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COUNTING ARGUMENTS FOR HOPF ALGEBRAS OF LOW DIMENSION

By

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Abstract. Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. We show that all Hopf algebras of dimension 15, 21 or 35 over $\mathbf{k}$ are necessarily semisimple. We also prove that Hopf algebras of dimension 25 or 49 are either semisimple or pointed. This concludes the full classification of Hopf algebras of the above mentioned dimensions. We also classify pointed Hopf algebras of dimension pq^2, where $p \neq q$ are prime numbers, and semisimple Hopf algebras of dimension 45.

§0. Introduction

In the last years there has been an intense activity in classification problems of finite dimensional Hopf algebras over an algebraically closed field $\mathbf{k}$ of characteristic 0. Many results have been found, containing mainly the semisimple case and the pointed non-semisimple case. The question of classifying all Hopf algebras of a fixed dimension, posed by I. Kaplansky in 1975, was solved in the Ph. D. thesis of R. Williams for dimension ≤ 11 [W]. An alternative proof of this result appears in [S1]. Apart from these, the complete classification is known only when the dimension is a prime number p; in this case there is only one isomorphism type, represented by the group algebra of the cyclic group of order p [Z].

In this paper we develope some ideas about the coradical filtration of a finite dimensional Hopf algebra, starting from a description that appears in an unpublished work of W. Nichols. These allow us to prove the following Theorem.

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THEOREM 0.1. (a). *A Hopf algebra of dimension* 15 *or* 35 *is semisimple and isomorphic to the group algebra of a cyclic group.*

(b). *A Hopf algebra of dimension* 21 *is semisimple and isomorphic to either* $\mathbf{k}\boldsymbol{Z}/(21)$, $\mathbf{k}G$ *or* $\mathbf{k}^G$, *where* G *is the only (up to isomorphisms) non-abelian group of order* 21.

(c). *Let* H *be a Hopf algebra of dimension* m^2, *where* $m = 5$ *or* 7. *Then* H *is either semisimple or pointed. Thus* H *is isomorphic to* $\mathbf{k}\boldsymbol{Z}/(m^2)$ *or* $\mathbf{k}\boldsymbol{Z}/(m) \oplus \boldsymbol{Z}/(m)$, *if* H *is semisimple; or to a Taft algebra* $T(\xi) \simeq T(\xi)^*$, *where* ξ *is a primitive* m*-th. root of unity, if* H *is pointed.*

Let ξ be a primitive m-th. root of unity. We recall that the Taft algebra $T(\xi)$ is defined as the algebra on two generators x and g, satisfying the relations

$$x^m = 0, \quad g^m = 1, \quad gx = \xi xg.$$

The Hopf algebra structure in $T(\xi)$ is determined by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g,$$

$$\varepsilon(x) = 0, \quad \varepsilon(g) = 1,$$

$$\mathscr{S}(g) = g^{-1}, \quad \mathscr{S}(x) = -xg^{-1}.$$

It is known that $T(\xi)$ is a pointed non-semisimple Hopf algebra of dimension m^2 whose proper Hopf subalgebras are semisimple and contained in $\mathbf{k}\langle g\rangle$. Also, we have $T(\xi) \simeq T(\xi)^*$ and $T(\xi) \simeq T(\xi')$ if and only if $\xi = \xi'$.

The paper is organized as follows: in § 1 we give a proof of the results of Nichols on the coradical filtration and a series of consequences of them. In section 2 we present some results on the possibilities for the dimensions of certain terms of the coradical filtration. We devote section 3 to prove Theorem 0.1 using the methods described in the previous sections.

We include an Appendix where we present the classification of pointed Hopf algebras of dimension pq^2, where $p \neq q$ are prime numbers; we use for this the "Lifting principle" from [AS2]. We also prove here that a semisimple Hopf algebra of dimension 45 is necessarily trivial.

Our references for the theory of Hopf algebras are [Sw], [Mo], [Sch]. The notation for Hopf algebras is standard: $\Delta, \mathscr{S}, \varepsilon$, denote respectively the comultiplication, the antipode, the counit; we use Sweedler notation but dropping the summation symbol. Throughout $\mathbf{k}$ denotes an algebraically closed field of characteristic zero.

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§1. Remarks on the Coradical Filtration

Let C be a coalgebra over $\mathbf{k}$. We denote by $\hat{C}$ the set of isomorphism types of simple left C-comodules and by $G(C)$ the set of group-like elements in C. We shall consider the coradical filtration of C,

$$C_0 \subset C_1 \subset \cdots;$$

so that C_0 is the coradical of C. We have $C_0 \simeq \bigoplus_{\tau \in \hat{C}} C_\tau$, where C_τ is a simple subcoalgebra of dimension d_τ^2, $d_\tau \in \boldsymbol{Z}$. It is convenient to introduce the notation

$$C_{0,d} := \bigoplus_{\tau \in \hat{C} : d_\tau = d} C_\tau;$$

for instance $C_{0,1} = \mathbf{k}G(C)$ and $C_{0,2}$ is the sum of all 4-dimensional simple subcoalgebras of C.

We have $C_n = ((\operatorname{Jac} C^*)^{n+1})^\perp$, $n \geq 0$, where $\operatorname{Jac} C^*$ denotes the Jacobson radical of C^* and for any subspace V of C^*, $V^\perp \subseteq C$ is the anihilator of V in C, *i.e.*, $V^\perp = \{c \in C : \langle v, c\rangle = 0, \forall v \in V\}$. See [Mo, 5.2.9].

We shall denote by V_τ (resp., V_τ^*) the simple left (resp. right) C-comodule corresponding to $\tau \in \hat{C}$. As usual, for $g, h \in G(C)$, $\mathscr{P}_{g,h}(C)$ denotes the space of (g,h)-skew primitive elements of C:

$$\mathscr{P}_{g,h}(C) := \{x \in C : \Delta(x) = x \otimes g + h \otimes x\};$$

a skew primitive element $x \in \mathscr{P}_{g,h}(C)$ will be called *trivial* if it belongs to the linear span of $g - h$.

By a C_0-bicomodule we understand a vector space endowed with left and right C_0-coactions $\rho_L : M \to C_0 \otimes M$ and $\rho_R : M \to M \otimes C_0$ such that $(\rho_L \otimes \mathrm{id})\rho_R = (\mathrm{id} \otimes \rho_R)\rho_L$. Any C_0-bicomodule is a direct sum of simple C_0-sub-bicomodules and a simple C_0-bicomodule is of the form $V_\tau \otimes V_\mu^*$ and has dimension $d_\tau d_\mu$ for some $\tau, \mu \in \hat{C}$. If M is a C_0-bicomodule, we set $M^{\tau,\mu}$ for the isotypic component of type $V_\tau \otimes V_\mu^*$.

We want to state a description of the coradical filtration due to Nichols, see [W]. Let C be a coalgebra; then its coradical is coseparable because $\mathbf{k}$ is algebraically closed. By [Mo, Th. 5.4.2] there exists a coalgebra projection π of C onto C_0; let $I := \ker \pi$. Then C is a C_0-bicomodule via $\rho_L := (\pi \otimes \mathrm{id})\Delta : C \to C_0 \otimes C$

and $\rho_R := (\mathrm{id} \otimes \pi)\Delta : C \to C \otimes C_0$. Clearly, I and C_n, $n \geq 0$, are sub-bicomodules of C. Let P_n be the sequence of subspaces defined recursively by

$$P_0 = 0,$$

$$P_1 = \{x \in C : \Delta(x) = \rho_L(x) + \rho_R(x)\} = \Delta^{-1}(C_0 \otimes I + I \otimes C_0),$$

$$P_n = \left\{ x \in C : \Delta(x) - \rho_L(x) - \rho_R(x) \in \sum_{1 \leq i \leq n-1} P_i \otimes P_{n-i} \right\}, \quad n \geq 2.$$

LEMMA 1.1 (W. Nichols). $P_n = C_n \cap I$.

PROOF. By induction on n, the case $n = 0$ being trivial. The inclusion $P_n \subseteq C_n \cap I$ follows from the induction hypothesis: indeed, clearly $P_n \subseteq C_n$ and if $x \in P_n$ then

$$\Delta(x) = (\pi \otimes \mathrm{id})\Delta(x) + (\mathrm{id} \otimes \pi)\Delta(x) + \sum_i x_i \otimes x_{n-i},$$

for some $x_i \in P_i$, $1 \leq i \leq n-1$. Applying $\pi \otimes \pi$, we obtain

$$\Delta(\pi(x)) = (\pi \otimes \pi)\Delta(x) = (\pi \otimes \pi)\Delta(x) + (\pi \otimes \pi)\Delta(x) = 2(\pi \otimes \pi)\Delta(x),$$

since by induction $P_i = C_i \cap I \subseteq I$, for all $i = 1, \dots, n-1$. Hence, $(\pi \otimes \pi)\Delta(x) = 0$ and $\pi(x) = 0$; so that $x \in C_n \cap I$.

Conversely, let $x \in C_n \cap I$. Then $\Delta(x) = \sum_{0 \leq i \leq n} x_i \otimes y_i$ with $x_i \in C_i$, $y_i \in C_{n-i}$. It is clear that $C_i = C_0 \oplus (C_i \cap I)$; accordingly we write $x_i = x_{i,0} + x_{i,+}$ with $x_{i,0} \in C_0$, $x_{i,+} \in C_i \cap I$ and similarly for the y_i's. It follows that

$$\Delta(x) - \rho_L(x) - \rho_R(x) = \sum_{0 \leq i \leq n} x_{i,+} \otimes y_{i,+} - \sum_{0 \leq i \leq n} x_{i,0} \otimes y_{i,0};$$

but the term $\sum_{1 \leq i \leq n-1} x_{i,0} \otimes y_{i,0}$ is 0 since $x \in I$. Hence $x \in P_n$ by induction. □

Observe that Lemma 1.1 implies that P_n is a C_0-sub-bicomodule of I, for all $n \geq 0$. The following Lemma relates the structure of P_1 with the first term of the coradical filtration of C.

LEMMA 1.2 (W. Nichols). *The first term of the coradical filtration can be expressed as* $C_1 = \sum_{\tau, \mu \in \hat{C}} C_\tau \wedge C_\mu$ *and* $C_\tau \wedge C_\mu = C_\tau \oplus C_\mu \oplus P_1^{\tau,\mu}$ *(only one simple coalgebra if* $\tau = \mu$*).*

We stress that $P_1^{\tau,\mu}$ is not intrinsic since it depends on the projection π.

PROOF. Clearly $P_1^{\tau,\mu} = \{x \in P_1 : \Delta(x) \in C_\tau \otimes C + C \otimes C_\mu\} \subseteq C_\tau \wedge C_\mu \subseteq C_1$. By Lemma 1.1, $C_1 = C_0 \oplus P_1 = (\bigoplus_\tau C_\tau) \oplus (\bigoplus_{\tau,\mu} P_1^{\tau,\mu}) \subseteq \sum_{\tau,\mu} C_\tau \wedge C_\mu$. The claim follows. □

Assume in what follows that $C = H$ is a finite dimensional Hopf algebra. Then $\mathcal{S}(C_\tau)$ is a simple subcoalgebra which we denote by C_{τ^d}; if $g \in G(H)$ then $g.C_\tau$ and $C_\tau.g$ are also simple subcoalgebras which we denote by $C_{g.\tau}$, $C_{\tau.g}$ respectively.

COROLLARY 1.3. $\dim P_1^{\tau,\mu} = \dim P_1^{\mu^d,\tau^d} = \dim P_1^{g.\tau,g.\mu} = \dim P_1^{\tau.g,\mu.g}$ *for any* $g \in G(H)$.

PROOF. As $\mathcal{S}(C_\tau \wedge C_\mu) = C_{\mu^d} \wedge C_{\tau^d}$, $g.(C_\tau \wedge C_\mu) = C_{g.\tau} \wedge C_{g.\mu}$ and $(C_\tau \wedge C_\mu).g = C_{\tau.g} \wedge C_{\mu.g}$, the claim follows from Lemma 1.2. □

COROLLARY 1.4. *If I is a direct sum of one-dimensional H_0-sub-bicomodules then* $H_1 = H_0 + \sum_{g,h \in G(H)} P_{g,h}(H)$. □

Consider the right action $\leftharpoonup : H^* \otimes H \to H^*$ given by $\alpha \leftharpoonup h = \langle \alpha_1, h\rangle \alpha_2$, $\forall h \in H$, $\alpha \in H^*$.

Let $\int \in H^*$ be a non-zero left integral and let $g_0 \in G(H)$ be the distinguished group-like element, so that

$$\alpha \int = \langle \alpha, 1\rangle \int \quad \text{and} \quad \int \alpha = \langle \alpha, g_0\rangle \int, \quad \forall \alpha \in H^*.$$

We shall assume in what follows that H is not cosemisimple, or equivalently, that $\langle \int, 1\rangle = 0$; in particular $\int^2 = 0$ and if $g \in G(H)$, also $(\int \leftharpoonup g)^2 = \int^2 \leftharpoonup g = 0$.

Observe that if $C \neq \mathbf{k}1$ is a simple subcoalgebra of H, and if $c \in C$, then

$$\left\langle \int, c \right\rangle 1 = \left\langle \int, c_2 \right\rangle c_1 \in C \cap \mathbf{k}1,$$

whence $\int|_C = 0$, *i.e.*, $\int$ belongs to the anihilator of H_0, $H_0^\perp = \operatorname{Jac} H^*$.

Let $g \in G(H)$. Since the left (and right) multiplication by g is a coalgebra automorphism of H, it preserves H_0. This implies that also $\int \leftharpoonup g$ belongs to $\operatorname{Jac} H^*$.

Also, for all $\alpha \in H^*$, we have

$$\alpha\left(\int \leftharpoonup g\right) = \langle \alpha, g^{-1}\rangle \int \leftharpoonup g, \quad \text{and} \quad \left(\int \leftharpoonup g\right)\alpha = \langle \alpha, g^{-1}g_0\rangle \int \leftharpoonup g.$$

Hence $\mathbf{k}(\int \leftharpoonup g)$ is a two-sided ideal of H^* and $\mathbf{k}(\int \leftharpoonup g) \subseteq \operatorname{Jac} H^*$. Moreover, since distinct group-like elements are linearly independent and the map $H \to H^*$, $h \mapsto \int \leftharpoonup h$, is injective, the ideals $\mathbf{k}(\int \leftharpoonup g)$ and $\mathbf{k}(\int \leftharpoonup g')$ are distinct if $g \neq g'$.

LEMMA 1.5. *Let H be a non-cosemisimple finite dimensional Hopf algebra. Let $L = (\int \leftharpoonup \mathbf{k}G(H))^{\perp}$. Then $L \subseteq H$ is a subcoalgebra of H containing H_0 and there is an H_0-bicomodule decomposition*

$$H = L \oplus \bigoplus_{j=1}^{s} I_j,$$

where $s = |G(H)|$ and I_j are one-dimensional H_0-sub-bicomodules of I, $\forall j = 1, \dots, s$.

PROOF. Call $L_g := \ker \int \leftharpoonup g \subseteq H$. Then $\forall g \in G(H)$, L_g is a subcoalgebra of H of codimension 1 containing H_0. Also, $L_g \neq L_{g'}$ if $g \neq g'$. Index $G(H)$ in the form $G(H) = \{1 = g_1, \dots, g_s\}$, where $s = |G(H)|$, and write $L_j := L_{g_j}$. Denote also by $L^{(j)} := \bigcap_{1 \leq i \leq j} L_i$. Then $L^{(j)}$ is a subcoalgebra of H and $H_0 \subseteq L^{(j)}$, $\forall j$. In particular, $(L^{(j)})_0 = H_0$, and $L^{(j)} = H_0 \oplus I^{(j)}$, where $I^{(j)} = \ker \pi|_{L^{(j)}}$. This gives a descending chain of H_0-sub-bicomodules

$$I^{(s)} \subseteq I^{(s-1)} \subseteq \cdots \subseteq I^{(1)} \subseteq I,$$

such that $\operatorname{codim}(I^{(j)}, I^{(j-1)}) = 1$, for all $j = 1, \dots, s$, where $I^{(0)} = I$. Hence, there exist one-dimensional H_0-sub-bicomodules I_j, $j = 1, \dots, s$, such that $I^{(j-1)} = I^{(j)} \oplus I_j$. We thus obtain

$$H = \left(\int \leftharpoonup \mathbf{k}G(H)\right)^{\perp} \oplus \bigoplus_{j=1}^{s} I_j,$$

as claimed. □

Combining Lemma 1.5 with Corollary 1.4, we obtain

COROLLARY 1.6. *Let H be a non-cosemisimple finite dimensional Hopf algebra. Suppose that* $\dim H - \dim H_0 = |G(H)|$. *Then $\mathscr{P}_{g,h} \supsetneq \mathbf{k}(g - h)$, for some $g, h \in G(H)$.* □

LEMMA 1.7. *Let H be a non-cosemisimple finite dimensional Hopf algebra.*

(i). *Suppose that $\mathscr{P}_{g,h} = \mathbf{k}(g - h)$, for all $g, h \in G(H)$. Then $\int \leftharpoonup \mathbf{k}G(H) \subseteq (\operatorname{Jac} H^*)^2$. In particular, $|G(H)| \leq \dim H - \dim H_1$.*

(ii). *If $H_1 = H$ then H has a non-trivial skew primitive element.*

Note that part (ii) of the Lemma above implies, since H is finite dimensional, that if $H = H_1$ then $G(H)$ is non-trivial.

PROOF. (i). Suppose that $\int \leftharpoonup \mathbf{k}G(H)$ is not contained in $(\operatorname{Jac} H^*)^2$. Then $\langle \int, H_1 \rangle \neq 0$. In the notation of Lemma 1.5, this implies that I_1 is an H_0-sub-bicomodule of $H_1 \cap I$, which in turn implies the claim.

(ii). Suppose that $H_1 = H$. Then the subcomodules I_j in Lemma 1.5 are necessarily contained in P_1, and thus spanned by non-trivial skew primitive elements of H. $\square$

Let M and N be non-negative integers such that M divides N and let $\xi \in \mathbf{k}^\times$ be a primitive M-th. root of unity. Consider the algebra $K_\mu(N, \xi)$, generated by elements x and g with relations

$$x^M = \mu(1 - g^M), \quad g^N = 1, \quad gx = \xi xg,$$

where $\mu = 0$, if $M = N$, and $\mu \in \{0, 1\}$, if $M \neq N$. The formulas

$$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g,$$
$$\varepsilon(x) = 0, \quad \varepsilon(g) = 1,$$
$$\mathcal{S}(g) = g^{-1}, \quad \mathcal{S}(x) = -xg^{-1},$$

determine a Hopf algebra structure in $K_\mu(N, \xi)$. It follows from [AS2, Thm. 5.5] that the dimension of $K_\mu(N, \xi)$ is MN. If $M = N$, then $K_\mu(N, \xi) \simeq T(\xi)$, where $T(\xi)$ is the Taft algebra corresponding to ξ.

Note that $\mathbf{k}\langle g^M \rangle$ is a central Hopf subalgebra of $K_\mu(N, \xi)$ and there is a short exact sequence of Hopf algebras $0 \to \mathbf{k}\langle g^M \rangle \to K_\mu(N, \xi) \xrightarrow{\pi} T(\xi) \to 1$.

Also, $K_\mu(N, \xi)$ is a non-semisimple pointed Hopf algebra over $\mathbf{k}$, whose coradical filtration is

$$K_\mu(N, \xi)_n = \bigoplus_{0 \le i \le N-1} \mathbf{k} g^i x^n, \quad 0 \le n \le M - 1.$$

Variations of the following Proposition appear in [N], [AS1], [S2].

PROPOSITION 1.8. *Let H be a non-semisimple finite dimensional Hopf algebra over $\mathbf{k}$. Suppose that $\mathbf{k}(g - h) \subsetneq \mathcal{P}_{g,h}$, for some $g, h \in G(H)$. Then H contains a Hopf subalgebra K isomorphic to $K_\mu(N, \xi)$, for some root of unity $\xi \in \mathbf{k}$, and some $\mu \in \{0, 1\}$.*

In particular, if $\dim H$ *is free of squares, then H does not contain non-trivial skew primitive elements.*

PROOF. We may assume that $\mathbf{k}(g-1) \subsetneq \mathscr{P}_{g,1}$, for some $1 \neq g \in G(H)$. Thus, the cyclic group $\Gamma = \langle g \rangle$ acts on $\mathscr{P}_{g,1}$ by conjugation and there exists a character $\chi \in \hat{\Gamma}$ and a non-zero $x \in \mathscr{P}_{g,1} - \mathbf{k}G(H)$ such that $gxg^{-1} = \chi(g)x$.

The subalgebra $K := \mathbf{k}\langle g, x \rangle$ of H is hence a Hopf subalgebra satisfying

$$(*) \qquad \Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes g + 1 \otimes x, \quad gx = \xi xg,$$

where $\xi = \chi(g)$. Moreover, ξ is a root of unity in $\mathbf{k}$ and $\xi \neq 1$, since otherwise, K would be a commutative Hopf subalgebra of H not contained in the coradical of H, which is not possible. Let M be the order of ξ and let N be the order of g, so that M divides N.

The relations in (*), together with the quantum binomial formula, imply that $\Delta(x^M) = x^M \otimes g^M + 1 \otimes x^M$ and $g^M x^M = x^M g^M$. Thus the subalgebra $\mathbf{k}\langle g^M, x^M \rangle$ is a commutative Hopf subalgebra and therefore it is contained in H_0. It then follows that $x^M = \mu(1 - g^M)$; if $M = N$ we may take $\mu = 0$, while if $\mu \neq 0$ and $M \neq N$, we can normalize x so that $\mu = 1$. We have then a Hopf algebra surjection $p : K_\mu(N, \xi) \to K$.

By choice of N, the restriction of p to the coradical of $K_\mu(N, \xi)$ is injective; since $x \neq 0$, it is not difficult to show that the restriction of p to $K_\mu(N, \xi)_1$ is also injective. Hence p is injective [Mo].

The last part of the Proposition follows from [NZ]. □

§2. Some General Results

In this section we give some results on the possible dimensions of the terms of the coradical filtration of a finite dimensional Hopf algebra H.

LEMMA 2.1. (i). *The order of* $G(H)$ *divides the dimension of* H_n, $n \geq 0$ *and of* $H_{0,d}$, $d \geq 1$.

(ii). *If* H *is neither pointed nor semisimple, then* $\dim H - |G(H)| \geq 6$. *If moreover* $|G(H)| > 1$ *is odd, then* $\dim H - |G(H)| \geq 11$.

PROOF. (i). All the H_n, as well as the $H_{0,d}$, are left $(\mathbf{k}G(H), H)$-Hopf modules by means of the comultiplication of H and the left multiplication by elements of $G(H)$. Hence [NZ] applies.

(ii). We have $\dim H > \dim H_0 = |G(H)| + \sum_{d \geq 2} \dim H_{0,d} > |G(H)|$. That is, $|G(H)| < |G(H)| + 4 < \dim H$. The case of codimension 5 is discarded by Lemma 2.2 below. If $|G(H)|$ is odd and $H_{0,2} \neq 0$ then $\dim H_{0,2} \geq 4|G(H)| \geq 12$. This implies the second claim: indeed, we have now $|G(H)| < |G(H)| + 9 < \dim H$ and the case of codimension 10 follows again by Lemma 2.2. □

LEMMA 2.2 [S1]. *If H is not cosemisimple*, $\dim H_0 + 1 < \dim H$.

We give an alternative proof that uses Lemma 1.5.

PROOF. Suppose that H is not cosemisimple and $\dim H_0 + 1 = \dim H$; in particular, $H = H_1$. By Lemma 1.7-(ii), H contains a non-trivial skew primitive element and *a fortiori* a non-trivial group-like element g, since it is finite dimensional. By Lemma 2.1-(i), the order of g divides both $\dim H_0$ and $\dim H$. This is a contradiction that finishes the proof of the Lemma. □

REMARK. The preceeding Lemma can be proved without using [NZ], as follows: suppose that $\dim H_0 + 1 = \dim H$. Write $H = H_0 \oplus I$ as in § 1, where I is the kernel of the coalgebra projection $H \to H_0$. Since I is coideal in H, $\Delta I \subseteq I \otimes H + H \otimes I$. On the other hand, since clearly $H = H_1$, $\Delta I \subseteq H_0 \otimes H + H \otimes H_0$. Thus, $\Delta I \subseteq H_0 \otimes I + I \otimes H_0$. Writting $I = \mathbf{k}x$, $x \in I$, we have

$$\Delta(x) = x \otimes b + a \otimes x,$$

for some $a, b \in H_0$. Let $0 \neq \int \in H^*$ be a left integral in H^*. Then we have $\langle \int, H_0 \rangle = 0$. We may assume that $\langle \int, x \rangle = 1$.

Let now $\alpha \in H^*$, so that $\alpha \int = \langle \alpha, 1 \rangle \int$ and $\int \alpha = \langle \alpha, g_0 \rangle \int$, where $g_0 \in H$ is the distinguished group-like element. Specializing in x, we have

$$\langle \alpha, a \rangle = \left\langle \alpha \int, x \right\rangle = \langle \alpha, 1 \rangle \left\langle \int, x \right\rangle = \langle \alpha, 1 \rangle,$$

and

$$\langle \alpha, b \rangle = \left\langle \int \alpha, x \right\rangle = \langle \alpha, g_0 \rangle.$$

Hence $a = 1$, $b = g_0$ and $\Delta(x) = 1 \otimes x + x \otimes g_0$. Also, $g_0 \neq 1$ since H is finite dimensional and the characteristic of $\mathbf{k}$ is zero.

Now write $g_0 x = y + tx$, where $y \in H_0$ and $t \in \mathbf{k}$. So that $\Delta(g_0 x) = \Delta y + t(1 \otimes x + x \otimes g_0)$ and on the other hand, $\Delta(g_0 x) = \Delta(g_0)\Delta(x) = g_0 \otimes g_0 x + g_0 x \otimes g_0^2$. This implies that

$$\begin{aligned} \Delta y &= g_0 \otimes g_0 x + g_0 x \otimes g_0^2 - t(1 \otimes x + x \otimes g_0) \\ &= g_0 \otimes y + y \otimes g_0^2 + t(g_0 \otimes x + x \otimes g_0^2 - 1 \otimes x - x \otimes g_0). \end{aligned}$$

But $\Delta y \in H_0 \otimes H_0$, then $t = 0$.

Thus $g_0 x = y \in H_0$ and since left multiplication by g_0^{-1} is a coalgebra automorphism of H, $g_0^{-1} H_0 = H_0$; in particular $x = g_0^{-1} g_0 x \in H_0$ which is an absurd.

LEMMA 2.3. (i). *Let P_n be as in* § 1. *Then $H_n = H_0 \oplus P_n$ and $|G(H)|$ divides* $\dim P_n$, $\forall n$.

(ii). *Suppose that H does not contain any non-trivial skew primitive element. Suppose that any simple subcoalgebra of H has dimension* 1 *or n^2, where $n > 1$ is a fixed integer. Then n divides* $\dim P_1$.

If moreover every irreducible H_0-sub-bicomodule of P_1 has dimension n, then $n|G(H)|$ divides $\dim P_1$.

The assumption that H does not contain any non-trivial skew primitive element is fulfilled, for instance, if either $\dim H$ is free of squares (by Proposition 1.8) or $\dim H = p^2$, p prime, and H is not pointed.

PROOF. Part (i) is an easy but useful consequence of Lemma 2.1.

If H does not contain any non-trivial skew primitive element, then any simple H_0-sub-bicomodule of P_1 has dimension n or n^2, whence n divides $\dim P_1$. If any such sub-bicomodule has dimension n, then Corollary 1.3 implies that $n|G(H)|$ divides $\dim P_1$. Hence part (ii) follows. □

LEMMA 2.4. *If H is pointed non-semisimple then* $\dim H$ *is divisible by p^2 for some prime number p.*

PROOF. This follows at once from the Theorem of Taft-Wilson (see *e.g.* [Mo, 5.4.1]) and Proposition 1.8. □

LEMMA 2.5 [Z]. *If H is not semisimple and* $\dim H$ *is odd, then either $G(H)$ or $G(H^*)$ is non-trivial.*

PROOF. Since the dimension of H is odd, Radford's formula for $\mathcal{S}^4$ implies that H and H^* can not be both unimodular; this implies the Lemma. See *e.g.* [Sch], [AS1, Lemma 2.2]. □

LEMMA 2.6. (i). *Let H be a non-cosemisimple Hopf algebra whose dimension is not divisible by* 4. *Then $H_1 \neq H$.*

(ii). *Let H be a non-cosemisimple non-pointed Hopf algebra of dimension* $3r$, *where r is an integer not divisible by* 4. *Then the order of $G(H)$ is not equal to r.*

PROOF. (i). Assume that $H_1 = H$. By Lemma 1.7, H has a non-trivial skew-primitive element. Therefore H contains a Hopf subalgebra K, of dimension NM, as in Proposition 1.8, where M and N are integers such that M divides N. Now, $K_1 = K \cap H_1 = K$ by [Mo, 5.2.12]. Then, since the coradical filtration of K has M terms, we have $M = 2$ and 4 divides $\dim H$, which is a contradiction.

(ii). Assume that the order of $G(H)$ equals r. By assumption and using Lemma 2.1, we find that $\dim H_0 = 2r$ and $H = H_1$. Now (i) applies. □

§3. Proof of Theorem 0.1

The proof of Theorem 0.1 will be carried out case by case. We will need the following Lemma.

Lemma 3.1. *Let p and q be prime numbers and let H be a Hopf algebra of dimension pq over* $\mathbf{k}$.

(i). [EG], [GW], [Ma]. *If H is semisimple, then H is either commutative or cocommutative.*

(ii). *If $p = q$ and H is pointed non-semisimple, then H is isomorphic to a Taft algebra $T(\xi)$, for some primitive p-th. root of unity $\xi \in \mathbf{k}$.* □

An alternative proof of part (i) of Lemma 3.1, in the case where p and q are distinct odd prime numbers, is given in [Na]. Part (ii) has been found independently by W. Nichols, W. Chin, D. Stefan and the first author. See [AS1] for a proof.

In what follows, H will denote a Hopf algebra of the prescribed dimension. We shall assume that H is neither pointed nor cosemisimple. By Lemma 2.5, we may also assume that $G(H) \neq 1$.

Dimension 15. By Lemma 2.6-(ii), $|G(H)| \neq 5$. Assume that $|G(H)| = 3$. Since $\dim H_0 - 3$ should be a sum of squares greater than 1, we discard all the possibilities except $\dim H_0 = 12$. In this case, $H_1 = H$ and this contradicts Lemma 2.6-(i). □

Dimension 21. By Lemma 2.6, $|G(H)| \neq 7$. If $|G(H)| = 3$ then arguing as for dimension 15, we eliminate all possibilities except $\dim H_0 = 12$ or 15.

If $|G(H)| = 3$ and $\dim H_0 = 15$ then $\dim H_1 = 18$, by Lemmas 2.1 and 2.6-(i). Thus, $\dim P_1 = 3$. But H_0 is the direct sum of $\mathbf{k}G(H)$ and three simple coalgebras of dimension 4. This contradicts Lemma (2.3)-(ii).

If $|G(H)| = 3$ and $\dim H_0 = 12$ then $\dim H_1 = 15$ or 18, by Lemmas 2.1 and 2.6-(i). Then $\dim P_1 = 3$ or 6. But in this case, the simple subcoalgebras of H have either dimension 1 or 9. Then Lemma 2.3-(ii) applies. □

DIMENSION 25. We can assume that $|G(H)| = 5$ and $\dim H_0 = 10$, 15 or 20; but neither 5 nor 10 nor 15 can be expressed as sums of squares greater than 1. □

DIMENSION 35. $|G(H)| = 7$ is not possible since 7 divides $\dim H_{0,d}$ for all d. If $|G(H)| = 5$ then arguing as for 15 we eliminate all the cases except $\dim H_0 = 25$. Necessarily, $\dim H_1 = 30$, $\dim P_1 = 5$ and H_0 is a direct sum of 1- or 4-dimensional simple coalgebras. Hence Lemma 2.3-(ii) applies. □

DIMENSION 49. We reduce by analogous considerations as above to the case $|G(H)| = 7$, $\dim H_0 = 35$ and H_0 is the direct sum of $\mathbf{k}G(H)$ and seven 4-dimensional simple coalgebras. By Lemma 1.5, $\dim H_1 = 42$; hence P_1 has dimension 7. Now Lemma 2.3-(ii) applies again. □

Appendix

In this section, we classify Hopf algebras under some additional hypothesis.

Let p and q be different prime numbers. Let $j = 1$ or pr, $1 \le r \le q-1$ and let $\mu = 0$ or 1, such that $\mu = 0$ when $j \neq 1$. Let ω be a root of 1 such that the order of ω is q if $j = 1$, and q divides the order of ω if $j \neq 1$. Let $\mathscr{A}(\omega, j, \mu)$ be the algebra generated by elements g and x with relations

$$g^{pq} = 1, \quad x^q = \mu(1 - g^q), \quad gx = \omega xg.$$

Then $\mathscr{A}(\omega, j, \mu)$ is a pointed Hopf algebra over $\mathbf{k}$, where the comultiplication is defined by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g^j \otimes x.$$

LEMMA A.1. *Let p and q be different prime numbers. Let H be a pointed non-semisimple Hopf algebra of dimension pq^2. Then $G(H)$ is a cyclic group of order pq and H is isomorphic to exactly one of the Hopf algebras in the following list*:

(i) $\mathscr{A}(\tau, 1, 0)$, *τ a primitive q-th. root of* 1.

(ii) $\mathscr{A}(\tau, 1, 1)$, *τ a primitive q-th. root of* 1.

(iii) $\mathscr{A}(\omega, pr, 0)$, $1 \le r \le q-1$, *where ω is a fixed primitive pq-th. root of* 1.

(iv) $\mathscr{A}(\tau, p, 0) \simeq T(\tau) \otimes \mathbf{k}\mathbf{Z}/p$, *where τ is a primitive q-th. root of* 1.

Conversely, all the Hopf algebras in the list have dimension pq^2.

That is, there are $4(q-1)$ isomorphism classes of pointed Hopf algebras of dimension pq^2 over $\mathbf{k}$.

PROOF. It follows from [AS2, Thm. 5.5] that the Hopf algebras in the list have dimension pq^2.

Suppose now that H is a pointed non-semisimple Hopf algebra of dimension pq^2. We shall apply the lifting principle in [AS2]. First, $|G(H)| \neq q$, resp. p by [AS3, Th. 1.3], for q, resp. p, odd, or [N, Th. 4.2.1] for $q = 2$; respectively, $p = 2$. Assume now that $|G(H)| = q^2$. By [AS1, Prop. 3.1] q^3 divides $\dim H$, which is impossible.

Assume finally that $|G(H)| = pq$. Let R be the diagram of H as in [AS2]; R is a braided Hopf algebra in the category of Yetter-Drinfeld modules over $G(H)$ of dimension q.

By the Taft-Wilson Theorem, there exists $0 \neq x \in \mathscr{P}_{1,u}(H) - \mathbf{k}G(H)$. We can assume that $uxu^{-1} = \xi x$ where ξ is a root of 1 of order N. Then N^2 divides $\dim H$, so that $N = q$. This shows that $R = \mathbf{k}\langle\bar{x}\rangle$, being $\bar{x}$ the class of x in $\operatorname{gr} H$ (see [AS2]), because both have the same dimension q. This implies in turn that u is central and hence that $G(H)$ is abelian and cyclic. It follows now readily that H is generated by g and x, where g is a group-like of order pq, x is a $(u, 1)$ skew primitive and $gxg^{-1} = \chi(g)x$, χ a character of the cyclic group generated by g and $\xi = \chi(u)$ has order q. Looking at the different possibilities for the orders of u and χ, we see that H is isomorphic to either of the Hopf algebras above. $\square$

Remarks on the Hopf algebras in the list. (a). All the Hopf algebras in the list can be presented as suitable extensions of Taft algebras and group algebras.

(b). It is not difficult to see that the dual of $\mathscr{A}(\tau, 1, 0)$ is isomorphic to a Hopf algebra of type (iii).

(c). The Hopf algebra $\mathscr{R}(\tau) := (\mathscr{A}(\tau, 1, 1))^*$ is not pointed, *cf.* [Ra]. More precisely, it is shown in *loc. cit.* that the coalgebra structure of $\mathscr{R}(\tau)$ is $T_\tau \oplus C_q \oplus C_q \oplus \cdots \oplus C_q$, $p - 1$ direct summands $C_q \simeq M_q(\mathbf{k})^*$.

All known examples of non-semisimple Hopf algebras of dimension pq^2 over $\mathbf{k}$ are either pointed or else dual of pointed Hopf algebras. The following Lemma gives insight into this question.

LEMMA A.2. *Let p and q be different prime numbers. Let H be a non-semisimple Hopf algebra of dimension pq^2. Suppose that the coradical of H is a Hopf subalgebra. Then H is pointed.*

PROOF. Suppose on the contrary that H_0 is not a group algebra. Then necessarily $H_0 \simeq \mathbf{k}^F$ as Hopf algebras, where F is the unique (up to isomorphisms) non-abelian group of order pq. Consider the coradical filtration of H.

The associated graded coalgebra $\operatorname{gr} H$ is a Hopf algebra whose coradical is isomorphic to $\mathbf{k}^F$. Moreover, $\operatorname{gr} H$ is isomorphic to a biproduct $\operatorname{gr} H = R\sharp\mathbf{k}^F$, where R is a braided Hopf algebra over $\mathbf{k}^F$. Then $(\operatorname{gr} H)^*$ is a non-semisimple Hopf algebra of dimension pq^2 and $(\operatorname{gr} H)^* \simeq R^*\sharp\mathbf{k}F$. Hence, $(\operatorname{gr} H)^*$ is a pointed Hopf algebra of dimension pq^2. This contradicts Lemma A.1. □

LEMMA A.3. *A semisimple Hopf algebra H of order* 45 *is necessarily trivial.*

PROOF. From the decomposition of H into simple subcoalgebras and [NR], we read $45 = |G(H)| + \sum_{j=3}^{6} n_j j^2$. Then $G(H)$ is non-trivial, by an easy calculation. The case $|G(H)| = 5$ is also impossible; for, 5 should divide n_3, n_4 and n_6 by Lemma 2.1. So all these numbers should be 0; but then 25 should divide 40, a contradiction. We discard similarly the cases $|G(H)| = 3$ or 15. Let us finally assume that $|G(H)| = |G(H^*)| = 9$. Let $\lambda \in \mathbf{k}G(H)$ be a normalized integral; then λ is an idempotent in $R(H^*)$, hence $\lambda = \Lambda + \sum_i e_i$ where Λ is a normalized integral and the e_i's are primitive idempotents in $R(H^*)$. Hence $H\lambda = \mathbf{k}\Lambda \oplus (\bigoplus_i He_i)$. Taking dimensions and using that $H\lambda$ is the representation induced from the trivial representation of $\mathbf{k}G(H)$, we see that $5 = 1 + \sum \dim He_i$. But $\dim He_i$ divides 45 by the class equation so it is either 1 or 3. Therefore at least one of the $\dim He_i$ is 1; but then there exists a non-trivial central group-like element in H^*, see *e.g.* [Sch, Lemma 4.14] and H is an extension of Hopf algebras. By [Na], H is trivial. □

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