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On Minimal Number of Singular Fibers in a Genus-2 Lefschetz Fibration

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Abstract. We show that the minimal number of singular fibers in a genus-2 Lefschetz fibration over a closed surface of genus h is equal to 5 if $h \geq 3$, 5 or 6 if $h = 2$ and 6 or 7 if $h = 1$.

1. Introduction

Let $N(g, h)$ denote the minimal number of singular fibers in a relatively minimal genus-g Lefschetz fibration (with at least one singular fiber) over a closed surface of genus h. Korkmaz and Ozbagci proved that (1) $N(g, h) = 1$ iff $g \geq 3$ and $h \geq 2$, (2) $2 \leq N(g, 1)$ for all $g \geq 1$, (3) $N(1, h) = 12$ for all $h \geq 0$ and (4) $5 \leq N(2, h) \leq 8$ for all $h \geq 0$ (see [8]). The proofs of (1) and (2) depend on a result of [14] which is the erratum to [13]. Moreover, the part (1) was proved by showing that a Dehn twist about a simple closed curve is written as a product of two commutator. Cadavid proved that $N(g, 0) \leq 2g + 4$ if g is even and $N(g, 0) \leq 2g + 10$ if g is odd (see [2]). This result was also discovered independently by Korkmaz (see [7]). Braungardt and Kotschick proved that $\frac{1}{5}(8g - 3) \leq N(g, 0)$ (see [1]). Korkmaz and Stipsicz showed that the fifth power of a Dehn twist about a separating curve is written as a product of 6 commutators (see [9]). Consequently, they proved that $N(2, h) = 5$ for all $h \geq 6$.

We show the following results by proving that a product of Dehn twists about 5 separating curves is written as a product of 3 commutators.

THEOREM 1. *For the number $N(2, h)$ the following holds.*

(1) $N(2, h) = 5$ *for all* $h \geq 3$,
(2) $N(2, 2) = 5$ *or* 6,
(3) $N(2, 1) = 6$ *or* 7.

It is well known that $N(2, 0) = 7$ or 8. Ozbagci proved that $7 \leq N(2, 0)$ (see [11]), and Matsumoto constructed a genus-2 Lefschetz fibration with 8 singular fibers over S^2 (see [10]). Sato prove that if a genus-2 Lefschetz fibration over S^2 has 7 singular fibers, then the

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total space is diffeomorphic to $S^2 \times T^2 \sharp 3\overline{C\mathbf{P}^2}$ (see [12]). However, the author does not know whether $S^2 \times T^2 \sharp 3\overline{C\mathbf{P}^2}$ admits a genus-2 Lefschetz fibration over S^2 with 7 singular fibers or not.

2. Preliminaries

Let Σ_h be a closed, connected, oriented surface of genus $h \geq 0$. Let $\mathcal{M}_2$ be the mapping class group of Σ_2, i.e., the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_2.

2.1. Lefschetz fibrations and the monodromy representation. We begin by recalling the definition and basic properties of Lefschetz fibrations. More details can be found in [5].

DEFINITION 2.1. Let X be a closed, connected, oriented smooth four manifold. A Lefschetz fibration is a map $f : X \to \Sigma_h$ such that f is injective on the set of critical points $C = \{x_1, \ldots, x_n\}$ and about each x_i and $f(x_i)$ there are complex local coordinate charts agreeing with the orientations of X and Σ_h on which f is of the form

$$f(z_1, z_2) = z_1^2 + z_2^2 .$$

Any fiber containing a critical point is called a singular fiber. Clearly all regular fibers are closed surfaces and are of the same diffeomorphism type. The *genus* of f is defined to be the genus of a regular fiber. We will assume that the generic fiber is connected and our fibration is relatively minimal, i.e., no fiber contains a (-1) sphere.

Hereafter, we assume that the genus of the fiber is equal to two and do not distinguish a diffeomorphism and its isotopy class. Moreover, we do not distinguish a curve and its isotopy class.

Each critical point of a genus-2 Lefschetz fibration corresponds to an embedded circle in a nearby regular fiber called a *vanishing cycle*, and the singular fiber is obtained by collapsing the vanishing cycle to a point. The boundary of a regular neighborhood of a singular fiber is a surface bundle over the circle. In fact, a singular fiber can be described by the monodromy of this surface bundle which turns out to be a right-handed Dehn twist along the corresponding vanishing cycle. Once we fix an identification of Σ_2 with the fiber over a base point of Σ_h, the topology of the Lefschetz fibration is determined by its *monodromy representation* $\Psi : \pi_1(\Sigma_h - \{f(x_1), \ldots, f(x_n)\}) \to \mathcal{M}_2$.

The monodromy of a Lefschetz fibration $f : X \to \Sigma_h$ comprises a factorization of $1 \in \mathcal{M}_2$ as

$$1 = \prod_{j=1}^{h} [a_j, b_j] \prod_{i=1}^{n} t_{v_i} ,$$

where v_i are the vanishing cycles of the singular fibers and t_{v_i} is the right handed Dehn twist about v_i. In particular, a product $\prod_{j=1}^{h}[a_j, b_j]$ of h commutators in $\mathcal{M}_2$ gives an Σ_2-bundle over the surface Σ_h^1 of genus h with one boundary component. The mapping classes a_j and b_j specify the monodromy along the obvious free generating system $\langle \alpha_1, \beta_1, \ldots, \alpha_h, \beta_h \rangle$ of $\pi_1(\Sigma_h^1)$.

An expression $\prod_{i=1}^n t_{v_i} \in \mathcal{M}_2$ provides a genus-2 Lefschetz fibration $X \to D^2$ over the disk with fiber Σ_2. If $\prod_{i=1}^n t_{v_i} = 1 \in \mathcal{M}_2$ then the fibration closes up to a fibration over the sphere S^2 and the closed-up manifold is uniquely determined by the word $\prod_{i=1}^n t_{v_i}$. By combining the above two constructions, a word

$$W = \prod_{j=1}^{h} [a_j, b_j] \prod_{i=1}^{n} t_{v_i}$$

gives a genus-2 Lefschetz fibration over $\Sigma_h - D^2$ and if $W = 1 \in \mathcal{M}_2$ we get a genus-2 Lefschetz fibration $X \to \Sigma_h$.

PROPOSITION 1 ([8]). *For all $h \geq 0$, $5 \leq N(2, h)$.*

PROOF. It is well known that $\mathcal{M}_2/[\mathcal{M}_2, \mathcal{M}_2]$ is generated by the class of a Dehn twist about a nonseparating curve and is equal to $\mathbf{Z}_{10}$. Since every right handed Dehn twist about a nontrivial separating curve is the product of 12 right handed Dehn twists about nonseparating curves, its image under the map $\mathcal{M}_2 \to \mathcal{M}_2/[\mathcal{M}_2, \mathcal{M}_2] \cong \mathbf{Z}_{10}$ is 2. Hence, if a product of right handed Dehn twists about n nonseparating and s separating curves is trivial (or equal to a product of commutators), then

$$n + 2s \equiv 0 \pmod{10}.$$

Therefore, there are at least 5 singular fibers in a genus-2 Lefschetz fibration over Σ_h. $\square$

2.2. Relations in the mapping class group.

We recall the following basic facts.

LEMMA 1. *Let c be a simple closed curve on Σ_2, let ρ be a self-diffeomorphism of Σ_g. Then $\rho t_c \rho^{-1} = t_{\rho(c)}$.*

LEMMA 2. *Let c and d be two simple closed curves on Σ_2.*
(a) *If c is disjoint from d, then $t_c t_d = t_d t_c$.*
(b) *If c intersects d in one point transversely, then $t_c t_d t_c = t_d t_c t_d$.*

The proof of the following lemma is based on an idea of Korkmaz [8].

LEMMA 3. *Let a, b, c, d denote nonseparating curves on Σ_2 such that a is disjoint from b, and c is disjoint from d. Morever, we assume that a is not isotopic to b, and c is not isotopic to d. Then $t_a^n t_b^{-n} t_c^n t_d^{-n}$ ($n \in \mathbf{Z}$) is a commutator.*

PROOF. Since $\Sigma_2 - \{a \cup b\}$ and $\Sigma_2 - \{d \cup c\}$ are connected surfaces, there exists an orientation-preserving diffeomorphism ρ_1 of Σ_2 such that $\rho_1(a) = d$ and $\rho_1(b) = c$. Then

$$t_a^n t_b^{-n} t_c^n t_d^{-n} = t_a^n t_b^{-n} t_{\rho_1(b)}^n t_{\rho_1(a)}^{-n}$$

$$= t_a^n t_b^{-n} \rho_1 t_b^n t_a^{-n} \rho_1^{-1}$$
$$= [t_a^n t_b^{-n}, \rho_1]\,. \qquad \square$$

LEMMA 4 ([9]). *Let a, b, c, d denote nonseparating curves on Σ_2 such that a intersects d transversely at one point and b intersects c transversely at one point. Then $t_a t_b t_c^{-1} t_d^{-1}$ is a commutator.*

PROOF. Since $t_d(b)$ and $t_d(c)$ intersect transversely at one point, by the classification of surfaces there exists an orientation-preserving diffeomorphism ρ_2 of Σ_2 such that $\rho_2(a) = t_d(c)$ and $\rho_2(d) = t_d(b)$. Then

$$\begin{aligned} t_a t_b t_c^{-1} t_d^{-1} &= t_a t_d^{-1}(t_d t_b t_d^{-1})(t_d t_c^{-1} t_d^{-1}) \\ &= t_a t_d^{-1} t_{t_d(b)} t_{t_d(c)}^{-1} \\ &= t_a t_d^{-1} t_{\rho_2(d)} t_{\rho_2(a)}^{-1} \\ &= t_a t_d^{-1} \rho_2 t_d t_a^{-1} \rho_2^{-1} \\ &= [t_a t_d^{-1}, \rho_2]\,. \end{aligned} \qquad \square$$

The following two relations in $\mathcal{M}_2$ are also well-known. The first one is the *lantern relation*. This relation was discovered by Dehn (see [3]) and was rediscovered by Johnson (see [6]). Let a_1, a_2, a_3, a_5, c and x be simple closed curves on Σ_2 in Figure 2. $t_{a_1}, t_{a_3}, t_{a_5}, t_c$ and t_x satisfy the *lantern relation*

$$t_{a_1}^2 t_{a_5}^2 = t_{a_3} t_c t_x\,.$$

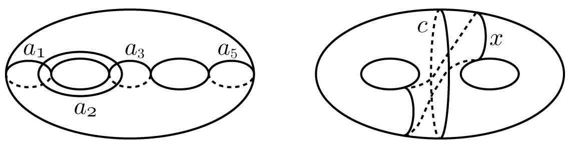

FIGURE 1. The curves a_1, a_2, a_3, a_5, c and x

The second relation is the *chain relation*. $t_{a_1}, t_{a_2}, t_{a_3}$ and t_{a_5} satisfy the *chain relation*

$$(t_{a_3} t_{a_2} t_{a_1})^4 = t_{a_5}^2\,.$$

The proof of Theorem 1 rests on the following lemmas.

LEMMA 5. $t_c t_{t_x(c)} \cdots t_{t_x^{i-1}(c)} = t_{a_1}^{2i} t_{a_5}^{2i} t_{a_3}^{-i} t_x^{-i}$ $(i = 1, \dots, 5)$.

PROOF. Since by the lantern relation we have $t_c t_x = t_{a_1}^2 t_{a_5}^2 t_{a_3}^{-1}$,

$$t_c t_{t_x(c)} \cdots t_{t_x^{i-1}(c)} t_x^i = t_c (t_x t_c t_x^{-1}) \cdots (t_x^{i-1} t_c t_x^{-i+1}) t_x^i$$

$$
\begin{aligned}
&= (t_c t_x)^i \\
&= t_{a_1}^{2i} t_{a_5}^{2i} t_{a_3}^{-i} .
\end{aligned}
$$

Therefore, $t_c t_{t_x(c)} \cdots t_{t_x^{i-1}(c)} = t_{a_1}^{2i} t_{a_5}^{2i} t_{a_3}^{-i} t_x^{-i}$. □

LEMMA 6. *$t_{a_1}^4 t_{a_3}^4 = t_b^{-1} t_{b'}^{-1}[\ ,\ ]$, where b is $(t_{a_1} t_{a_3})^3(a_2)$, b' is $(t_{a_1} t_{a_3})^2(a_2)$, and $[\ ,\ ]$ is a commutator.*

PROOF. By the chain relation we have

$$
\begin{aligned}
t_{a_5}^2 &= t_{a_3} t_{a_2} t_{a_1} t_{a_3} t_{a_2} t_{a_1} t_{a_3} t_{a_2} t_{a_1} t_{a_3} t_{a_2} t_{a_1} \\
&= t_{a_3} t_{a_2} \{(t_{a_1} t_{a_3}) t_{a_2} (t_{a_1} t_{a_3})^{-1}\}\{(t_{a_1} t_{a_3})^2 t_{a_2} (t_{a_1} t_{a_3})^{-2}\}\{(t_{a_1} t_{a_3})^3 t_{a_2} (t_{a_1} t_{a_3})^{-3}\}(t_{a_1} t_{a_3})^3 t_{a_1} \\
&= t_{a_3} t_{a_2} t_{t_{a_1} t_{a_3}(a_2)} t_{(t_{a_1} t_{a_3})^2(a_2)} t_{(t_{a_1} t_{a_3})^3(a_2)} (t_{a_1} t_{a_3})^3 t_{a_1} \\
&= t_{a_3} t_{a_2} t_{t_{a_1} t_{a_3}(a_2)} t_{b'} t_b (t_{a_1} t_{a_3})^3 t_{a_1} .
\end{aligned}
$$

Since a_1, a_3 and a_5 are disjoint from each other, we have

$$
\begin{aligned}
t_{a_5}^2 &= t_{a_2} t_{t_{a_1} t_{a_3}(a_2)} t_{b'} t_b (t_{a_1} t_{a_3})^3 t_{a_1} t_{a_3} \\
&= t_{a_2} t_{t_{a_1} t_{a_3}(a_2)} t_{b'} t_b t_{a_1}^4 t_{a_3}^4 .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
t_{a_1}^4 t_{a_3}^4 &= t_b^{-1} t_{b'}^{-1} t_{t_{a_1} t_{a_3}(a_2)}^{-1} t_{a_2}^{-1} t_{a_5}^2 \\
&= t_b^{-1} t_{b'}^{-1} (t_{a_5} t_{t_{a_1} t_{a_3}(a_2)}^{-1} t_{a_5} t_{a_2}^{-1}) .
\end{aligned}
$$

By lemma 3 we see that $t_{a_5} t_{t_{a_1} t_{a_3}(a_2)}^{-1} t_{a_5} t_{a_2}^{-1}$ is written as a commutator. This completes the proof of Lemma 6. □

LEMMA 7. *$t_{a_1}^5 t_{a_3}^5$ is written as a product of two commutators.*

PROOF. From Lemma 6 we have

$$
t_{a_1}^5 t_{a_3}^5 = t_{a_1} t_{a_3} t_b^{-1} t_{b'}^{-1}[\ ,\ ],
$$

where b is $(t_{a_1} t_{a_3})^3(a_2)$, b' is $(t_{a_1} t_{a_3})^2(a_2)$, and $[\ ,\ ]$ is a commutator. By lemma 4 we see that $t_{a_1} t_{a_3} t_b^{-1} t_{b'}^{-1}$ is written as a commutator. This completes the proof of Lemma 7. □

3. Proofs

THEOREM 2. *$t_c t_{t_x(c)} t_{t_x^2(c)} t_{t_x^3(c)} t_{t_x^4(c)}$ is written as a product of three commutators.*

PROOF. By Lemma 5 we have

$$\begin{aligned} t_c t_{t_x(c)} t_{t_x^2(c)} t_{t_x^3(c)} t_{t_x^4(c)} &= t_{a_1}^{10} t_{a_5}^{10} t_{a_3}^{-5} t_x^{-5} \\ &= (t_{a_1}^5 t_{a_5}^5)(t_{a_1}^5 t_{a_3}^{-5} t_{a_5}^5 t_x^{-5}) . \end{aligned}$$

By Lemma 3 we see that $t_{a_1}^5 t_{a_3}^{-5} t_{a_5}^5 t_x^{-5}$ is written as a commutator. Let ρ denote the rotation by $2\pi/3$ about the axis indicated in Figure 2. Since $\rho(a_1) = a_5$ and $\rho(a_3) = a_1$, we have

$$\begin{aligned} t_{a_1}^5 t_{a_5}^5 &= \rho t_{a_3}^5 t_{a_1}^5 \rho^{-1} \\ &= \rho t_{a_1}^5 t_{a_3}^5 \rho^{-1} . \end{aligned}$$

By Lemma 7 we see that $t_{a_1}^5 t_{a_5}^5 = \rho t_{a_1}^5 t_{a_3}^5 \rho^{-1}$ is a product of two commutators. This completes the proof of Theorem 2. □

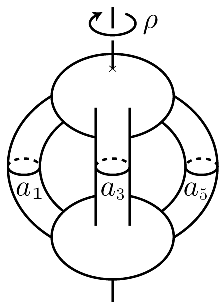

FIGURE 2. The rotation by $2\pi/3$

COROLLARY 1. *$N(2, h) = 5$ for $h \geq 3$.*

PROOF. Theorem 2 gives a genus-2 Lefschetz fibration with 5 singular fibers over Σ_3. By taking fiber sum with the trivial Lefschetz fibration, it is clear to see that for $h \geq 3$ $N(2, h) = 5$. □

THEOREM 3. *$N(2, 2) = 5$ or 6.*

PROOF. By Proposition 1 it is clear that $5 \leq N(2, 2)$. Hence, we show $N(2, 2) \leq 6$. From the proof of Proposition 1 it suffices to show that there exists a product of Dehn twists about 2 nonseparating and 4 separating curves such that it is written as a product of two commutators.

By Lemma 5 we have

$$\begin{aligned} t_c t_{t_x(c)} t_{t_x^2(c)} t_{t_x^3(c)} &= t_{a_1}^8 t_{a_5}^8 t_{a_3}^{-4} t_x^{-4} \\ &= (t_{a_1}^4 t_{a_5}^4)(t_{a_1}^4 t_{a_3}^{-4} t_{a_5}^4 t_x^{-4}) . \end{aligned}$$

By Lemma 3 we see that $t_{a_1}^4 t_{a_3}^{-4} t_{a_5}^4 t_x^{-4}$ is written as a commutator. Since $t_{a_1}^4 t_{a_5}^4 = \rho t_{a_3}^4 t_{a_1}^4 \rho^{-1} = \rho t_{a_1}^4 t_{a_3}^4 \rho^{-1}$, by Lemma 6 we have $t_{a_1}^4 t_{a_5}^4 = t_{\rho(b)}^{-1} t_{\rho(b')}^{-1} [\ ,\ ]^{\rho}$, where $[\ ,\ ]^{\rho} = \rho[\ ,\ ]\rho^{-1}$. Therefore, we see that $t_{\rho(b')} t_{\rho(b)} t_c t_{t_x(c)} t_{t_x^2(c)} t_{t_x^3(c)}$ is a product of two commutators. This completes the proof of theorem 3. $\square$

In order to prove $6 \leq N(2,1)$, we will need the following result.

THEOREM 4 ([4]). *If a genus-$g \geq 2$ Lefschetz fibration over Σ_h $(h \geq 1)$ have s separating and n nonseparating singular fibers, then*

$$s \leq 6(3g-1)(h-1) + 5n\,.$$

THEOREM 5. $N(2,1) = 6$ *or* 7.

PROOF. First, we show $6 \leq N(2,1)$. Suppose that $N(2,1) = 5$. Thus, from the proof of Proposition 1 there exists a genus-2 Lefschetz fibration with 5 separating singular fibers and no nonseparating singular fibers over T^2. However, this contradicts to Theorem 4.

We show $N(2,1) \leq 7$. From the proof of Proposition 1 it suffices to show that there exists a product of Dehn twists about 4 nonseparating and 3 separating curves such that it is written as a commutator. By the chain relation

$$\begin{aligned}
t_{a_5}^2 &= t_{a_3} t_{a_2} \underline{t_{a_1} t_{a_3}} t_{a_2} t_{a_1} t_{a_3} t_{a_2} \underline{t_{a_1} t_{a_3}} t_{a_2} t_{a_1} \\
&= t_{a_3} t_{a_2} t_{a_3} \underline{t_{a_1} t_{a_2} t_{a_1}} t_{a_3} t_{a_2} t_{a_3} t_{a_1} t_{a_2} t_{a_1} \\
&= t_{a_3} \underline{t_{a_2} t_{a_3} t_{a_2}} t_{a_1} \underline{t_{a_2} t_{a_3} t_{a_2}} t_{a_3} t_{a_1} t_{a_2} t_{a_1} \\
&= t_{a_3} t_{a_3} t_{a_2} t_{a_3} t_{a_1} t_{a_3} t_{a_2} t_{a_3} t_{a_3} t_{a_1} t_{a_2} t_{a_1} \\
&= t_{a_3}^6 (t_{a_3}^{-4} t_{a_2} t_{a_3}^4)(t_{a_1} t_{a_3}^{-2} t_{a_2} t_{a_3}^2 t_{a_1}^{-1})(t_{a_1}^2 t_{a_2} t_{a_1}^{-2}) t_{a_1}^3 \\
&= t_{a_3}^6 t_{t_{a_3}^{-4}(a_2)} t_{t_{a_1} t_{a_3}^{-2}(a_2)} t_{t_{a_1}^2(a_2)} t_{a_1}^3\,.
\end{aligned}$$

Since $t_{t_{a_3}^{-4}(a_2)} t_{t_{a_1} t_{a_3}^{-2}(a_2)} t_{t_{a_1}^2(a_2)} = t_{a_3}^{-6} t_{a_1}^{-3} t_{a_5}^2$, we have

$$t_{t_{a_3}^{-4}(a_2)} t_{t_{a_1} t_{a_3}^{-2}(a_2)} t_{t_{a_1}^2(a_2)} t_{a_5} = t_{a_3}^{-6} t_{a_1}^{-3} t_{a_5}^3\,.$$

Therefore, by $\rho(a_1) = a_5$, $\rho(a_3) = a_1$ and $\rho(a_5) = a_3$,

$$\begin{aligned}
t_{\rho t_{a_3}^{-4}(a_2)} t_{\rho t_{a_1} t_{a_3}^{-2}(a_2)} t_{\rho t_{a_1}^2(a_2)} t_{\rho(a_5)} &= \rho t_{t_{a_3}^{-4}(a_2)} t_{t_{a_1} t_{a_3}^{-2}(a_2)} t_{t_{a_1}^2(a_2)} t_{a_5} \rho^{-1} \\
&= \rho t_{a_3}^{-6} t_{a_1}^{-3} t_{a_5}^3 \rho^{-1} \\
&= t_{a_1}^{-6} t_{a_5}^{-3} t_{a_3}^3\,.
\end{aligned}$$

Since a_5 and x are nonseparating curves, there exists $\phi \in \mathcal{M}_2$ such that $\phi(a_5) = x$. By Lemma 5 we have

$$(t_{\rho t_{a_3}^{-4}(a_2)} t_{\rho t_{a_1} t_{a_3}^{-2}(a_2)} t_{\rho t_{a_1}^2(a_2)} t_{\rho(a_5)})(t_c t_{t_x(c)} t_{t_x^2(c)}) = (t_{a_1}^{-6} t_{a_5}^{-3} t_{a_3}^3)(t_{a_1}^6 t_{a_5}^6 t_{a_3}^{-3} t_x^{-3})$$

$$
\begin{aligned}
&= t_{a_5}^3 t_x^{-3} \\
&= t_{a_5}^3 \phi t_{a_5}^{-3} \phi^{-1} \\
&= [t_{a_5}^3, \phi] .
\end{aligned}
$$

This completes the proof of Theorem 5 □

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