$\hat{f}(\lambda)=|\hat{g}(\lambda)|^2\geq 0\qquad(\lambda\in\boldsymbol{R})\,. \tag{5}$$

We define $A(x)=(2\gamma/|x|)^{1/\beta}$. Looking at the support of g^n, we see that $g^n(\cdot)g^n(\cdot-x)=0$ for n and x satisfying $n\geq[A(x)]+1$, where $[a]$ denotes the greatest integer not exceeding $a\in\boldsymbol{R}$, and moreover, $g^n(\cdot)g^m(\cdot-x)=0$ $(n\neq m)$, if $|x|\leq\delta\leq 1-2\gamma$. Therefore, we can deduce that

$$f(x)\leq\sum_{n=1}^{[A(x)]} 2\gamma n^{2\alpha-\beta}\leq\int_1^{[A(x)]} 2\gamma y^{2\alpha-\beta}dy+2\gamma\leq c_1|x|^{-(2\alpha-\beta+1)/\beta}+c_2$$

by (3). Since $2\alpha-\beta+1<\beta/2$ by (1) and (3), it follows that

$$\int_{-\delta}^{\delta}|f(x)|^2dx<\infty\,. \tag{6}$$

We next obtain an estimate for f on the neighborhood $I_l=[l-c_3l^{-\beta}, l+c_3l^{-\beta}]$ of $l\in\boldsymbol{N}$, where $c_3=\gamma((\beta-2\alpha)/\beta)^{\beta+1}$. For $x\in I_l$, we put $B_l(x)=\gamma^{1/(\beta+1)}(l/|x-l|)^{1/(\beta+1)}-l$. Obviously, $B_l(x)\geq 2\alpha l/(\beta-2\alpha)$ on I_l and the inequality $n\leq B_l(x)$ $(l\geq 1)$ implies that $|x-l|\leq\gamma l(n+l)^{-\beta-1}<\gamma l n^{-1}(n+l)^{-\beta}<\gamma\{n^{-\beta}-(n+l)^{-\beta}\}$, because $\beta>1$ by (1). Therefore, $\operatorname{supp}(g^{n+l}(\cdot+x))\subset\operatorname{supp}(g^n(\cdot))$ for n, x satisfying $n\leq B_l(x)$ $(l\geq 1)$, so we obtain that if $x\in I_l$

$$f(x)=\sum_{n,m}\int_{-\infty}^{\infty} g^n(y)g^m(y+x)dy\geq \sum_{n\leq B_l(x)}\int_{-\infty}^{\infty} g^n(y)g^{n+l}(y+x)dy$$

$$\geq 2\gamma\sum_{n=1}^{[B_l(x)]} n^\alpha(n+l)^{\alpha-\beta}.$$

We note that the function $y^\alpha(y+l)^{\alpha-\beta}$ is monotone decreasing on $y\geq B_l=\alpha l/(\beta-2\alpha)$ and, since α and $\beta-2\alpha$ are positive (see (3)), there exists an $\varepsilon>0$ such that $\alpha>\varepsilon(\beta-2\alpha)$. Then, for large $l\geq L=(\alpha/(\beta-2\alpha)-\varepsilon)^{-1}$ and $x\in I_l$, we have $B_l(x)-(B_l+1)\geq \alpha l/(\beta-2\alpha)-1\geq \varepsilon l$, and thus, the last summation is estimated below as

$$\geq 2\gamma\int_{B_l+1}^{B_l(x)} y^\alpha(y+l)^{\alpha-\beta}\,dy\geq 2\gamma B_l(x)^\alpha(B_l(x)+l)^{\alpha-\beta}\int_{B_l+1}^{B_l(x)} dy$$

$$\geq c_4 l^{\alpha+1}(l/|x-l|)^{(\alpha-\beta)/(\beta+1)}.$$

Taking the square of this inequality and integrating it over $I_l(l\geq L)$, we can deduce that

$$\int_{x\in I_l} |f(x)|^2\,dx\geq 2c_4^2\,l^{2\alpha+2+2(\alpha-\beta)/(\beta+1)}\int_0^{c_3 l^{-\beta}} x^{-2(\alpha-\beta)/(\beta+1)}dx=c_5 l^{4\alpha-3\beta+2},$$

and

$$\|f\|_2^2\geq\sum_{l\geq L}\int_{x\in I_l}|f(x)|^2dx\geq c_5\sum_{l\geq L} l^{4\alpha-3\beta+2}=\infty \tag{7}$$

by (2). Therefore, (4)–(7) imply that $f_{\alpha,\beta,\gamma}\in L^1(\boldsymbol{R})$ with $\hat{f}_{\alpha,\beta,\gamma}\geq 0$ and the restriction of $f_{\alpha,\beta,\gamma}$ to $(-\delta,\delta)$ belongs to $L^2(\boldsymbol{R})$ for $\delta\leq 1-2\gamma$. However, $f_{\alpha,\beta,\gamma}$ does not belong to $L^2(\boldsymbol{R})$.

COUNTEREXAMPLE 2. Let $\boldsymbol{b}=(b_n)_{n\geq 1}$ be a sequence satisfying

$$0<b_n<1 \qquad \text{for all} \quad n, \tag{8}$$

$$\sum_{n=1}^{\infty} b_n<\infty, \tag{9}$$

$$\sum_{n=1}^{\infty} 2^{-n}b_n^{-1}<\infty. \tag{10}$$

We let $d_l=(1-b_l^2)^{1/2}$ $(l\in\boldsymbol{N})$, and for $j\in 2\boldsymbol{N}, k\in\boldsymbol{Z}$,

$$a_j^k=\begin{cases} b_l & k=0,\ j=2l\ (l\in\boldsymbol{N}),\\ 2^{-1}b_l d_l^n & |k|=n2^j,\ j=2l\ (l,n\in\boldsymbol{N}),\\ 0 & \text{otherwise}.\end{cases} \tag{11}$$

We now put

$$f^{b}(x)=\sum_{\substack{j\in 2N\\ k\in Z}} a_j^k\psi_j^k(x+2^{-(j+1)}),$$

where $\psi_j^k(x)=2^{j/2}\psi(2^jx-k)$ $(j,k\in Z)$ are wavelets constructed by Meyer [6, p. 74]. We see from (8)–(11) that

$$(12)\qquad \|f^b\|_1\le c\sum_{\substack{j\in 2N\\ k\in Z}}|a_j^k|2^{-j/2}\le c\sum_{l=1}^{\infty}b_l2^{-l}+c\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}b_ld_l^n2^{-l}$$

$$\le c\sum_{l=1}^{\infty}b_l+2c\sum_{l=1}^{\infty}2^{-l}b_l^{-1}<\infty,$$

where $c=\|\psi\|_1$ and we use $\sum_{n=1}^{\infty}d_l^n=d_l(1-d_l)^{-1}=d_l(1+d_l)(1-d_l^2)^{-1}\le 2b_l^{-2}$. Moreover, we can deduce that

$$\left(\int_{-\delta}^{\delta}|f^b(x)|^2dx\right)^{1/2}\le\sum_{\substack{j\in 2N\\ k\in Z}}|a_j^k|2^{j/2}\left(\int_{-\delta}^{\delta}|\psi(2^jx+2^{-1}-k)|^2dx\right)^{1/2}$$

$$\le C_m\sum_{\substack{j\in 2N\\ k\in Z}}|a_j^k|\left(\int_{-2^j\delta-(k-1/2)}^{2^j\delta-(k-1/2)}(1+|x|)^{-2m}dx\right)^{1/2},$$

for $m\ge 1$ (see [6, Théorème 1 in p. 70]). We here recall that $a_j^k=0$ unless $k=0$ or $|k|=n2^j$, especially, $a_j^k=0$ if $j\in 2N$ and $0<|k|<2^j$ (see (11)). Therefore, if $\delta<1/4$, the last expression is bounded by

$$(13)\qquad C_m\sum_{l=1}^{\infty}b_l+C_m\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}b_ld_l^n2^l(1+|k|-2^{2l-2})^{-m}$$

$$\le C_m\sum_{l=1}^{\infty}b_l+C_m2^{2m}\sum_{l=1}^{\infty}2^{(1-2m)l}b_l\sum_{n=1}^{\infty}d_l^n<\infty$$

as in (12). We next note that $\hat{\psi}_j^k(\cdot+2^{-(j+1)})(\xi)=2^{-j/2}\hat{\psi}(2^{-j}\xi)e^{-i2^{-j}k\xi}e^{i2^{-(j+1)}\xi}$ and $\hat{\psi}(\xi)=\theta_1(\xi)e^{-i\xi/2}$ for $\theta_1\ge 0$ (see [6, p. 74]). Therefore, we have

$$(14)\qquad \hat{f}^b(\xi)=\sum_{\substack{j\in 2N\\ k\in Z}}a_j^k\hat{\psi}_j^k(\cdot+2^{-(j+1)})(\xi)=\sum_{j\in 2N}2^{-j/2}\theta_1(2^{-j}\xi)\sum_{k\in Z}a_j^ke^{-i2^{-j}k\xi}$$

$$=\sum_{j\in 2N}2^{-j/2}\theta_1(2^{-j}\xi)\,b_l\frac{1-d_l\cos\xi}{1-2d_l\cos\xi+d_l^2}\ge 0.$$

Since $j\in 2N$ and the support of $\theta_1(2^{-j}\xi)$ is contained in $[-2^{j+3}\pi/3,\ -2^{j+1}\pi/3]\cup[2^{j+1}\pi/3,\ 2^{j+3}\pi/3]$ (see [6, p. 74]), it is easy to see that $\psi_j^k(x+2^{-(j+1)})$ $(j\in 2N,\ k\in Z)$ are orthonormal in $L^2(R)$. Then it follows that

(15) $$\|f^b\|_2^2=\sum_{\substack{j\in 2N\\ k\in Z}}|a_j^k|^2\geq 2^{-1}\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}b_l^2d_l^{2n}=2^{-1}\sum_{l=1}^{\infty}d_l^2=\infty\,,$$

because $d_l^2=1-b_l^2\to 1$ $(l\to\infty)$. Therefore, (12)–(15) imply that $f^b\in L^1(\boldsymbol{R})$ with $\hat{f}^b\geq 0$ and the restriction of f^b to $(-\delta,\delta)$ belongs to $L^2(\boldsymbol{R})$ for $\delta<1/4$. However, f^b does not belong to $L^2(\boldsymbol{R})$.

3. Some criteria for square-integrability. As an application of $C1$-summability and Riemann-Lebesgue's lemma, we obtain the following theorem, which can be regarded as a special case of [3, Lemma 4.3].

THEOREM 3.1. *Let $f\in L^1(\boldsymbol{R})$ and $\hat{f}(\xi)\geq 0$ for all $\xi\in\boldsymbol{R}$. Suppose that there is a $\delta>0$ such that $f\in L^\infty(-\delta,\delta)$. Then $\hat{f}(\xi)\in L^1(\boldsymbol{R})$ and in particular, $f\in L^2(\boldsymbol{R})$.*

Let $f\in L^1(\boldsymbol{R})$. We note that $f*f\in L^1(\boldsymbol{R})$, and $(f*f)^\wedge=(\hat{f})^2\geq 0$ is equivalent to the fact that $\hat{f}$ is real-valued. Therefore, applying Theorem 3.1 to $f*f$, we can deduce the following:

THEOREM 3.2. *Let $f\in L^1(\boldsymbol{R})$ with the real-valued Fourier transform $\hat{f}$. Suppose that there is a $\delta>0$ such that $f*f\in L^\infty(-\delta,\delta)$. Then $f\in L^2(\boldsymbol{R})$.*

Since the convolution of two functions with supports far from the origin may have its support near the orign, this theorem suggests that to obtain the global square-integrability of f a local one may not be sufficient. From this point of view we prove the following:

THEOREM 3.3. *Let $f\in L^1(\boldsymbol{R})$ and $\hat{f}(\xi)\geq 0$ for all $\xi\in\boldsymbol{R}$. We suppose that*

(16) $$f(x)\cdot\sum_{k\in Z}\mathbf{1}_{(2Tk-\delta,\,2Tk+\delta)}(x)\in L^2(\boldsymbol{R})$$

for some T and δ with $0<\delta<T$, where $\mathbf{1}_A(x)$ denotes the characteristic function of a measurable set A. Then $f\in L^2(\boldsymbol{R})$.

For the proof we use the following lemma, which is a simple modification of Theorem in [1].

LEMMA 3.4. *Let $f\in L^1(-T,T)$. Suppose that $c_n=(2T)^{-1}\int_{-T}^{T}f(x)e^{-in\pi T^{-1}x}dx\geq 0$ for all $n\in\boldsymbol{Z}$ and $f\in L^2(-\delta,\delta)$ for some δ, $0<\delta<T$. Then $f\in L^2(-T,T)$, in particular,*

$$\int_{-T}^{T}|f(x)|^2dx\leq\frac{4T^2}{\delta^2}\int_{-\delta}^{\delta}|f(x)|^2dx\,.$$

PROOF OF THEOREM 3.3. Define

$$G(x,s)=\sum_{l\in Z}f(x+2Tl)e^{-i\pi T^{-1}s(x+2Tl)}$$

for x with $-T \leq x \leq T$ and s with $0 \leq s \leq 1$. Then, for a fixed s

$$\int_{-T}^{T} |G(x,s)|dx \leq \sum_{l \in \boldsymbol{Z}} \int_{-T}^{T} |f(x+2Tl)|dx \leq \int_{-\infty}^{\infty} |f(x)|dx < \infty \tag{17}$$

and the Fourier coefficients of $G(x, s)$ are given as follows: for $n \in \boldsymbol{Z}$,

$$\begin{aligned} (2T)^{-1} \int_{-T}^{T} G(x,s) e^{-in\pi T^{-1}x} dx &= (2T)^{-1} \sum_{l \in \boldsymbol{Z}} \int_{-T}^{T} f(x+2Tl) e^{-i\pi T^{-1}(s+n)(x+2Tl)} dx \\ &= (2T)^{-1} \int_{-\infty}^{\infty} f(x) e^{-i\pi T^{-1}(s+n)x} dx = (2T)^{-1} \hat{f}(\pi T^{-1}(s+n)) \geq 0 . \end{aligned} \tag{18}$$

On the other hand the assumption (16) on f implies that

$$\begin{aligned} \infty > \int_{-\delta}^{\delta} \sum_{l \in \boldsymbol{Z}} |f(x+2Tl)|^2 dx &= \int_{-\delta}^{\delta} \left(\int_0^1 |G(x,s)|^2 ds \right) dx \\ &= \int_0^1 \left(\int_{-\delta}^{\delta} |G(x,s)|^2 dx \right) ds . \end{aligned} \tag{19}$$

Therefore, (17)–(19) imply that $G(x, s)$ satisfies the assumption of Lemma 3.4 for almost all s. Then, Lemma 3.4 yields that the last integral is estimated as

$$\begin{aligned} &\geq \int_0^1 \left(\frac{\delta^2}{4T^2} \int_{-T}^{T} |G(x,s)|^2 dx \right) ds = \frac{\delta^2}{4T^2} \int_{-T}^{T} \left(\int_0^1 |G(x,s)|^2 ds \right) dx \\ &= \frac{\delta^2}{4T^2} \int_{-T}^{T} \sum_{l \in \boldsymbol{Z}} |f(x+2Tl)|^2 dx = \frac{\delta^2}{4T^2} \int_{-\infty}^{\infty} |f(x)|^2 dx . \end{aligned}$$

■

4. An analogue of the Question. We now give a modification of the Question. We let $\psi = \psi_0^0$ (see [6, p. 74]) and for a real valued $h \in L^\infty(\boldsymbol{R})$ we define the Ψ-coefficients of h by

$$\Psi_n^0(h) = \int_{\boldsymbol{R}} |\hat{\psi}(\lambda)|^2 h(\lambda) e^{in\lambda} d\lambda \tag{20a}$$

and

$$\Psi_n^1(h) = \sqrt{2} \int_{\boldsymbol{R}} \hat{\psi}(\lambda) \bar{\hat{\psi}}(2\lambda) h(\lambda) e^{in\lambda} d\lambda \tag{20b}$$

for $n \in \boldsymbol{Z}$. We say that h has *nonnegative Ψ-coefficients* if $\Psi_n^i(h) \geq 0$ for all $n \in \boldsymbol{Z}$ and $i = 0$, 1. Moreover, we say that h is *dyadically invariant* if $h(x) = h(2x)$. We now fix a dyadically invariant L^∞-function h on $\boldsymbol{R}$ with nonnegative Ψ-coefficients and $\Psi_0^0(h) > 0$. Then, looking at the support of $\hat{\psi}$, we deduce that

(21)
$$h_{j_1j_2}^{k_1k_2}=(\hat\psi_{j_1}^{k_1}, h\hat\psi_{j_2}^{k_2})=2^{-(j_1+j_2)/2}\int_{\mathbf{R}}\hat\psi(\lambda 2^{-j_1})\bar{\hat\psi}(\lambda 2^{-j_2})h(\lambda)e^{-i(k_1 2^{-j_1}-k_2 2^{-j_2})\lambda}d\lambda$$
$$=\begin{cases}\Psi_{k_2-k_1}^0(h) & j_1=j_2\\ \Psi_{2k_2-k_1}^1(h) & j_1=j_2+1\\ \bar\Psi_{k_2-2k_1}^1(h) & j_1=j_2-1\\ 0 & |j_1-j_2|>1\,.\end{cases}$$

As an application of this property, we obtain the following:

THEOREM 4.1. *Let h be a real valued, even, piecewise-differentiable, dyadically invariant L^∞-function on $\mathbf{R}$ with nonnegative Ψ-coefficients and $\Psi_0^0(h)>0$. Let $f\in L^1(\mathbf{R})$ with $(\hat f, \psi_j^k)\geq 0$ for all $j, k\in\mathbf{Z}$ and $f(x)\cdot h(x)\in L^2(\mathbf{R})$. Then f belongs to $L^2(\mathbf{R})$.*

PROOF. We note that $\hat f=\sum_{j,k\in\mathbf{Z}}a_j^k\psi_j^k$ with $a_j^k\geq 0$, as a wavelet decomposition of functions in BMO (see [6, p. 150]), and $((fh)^\wedge, \psi_{j_2}^{k_2})=(\tilde f h, \psi_{j_2}^{k_2})=\sum_{j_1,k_1\in\mathbf{Z}}a_{j_1}^{k_1}h_{j_1j_2}^{k_1k_2}$, where $\tilde f(x)=f(-x)$. Since h is piecewise-differentiable, we easily see that $(h\hat\psi_{j_2}^{k_2})^\wedge$ is a $(1, 2, 0)$-molecule on $\mathbf{R}$ and thus, it is in $H^1(\mathbf{R})$. Therefore, the above calculation makes sense, because $\hat f$ is in BMO. Since $\{\psi_j^k; j, k\in\mathbf{Z}\}$ is a complete orthonormal system of $L^2(\mathbf{R})$, we see that

$$\infty>\|fh\|_2^2=\|(fh)^\wedge\|_2^2=\sum_{j_2,k_2\in\mathbf{Z}}|((fh)^\wedge, \psi_{j_2}^{k_2})|^2=\sum_{j_2,k_2\in\mathbf{Z}}\left|\sum_{j_1,k_1\in\mathbf{Z}}a_{j_1}^{k_1}h_{j_1j_2}^{k_1k_2}\right|^2.$$

Since $a_j^k\geq 0$, $h_{j_1j_2}^{k_1k_2}\geq 0$ and $h_{jj}^{kk}=\Psi_0^0(h)>0$ (see (20)), the last summation is estimated as

$$\infty>\sum_{j,k\in\mathbf{Z}}|a_j^kh_{jj}^{kk}|^2=\Psi_0^0(h)^2\|\hat f\|_2^2=\Psi_0^0(h)^2\|f\|_2^2\,.$$

■

Let $0<\delta<2\pi/3$ and for a measurable set S in $\mathbf{R}$ let $\mathbf{1}_{\pm S}$ be the characteristic function of $(-S)\cup S$. Then we see the following:

COROLLARY 4.2. *Let $f\in L^1(\mathbf{R})$ with $(\hat f, \psi_j^k)\geq 0$ for all $j, k\in\mathbf{Z}$. If*

(22)
$$f(x)\cdot\sum_{j\in\mathbf{Z}}\mathbf{1}_{\pm((2\pi-\delta)2^j,(2\pi+\delta)2^j)}(x)\in L^2(\mathbf{R})\,,$$

then $f\in L^2(\mathbf{R})$.

PROOF. Let k_δ be the function on $[-\pi, \pi]$ defined by

$$k_\delta(x)=\begin{cases}1-|x|/\delta & |x|\leq\delta\\ 0 & \delta<|x|\leq\pi\end{cases}$$
$$=\frac{\delta}{2\pi}+\frac{2}{\pi\delta}\sum_{n=1}^{\infty}\frac{1}{n^2}(1-\cos(n\delta))\cos(nx)$$

and $h_0(x)=k_\delta(x-2\pi)$. Then, since $\delta<2\pi/3$, $h_0(x)$ can be regarded as a function on $[2\pi/3, 8\pi/3]$ with the same Fourier series as that of k_δ and supported on $[2\pi-\delta, 2\pi+\delta]$. As a function on $[2\pi/3, 8\pi/3]$, we put $h_1(x)=h_0(2x)+h_0(x)$ and we denote the Fourier series of $h_1(x)$ as $h_1(x)=\sum_{n\in \boldsymbol{Z}} a_n \cos(nx)$. Then, it is easy to see that h_1 is supported on $[\pi-2^{-1}\delta, \pi+2^{-1}\delta]\cup[2\pi-\delta, 2\pi+\delta]$, $a_n\geq 0$ for all $n\in \boldsymbol{Z}$ and $a_0=\delta/\pi>0$. We finally put $h(x)=\sum_{j\in \boldsymbol{Z}}\{h_1(-2^{2j}x)+h_1(2^{2j}x)\}$. Obviously, h is a dyadically invariant L^∞-function on $\boldsymbol{R}$. To show that h has nonnegative Ψ-coefficients we note that

$$\delta_{n0}=(\psi_0^n, \psi_0^0)=\int_{\boldsymbol{R}} |\hat{\psi}(\lambda)|^2 e^{-in\lambda}\, d\lambda=2\int_{2\pi/3}^{8\pi/3} |\hat{\psi}(\lambda)|^2 \cos(n\lambda)d\lambda$$

and

$$\begin{aligned}0=(\psi_1^n, \psi_0^0)&=\sqrt{2}\int_{\boldsymbol{R}} \hat{\psi}(\lambda)\bar{\hat{\psi}}(2\lambda)e^{-in\lambda}d\lambda\\ &=2\sqrt{2}\int_{2\pi/3}^{8\pi/3} \hat{\psi}(\lambda)\bar{\hat{\psi}}(2\lambda)e^{-i\lambda/2}\cos((n-1/2)\lambda)d\lambda\,.\end{aligned}$$

Then, since $\operatorname{supp}(\hat{\psi})\cap\operatorname{supp}(h)=\operatorname{supp}(\hat{\psi})\cap\operatorname{supp}(h_1)$, these relations imply that

$$\begin{aligned}\Psi_n^0(h)&=2\int_{2\pi/3}^{8\pi/3} |\hat{\psi}(\lambda)|^2 h_1(\lambda)\cos(n\lambda)d\lambda\\ &=\sum_{m\in \boldsymbol{Z}} a_m\int_{2\pi/3}^{8\pi/3} |\hat{\psi}(\lambda)|^2\{\cos((n+m)\lambda)+\cos((n-m)\lambda)\}d\lambda\\ &=\frac{1}{2}(a_n+a_{-n})\end{aligned}$$

and

$$\Psi_n^1(h)=2\sqrt{2}\int_{2\pi/3}^{8\pi/3} \hat{\psi}(\lambda)\bar{\hat{\psi}}(2\lambda)h_1(\lambda)e^{-i\lambda/2}\cos((n+1/2)\lambda)d\lambda=0\,.$$

Since $a_n\geq 0$ for all $n\in \boldsymbol{Z}$ and $a_0>0$, it follows that h has nonnegative Ψ-coefficients and $\Psi_0^0(h)>0$. Furthermore, the assumption (22) on f easily yields that $f(x)\cdot h(x)\in L^2(\boldsymbol{R})$. Therefore, the desired result follows from Theorem 4.1. ■

REMARK 4.3. Although the nonnegativity of the wavelet coefficients of the Fourier transform $\hat{f}$ of $f\in L^1(\boldsymbol{R})$ looks unrelated to the other properties of f, it is deeply related to those of the Fourier coefficients. Indeed, for $f=\sum_{n\in \boldsymbol{Z}} a_n e^{inx}\in L^1([-\pi, \pi])$ with $a_n\geq 0$ $(n\in \boldsymbol{Z})$, we put $g(x)=f(x)\cdot\hat{\psi}(-x)$ $(x\in \boldsymbol{R})$, where we regard f as a 2π-periodic function on $\boldsymbol{R}$. Then, since $\hat{\psi}$ has compact support on $\boldsymbol{R}$ (see [6, p. 74]) and $g(x)=\sum_{n\in \boldsymbol{Z}} a_n\hat{\psi}_0^{-n}(-x)$, it follows that $g\in L^1(\boldsymbol{R})$ and $(\hat{g}, \psi_j^k)\geq 0$ for all $j, k\in \boldsymbol{Z}$. As an application of this idea and Corollary 4.2, we can give another proof of Wiener's result stated in §1. Let

$f=\sum_{n\in Z}a_n e^{inx}$ be in $L^1([-\pi,\pi])$ with $a_n\geq 0$ for all $n\in Z$ and f restricted to a neighborhood $(-\delta,\delta)$ of $x=0$ belongs to $L^2([-\pi,\pi])$ for some δ with $0<\delta<\pi$. As stated above, if we put $g(x)=f(2x)\cdot\hat{\psi}(-x)$, it follows that $g\in L^1(R)$ and $(\hat{g},\psi_j^k)\geq 0$ for all $j,k\in Z$. Since the support of $\hat{\psi}$ is contained in $[-8\pi/3,-2\pi/3]\cup[2\pi/3,8\pi/3]$ (see [6, p. 74]) and $0<\delta/2<2\pi/3$, the terms in the summation $g(x)\cdot\sum_{j\in Z}\mathbf{1}_{\pm((2\pi-\delta/2)2^j,(2\pi+\delta/2)2^j)}(x)$ vanish except when $j=0,-1$. Especially, it follows from the assumption on f that

$$g(x)\cdot\sum_{j\in Z}\mathbf{1}_{\pm((2\pi-\delta/2)2^j,(2\pi+\delta/2)2^j)}(x)\in L^2(R)\,.$$

Therefore, Corollary 4.2 yields that $g(x)$ belongs to $L^2(R)$ and thus, $\int_{-\pi}^{\pi}|f(x)|^2dx=2\pi\sum_{n\in Z}|a_n|^2=2\pi\int_R|g(x)|^2dx<\infty$ by the orthonormality of $\{\psi_j^k;\,j,k\in Z\}$.

REMARK 4.4. We cannot replace the condition (22) of Corollary 4.2 by a weaker one like local square-integrability of f or square-integrability of a finite sum of j in (22). Indeed, look at the following function:

$$f(x)=(2\sin(x/2))^{-1/2}\cos\left(\frac{\pi-x}{4}\right)-1=\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\cos(nx)\qquad(0<x<2\pi)\,.$$

Obviously, $f\in L^1(T)$ has nonnegative Fourier coefficients. However it does not belong to $L^2(T)$. We now regard this function as a 2π-periodic function on R and we put for a fixed $j_0\in Z$

$$f_{j_0}(x)=\hat{\psi}\left(\frac{x}{2^{j_0}}\right)f\left(\frac{x}{2^{j_0}}\right)=\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\hat{\psi}\left(\frac{x}{2^{j_0}}\right)\cos\left(n\frac{x}{2^{j_0}}\right)$$

$$=2^{j_0/2-1}\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}(\hat{\psi}_{j_0}^{n}(x)+\hat{\psi}_{j_0}^{-n}(x))\,.$$

Then, $(\hat{f}_{j_0},\psi_j^k)\geq 0$ for all $j,k\in Z$ and f_{j_0} vanishes on a neighborhood of $x=0$, because the support of f_{j_0} is contained in $[-2^{j_0+3}\pi/3,-2^{j_0+1}\pi/3]\cup[2^{j_0+1}\pi/3,2^{j_0+3}\pi/3]$. However, f_{j_0} does not belong to $L^2(R)$.

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